FRACTIONAL REPLICATION IN ASYMMETRICAL FACTORIAL DESIGNS AND PARTIALLY BALANCED ARRAYS

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1. Introduction

Hypercubes of strength "d" were defined by Rao (1946). Later Rao (1947) extended the definition of hypercubes of strength d to cover a wider class of arrays called orthogonal arrays. Rao (1946) has shown how these hypercubes may be used in the construction of a system of confounded designs which accommodates maximum number of factors and preserves main effects and interactions up to the order $(d-1)$ provided a hypercube of strength "d" exists in the case of a symmetrical factorial experiment. It has also been shown there that hypercubes of strength 2 supply balanced confounded designs for asymmetrical factorial experiments defined by Nair and Rao (1941, 1942a, 1942b) and later (Nair and Rao, 1948) treated in detail by the same authors.

Plackett and Burman (1946) constructed a class of designs called multifactorial designs which accommodate maximum number of factors and preserve only the main effects.

Rao (1947) has shown how orthogonal arrays of strength d may be made to yield multifactorial designs which will allow estimation of main effects and interactions of order upto k $(d > k)$ when higher order interactions are absent. He has also used orthogonal arrays in the construction of block designs for symmetrical factorial experiments involving only a sub-set of treatment combinations and preserving main effects and interaction up to a given order, when higher order interactions are absent.

The existence of block designs allows construction of fractional replication in the case of symmetrical factorial experiments. The method of actual construction of fractional replicates using orthogonal arrays has been treated fully by Rao (1950).

In this paper, the problem of construction of arrangements of fractional replication in asymmetrical factorial designs has been considered. It has been shown that orthogonal arrays may be used to obtain fractional replications in some of the important asymmetrical factorial experiments which find ready application in actual fields of research like industrial experimentation. These fractional replicate designs lead to a considerable saving in the number of experiments to be conducted or observations to be made. Method of construction of these designs are flexible to a certain extent to suit the needs of the varying nature of experimental enquiries. Experimental situations which have actually occurred in practice in the fields of industrial experimentation are considered. A list of useful designs has been supplied.

A new class of arrays called partially balanced arrays has been defined. The combinatorial problem and analysis of designs derived from these partially balanced arrays are given. These designs economise considerably the amount of experimental material to be used in the experiment. These will be found useful in those situations where the most economic design does not exist.

2. Orthogonal arrays and fractional replication

Let A be a matrix with m rows and N columns, elements of the matrix being the integers 0, 1, 2,..., $s-1$. If amongst the N columns of any of $\binom{m}{d}$ d-row submatrices from A, all s^d d-tuples occur equal number of times, say λ times, then A is an orthogonal array with N assemblies, m constraints, strength d and index λ, symbolically denoted by (N, m, s, d, λ). Then it follows that $N = \lambda s^d$. If $N = s^t$ then $\lambda = s^{t-d}$ and such an array is called a hypercube. The following two general inequalities due to Rao (1947) connect the parameters

$$N-1 \geqslant \binom{m}{1}(s-1)+\binom{m}{2}(s-1)^2+\cdots+\binom{m}{\frac{1}{2}d}(s-1)^{\frac{1}{2}d}$$

when d is even and

$$N-1 \geqslant \binom{m}{1}(s-1)+\cdots+\binom{m}{\frac{1}{2}(d-1)}(s-1)^{\frac{1}{2}(d-1)}+\binom{m-1}{\frac{1}{2}(d-1)}(s-1)^{\frac{1}{2}(d+1)}$$

when d is odd.

For a symmetrical factorial experiment involving m factors each at s levels, N columns of an array are identifiable with N treatment combinations or assemblies, m rows stand for m factors and an entry stands for the level of a factor against which it is shown. These N assemblies then form a sub-set of s^m possible assemblies of the complete factorial experiment. From a complete factorial experiment all main effects and interactions of all orders upto $(m-1)$ are estimable but these take up all the s^m-1 degrees of freedom leaving none for error. In such situations, one may use estimates of error variance from previous experiences or one may derive a valid estimate of error variance assuming certain higher order interactions to be absent. Sometimes, it is not possible to set up even a single complete replication of a factorial experiment. To get over this difficulty Finney (1945) introduced fractionally replicated designs which using only a sub-set (properly chosen) of s^m assemblies provide estimates of main effects and lower order interactions on the assumption that higher order interactions are absent.

Rao (1947) has proved that a sub-set of N assemblies forming an orthogonal array $(N, m, s, d+k-1, \lambda)$ yields a fractionally replicated design from which :

(i) all the main effects and interactions upto order $(k-1)$ can be measured when interactions of order equal to and greater than $d-1$ are absent,

(ii) the expressions for main effects and interactions are simply obtained from the usual definitions by retaining only the treatment combinations present in the array and these expressions belonging to different contrasts are orthogonal. Later Rao (1950) gives an elegant method of construction of fractional factorial experiments in blocks from orthogonal arrays when s is a prime or a prime power.

3. Asymmetrical factorial experiment

An experiment involving m factors $F_1, F_2, \ldots, F_m$ occurring at $s_1, s_2, \ldots, s_m$ levels respectively is called an asymmetrical factorial experiment, provided not all s_i's are equal.

But the situation that occurs most often in practice is that there are g groups of factors—there being m_i factors in the i-th group, each occuring at s_i levels ($i = 1, 2, \ldots, g$; $\sum_{i=1}^{g} m_i = m$). So the experiment $s_1^{m_1} \times s_2^{m_2} \times \cdots \times s_g^{m_g}$ may be regarded as a compound of g symmetrical factorial experiments. The problem of finding a suitable sub-set of the assemblies of the complete experiment, which preserves interactions upto desired order is important here for the same reasons as in a symmetrical factorial experiment and besides, it has an added interest because of its general nature.

A solution to the above problem is provided by the following.

