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NOTES AND COMMENTS

A NOTE ON DISTRIBUTIONAL PROPERTIES OF THE JÖRESKOG-SÖRBOM FIT INDICES

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In introducing the LISREL model for systems of linear structural equations, Jöreskog and Sörbom proposed two goodness-of-fit indices, GFI and AGFI. Their asymptotic distributions and some statistical properties are discussed.

Key words: Jöreskog-Sörbom fit indices, central and noncentral chi-square variables, structured covariance matrix, biasedness of GFI and AGFI, fit index for GLS estimate.

In this note, we derive the asymptotic distribution of a goodness-of-fit index, GFI, proposed by Jöreskog and Sörbom (1981). Jöreskog and Sörbom (1981, 1982, 1988) discussed only its properties and limitations in the implementation of the LISREL program, but its sampling distribution could not be derived "even under idealized assumptions" (Jöreskog & Sörbom, 1982, p. 409). Monte Carlo experiments were conducted by various authors (cited in Marsh, Balla, & McDonald, 1988) for studying the effect of sample size on this fit index. Here we attempt to discuss some distributional properties of GFI in the context of a wide class of both linearly and nonlinearly structured covariance matrices.

Let the population covariance matrix $\mathbf{\Sigma}_{p \times p}$ (the null model) be structured in the sense that the $p(p + 1)/2$ nonduplicated elements of the positive definite (pd) matrix $\mathbf{\Sigma}$ depend only on k-component parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)'$, $k < p(p+1)/2$. We assume that the structure is identifiable, that is, equality of two matrices $\mathbf{\Sigma}(\theta^*)$ and $\mathbf{\Sigma}(\theta^{**})$ implies $\boldsymbol{\theta}^* = \boldsymbol{\theta}^{**}$. Under the usual multinormal assumptions, let the maximum likelihood (ML) estimate of $\boldsymbol{\theta}$ be denoted by $\hat{\boldsymbol{\theta}}$. Consequently, $\hat{\mathbf{\Sigma}} = \mathbf{\Sigma}(\hat{\theta})$ is called the ML estimate of $\mathbf{\Sigma}(\theta)$ under the null model. Note that S, the sample covariance matrix based on sample size N, is the ML estimate of $\mathbf{\Sigma}$ when we do not impose any restrictions on its elements. We now define the following

$$\mathbf{Q} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{S}\mathbf{\Sigma}^{-1}; \tag{1}$$

$$\hat{\mathbf{Q}} = \hat{\mathbf{\Sigma}}^{-1} - \hat{\mathbf{\Sigma}}^{-1}\mathbf{S}\hat{\mathbf{\Sigma}}^{-1}, \tag{2}$$

and

$$\hat{f} = \tfrac{1}{2} \operatorname{tr} (\hat{\mathbf{\Sigma}}^{-1}\mathbf{S} - \mathbf{I})^2$$
$$= \tfrac{1}{2} \operatorname{tr} [(\hat{\mathbf{\Sigma}} - \mathbf{S})\hat{\mathbf{Q}}]. \tag{3}$$

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By utilizing Browne's (1974) Proposition 7, it may be shown that under the null model, $N\hat{f}$ is asymptotically distributed as a central χ^2-variable with parameter ν, where $\nu = (p(p+1)/2 - k)$ degrees of freedom (df).

When $\mathbf{\Sigma}$ satisfies some general conditions as suggested by Browne (1974, Proposition 8), its ML estimate $\hat{\mathbf{\Sigma}}$ would satisfy the condition

$$\text{tr}\,(\hat{\mathbf{Q}}\hat{\mathbf{\Sigma}}) = 0, \tag{4}$$

or consequently,

$$\text{tr}\,(\hat{\mathbf{\Sigma}}^{-1}\mathbf{S}) = p. \tag{5}$$

We shall designate such a $\mathbf{\Sigma}$ satisfying (4) or (5) as belonging to a wide class of covariance structures, say $\mathscr{C}$. (According to the Associate Editor, this class $\mathscr{C}$ satisfies the following property: If $\boldsymbol{\theta}^*$ is any admissable value of the parameter vector $\boldsymbol{\theta}$ and α is any positive scalar, then there exits a $\boldsymbol{\theta}^{**}$ such that $\mathbf{\Sigma}(\boldsymbol{\theta}^{**}) = \alpha\mathbf{\Sigma}(\boldsymbol{\theta}^*)$).

The afore-said wide class includes most models used in the analysis of covariance structures that would be employed in practice. It includes all models given in Mukherjee (1970, 1981) as well as all particular examples of the LISREL model considered in Jöreskog (1978). Recalling (3), when $\mathbf{\Sigma} \in \mathscr{C}$, we write

$$\hat{f}_{\mathscr{C}} = -\tfrac{1}{2}\,\text{tr}\,(\hat{\mathbf{Q}}\mathbf{S}), \tag{6}$$

so that the statistic $N\hat{f}_{\mathscr{C}}$ is asymptotically a central χ^2-variable with ν df (see Maiti & Mukherjee, 1988).

