INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2010-11 (First Semester)

M. MATH. I YEAR
Algebra I

Date : 31.08.10 Maximum Marks : 60 Duration : $3\frac{1}{2}$ Hours

Note: R will denote a commutative ring with unity.
$\mathbb{Z}$: the ring of integers,
$\mathbb{Q}$: the field of rational numbers and
$\mathbb{C}$: the field of complex numbers.

1. Let k be an infinite field and $f(X_1, \cdots, X_n)$ be a non-zero element of the polynomial ring $k[X_1, \cdots, X_n]$. Show that there exist $a_1, \cdots, a_n$ such that $f(a_1, \cdots, a_n) \neq 0$. [6]
2. Prove that the polynomial ring $R[X]$ has infinitely many maximal ideals. [7]
3. Let $A = \mathbb{C}[X, Y, Z]/(XY - Z^2)$. Denote the images of X, Y, Z in A by x, y, z respectively.
   (i) Which of the ideals xA, $(x, y)A$ and $(x, z)A$ are prime ideals of A?
   (ii) Explicitly describe two maximal ideals of A. [6+4=10]
4. Let $R = \mathbb{Z}[\sqrt{-5}]$.
   (i) Prove that $R \cong \mathbb{Z}[X]/(X^2 + 5)$.
   (ii) Prove that $R/3R \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.
   (iii) Show that 3 is not a prime in R.
   (iv) Examine whether 3 is an irreducible element of R.
   (v) Give an example of an idempotent element of the ring $R/3R$. [4+4+1+3+2=14]
5. Describe all units in the Eisenstein ring $\mathbb{Z}[\omega]$. [4]
6. Let $R = \mathbb{Z}[i]$, $u = -3 + 11i$ and $v = 8 - i$. Find d, a, b in R such that $(u, v)R = dR$ and $d = au + bv$. [7]
7. State whether the following statements are TRUE or FALSE with brief justification.
   (i) The ring $C[0, 1]$ of real-valued continuous functions on $[0, 1]$ has no non-zero nilpotents.
   (ii) If P is a prime ideal of R and I any ideal of R, then the image of P in R/I is a prime ideal of R/I.
   (iii) If R is any commutative ring and $f(X) \in R[X]$ is a polynomial of degree n, then $f(X)$ has at most n roots in R.
   (iv) If R has exactly one maximal ideal then R cannot contain any non-trivial idempotent.
   (v) $(X - 1)\mathbb{Q}[[X]]$ is a maximal ideal of $\mathbb{Q}[[X]]$.
   (vi) Any subring of a PID is a PID. [$3 \times 6 = 18$]

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination : 2010-2011
M.Math.-I & M.Stat-II
Topology

Date : 02.09. 2010 | Maximum Score : 40 | Time : $2\frac{1}{2}$ Hours

Answer all questions.
Any result that you use should be stated clearly.

1.
- Define the notion of a basis for a topology on a non-empty set X.
- Prove that a basis generate a topology on X.

[2+4=6]

2.
- Describe the Hausdorff property of a topological space.
- Let X denote the product space of an indexed family of Hausdorff topological spaces $\{X_\alpha;\ \alpha \in J\}$, J being the indexing set. Prove that X is Hausdorff.
- Prove that a topological space X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) : x \in X\}$$

is closed in $X \times X$.

[2+4+6=12]

3.
- Define limit point of a subset A of a topological space X.
- Let X be the set of all real numbers with the finite complement topology and let A be the subset of all integers. What is the closure of A in X? Justify your answer.

[2+4=6]

P.T.O

4. Let R be an equivalence relation on a topological space X and $q : X \longrightarrow X/R$ be the quotient map.

- Prove that if the quotient space X/R is Hausdorff then R is a closed set in $X \times X$.
- Suppose q is an open map and R is closed in $X \times X$. Then prove that X/R is Hausdorff.

[4+4=8]

5. Let X be a G-space, G being a finite group.

- Explain the notion of the orbit space X/G.
- Prove that the map $X \longrightarrow X/G$, sending an element $x \in X$ to its orbit, is an open map.
- Prove that if X is Hausdorff then X/G is also Hausdorff.

[2+4+10=16]

Indian Statistical Institute

Mid-semsetral examination : (2010-11)

M. Math I year

Analysis of Several Variables

Date : 6.9.10 Maximum marks : 40 Duration : 2 hours.

Answer ALL questions. Marks are indicated in brackets.