Theorem 1: *If orthogonal arrays* $(N_i, m_i, s_i, d_i+k_i-1, \lambda_i)$ $i = 1, 2, \ldots, g$ *exist, then there exists an array which yields a fractional replicate design in* $\prod_{i=1}^{g} N_i = \prod_{i=1}^{g} \lambda_i s_i^{d_i+k_i-1}$ *assemblies of the asymmetrical factorial experiment* $s_1^{m_1} \times \cdots \times s_g^{m_g}$. *Further, if i-th orthogonal array* ($i = 1, 2, \ldots, g$) *preserves all main effects and interactions of order upto* (k_i-1) *on the assumption that interactions of order* (d_i-1) $(d_i > k_i)$ *and higher are absent, then in the derived array, all main effects and interactions involving* $r = \sum_{i=1}^{g} r_i$ *factors* $(0 < r \leqslant gk, 0 \leqslant r_i < k)$ *become measurable, where* r_1 *factors are chosen from the first group of* m_1 *factors,* r_2 *factors from the second group and so on.*

Proof: The theorem will be proved for the case $g = 2$. The extension for any integer $g > 2$ is almost immediate. Consider the two orthogonal arrays $(N_1, m_1, s_1, d_1+k_1-1, \lambda_1)$, $(N_2, m_2, s_2, d_2+k_2-1, \lambda_2)$. A column is taken from the first array and just below it is put a column from the second array. As there are N_1 columns in the first array and N_2 columns in the second, the above method of combining the columns of the two arrays will generate $N_1 N_2$ columns with no combination repeated and each of these new columns will have (m_1+m_2) rows. In this new array, any combination of levels of t_1 factors of the first group and t_2 factors from the second group will be repeated in $\lambda_1 s_1^{d_1+k_1-1-t_1}$, $\lambda_2 s_2^{d_2+k_2-1-t_2}$ columns $\{t_1 \leqslant d_1+k_1-1;$ $t_2 \leqslant d_2+k_2-1\}$.

Let $F_1, F_2, \ldots, F_{m_1}$ represent the m_1 factors of the first group and $G_1, G_2, \ldots, G_{m_2}$ represent the m_2 factors of the second group. Defining symbolically

$$F_i^a = m_{0a} F_{i0} + \ldots + m_{s_1-1a} F_{i s_1-1} \quad (i = 1, 2, \ldots, m_1)$$

$$G_j^a = l_{0a} G_{j0} + \ldots + l_{s_2-1a} G_{j s_2-1} \quad (j = 1, 2, \ldots, m_2)$$

$$\Sigma\, m_{ra} = 0 \quad \text{when} \quad a \neq 0$$

$$\Sigma\, m_{ra} m_{rb} = 0 \quad \text{when} \quad a \neq b$$

$$m_{ra} = 1 \quad \text{for all } r \quad \text{when} \quad a = 0$$

$$\Sigma\, l_{u\alpha} = 0 \quad \text{when} \quad \alpha \neq 0$$

$$\Sigma\, l_{u\alpha} l_{u\beta} = 0 \quad \text{when} \quad \alpha \neq \beta$$

$$l_{u\alpha} = 1 \quad \text{for all } u \quad \text{when} \quad \alpha = 0.$$

$F_{i0}, F_{i1}, \ldots, F_{i s_1-1}$ and $G_{j0}, G_{j1}, \ldots, G_{j s_2-1}$ representing the levels of F_i and G_j respectively. Then the symbolic product $F_1^a F_2^b \ldots F_{m_1}^k G_1^\alpha G_2^\beta \ldots G_{m_2}^\rho$ may be taken to represent the interaction $[a, b, \ldots; \alpha, \beta, \ldots]$ of the factors for which the values are not zero. The expression obtained from $[a, b, \ldots; \alpha, \beta, \ldots]$ by retaining only the assemblies occuring in the derived array may be denoted by $\{a, b, \ldots; \alpha, \beta, \ldots\}$ and a set of necessary and sufficient conditions (Rao, 1947) that this will measure the corresponding interaction is that it is not orthogonal to $[a, b, \ldots; \alpha, \beta, \ldots]$ but is orthogonal to every other function of this type including the interactions which are absent.

Consider an expression $\{a_1, a_2, \ldots, a_{r_1}, 0, \ldots, 0; \alpha_1, \alpha_2, \ldots, \alpha_{r_2}, 0, \ldots, 0\}$ where $a_i \neq 0$ $(i = 1, 2, \ldots, r_1)$, $\alpha_j \neq 0$ $(j = 1, 2, \ldots, r_2)$ for $0 \leqslant r_1 \leqslant k_1$, $0 < r_2 < k_2$ and $0 < r_1 + r_2 < 2k$. This is evidently a contrast and this is not orthogonal to $[a_1, a_2, \ldots, a_{r_1}, 0, \ldots, 0; \alpha_1, \alpha_2, \ldots, \alpha_{r_2}, \ldots, 0]$.

Let in the expression $\{a, b, \ldots; \alpha, \beta, \ldots\}$ there be t_1 non-zero coordinates among the first m_1 and t_2 non zero coordinates among the second m_2 and consider the case when this has got no non-zero coordinate in common with those of $\{a_1, a_2, \ldots, a_{r_1}, 0, \ldots, 0; \alpha_1, \alpha_2, \ldots, \alpha_{r_2}, 0, \ldots, 0\}$. Then, since any assembly of a given set of $(r_1 + t_1) \leqslant d_1 + k_1 - 1$ factors from the first group will be repeated the same number of times with all the assemblies of another given set of $(r_2 + t_2) \leqslant d_2 + k_2 - 1$ factors from the second group, it follows that $\{a_1, a_2, \ldots, a_{r_1}, 0, \ldots, 0; \alpha_1, \alpha_2, \ldots, \alpha_{r_2}, 0, \ldots, 0\}$ is orthogonal to $\{a, b, \ldots; \alpha, \beta, \ldots\}$ and hence to $[a, b, \ldots; \alpha, \beta, \ldots]$. When $\{a, b, \ldots; \alpha, \beta, \ldots\}$ has some (not all) non-zero coordinates in common with those of $\{a_1, a_2, \ldots, a_{r_1}, 0, \ldots, 0; \alpha_1, \alpha_2, \ldots, \alpha_{r_2}, 0, \ldots, 0\}$ it follows from similar considerations as above, that $\{a_1, a_2, \ldots, a_{r_1}; 0, \ldots, 0; \alpha_1, \alpha_2, \ldots, \alpha_{r_2}; 0, \ldots 0\}$ is orthogonal to $[a, b, \ldots; \alpha, \beta, \ldots]$. So $\{a_1, a_2, \ldots, a_{r_1}; 0, \ldots, 0; \alpha_1, \alpha_2, \ldots, \alpha_{r_2}, 0, \ldots, 0\}$ by Rao's theorem, defines the corresponding interaction involving the factors $F_1, F_2, \ldots, F_{r_1}$ of first group and factors $G_1, G_2, \ldots, G_{r_2}$ from the second group. This establishes the theorem for the case $g = 2$.

If now, there exists an orthogonal array $(N_3, m_3, s_3, d_3+k_3-1, \lambda_3)$ then by taking a column from the derived matrix and putting below it a column from the third orthogonal array and repeating this operation so that no combination of any two columns is repeated we get an array in $N_1\, N_2\, N_3$ columns and $(m_1+m_2+m_3)$ rows. And thus taking a new orthogonal array at each stage and combining the columns of these with those of the derived matrix obtained at the preceding stage, we will finally get an array in $N_1\, N_2 \ldots N_g$ columns and $m_1+m_2+\cdots+m_g$ rows with the stated properties.

