ISRAEL JOURNAL OF MATHEMATICS 76 (1991), 229–235

A COROLLARY TO KODAIRA–SPENCER'S THEOREM ON CONTINUITY OF EIGENVALUES

BY

S. SRINIVAS RAU

Stat-Math Unit, Indian Statistical Institute, 8th mile, Mysore Road, Bangalore 560 059, India

ABSTRACT

We give an elementary proof of continuity of the determinant in the parameter for a smooth family of laplacians (of the same nullity) on a smooth family of holomorphic vector bundles over a compact complex manifold. Families of unitary flat bundles over a compact Riemann surface are discussed, as an example.

Introduction

Kodaira and Spencer introduced around 1960 the notion of differential families F of holomorphic vector bundles F_p over a compact complex manifold X, for p a parameter varying in an open subset U of a Euclidean space ([2], p. 324). Suppose X has a given Riemannian metric. Then Kodaira–Spencer define also differentiable families of hermitian metrics h_p on F_p and of associated laplacians Δ_p acting on the space of C^∞ sections of $F_p \to X, C^\infty(F_p)$. Each Δ_p has a spectrum of the form ([2], p. 351)

$$0 \le \lambda_1(p) \le \lambda_2(p) \le \cdots \le \lambda_m(p) \le \cdots, \quad \lambda_m(p) \to \infty \quad \text{as } m \to \infty.$$

Kodaira and Spencer showed the continuity of each eigenvalue λ_m in $p \in U$ ([2]), Theorem 7-2, 7-7) (see Section 1 for details).

It is natural to ask if the determinant of Δ_p defined by zeta function regularisation is continuous in p. Under the assumption that the dimension of the kernel

Received February 1, 1990 and in revised form March 20, 1991

of Δ_p is constant over U we prove in Proposition 1, Section 1 below that $\det \Delta_p$ is continuous in p. In Section 2 we discuss the example of families of unitary flat holomorphic vector bundles over a compact Riemann surface of genus > 1. In this case continuity of the determinant is readily deduced also from the results in [7].

1. Continuity of det Δ_p

We recall the definitions first. Let X be a compact complex manifold of complex dimension n and U an open subset of $\mathbb{R}^N$.

Suppose $F \to X \times U$ is a $\mathcal{C}^\infty$ complex vector bundle of rank r. Then for each $p \in U$, the restriction of F to $X \times (p)$ is a smooth ($\mathcal{C}^\infty$) complex vector bundle $F_p \to X \times (p)$ of the same rank as F.

F or (F_p) is called a differentiable family of holomorphic vector bundles of rank r over X if there exist local trivializations of F

$$\pi^{-1}(U_j \times U) = \mathbb{C}^r \times U_j \times U$$

such that the transition functions

$$(\zeta_j, z, p) \to (\zeta_k, z, p)$$

are holomorphic in $z \in U_j$ and $\mathcal{C}^\infty$ in $p \in U$.

In particular, for a given $p \in U$ one has thus local trivializations of F_p. The fibre coordinates ζ_j thus obtained are called admissible fibre coordinates.

Suppose we are given a smooth function ψ_p of F_p for each $p \in U$. One says that ψ_p is $\mathcal{C}^\infty$ differentiable in p if each admissible fibre coordinate of $\psi_p(z)$ is a $\mathcal{C}^\infty$ function of (z, p).

Suppose we are given a linear operator $L_p : \mathcal{C}^\infty(F_p) \to \mathcal{C}^\infty(F_p)$ for each $p \in U$. $(L_p)_{p\in U}$ is called a differentiable family of linear operators if $L_p\psi_p$ is $\mathcal{C}^\infty$ differentiable in p whenever $\psi_p \in \mathcal{C}^\infty(F_p)$ is so. If each L_p is a linear differential operator, then $(L_p)_{p\in U}$ is a differentiable family if and only if in the admissible local trivializations one has

$$(L_p\psi_p)(z) = (\phi^1(z,p), \ldots, \phi^r(z,p))$$

where $\phi^\lambda(z,p) = \Sigma_{\mu=1}^{p} L_\mu^\lambda(z, p, \partial/\partial x_i, \partial/\partial y_i)\psi_p^\mu(z)$,

$$z = (z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$$

with L^p_μ polynomials in the $\partial/\partial x_i, \partial/\partial y_j$ with the coefficients C^∞ in (z,p).