Now to assess the overall fit of some hypothesized structure of $\mathbf{\Sigma}$ to $\mathbf{S}$, Jöreskog and Sörbom (1981) have proposed the following goodness-of-fit index, GFI, when $\mathbf{\Sigma}$ is estimated by ML procedure:

$$g = 1 - \frac{\text{tr}\,(\hat{\mathbf{\Sigma}}^{-1}\mathbf{S} - \mathbf{I}_p)^2}{\text{tr}\,(\hat{\mathbf{\Sigma}}^{-1}\mathbf{S})^2}. \tag{7}$$

They also proposed another index, AGFI, adjusted for the degrees of freedom (ν) of the hypothesized structure of $\mathbf{\Sigma}$, given by

$$a = 1 - \frac{p(p+1)}{2\nu}(1-g). \tag{8}$$

Since a and g are linearly related, it is sufficient to obtain the distribution of g only, and we shall do so when $\mathbf{\Sigma} \in \mathscr{C}$. Once the sampling distribution of GFI is known, we will have a standard with which to compare this index for analysis of a wide class of covariance structures.

Because of (4) or (5), (7) reduces to the simpler form

$$g_{\mathscr{C}} = \frac{p}{p - \text{tr}\,(\hat{\mathbf{Q}}\mathbf{S})} = \frac{p}{p + 2\hat{f}_{\mathscr{C}}}. \tag{9}$$

Hence, by recalling the asymptotic distribution of $N\hat{f}$, the asymptotic distribution of $g_{\mathscr{C}}$ is given by

$$\text{Prob}\,[g_{\mathscr{C}} \le g_0] = \text{Prob}\,[\chi^2_\nu \ge \chi^2_0], \tag{10}$$

where χ^2_ν denotes a central χ^2-variable with ν df, and

$$\chi_0^2 = \frac{Np}{2}\left(\frac{1}{g_0} - 1\right). \tag{11}$$

Similarly, for AGFI (see (8)),

$$\text{Prob}\,[a_{\mathscr{C}} \leq a_0] = \text{Prob}\,[\chi_\nu^2 \geq \chi_0^{*2}], \tag{12}$$

where

$$\chi_0^{*2} = \frac{Np}{2}\left[\frac{p(p+1)}{2\nu(1-a_0)} - 1\right]. \tag{13}$$

Thus, both the GFI and AGFI are exact functions of the χ^2-variable.

Involvement of N in (11) or (13) provides a clear insight into the dependence of the distribution of GFI or AGFI on sample size. The fact that GFI is not independent of sample size was previously noted by a number of authors such as Bearden, Sharma, and Teel (1982), Hoelter (1983), Anderson and Gerbing (1984), as well as Marsh, Balla, and McDonald (1988) on the basis of different Monte Carlo experiments. Although Jöreskog and Sörbom (1982, p. 408) stated that "unlike χ^2, GFI is independent of the sample size", they meant that GFI does not depend upon N in its calculation. The sampling distribution of both GFI and AGFI must, of course, depend on N since the distribution of any function of sample moments (covariances in this case) is dependent on sample size. This distribution was not then known but has now been obtained for covariance structures in class $\mathscr{C}$.

Since the fit indices are usually intended for use in situations where a model serves as an approximation, not necessarily when it is correct and the null hypothesis holds, we are also interested in the non-null distribution of $g_{\mathscr{C}}$. When the null model is not true, let $\boldsymbol{\Sigma}_0$ be the true population covariance matrix and let $\hat{\boldsymbol{\Sigma}}_0$ be the reproduced covariance matrix obtained if the model were fitted to $\boldsymbol{\Sigma}_0$ by *maximum likelihood*. Since the model is not assumed to fit, there is no assumption that $\hat{\boldsymbol{\Sigma}}_0 = \boldsymbol{\Sigma}_0$.

Let

$$f_0 = -\tfrac{1}{2}\,\text{tr}\,[\mathbf{U}(\hat{\boldsymbol{\Sigma}}_0 - \boldsymbol{\Sigma}_0)], \tag{14}$$

where

$$\mathbf{U} = \hat{\boldsymbol{\Sigma}}_0^{-1} - \hat{\boldsymbol{\Sigma}}_0^{-1}\boldsymbol{\Sigma}_0\hat{\boldsymbol{\Sigma}}_0^{-1}. \tag{15}$$

Then, if N is large, the distribution of $N\hat{f}$ is approximately noncentral chi-square with ν df and noncentrality parameter Nf_0 (see Browne, 1984, Corollary 4.1). In this connection, we refer to the recent results of McDonald (1989) as well as McDonald and Marsh (in press) showing how a number of goodness-of-fit indices may be expressed as functions of their corresponding noncentrality parameters. Under nonnull situations, $\mathrm{E}(\hat{f}) \approx f_0 + \nu/N$, and $\mathrm{Var}\,(\hat{f}) \approx 2/N\,(2f_0 + \nu/N)$ (Kendall & Stuart, 1968, p. 229).