To illustrate the general theorem proved above, let us consider an

Example : Fractional Replication in $2^2.3^2 = 36$ assemblies of a $2^3 \cdot 3^4$ experiment.

It is known that the hypercubes $(2^2, 3, 2, 2,)$ and $(3^2, 4, 3, 2)$ exist and each one of them accommodates maximum number of factors and preserves main effects on the assumption that higher order interactions are absent. The arrays are

TABLE 1. ARRAY: $(2^2, 3, 2, 2)$ ARRAY: $(3^2, 4, 2, 2)$

	assemblies					assemblies								
factors	1	2	3	4	factors	1	2	3	4	5	6	7	8	9
F_1	0	1	1	0	G_1	0	0	0	1	1	1	2	2	2
F_2	0	1	0	1	G_2	0	1	2	1	2	0	2	0	1
F_3	0	0	1	1	G_3	0	2	1	1	0	2	2	1	0
					G_4	0	1	2	0	1	2	0	1	2

TABLE 2. DERIVED ARRAY: $(2^2\ 3, 2, 2)\times(3^2, 4, 2, 2)$

	assemblies											
factors	1	2	3	4	5	6	7	8	9	10	11	12
F_1	0	1	1	0	0	1	1	0	0	1	1	0
F_2	0	1	0	1	0	1	0	1	0	1	0	1
F_3	0	0	1	1	0	0	1	1	0	0	1	1
G_1	0	0	0	0	0	0	0	0	0	0	0	0
G_2	0	0	0	0	1	1	1	1	2	2	2	2
G_3	0	0	0	0	2	2	2	2	1	1	1	1
G_4	0	0	0	0	1	1	1	1	2	2	2	2

assemblies (continued)

factors	13	14	15	16	17	18	19	20	21	22	23	24
F_1	0	1	1	0	0	1	1	0	0	1	1	0
F_2	0	1	0	1	0	1	0	1	0	1	0	1
F_3	0	0	1	1	0	0	1	1	0	0	1	1
G_1	1	1	1	1	1	1	1	1	1	1	1	1
G_2	1	1	1	1	2	2	2	2	0	0	0	0
G_3	1	1	1	1	0	0	0	0	2	2	2	2
G_4	0	0	0	0	1	1	1	1	2	2	2	2

assemblies (continued)

factors	25	26	27	28	29	30	31	32	33	34	35	36
F_1	0	1	1	0	0	1	1	0	0	1	1	0
F_2	0	1	0	1	0	1	0	1	0	1	0	1
F_3	0	0	1	1	0	0	1	1	0	0	1	1
G_1	2	2	2	2	2	2	2	2	2	2	2	2
G_2	2	2	2	2	0	0	0	0	1	1	1	1
G_3	2	2	2	2	1	1	1	1	0	0	0	0
G_4	0	0	0	0	1	1	1	1	2	2	2	2

This design in 36 experimental units will allow estimation of all the main effects on the assumption that interactions involving any two members or more of the same group are absent and besides, the first order interactions involving any member of the first group and any member of the second will also become measurable.

The analysis of variance table may be set up as follows:

TABLE 3. ANALYSIS OF VARIANCE

		degrees of freedom
main effects	F	$3.(2-1) = 3$
	G	$4.(3-1) = 8$
first order interaction	FG	$3\times 4(2-1)(3-1) = 24$
total ...	...	35

Then, this design has no degrees of freedom left for estimation of error. If the first order interactions FG are absent then these will provide an estimate of error. Sometimes estimates of error variance based on previous experience may be available and these may be used in such situations. When, however, such estimates are not available and there are no *a priori* reasons to assume that interactions FG will not produce any effect we may repeat the fractional replicate designs to get an estimate of error variance.

Corollary to Theorem 1: *If $s_1, s_2, \ldots, s_g$ are primes or powers of primes, then the hypercubes $\left(s_i^{t_i}, n_i = \frac{s_i^{t_i}-1}{s_i-1}, s_i \;.\; 2\right)$ $i = 1, 2, \ldots g$ can be combined in the manner of Theorem 1 to get a fractional replicate design for the asymmetrical factorial experiment $s_1^{n_1} \times s_2^{n_2} \times \ldots \times s_g^{n_g}$ and this design will preserve (i) all main effects on the assumption that interactions involving two factors or more from the same group are absent, (ii) all interactions involving upto g factors but no two factors present in the interaction should come from the same group.*

Proof: If $s_1, s_2, \ldots, s_g$ are primes or powers of primes, then according to Rao's theorem (1946) the hypercubes$\left(s_i^{t_i}, n_i = \frac{s_i^{t_i}-1}{s_i-1}, s_i, 2\right)$exist and the i-th hypercube in $s_i^{t_i}$ assemblies accommodate the maximum number of factors

$$n_i = \frac{s_i^{t_i}-1}{s_i-1} \quad (i = 1, 2, \ldots, g).$$

Further, all the n_i main effects from the i-th hypercube are estimable on the assumption that interaction of order $d \geqslant 1$ are absent and since each one of them carries (s_i-1) degrees of freedom, these exhaust the $(s_i^{t_i}-1)$ degrees of freedom associated with the i-th hypercube in $s_i^{t_i}$ assemblies. When these hypercubes are combined in the manner of Theorem 1, an array in $s_1^{t_1} \times s_2^{t_2} \times \ldots \times s_g^{t_g}$ assemblies is obtained and it follows from Theorem 1 that this will have the properties as stated in the enunciation.

Example 1 illustrates the corollary just proved, for the case $g = 2$, $s_1 = 2$, $t_1 = 2$, $s_2 = 3$ and $t_2 = 2$.

4. Construction of some important designs

Using the theorem proved above, it is now possible to construct fractional replicate designs for asymmetrical factorial experiments from the orthogonal arrays already constructed and listed in Rao (1946, 1947), Plackett and Burman (1946), Bose and Bush (1952), Bush (1952). A list of designs for some of the important experiments likely to occur in practice is given here. In a later section, two other methods of construction of fractional replicate designs for asymmetrical factorial experiments will be described. These methods are useful only for limited values of g's and s_i's. But such designs effect a saving in the number of experimental units, otherwise required by the general method described above.

The class of experiments $2^{4\lambda-1} . s$, where λ a positive integer, requires $4\lambda.s$ assemblies for a fractional replicate design which preserves all the 4λ main effects and all $(4\lambda-1)$ first order interactions between any member of the first group and the factor with s levels. Since orthogonal arrays of strength 2 for $s = 2$ and $\lambda = 25$ are available in Plackett and Burman (1946), such designs can be easily constructed. Experiments $2^k.s$, $4\lambda-4 \leqslant k < 4\lambda-1$ also require $4\lambda s$ assemblies for a fractional replicate design.