(1) Suppose that F is a continuous 1-1 map from an open subset U of $\mathbb{R}^n$ onto an open subset V of $\mathbb{R}^n$, and the inverse map $F^{-1} : V \to U$ is continuous too. Let $a \in U$, $b = F(a)$. Assume furthermore the following:

(i) F is differentiable at the point a and $F'(a)$ is invertible;

(ii) $\exists\ \delta > 0$ and $C > 0$ such that $B_\delta(b) \subseteq V$ and $\|F^{-1}(b_1) - F^{-1}(b_2)\| \leq C\|b_1 - b_2\|$ for all b_1, b_2 in $B_\delta(b)$.

Prove that F^{-1} is differentiable at the point b.

[10]

(2) Let S be the solid sphere given by $\{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \leq 1\}$. Compute its volume given by the integral $\int_S dxdydz$.

(Hint: use a suitable change of variable and justify it.)

[10]

(3) Prove that a continuously differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^m$ cannot be one-to-one unless $m \geq n$.

(Hint: recall that rank of an $m \times n$ matrix A is the maximum integer r for which A has $r \times r$ invertible *principal* submatrix.)

[10]

(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by:

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

Find out all the points where f attains a local maximum or local minimum, with justifications.

[10]

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination: 2010-2011

M. Math. I Year

Linear Algebra

Date: Sept 3, 2010 **Maximum Marks: 40** **Duration: 2 hours**

Note: There are 5 questions on this test. Answer all questions. JUSTIFY your answers. Each question carries 8 marks.

1. Let V be a vector space over the field F. [4+4]

   (a) Define the algebraic dual V^* of V. Show that V^* is isomorphic to V if $\dim_F(V)$ is finite.

   (b) Consider the map $\iota : V \to (V^*)^*$ defined by $\iota(v)(\phi) = \phi(v)$ for each $v \in V$ and each $\phi \in V^*$. Show that ι is an injective linear map which is an isomorphism if $\dim_F(V)$ is finite.

2. Suppose V and W are finite dimensional vector spaces over the field F. [5+3]

   (a) Show that $\dim_F(V \otimes_F W) = \dim_F(V)\dim_F(W)$.

   (b) If $B(V,W)$ is the space of all bilinear maps from $V \times V$ to W, find the dimension of $B(V,W)$ in terms of $\dim_F(V)$ and $\dim_F(W)$ using (a).

3. Consider $\mathbb{R}^3$ to be equipped with the Euclidean inner product. Let W be the subspace of $\mathbb{R}^3$ defined by $W = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$. [2+3 +3]

   (a) Show that $\mathcal{A} = \{(1,-1,0), (1,0,-1)\}$ and $\mathcal{B} = \{(1,1,1)\}$ are ordered bases for W and the orthogonal complement $W^\perp$ respectively.

   (b) Suppose $P : \mathbb{R}^3 \to W$ is the orthogonal projection map. Find the matrix representing P with respect to the ordered bases $\{(1,-1,0), (1,0,-1), (1,1,1)\}$ and $\mathcal{A}$.

   (c) Then find the matrix representing P with respect to the standard basis of $\mathbb{R}^3$ and the basis $\mathcal{A}$.

4. Let $||x||$ denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. Show that an $n \times n$ real matrix P is orthogonal if and only if $||Px|| = ||x||$ for all $x \in \mathbb{R}^n$. [8]

5. Suppose $A \in M_{n,n}(\mathbb{C})$. Denote the conjugate transpose of A by A^*. Then prove that [1+3+4]

   (a) A^*A is Hermitian

   (b) All eigenvalues of A^*A are real and nonnegative

   (c) $\det(I + A^*A)$ is real and positive.

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination – Semester I : 2010-2011

M. Math. I Year

Measure Theoretic Probability

08·09·10

Date : [redacted] | Maximum Score : 70 | Time : 3 Hours

Note : This paper carries questions worth a **total** of **84 MARKS**. Answer as much as you can. The **MAXIMUM** you can score is **70**.

1. When is a class $\mathcal{S}$ of subsets of a non-empty set Ω called a semi-field? Show that if $\mathcal{S}$ is a semi-field, then the class of subsets obtained by forming all possible finite disjoint unions of sets in $\mathcal{S}$ is closed under finite intersections and complementations.

(4+8)=[12]

2. Let $A_1, \ldots, A_n$ be non-empty subsets of a set Ω. Consider sets obtained by taking the intersections $\bigcap_{i=1}^{n} B_i$ where each B_i equals either A_i or A_i^c. Let $C_1, \ldots, C_k$ be the non-empty sets obtained this way. Show that $C_1, \ldots, C_k$ form a partition of Ω and that the σ-field generated by the sets $A_1, \ldots, A_n$ is the same as that generated by the sets $C_1, \ldots, C_k$, which in turn consists of all possible unions from the sets $C_1, \ldots, C_k$.