It is worthwhile to examine here the biasedness of the sample values of GFI for $\boldsymbol{\Sigma} \in \mathscr{C}$. Since $\mathrm{E}(\hat{f}_{\mathscr{C}}) \approx f_0 + \nu/N$,

$$\mathrm{E}(g_{\mathscr{C}}) \approx \frac{p}{p + 2f_0 + \dfrac{2}{N}\nu}, \tag{16}$$

while the population value of GFI is given by

$$\gamma_{0\mathscr{C}} = \frac{p}{p + 2f_0}, \tag{17}$$

so that

$$\mathrm{E}(g_{\mathscr{C}}) \le \gamma_{0\mathscr{C}}. \tag{18}$$

Thus, $g_{\mathscr{C}}$ is a consistent but biased estimator of $\gamma_{0\mathscr{C}}$. Obviously, $\gamma_{0\mathscr{C}}$ is low for a poor fit of the model, and it increases toward 1 as goodness-of-fit improves. A bias-corrected estimator of $\gamma_{0\mathscr{C}}$ is given by

$$\tilde{g}_{\mathscr{C}} = \frac{p}{p + 2\hat{f}_{\mathscr{C}} - \dfrac{2\nu}{N}} = \frac{p}{p - \mathrm{tr}\,(\hat{\mathbf{Q}}\mathbf{S}) - \dfrac{2\nu}{N}}$$

$$= 1 - \frac{\mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{S} - \mathbf{I}_p)^2 - \dfrac{2\nu}{N}}{\mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{S})^2 - \dfrac{2\nu}{N}}. \tag{19}$$

(We note that Steiger and Lind, 1980, introduced the population noncentrality index as a measure of badness-of-fit. More recently, Steiger, 1989, Equations (51) and (47), has referred to $\gamma_{0\mathscr{C}}$ as the population gamma index Γ, of which g is the asymptotic ML estimator.)

Because $\mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{S})^2 > [\mathrm{tr}^2\,(\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{S})/p]$, the biased estimator $g_{\mathscr{C}}$ lies in the interval $[1/p,\ 1]$. But the (approximately) unbiased estimator $\tilde{g}_{\mathscr{C}}$ as obtained from (19) can assume values outside this interval when $\mathrm{tr}\,(\hat{\mathbf{Q}}\mathbf{S})$ is either greater than $-2\nu/N$, or less than $-[2\nu/N + p(p-1)]$. However, for practical purposes, the approximately unbiased estimate (19) can be used as a measure of goodness-of-fit (see McDonald, 1989).

As in the case of GFI, we observe that the (approximately) unbiased estimator, $\tilde{a}_{\mathscr{C}}$, obtainable from the biased estimator AGFI, $a_{\mathscr{C}}$, may even assume (apart from values greater than unity) negative values when $\mathrm{tr}\,(\hat{\mathbf{Q}}\mathbf{S})$ is less than $-[2\nu/N + [2\nu p/p(p+1) - 2\nu]$, although the population value of AGFI never assumes negative values.

The GFI or AGFI is analogous (see Tanaka & Huba, 1985) to the squared multiple correlation coefficient (SMC). It is interesting that the sample GFI (or AGFI) is a *negatively biased* estimator of the population GFI (or AGFI), whereas the sample SMC is a *positively biased* estimator of the population SMC (see Kendall & Stuart, 1967, p. 341).

When $\hat{f}_{\mathscr{C}}$ is small, an approximation of $g_{\mathscr{C}}$ can be given by a first-order Taylor series as

$$g_{\mathscr{C}} \approx 1 - \frac{\hat{f}_{\mathscr{C}}}{\dfrac{p}{2}}, \tag{20}$$

which is greater than $1 - \hat{f}_{\mathscr{C}}$ so that GFI ($g_{\mathscr{C}}$) gives a more optimistic impression of the closeness of model fit than does the residual quadratic form

$$\hat{f}_{\mathscr{C}} = -\tfrac{1}{2}\,\mathrm{tr}\,(\hat{\mathbf{Q}}\mathbf{S}) = \tfrac{1}{2}\,\mathrm{tr}\,(\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{S} - \mathbf{I}_p)^2. \tag{21}$$

The first-order approximation (20) of $g_{\mathscr{C}}$ is comparable to the fit index (g^*) for the generalized least squares (GLS) estimate ($\hat{\mathbf{\Sigma}}^*$) of $\mathbf{\Sigma}$ that is given by (see Tanaka & Huba, 1985):

$$g^* = 1 - \frac{\hat{f}^*}{\frac{p}{2}}, \tag{22}$$

where

$$\hat{f}^* = \tfrac{1}{2} \operatorname{tr} (\mathbf{I}_p - \mathbf{S}^{-1}\hat{\mathbf{\Sigma}}^*)^2, \tag{23}$$

and $\hat{\mathbf{\Sigma}}^*$ is obtained by minimizing the GLS fit function

$$f^* = \tfrac{1}{2} \operatorname{tr} (\mathbf{I}_p - \mathbf{S}^{-1}\mathbf{\Sigma})^2. \tag{24}$$

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