TABLE 4. LIST OF SOME IMPORTANT FRACTIONAL REPLICATE DESIGNS FOR ASYMMETRICAL FACTORIAL EXPERIMENTS

(designs based on orthogonal arrays of strength 2)

	description of the complete factorial experiment	no. of assemblies required	orthogonal arrays which combine to give the design	nature of effects measurable
	(1)	(2)	(3)	(4)
1.	2^k $(4\lambda-5 < k \leqslant 4\lambda-1)$	$4\lambda s$	$(4\lambda k, 2, 2, \lambda) \times (s)$	all main effects and interactions involving two factors one from each group. (On the assumption interaction of order $d > 1$ within each group absent.)
2.	$2^k . 3^l$ $\begin{bmatrix} 4\lambda-5 < k \leqslant 4\lambda-1, \\ 3^{l-1} < 2l+1 \leqslant 3^l \end{bmatrix}$	$4\lambda . 3^l$	$(4\lambda, k, 2, 2, \lambda)$ $\times$ $(3^l, l, 3, 2, 3^{l-2})$	all main effects and interactions involving two factors one from each group. (On the assumption interactions of order $d > 1$ within each group absent.)
3.	$2^k . 4^m$ $\begin{bmatrix} 4\lambda-3 < k \leqslant 4\lambda-1 \\ 4^{l-1} < 3m+1 \leqslant 4^l \end{bmatrix}$	$4\lambda . 4^l$	$(4\lambda, k, 2, 2, \lambda)$ $\times$ $(4^l, m, 4, 2, 4^{l-2})$	-do-
4.	$2^k . 3^l . 4^m$ $\begin{bmatrix} 4\lambda-5 < k \leqslant 4\lambda-1 \\ 3^{l_1-1} < 2l+1 \leqslant 3^{l_1} \\ 4^{l_2-1} < 3m+1 \leqslant 4^{l_2} \end{bmatrix}$	$4\lambda . 3^{l_1} . 4^{l_2}$	$(4\lambda, k, 2, 2, \lambda)$ $\times$ $(3^{l_1}, l, 3, 2, 3^{l_1-2})$ $\times$ $(4^{l_2}, m, 4, 2, 4^{l_2-2})$	all main effects and all first order interactions and second order interactions involving not more than one from each group are estimable. (on the assumption interactions of order $d > 1$ within each group absent.)

In the above list only orthogonal arrays of strength 2, have been considered. Orthogonal arrays of strength higher than 2 may also be combined in a similar manner but the designs obtained from them will require comparatively large number of assemblies.

5. Fractional replicate designs for $s_1^n.s_2$ experiments where $s_2=s_1^t$ and s_1 is a prime or power of a prime

Fractional replicate designs for asymmetrical factorial exprimments $s_1^n . s_2$ where s_1 is a prime or power of a prime and $s_2 = s_1^t$ can be constructed in considerably reduced number of assemblies. These designs are made available by a theorem (Rao, 1950): The necssary and sufficient conditions that the s^r combinations obtained as a solution of the set of $(n-r)$ independent homogeneous equations

$$a_{i1} x_1 + \cdots + a_{in} x_n = 0 \quad i = 1, 2, \ldots, n-r \qquad \ldots \quad (5.1)$$

define an array of strength $(d+1)$ is that no equation derivable as a linear combination from the $(n-r)$ equations determining the subset contains less than $(d+2)$ non-null coefficients.

He, then, uses this theorem to construct block designs for fractional factorial experiments which preserve main effects and first order interactions on the assumption that interactions of order $d-1$ or higher are absent.

For that, he proceeds as follows: Another set of t homogeneous equations (properly chosen)

$$a_{j1} x_1 + \cdots + a_{jn} x_n = 0 \qquad j = n-r+1, \ldots, n-r+t \qquad \ldots \quad (5.2)$$

are considered together with (5.1), so that the s^{r-t} assemblies obtained as solutions of this set of $(n-r+t)$ equations will define an array of strength 2. This set of s^{r-t} assemblies may be called the key array. The s^r assemblies obtained as a solution of (5.1) are divisible into s^t such arrays each of strength 2. These are obtained by adding to each member of the key array, a solution of (5.1) which does not already occur in the key array. Now if there are n factors each at s_1 levels and another factor at s_1^t levels, then to each one of the s_1^{r-t} assemblies of a group is added one level of the additional factor. The s_1^t groups will each have a different level of the additional factor. Thus we have in s_1^r assemblies, a fractional replicate design of an asymmetrical factorial experiment $s_1^n . s_2$ where $s_2 = s_1^t$ and s_1 is a prime or power of a prime.

This design will allow estimation of all the $(n+1)$ main effects, and $\binom{n}{2}$ first order interactions between any two factors of the first group on the assumption that interactions involving d factors or more of the first group are absent. The analysis of variance table may be set up as follows:

TABLE 5. ANALYSIS OF VARIANCE FOR FRACTIONAL REPLICATE DESIGN

factors		degrees of freedom
main effects ..	first group of factors	$n(s_1-1)$
	extra factor	(s_2-1)
first order interaction	between factors of first group	$\binom{n}{2}(s_1-1)^2$
error		*(obtained by substraction)*
total		s^r-1

A further reduction in the number of assemblies can be achieved if we are not interested in preserving all the $\binom{n}{2}$ first order interactions between factors of the first group but only in a selected sub-set of first order interactions. Then each group of s_1^{r-t} assemblies need not be an orthogonal array of strength 2 but it will do simply if each group of s_1^{r-t} assemblies satisfies the following less restrictive properties

(i) all the levels of any factor of first set occur equal number of times, this number being same for all factors and groups,

(ii) all combinations of levels of any two factors whose interaction we want to preserve, should occur equal number of times, this number being same for all pairs and groups and all the s_1^r assemblies together should form an orthogonal array of strength $d \geqslant 4$.

Consider the factorial experiment $2^5 \times 4$. For five factors each at 2 levels, a hypercube of strength 4 in 16 assemblies exists and this allow estimation of main effects and first order interactions on the assumption that interactions of order $d \geqslant 2$ are absent. Let us denote the first five factors by $F_1, F_2, \ldots, F_5$ and the one at 4 levels by G. Now, since each level of G is to occur equal number of times, in order to preserve all first order interactions of any two F's it is necessary to construct an orthogonal array of strength 2 in 4 assemblies. But an orthogonal array of strength 2 in 4 assemblies can accommodate only upto three factors. And since 16 assemblies provide us with only 15 degrees of freedom and 8 degrees of freedom are taken up by the 6 main effects, only 7 degrees of freedom are left and we can at best estimate 7 first order interactions involving F's alone. Then no degrees of freedom will be left for estimation of error variance. If a previous estimate of error variance is available then this need not worry us. The interactions to be preserved will be decided by the nature of the experiment. Having decided on the interactions to be preserved the problem is to divide the 16 assemblies into four groups so that conditions (i) and (ii) are satisfied.