(8+8+8)=[24]

3. Let $\mathcal{A}$ is a σ-field on a non-empty set Ω. Show that, if $f : \Omega \to \mathbb{R}$ is a function such that, for every $x \in \mathbb{R}$, the set $\{\omega \in \Omega : f(\omega) \leq x\}$ belongs to $\mathcal{A}$, then for any borel set B, the set $\{\omega \in \Omega : f(\omega) \in B\}$ also belongs to $\mathcal{A}$.

[8]

4. Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function and μ be the associated Radon measure on $\mathcal{B}(\mathbb{R})$. Suppose $a < b$ are two real numbers. Show that the function G defined by $G(x) = F(x)$ for $a \leq x < b$, $G(x) = F(a)$ for $x < a$ and $G(x) = F(b)$ for $x \geq b$ is a distribution function and that the associated measure ν on $\mathcal{B}(\mathbb{R})$ is given by $\nu(B) = \mu(B \cap (a, b])$.

(8+8)=[16]

5. Let μ be a measure on $\mathcal{B}(\mathbb{R})$ satisfying the property that $\mu(B + x) = \mu(B)$ for all borel sets B and all $x \in \mathbb{R}$. Suppose $\mu((0, 1]) = c < \infty$.
(a) Show that, for any two integers $a < b$, $\mu((a, b]) = c \cdot (b - a)$.
(b) Show that, for any two rational numbers $a < b$, $\mu((a, b]) = c \cdot (b - a)$. [Show first that $\mu((0, \frac{1}{n}]) = c \cdot \frac{1}{n}$]
(c) Conclude that $\mu \equiv c \cdot \lambda$ on $\mathcal{B}(\mathbb{R})$, where λ denotes the lebesgue measure.

(8+8+8)=[24]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2010-11 (First Semester)

M. MATH. I YEAR
Algebra I

Date : 22.11.10 | Maximum Marks : 70 | Duration : $3\frac{1}{2}$ Hours

ANSWER Q. 1 AND ANY FIVE FROM THE REST.
Clearly state the results that you use.
R will denote a commutative ring *with unity.*
$\mathbb{Z}$: the ring of integers.
$\mathbb{Q}$: the field of rational numbers
$\mathbb{R}$: the field of real numbers
$\mathbb{C}$: the field of complex numbers.

1. State whether the following statements are TRUE or FALSE with brief justification.

   (i) If B is a subring of A such that B and A are isomorphic as rings, then $B = A$.

   (ii) The ring $\mathbb{Z}[i]/(2)$ has exactly one non-zero nilpotent element.

   (iii) If a and b are coprime elements in a UFD R, then $aX - b$ is a prime element of $R[X]$.

   (iv) In $R = \mathbb{Z}[X]$, the ideal $(5X, 9X)R$ is a free R-module.

   (v) If N and P are internal direct summands of M, then $N + P$ is necessarily an internal direct summand of M. [$5 \times 3 = 15$]

2. Show that the following rings are Noetherian *integral domains.*

   (i) $\mathbb{Q}[X]/(f)$, where $f = X^5 + 5X^4 + 10X^3 + 10X^2 + 7X + 5$.

   (ii) $\mathbb{R}[X, Y]/(g)$, where $g = X^4 + 3X^2Y^2 + X^2 + 2XY^2 + 7X + Y^3$.

   (iii) $\mathbb{C}[X, Y, Z]/(h)$, where $h = X^2 + Y^2 + Z^2 - 1$. [12]

3. (i) Prove that any non-zero commutative ring R has a maximal ideal.

   (ii) Deduce that any two bases of a finitely generated free module over a commutative ring have the same cardinality.

   (iii) Give an example of a non-zero module over a PID which does not have any maximal submodule. [5+3+4=12]

4. (i) Let $I_1, \cdots, I_n$ be ideals in R such that $I_1 \cap \cdots \cap I_n = 0$. If each R/I_j is a Noetherian ring, then show that R is a Noetherian ring.

   (ii) Let $\phi : M \to F$ be a surjective R-linear map from a finitely generated R-module M onto a free R-module F. Show that the kernel of ϕ is finitely generated. [6+6=12]

P. T. O.

5. Let M and N be modules over a commutative ring R.

(i) Define the tensor product $M \otimes_R N$.

(ii) If M and N are Noetherian, then show that $M \otimes_R N$ is Noetherian.