Let h be an hermitian metric on F given in admissible coordinates over U_j by $\Sigma_i^\infty h_{j\lambda\mu}(z,p)\zeta_j^\lambda\bar{\zeta}_j^\mu$. In particular the $h_{j\lambda\mu}$ are smooth in (z,p).

Suppose the base manifold X has a Riemannian metric $g_{\alpha,\beta}$ and an associated volume element dv, then for $\psi_p, \phi_p \in C^\infty(F_p)$ there is an inner product defined by

$$(\psi_p, \phi_p)_p = \int_X (\Sigma h_{j\lambda\mu}(z,p)\psi^\lambda_{pj}(z)\bar{\phi}^\mu_{pj}(z))dv.$$

Let Δ_p be the laplacian of the hermitian metric h_p on F_p induced by h. Then $(\Delta_p)_{p\in U}$ form a differentiable family of linear elliptic differential operators of order 2. Each Δ_p is formally self-adjoint and strongly elliptic for the inner product $(\ ,\ )_p$. It is known that the spectrum of Δ_p is discrete and has the form

$$0 \leq \lambda_1(p) \leq \lambda_2(p) \leq \cdots \leq \lambda_m(p) \leq \cdots, \quad \lambda_m(p) \to \infty \quad \text{as } m \to \infty.$$

KODAIRA–SPENCER'S THEOREM ([2], Theorems 7.2, 7.3, 7.7, 7.8.): *Let X be a compact complex manifold of complex dimension n with a Riemannian metric g. Let U be an open subset of $\mathbb{R}^N$. Let F be a differentiable family of holomorphic vector bundles F_p of rank r over $x, p \in U$. Let h be a smooth hermitian metric on F and let $(\Delta_p)_{p\in U}$ be the differentiable family of laplacians corresponding to the (h_p). Then*

(i) *each eigenvalue $\lambda_m(p)$ is continuous in p,*

(ii) *given $p_0 \in U$ there exists a small neighbourhood $N(p_0)$ of p_0 such that for each $p \in N(p_0)$*

$$\dim\ker\Delta_p \leq \dim\ker\Delta_{p_0} < \infty.$$

Now the determinant of Δ_p is defined as follows. First set, for $s \in \mathbb{C}$,

$$\zeta_p(s) = 1/\Gamma(s)\int_0^\infty t^{s-1}(\Sigma e^{-\lambda_m(p)t} - m_0(p))dt;$$

ζ_p is analytically continued to zero ([5], Theorem 13.1). Define the determinant of Δ_p by

$$\det\Delta_p = |\exp -\zeta_p'(0)|.$$

PROPOSITION 1: *In addition to Kodaira–Spencer's hypotheses assume that* $\dim\ker\Delta_p$ *is the same number* h_0 *for all* $p\in U$. *Then* $\zeta_p'(0)$ *is continuous in* $p\in U$ *and so* $\det\Delta_p$ *is continuous in* p.

Proof: STEP 1: We first prove the continuity of $p\to\zeta_p'(s)$ for $\mathrm{Re}(s)\gg 0$.

Choose $p_0\in U$ and let K be a (small) compact neighbourhood of p_0. It suffices to show that $\Sigma e^{-\lambda_m(p)t}$ is continuous in $p\in K$. We show that the series is uniformly convergent in p and invoke the continuity of each λ_m in $p\in K$. Set

$$\mu_m=\inf_{p\in K}\lambda_m(p),\quad m=h_0+1,h_0+2,\ldots.$$

Each μ_m is positive by our hypothesis. The preceding series is majorised by

$$(*)\qquad \Sigma e^{-\mu_m t}.$$

Thus if $(*)$ converges uniformly for $t>0$, then by Weierstrass's M-test the earlier series converges uniformly in $p\in K$.