Example 2: An example of this experiment taken from Davies and Hay (1950) is given below. The experiment concerned investigation of effects of several factors on yield of penicillin. Five factors A, B, C, D, E each at two levels and a factor

F at four levels were chosen. The factors were

TABLE 6. DESCRIPTION OF EXPERIMENTAL STAGES

stage 1	preparation of inoculum
A	concentration of corn steep liquor
B	amount of sugars
C	quality of sugars
stage 2	fermentation
D	concentration of corn steep liquor obtained from first stage.
E	quality of corn steep liquor
F	4 fermenters

A design in 16 assemblies preserving the main effects and the 7 first order interactions *AB*, *AC*, *AD*, *AE*, *BD*, *CD* and *DE* was adopted and the detailed analysis is also given in Davies and Hay (1950), Davies (1954). They have constructed the design from different considerations. We have constructed the design with the help of orthogonal arrays and in the manner described in an earlier paragraph. The design happens to be the same as that obtained by Davies and Hay. The design is given below:

TABLE 7. DESIGN OF THE EXPERIMENT

factors	assemblies															
	first group				second group				third group				fourth group			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0
B	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0
C	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
D	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
E	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
G	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3

The splitting up of degrees of freedom will be as follows:

TABLE 8. ANALYSIS OF VARIANCE

factors		degrees of freedom
main effects	*A*	1
	B	1
	C	1
	D	1
	E	1
	G	3
first order interactions	*AB*	1
	AC	1
	AD	1
	AE	1
	BD	1
	CD	1
	DE	1
	total	15

6. DESIGNS FOR 2×3^k AND EXPERIMENTS IN STAGES

If in experiments 2×3^k, one is allowed to relax the condition that all the levels of a factor should occur equal number of times, fractional replicate designs for such experiments can be constructed by taking an orthogonal array with $(k+1)$ factors each at three levels and then replacing the highest level of one factor by one of the two lower levels, all through the array. So that for this particular factor, one level will occur twice as many times as the other. This method of construction sometimes effects a saving in the number of experimental units to be used e.g. in the case of orthogonal arrays of strength 2, if $2k+1 > 3^{t-1}$ and $2k-3 \leqslant 3^t$ where t is some positive integer.

Experiments in industries are often conducted in several stages and different number of factors have to be introduced at different stages. The total number of units to be used in the experiment is sometimes decided by the varying costs of setting up different stages of the experiment. If products of an earlier stage, which have to be used in a later stage, are costly, we may have to be content only with the estimates of main effects of the factors introduced at the earlier stage, while possibly for the factors introduced at a later stage, it may be possible to estimate interactions upto a certain order. This restriction on the experiment may be taken care of, in the design in the case of a symmetrical factorial experiment by taking an orthogonal array of lower strength for the factors of the earlier stage and an orthogonal array of higher strength for the factors of the later stage and combining the two arrays in a manner so that the derived array remains an orthogonal array. Or alternatively, starting with an orthogonal array of suitable strength for all the factors of an experiment, it is sometimes possible, to find out within the array a sub-group of factors which amongst themselves form an orthogonal array of lower strength say 2, while the remaining factors form one of a higher strength. Now, if each of the distinct sub-assemblies of the factors of the sub-group forming an array of strength 2 occurs more than once, then these may serve as factors of the earlier stage of the experiment while the rest of the factors may be introduced in the later stage of the experiment. As an example, we may consider the following array.

TABLE 9. ARRAY: $(2^3, 7, 2, 2,)$

factors	assemblies							
	1	2	3	4	5	6	7	8
A	1	1	1	1	0	0	0	0
B	1	1	0	0	1	1	0	0
C	1	0	1	0	1	0	1	0
D	1	0	0	1	0	1	1	0
E	1	1	0	0	0	0	1	1
F	1	0	1	0	0	1	0	1
G	1	0	0	1	1	0	0	1

Here the factors E, F, G form an array of strength 2 and each of 4 distinct sub-assemblies occur twice while the four factors A, B, C, D form an array of strength 3. So in an experiment which has to be conducted in two stages and where it is costly to have many experimental units in the first stage, the factors E, F, G may be used at the first stage which will require only 4 units and the factors A, B, C, D may be introduced in the following stage. Examples of experimental situations where the problems discussed above, have occurred are provided by Taguchi (1955). On behalf of the Quality Control Unit of Indian Statistical Institute, he had conducted several experiments in different industries in South India. His reports submitted to the Indian Statistical Institute contain description of designs, analysis and inferences drawn. But he does not discuss how he constructed these designs. A brief description of one of the experiments together with the design adopted by him is given here to illustrate the applicability of the designs constructed in this paper.

Experiment: *A* 2×3^7 experiment to find an optimum operational standard for anodising aluminium alloy parts: All aluminium base alloy parts have to be given an anodic tratment for forming a thin film of oxide coating on the metal by an electrolyte oxidation process. The problem was to design a suitable experiment in two stages, four factors to be introduced in one stage and four factors in the other stage; all the factors chosen excepting one, were at 3 levels. The first stage consists in preparation of an anodising bath which is controlled by the four facors.

TABLE 10. FACTORS OF THE FIRST STAGE

			levels	
C	concentration of bath (acid content)		50 gms/litre	(1)
			40 gms/litre	(2)
			30 gms/litre	(3)
D	voltage cycle		38V−50V	(1)
			40V−50V	(2)
			45V−52V	(3)
E	time cycle	D(1)	D(2)	D(3)
		(1) 10−30−5−5 (50m)	10−25−5−5 (45m)	10−20−4−4 (38m)
		(2) 10−35−5−5 (55m)	10−30−5−5 (50m)	10−25−4−4 (43m)
		(3) 10−40−5−5 (60m)	10−35−5−5 (55m)	10−30−4−4 (48m)
F	temperature of bath		(1) 100°F	
			(2) 106°F	
			(3) 103°F	

The symbol 10-30-5-5 (50m) under D(1) implies that the voltage has to be raised from 0 to 38 volts in 10 minutes, maintained at 38 volts for 30 minutes and again raised to 50 volts in 5 minutes and kept at 50 volts for 5 minutes—total time taken for the entire operation being 50 minutes. The second stage of the experiment concerns the alloy parts that have to be suspended from the anode into the bath by pure aluminium. Factors which control this stage of the experiment are

TABLE 11. FACTORS OF THE SECOND STAGE

		levels	
A	degreasing operation time	8 minutes	(1)
		10 minutes	(2)
		12 minutes	(3)
B	type of parts	big	(1)
		medium	(2)
		small	(3)
G	rinsing time in cold water swill	2 minutes	(1)
		3 minutes	(2)
H	rinsing time in hot water bath (150°F)	3 minutes	(1)
		4 minutes	(2)
		5 minutes	(3)