(iii) Compute $M \otimes_R N$ when $R = \mathbb{Z}$, $M = \mathbb{Q}[\omega]$ and $N = \mathbb{Z}/3\mathbb{Z}$. [3+6+3=12]

6. (i) Let G be a finite group which acts transitively on X. Let N be a normal subgroup of G. Show that all the orbits of the induced action of N on X have the same size.

(ii) Show that no group of order 24 can be simple. [7+5=12]

7. (i) Prove that any group with more than two elements has a non-trivial automorphism.

(ii) How many non-isomorphic Abelian groups are there of order 90000? [9+3=12]

Indian Statistical Institute

Semsetral examination : (2010-11)

M. Math I year

Analysis of Several Variables

Date : 18-11-10 Maximum marks : 60 Duration : 3 hours.

Answer ALL questions. Marks are indicated in brackets.

(1) Let Φ, ω and f be a smooth k-chain in $\mathbb{R}^n$, smooth $(k-1)$-form in $\mathbb{R}^n$ and smooth function on $\mathbb{R}^n$ respectively. Prove that

$$\int_\Phi f d\omega = \int_{\partial\Phi} f\omega - \int_\Phi (df) \wedge \omega.$$

As a corollary to this, derive the formula for integration by parts of calculus (of one variable). [9+3=12]

(2)Let E be an open convex subset of $\mathbb{R}^n$ and ω is a smooth 1-form in E. prove that ω is exact, i.e. $\omega = df$ for some smooth function f on E, if and only if $\int_\gamma \omega = 0$ for every smooth closed curve γ in E (closed means $\gamma(0) = \gamma(1)$). [10+5=15]

(3) Consider the smooth curve γ in $\mathbb{R}^3$ given by the intersection of the sphere $S = \{(x,y,z) : x^2 + y^2 + z^2 = 2ax\}$ and the infinite cylinder $B = \{(x,y,z) : z \geq 0, x^2 + y^2 = 2bx\}$, where $a > b > 0$ are constants. The orientation of the curve is chosen in such a way that the curve begins from the origin and moves first in the positive octant $\{(x,y,z) : x \geq 0, y \geq 0, z \geq 0\}$.

(i) Write down a parametrization of the above curve γ.

(ii) Prove that $\int_\gamma \omega = -2\pi ab^2$, where ω is the one-form given by

$$\omega = (y^2 + z^2)dx + (z^2 + x^2)dy + (x^2 + y^2)dz.$$

[5+15=20]

(4) We say that an $n \times n$ matrix A with real entries is conformal if satisfies $A'A = AA' = cI$ for some constant $c > 0$, where A' denotes the transpose of A. Moreover, a smooth map f from some open set of $\mathbb{R}^n$ to $\mathbb{R}^n$ is called conformal if $Df(x)$ is conformal for every x in the domain of f. Now, let $n \geq 2$ and suppose that $g : (0,\infty) \to \mathbb{R}$ is a smooth map, and $f(x) := g(\|x\|)x$ for $x \in \mathbb{R}^n$, $x \neq 0$. Prove that f is conformal if and only if g is either a nonzero constant map or of the form $g(t) = \frac{c}{t^2}$ for some $c \neq 0$.

[18]

INDIAN STATISTICAL INSTITUTE
Semester Examination : 2010-2011
M.Math.-I & M.Stat-II
Topology

Date :29.11.2010 | Maximum Score : 60 | Time : 3 Hours

Answer all questions.
Any result that you use should be stated clearly.

1. Prove that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

[8]

2.
   - Explain the notion of local path connectedness of a topological space X.
   - Prove that a space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

[2+10=12]

3.
   - What is a normal topological space?
   - Prove that every regular second countable space is normal.

[2+12=14]

4.
   - Let $f : X \longrightarrow Y$ be a continuous surjection and X be compact. Prove that Y is also compact.
   - Explain what is the real projective space $\mathbb{R}P^2$. Prove that it is a compact space.

[4+5=9]

P.T.O

5.
- Explain the notions of a filter and an ultrafilter on a set X.
- Prove that a filter $\mathcal{F}$ is an ultrafilter if and only if for any subset A of X either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

[4+8=12]

6.
- What is a contractible space? Prove that a space X is contractible if and only if for any space Y any two maps $f, g : X \longrightarrow Y$ are homotopic.
- Compute fundamental group of S^1.

[2+5+8=15]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination: 2010-2011

M. Math. Year I

Linear Algebra

Date: December 2, 2010 Maximum Marks: 60 Duration: 3 hours

Note: There are TEN questions in this test. Answer ANY EIGHT. JUSTIFY your answers. Each question carries 8 marks.