To verify uniform convergence of $(*)$ to the right of zero, it suffices by Widder ([8], Theorem 3.3, pp. 47–48) to check that

$$\operatorname*{Lim}_{m\to\infty}(\log m/\mu_m)=0.$$

For this we use the asymptotic behaviour of $\lambda_m(p)$. It is known ([5], p. 291) that

$$(m/\alpha(p)(\lambda_m(p))^n)\to 1\quad\text{as } m\to\infty$$

with

$$\alpha(p)=(1/(2n)(2\pi)^{2n})\int_X\int_{|\varepsilon'|=1}\text{Trace}\,[a_2(z,\varepsilon',p)^{-n}]d\varepsilon' dv,$$

where the inner integration is with respect to the natural measure on the unit sphere of T^*X given by the symplectic structure of T^*X. By the strong ellipticity of Δ_p, the integrand is everywhere positive. One sees that the integrand is continuous in p, because the principal symbol a_2 is even smooth in p, by hypothesis. Thus $\alpha(p)$ is seen to be continuous in p. Thus there exist positive constants c_1,c_2 such that $c_1\le\alpha(p)\le c_2$ for all $p\in K$. Thus

$$\begin{aligned}0&=[\operatorname*{Lim}_{m\to\infty}(\log m/m^{1/n})(c_2)^{1/n}]\le[\lim_{m\to\infty}\log m/\mu_m)]\\&\le[\operatorname*{Lim}_{m\to\infty}((\log m)(c_1)^{1/n}/m^{1/n})]=0.\end{aligned}$$

Thus the middle term is zero and we are done.

STEP 2: Having verified that $p \to \zeta_p'(s)$ is continuous in p for $\mathrm{Re}(s) \gg 0$ we turn our attention to proving the same for $p \to \zeta_p'(0)$. For this purpose we consider how $\zeta_p(s)$ is defined in a neighbourhood of 0. To begin with the integral ([0], p. 79)

$$\int_1^\infty t^{s-1}\Big(\sum_{\lambda_n>0} e^{-\lambda_n(p)t}\Big)dt = r(s,p)$$

is an entire function in s. $r(s,p)$ has derivative at 0 continuous in p.

For $0 < t < 1$ consider the relation ([0], p. 79)

$$\sum_{\lambda_n>0} e^{-\lambda_n(p)t} = \sum_{k\le n_0} t^{(k-m)/2} a_k(p) + O(t^{(n_0-m)/2})$$

where $a_k(p) = \int_X a_k(x,\Delta_p)d\,\mathrm{vol}$, $m = \dim_{\mathbb{R}} X$ and $a_k(x,\Delta_p)$ is a local scalar invariant of the jets of the total symbol of Δ_p. Thus $a_k(p)$ is continuous in p, because the total symbol varies continuously in p.

We consider the error term $E(t,p)$. Let p vary in a compact ball K of positive radius. By an application of Garding's inequality the kernel $K_p(t,x,x)$ of $e^{-t\Delta_p}$ is continuous on $K \times (0,\infty) \times X$. Thus the bound ([0], p. 54)

$$\|K_p(t,x,x) - \sum_{n=0}^{n(k)} t^{(n-d)/2} a_n(x,p)\|_{\infty,k} < Ct^k$$

is uniform in p for k small enough, $n(k)$ being the same integer for all p in K. This yields a uniform bound for the error term $E(t,p)$ (which is the trace of the above difference and so is continuous).

Choose k large enough and use the Dominated Convergence Theorem to derive the continuity of $\int_0^1 t^{s-1}E(t,p)dt$. The continuity of the derivative at $s = 0$ follows from standard criteria.

The integral is analytic on $\{\mathrm{Re}(s) > -(n_0 - m)/2\}$. Thus considering the expansion ([0], p. 79) near the origin

$$\Gamma(s)\zeta_p(s) = \sum_{n\le n_0} 2(2s+n-m)^{-1}a_n(p) + r_{n_0}(s,p)$$

(with $r_{n_0}(s,p)$ analytic in $s \in \{\mathrm{Re}(s) > -(n_0-m)/2\}$ with derivative continuous in p) we have $p \to \zeta_p'(0)$ is continuous. ∎

2. An Example

Let X be a compact Riemann surface of genus $g > 1$ endowed with the canonical (Poincaré) metric. Narasimhan and Seshadri ([3]) considered the space

$$M_n = \{\text{equivalence classes } \tilde{\rho} \text{ of irreducible unitary representations } \rho : \pi_1(X) \to U(n) \text{ of dimension } n \text{ of } \pi_1(X)\}.$$

They showed that M_n is a complex manifold of complex dimension $n^2(g-1)-1$.