Cost and difficulty of operation limit the number of baths that can be set up. The design has to take care of this feature of the experiment. So the different sub-assemblies of the factors *C, D, E, F* which we can introduce in the experiment are limited while we may have sufficiently large number of sub-assemblies for *A*, *B*, *G*, *H*. Taguchi's design is given below:

TABLE 12. DESIGN FOR 2×3^7 EXPERIMENT

assemblies	factors								
	C	*D*	*E*	*F*		*B*	*A*	*G*	*H*
						2	3	1	2
1—3	3	2	1	1	—	1	1	1	2
						3	2	2	1
						2	1	1	1
4—6	3	1	2	2	—	1	2	2	3
						3	3	1	2
						2	2	2	2
7—0	3	3	3	3	—	1	3	1	1
						3	1	1	3
						2	3	1	1
10–12	2	3	2	1		1	1	2	3
						3	2	1	2
						2	1	2	2
13–15	2	2	3	2	—	1	2	1	1
						3	3	1	3
						2	2	1	3
16–18	2	1	1	3	—	1	3	1	2
						3	1	2	1
						2	3	2	2
19–21	1	1	3	1	—	1	1	1	1
						3	2	1	3
						2	1	1	3
22–24	1	3	1	2	—	1	2	1	2
						3	3	2	1
						2	2	1	1
25–27	1	2	2	3		1	3	2	3
						3	1	1	2

If we suppose that G had also three levels to start with (and later 3 has been converted into 1), then it is easily seen that the 27 sub-assemblies of A, B, G, H form an orthogonal array of strength 3. Further, each of the groups of the sub-assemblies *viz.*, 1-9, 10-18, 19-27 form an orthogonal array of strength 2. The 9 distinct sub-assemblies involving C, D, E, F alone, define an orthogonal array of strength 2 and each of these sub-assemblies was tagged on to three sub-assemblies involving A, B, G, H so that this becomes an orthogonal array of strength 2 in 27 assemblies involving 8 factors. It is further seen that the interaction BC is preserved, the condition for which given by Rao, (1947), states that all combinations of 3 factors involving both B and C should occur an equal number of times in an orthogonal array $(N, n, s, 2)$. Taguchi (1955) has not discussed the features of the design, that are given above. The splitting up of degrees of freedom was done as follows:

TABLE 13. ANALYSIS OF VARIANCE

	factors	degrees of freedom
main effects	C	2
	D	2
	E	2
	F	2
	B	2
	A	2
	G	1
	H	2
interaction	BC	4
	error	7
	total	26

7. Analysis of fractional replicate designs

Systematic methods (Yates' technique) of analysis of fractional replicate designs are available in the case of symmetrical factorial experiments (Davies 1954). If, however, the interest is mainly on main effects and first order interactions, systematic methods may prove to be laborious and it may be advantageous to obtain estimates of main effects and lower order interactions by writing down the contrasts omitting those assemblies which do not occur in the array A great simplicity results in the analysis of designs if for any two mutually orthogonal estimable linear functions of parameters, the corresponding best estimates are uncorrelated.

If we consider the set up

$$E(y) = \tau A'$$

where y is the vector of n observations, τ is a vector of k parameters and A' is the design matrix of the form $k \times n$ and of rank r, then a linear function $\tau p'$ is estimable if there exists an l such that

$$lA'A = p \qquad \text{(Rao, 1952)}$$

Theorem: *The necessary and sufficient condition that the best linear estimates of two estimable linear functions* $\tau p_1'$ and $\tau p_2'$ $(p_1 p_2' = 0)$, are uncorrelated is that

$$(A'A).(A'A) = \mu . A'A$$

i.e. that non-zero roots of $A'A$ *are all equal.*

Proof: Sufficiency: Since $\tau p_1'$ and $\tau p_2'$ are estimable, there exist l_1 and l_2 so that

$$l_1 A'A = p_1$$

and

$$l_2 A' A = p_2$$

and covariance of the best estimates $l_1 Q'$ and $l_2 Q'$ is given by

$$l_1 A'A\, l_2' \sigma^2$$

where $Q = yA$.

Now
$$p_1 p_2' = l_1 A'A . A'A l_2' = 0. \quad \text{... (7.1)}$$

So if
$$A'A . A'A = \mu A'A \quad \text{... (7.2)}$$

then (7.1) would imply that

$$l_1 A'A\, l_2' = 0 \quad \text{... (7.3)}$$

Necessity: To prove that the condition is necessary, we will have to show that

if
$$p_1 p_2' = 0 \quad \text{... (7.4)}$$

implies
$$l_1 A' A\, l_2' = 0 \quad \text{... (7.5)}$$

then
$$(A'A)^2 = \mu A'A$$

or the non-zero roots of $A'A$ are all equal.

In order that $p_1.\tau'$ and $p_2.\tau'$ are estimable, it is known that

$$l_1 A'A = p_1 = \alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_r C_r \quad \text{... (7.6)}$$

$$l_2 A'A = p_2 = \alpha_1' C_1 + \alpha_2' C_2 + \cdots + \alpha_r' C_r \quad \text{... (7.7)}$$

where $C_1, C_2, \ldots, C_r$ are the normalized latent vectors of $A'A$ corresponding to its r positive latent-roots $\lambda_1, \lambda_2, \ldots, \lambda_r$ and $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $(\alpha_1', \alpha_2', \ldots, \alpha_r')$ are arbitrary constants.

We get from (7.4).

$$\alpha_1\alpha_1' + \alpha_2\alpha_2' + \cdots + \alpha_r\alpha_r' = 0. \quad \text{... (7.8)}$$

Now, since $C_iA'A = \lambda_i C_i$ $(i = 1, 2, \ldots, r)$ from (7.5), (7.6) and (7.7) we get

$$l_1A'Al_2' = (\alpha_1C_1+\cdots+\alpha_rC_r)\,l_2'$$

$$= \left(\frac{\alpha_1}{\lambda_1}C_1A'A+\cdots+\frac{\alpha_r}{\lambda_r}C_rA'A\right)l_2'$$

$$= \left(\frac{\alpha_1}{\lambda_1}C_1+\cdots+\frac{\alpha_r}{\lambda_r}C_r\right).\,A'Al_2'$$

$$= \left(\frac{\alpha_1}{\lambda_1}C_1+\cdots+\frac{\alpha_r}{\lambda_r}C_r\right)(\alpha_1'C_1+\alpha_2'C_2+\cdots+\alpha_r'C_r)'$$

$$= \frac{\alpha_1\alpha_1'}{\lambda_1}+\cdots+\frac{\alpha_r\alpha_r'}{\lambda_r} = 0 \qquad \ldots \quad (7.9)$$

Since α_i's and α_i''s can be arbitrarily fixed subject to (7.8), it follows that $\lambda_1 = \lambda_2 = \ldots = \lambda_r = \mu$

Hence the theorem.

Analysis of a fractional replicate design of an asymmetrical factorial experiment is given here.

Analysis of the derived array $(2^2, 2, 2, 2)\times(3^2, 4, 3, 2)$: An artifical example constructed using the above design for an asymmetrical factorial experiment $2^4\times 3^4$ is analysed here. Since this design preserves the main effects and interactions involving two factors—only one factor being taken from one group and as their estimates are uncorrelated, the analysis is easily done by forming 2×3 tables like,

TABLE 14. $F_1\times G_1$

		levels of G_1			
		0	1	2	total
levels of F_1	0				F_{01}
	1				F_{11}
	total	G_{01}	G_{11}	G_{12}	T

Then, sum of squares due to $F_1 = \dfrac{F_{01}^2+F_{11}^2}{18}-\dfrac{T^2}{36}$

Sum of squares due to $G_1 = \dfrac{G_{01}^2+G_{11}^2+G_{12}^2}{12}-\dfrac{T^2}{36}$

Sum of squares due to

interaction $F_1\,.\,G_1$ = Total corrected s.s. due to the table $F_1\times G_1$ —s.s. due to F_1—s.s. due to G_1

and similar computations for 11 other tables will complete the analysis which is given below. Since we do not have any degrees of freedom left for estimation of error, tests of significance for the main effects may be performed on the assumption that FG interactions are absent.

TABLE 15. DESIGN AND YIELD OF THE EXPERIMENT

sl. no.	design							yield
	G_4	G_3	G_2	G_1	F_3	F_2	F_1	y
1	0	0	0	0	0	0	0	48
2	0	0	0	0	0	1	1	53
3	0	0	0	0	1	1	0	65
4	0	0	0	0	1	0	1	50
5	0	1	1	1	0	0	0	67
6	0	1	1	1	0	1	1	74
7	0	1	1	1	1	1	0	75
8	0	1	1	1	1	0	1	74
9	1	2	1	0	0	0	0	72
10	1	2	1	0	0	1	1	76
11	1	2	1	0	1	1	0	78
12	1	2	1	0	1	0	1	84
13	1	0	2	1	0	0	0	72
14	1	0	2	1	0	1	1	73
15	1	0	2	1	1	1	0	82
16	1	0	2	1	1	0	1	76
17	1	1	0	2	0	0	0	62
18	1	1	0	2	0	1	1	79
19	1	1	0	2	1	1	0	78
20	1	1	0	2	1	0	1	69
21	2	1	2	0	0	0	0	95
22	2	1	2	0	0	1	1	87
23	2	1	2	0	1	1	0	88
24	2	1	2	0	1	0	1	81
25	2	2	0	1	0	0	0	85
26	2	2	0	1	0	1	1	85
27	2	2	0	1	1	1	0	82
28	2	2	0	1	1	0	1	77
29	2	0	1	2	0	0	0	72
30	2	0	1	2	0	1	1	80
31	2	0	1	2	1	1	0	81
32	2	0	1	2	1	0	1	80
33	0	2	2	2	0	0	0	84
34	0	2	2	2	0	1	1	79
35	0	2	2	2	1	1	0	83
36	0	2	2	2	1	0	1	88

TABLE 16. ANALYSIS OF VARIANCE

factors		d.f.	s.s.	mean square	F
	F_1	1	2.25	2.25	
main effects	F_2	1	124.69	124.69	5.08*
	F_3	1	78.03	78.03	3.18
	G_1	2	175.50	87.75	3.58*
	G_2	2	1066.67	533.33	21.74**
	G_3	2	920.16	460.08	18.70**
	G_4	2	930.66	465.33	18.97**
interaction	F_1G_1	2	26.16		
	F_1G_2	2	97.89		
	F_1G_3	2	3.17		
	F_1G_4	2	32.67		
	F_2G_1	2	7.72		
	F_2G_2	2	110.89		
	F_2G_3	2	74.40		
	F_2G_4	2	13.56		
	F_3G_1	2	14.38		
	F_3G_2	2	22.68		
	F_3G_3	2	51.30		
	F_3G_4	2	133.56		
sub-total for interactions		24	588.77		
total		35	3866.73		

Main effects G_2, G_3 and G_4 are significant at both the levels 5% and 1% while F_2 and G_1 are significant at 5% only. F_1 and F_3 are not significant at all.

8. Partially balanced arrays

In an orthogonal array $(s^t, n, s, 2)$ where s is a prime or power of a prime, it is known (Rao 1946) that it can accommodate upto $n = \frac{s^t - 1}{s-1}$ factors. In this case, all the n main effects take up the $(s^t - 1)$ degrees of freedom and if an estimate of error variance based on previous experience is available this may be considered as the most economic design. But if $\frac{s^{t-1}-1}{s-1} < n < \frac{s^t-1}{s-1}$ then also we have to use an orthogonal array in s^t assemblies. If our interest is in the estimation of main effects only, then the observations $s^t - n(s-1) - 1$ may be regarded as unnecessary. It is possible to economise the number of observations in such cases if we relax the conditions of an orthogonal array, to certain extent. But this will introduce a little more complication in the analysis of the design.

An array involving n factors $F_1, F_2, \ldots, F_n$ each with s levels will be called a *partially balanced array* of strength d if for any group of d factors $(d < n)$ a combination of levels of d factors, $F_{1i_1}, F_{2i_2}, \ldots, F_{di_d}$ occurs $\lambda_{i_1 i_2 \ldots i_d}$ times, where $\lambda_{i_1 i_2 \ldots i_d}$ remains the same for all permutations of a given set $(i_1, i_2, \ldots, i_d)$ and for any group of d factors, i_j ranging from 0 to $s-1$ for all j. Then, it is obvious that this property holds also for any $k < d$. Let amongst the d integers $(i_1, i_2, \ldots, i_d)$, 0 occur r_0 times, 1 occur r_1 times and so on;—or, in other words, let in the treatment combination $F_{1i_1} F_{2i_2} \ldots F_{di_d}$ there be r_0 factors which occur at 0-th level. r_1 factors which occur at level 1 and so on.

Then,
$$r_0 + r_1 + \ldots + r_{s-1} = d$$

and each of the $\frac{d!}{r_0!\, r_1! \ldots r_{s-1}!}$ treatment combinations obtained by permuting $(i_1, i_2, \ldots, i_d)$ will have the same $\lambda_{i_1 i_2 \ldots i_d}$ attached to it. Value of $\lambda_{i_1 i_2 \ldots i_r}$, where $r < d$, is easily obtained by summing $\lambda_{i_1 i_2 \ldots i_d}$ over $(i_{r+1}, \ldots, i_d)$ where each i of $(i_{r+1}, \ldots, i_d)$ ranges from 0 to $s-1$.