Let U be a (small) open subset of M_n. It is known that the associated holomorphic vector bundles F_ρ over x determined by $\tilde{\rho} \in U$ give a differentiable family of homomorphic vector bundles ([3], Remark on p.80). Further, the natural flat metrics form a differentiable family too ([6], Remark (iv), p. 17). One notes that equivalent unitary representations ρ, ρ' correspond to isometrically isomorphic flat bundles which have thus the same spectrum for their laplacians $\Delta_\rho, \Delta_{\rho'}$. Thus the eigenvalues $\lambda_m(\tilde{\rho})$ are functions defined on the whole of M_n.

Remark 1: By Kodaira–Spencer's Theorem each λ_m is continuous on M_n (for each n). ∎

Remark 2: For $n = 1, M_n = \text{Pic}(X)$, for the trivial character $\tau \in \text{Pic}(X), \tau \equiv 1, \Delta_\tau = \Delta_X$, the Laplace–Beltrami operator of X, 0 is an eigenvalue of Δ_p for $\tilde{\rho} \in \text{Pic}(X)$ if and only if $\tilde{\rho} = \tau$ ([11], p. 353). ∎

Remark 3: For $n > 1, \tilde{\rho} \in M_n$, 0 is never an eigenvalue of Δ_p ([11], p. 353). ∎

Remark 4: $\det \Delta_p$ is continuous on M_n for $n > 1$ and on $\text{Pic}(X) \setminus (\tau)$ by Proposition 1 above. ∎

Remark 5: Remark 4 can be deduced also from [7]. There it was shown that the evaluation of the Selberg Zeta Function at 1

$$Z_\Gamma(1, .) : \tilde{\rho} \to Z_\Gamma(1, \tilde{\rho})$$

is continuous on M_n for each n. For nontrivial $\tilde{\rho}_1, \tilde{\rho}_2$ the formula of Ray–Singer ([4], [7]) says

$$\det \Delta_{\tilde{\rho}_1} / \det \Delta_{\tilde{\rho}_2} = Z_\Gamma(1, \tilde{\rho}_1)/Z_\Gamma(1, \tilde{\rho}_2).$$

Fixing ρ_2 and varying ρ_1 one has the continuity of $\det \Delta_{\tilde{\rho}_1}$ in $\tilde{\rho}_1$ on M_n if $n > 1$ and $\text{Pic}(X) \setminus (\tau)$ if $n = 1$. ∎

Remark 6: det $\Delta_{\bar{\rho}}$ as defined in Section 1 is not continuous at the trivial character $\tau \in \mathrm{Pic}(X)$, for $Z_\Gamma(1,\tau) = 0$ ([1], p. 72), while by definition the determinant is positive. ∎

ACKNOWLEDGEMENT: The author would like to thank Professor Abdus Salam, the IAEA and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. He is grateful to Professor A. Verjovsky, Dr. Y. Hantout and Professor V. Pati for discussions and encouragement. He thanks the referee for pointing out the need for Step 2 in the proof of Proposition 1.

References

[0] P. B. Gilkey, *Invariance Theory, The Heat Equation and the Atiyah–Singer Index Theorem*, Publish or Perish Inc, 1984.

[1] D. Hejhal, *The Selberg Trace Formula for* PSL(2, $\mathbb{R}$), Lecture Notes in Math. **548**, Springer-Verlag, Berlin, 1976.

[2] K. Kodaira, *Complex Manifolds and Deformations of Complex Structures*, Springer-Verlag, Berlin, 1986.

[3] M. S. Narasimhan and C. S. Seshadri, *Holomorphic vector bundles on a compact Riemann surface*, Math. Ann. **155**(1964), 69–80.

[4] D. B. Ray and I. M. Singer, *Analytic torsion for complex manifolds*, Ann. of Math. **98** (1973), 154–177.

[5] R. T. Seeley, *Complex Powers of an Elliptic Operator*, Proceedings of Symposia in Pure Mathematics, Vol. X, AMS, 1967.

[6] Y.-T. Siu, *Lectures on Hermitian–Einstein Metrics for Stable Bundles*, DMV Lecture Series, Birkhäuser, Basel, 1987.

[7] S. Srinivas Rau, Ph.D. Thesis, University of Hyderabad, 1988.

[8] D. V. Widder, *The Laplace Transform*, Princeton University Press, 1946.