9. Examples of partially balanced arrays and analysis

Consider the following orthogonal array

TABLE 17. ARRAY: $(2^3, 4, 2, 2)$

	assemblies							
	1	2	3	4	5	6	7	8
A	1	1	1	0	0	0	1	0
B	1	0	0	0	1	1	1	0
C	0	1	0	1	1	0	1	0
D	0	0	1	1	0	1	1	0

It is known that 4 factors each with 2 levels require 8 assemblies for constructing an orthogonal array of strength 2. If from the above array we omit assemblies 7 and 8, we get an arrangement in 6 assemblies, which has the properties that for any two factors say A, B the combination of levels of the type (0,0) or (1,1) occurs once while (1, 0) or (0,1) occurs twice. So this satisfies the properties of a partially balanced array.

This design can be analysed by the method of least squares as follows.

Minimising the expression

$$L = (y_1 - a_1 - b_1 - c_0 - d_0)^2 + \ldots + (y_6 - a_0 - b_1 - c_0 - d_1)^2$$

Where y_i denotes the observation corresponding to the i-th assembly and a_i denotes the effect of i-th level of a and similarly for other constants, we get

$$3a_0 + (b_0 + 2b_1) + (c_0 + 2c_1) + (d_0 + 2d_1) = y_4 + y_5 + y_6 \quad \ldots \quad (9.1)$$
$$3a_1 + (2b_0 + b_1) + (2c_0 + c_1) + (2d_0 + d_1) = y_1 + y_2 + y_3 \quad \ldots \quad (9.2)$$
$$(a_0 + 2a_1) + 3b_0 + (c_0 + 2c_1) + (d_0 + 2d_1) = y_2 + y_3 + y_4 \quad \ldots \quad (9.3)$$
$$(2a_0 + a_1) + 3b_1 + (2c_0 + c_1) + (2d_0 + d_1) = y_1 + y_5 + y_6 \quad \ldots \quad (9.4)$$
$$(a_0 + 2a_1) + (b_0 + 2b_1) + 3c_0 + (d_0 + 2d_1) = y_1 + y_3 + y_6 \quad \ldots \quad (9.5)$$
$$(2a_0 + a_1) + (2b_0 + b_1) + 3c_1 + (2d_0 + d_1) = y_2 + y_4 + y_5 \quad \ldots \quad (9.6)$$
$$(a_0 + 2a_1) + (b_0 + 2b_1) + (c_0 + 2c_1) + 3d_0 = y_1 + y_2 + y_5 \quad \ldots \quad (9.7)$$
$$(2a_0 + a_1) + (2b_0 + b_1) + (2c_0 + c_1) + 3d_1 = y_3 + y_4 + y_6 \quad \ldots \quad (9.8)$$

Now taking one equation from each pair, and putting $a_0 = 0$, $b_0 = 0$, $c_0 = 0$, $d_0 = 0$, we get

$$
\begin{aligned}
3a_1+b_1+c_1+d_1 &= y_1+y_2+y_3 = Q_1 \\
a_1+3b_1+c_1+d_1 &= y_1+y_5+y_6 = Q_2 \\
a_1+b_1+3c_1+d_1 &= y_2+y_4+y_5 = Q_3 \\
a_1+b_1+c_1+3d_1 &= y_3+y_4+y_6 = Q_4.
\end{aligned}
$$

Solving we get,

$$a_1+b_1+c_1+d_1 = \frac{1}{3}\sum_{i=1}^{6} y_i$$

$$\hat{a}_1 = \frac{1}{2}\left[(y_1+y_2+y_3)-\frac{1}{3}\sum_{i=1}^{6} y_i\right]$$

$$\hat{b}_1 = \frac{1}{2}\left[(y_1+y_5+y_6)-\frac{1}{3}\sum_{i=1}^{6} y_i\right]$$

$$\hat{c}_1 = \frac{1}{2}\left[(y_2+y_4+y_5)-\frac{1}{3}\sum_{i=1}^{6} y_i\right]$$

$$\hat{d}_1 = \frac{1}{2}\left[(y_3+y_4+y_6)-\frac{1}{3}\sum_{i=1}^{6} y_i\right].$$

Sum of squares due to a factor A is then, equal to $\frac{12}{5}\hat{a}_1^2$. Sum of squares due to main effects is $\hat{a}_1Q_1+\hat{b}_1Q_2+\hat{c}_1Q_3+\hat{d}_1Q_4$ and in this case we are left with 1 degree of freedom which may be used to get an estimate of error but this, however, may not be, very reliable.

In the general case of k factors each with s levels if we were trying out a partially balanced array of strength 2 the normal equations (on the assumption that all interaction are absent) may be seen to be

$$yA = (\tau_1 \,\vdots\, \tau_2 \,\vdots\, \dots \,\vdots\, \tau_k)\begin{bmatrix} D & \Lambda & \Lambda & \dots & \Lambda \\ & D & \Lambda & \dots & \Lambda \\ & & & \dots & \\ & & & & D \end{bmatrix}$$

where y is the vector of observations, $\tau_1, \tau_2, \dots, \tau_k$ are the k groups of parameters, A is the transpose of the design matrix and in the partitioned matrix which is $A'A$, D is a diagonal matrix with μ_i's in the diagonal line,

where μ_i represents the number of times the i-th level of a factor occurs in the array and

$$\Lambda = \begin{bmatrix} \lambda_{00} & \lambda_{01} & \dots & \lambda_{0s-1} \\ & \lambda_{11} & \dots & \lambda_{1s-1} \\ & & \dots & \dots \\ & & & \lambda_{s-1\,s-1} \end{bmatrix}$$

where an element λ_{ij} represents the number of times i-th level of a factor occurs with the j-th level of another factor. Solving these equations, together with some more suitably chosen equations (since all the parameters can not be estimated) one would be able to calculate sum of squares due to each main effect and also total sum of squares due to all main effects which on substraction from total sum of squares will give error sum of squares.

Two examples of partially balanced arrays are given below:

TABLE 18. PARTIALLY BALANCED ARRAYS

(1)						(2)						
factors	assemblies					factors	assemblies					
	1	2	3	4	5		1	2	3	4	5	6
A	0	1	0	1	1	*A*	0	1	0	1	1	1
B	0	0	1	1	1	*B*	0	0	1	1	1	1
C	0	1	1	0	1	*C*	0	1	1	0	1	1
D	0	1	1	1	0	*D*	0	1	1	1	0	1
						E	0	1	1	1	1	0

In the first array, four factors have been accommodated in 5 assemblies, while in the second 5 factors have been accommodated in 6 assemblies.

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