1. Let V be a finite dimensional vector space. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two ordered bases for V. Define the transition matrix S from $\mathcal{A}$ to $\mathcal{B}$. Suppose a linear transformation $T : V \to V$ is represented by the matrix A in the basis $\mathcal{A}$. Find the matrix representing T in the basis $\mathcal{B}$ in terms of S and A. [3+5]

2. Define the tensor product of two finite dimensional vector spaces V and W over the field F. Show that $Hom_F(V, W) \cong V^* \otimes_F W$. [4+4]

3. Consider $\mathbf{R}^4$ to be equipped with the Euclidean inner product. Let V be the subspace of $\mathbf{R}^4$ defined by $V = \{(x, y, z, w) \in \mathbf{R}^4 : x + y + z = 0, w - z = 0\}$. [4+4]

   (a) Find an (ordered) orthogonal basis $\mathcal{A}$ for V.

   (b) Find the matrix representation of the orthogonal projection operator $P : \mathbf{R}^4 \to V$ in terms of the standard basis of $\mathbf{R}^4$ and the basis $\mathcal{A}$ of V.

4. Suppose λ is an eigenvalue of a Hermitian matrix A. Define the algebraic and geometric multiplicities of λ. Show that they are equal. [8]

5. Let V be a finite dimensional vector space over $\mathbf{R}$. [4+4]

   (a) Verify that the pairing $(,) : \bigwedge^k(V^*) \times \bigwedge^k(V) \to \mathbf{R}$ defined by
   $$(v_1^* \wedge \ldots \wedge v_k^*, u_1 \wedge \ldots \wedge u_k) = \det(v_i^*(u_j)),$$
   on decomposable elements, is nonsingular.

   (b) Use the pairing to define a natural isomorphism between $\bigwedge^k(V^*)$ and $(\bigwedge^k(V))^*$.

6. Suppose R is an integral domain and M is an R-module. Define $\mathrm{Tor}(M)$ and $\mathrm{rank}(M)$. If $R = \mathbf{Z}[x]$, find a rank one torsion-free R-module which is not free. [3+5]

7. (a) When is an R-module M said to be finitely generated? Give an example of a finitely generated module such that it has a submodule which is not finitely generated. [2+3]

   (b) If every submodule of M is finitely generated, then show that M must satisfy the ascending chain condition on submodules. [3]

[Please turn over]

8. Let V be a vector space over a field F and let S and T be linear transformations of V. Prove that S and T are similar linear transformations if and only if the $F[x]$-module structures on V defined by S and T are isomorphic. [8]

9. Find the invariant factors and the rational canonical form of the following matrix A (over the field $\mathbf{Q}$). Determine if the matrix A is diagonalizable. [6 + 2]

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

10. Let R be a commutative ring and M be a finitely generated R-module with trivial annihilator. Show that R is isomorphic to a submodule of M^n for some integer n. [8]

INDIAN STATISTICAL INSTITUTE
Semestral Backpaper Examination: 2010-11 (First Semester)

M. MATH. I YEAR
Algebra I

Date : 10/01/11 Maximum Marks : 100 Duration : 3 Hours

$\mathbb{Z}$: the ring of integers.
$\mathbb{Q}$: the field of rational numbers
$\mathbb{R}$: the field of real numbers
$\mathbb{C}$: the field of complex numbers.

1. Prove that any finitely generated subgroup of $\mathbb{R}$ is countable. Deduce that $\mathbb{R}$ is not finitely generated. [12]

2. Let G be a simple group of order 60. Show that if X is a set with $|X| \leq 4$, then the only group action of G on X is the trivial action. [10]

3. Let G be a finite Abelian group. Let $f : G \to G$ be defined by $f(x) = x^n$, where n is coprime to $|G|$. Show that $f \in Aut\ G$. [10]

4. Let $R = \mathbb{Q}[X]/(X^3 - X)$.

   (i) Prove that R is isomorphic to the product ring $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

   (ii) Which element of $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ corresponds to the element $\overline{X}$ of R?

   (iii) Describe all prime and maximal ideals of R. [15]

5. Show that if R is a Noetherian ring, then so is $R[X]$. [16]

6. Let $M_1, \cdots, M_n$ be submodules of a module M such that each M/M_i is Noetherian. Show that $M/M_1 \cap \cdots \cap M_n$ is Noetherian. [10]

7. Define the tensor product $M \otimes_R N$ of modules M and N over a commutative ring R. Show that the tensor product exists and is unique up to isomorphism. [12]

8. State whether the following statements are TRUE or FALSE with brief justification.

   (i) If the groups G and H have the same composition series, then they are isomorphic.

   (ii) The image of X in $\mathbb{Z}[X,Y]/(XY - 1)$ is a prime element.

   (iii) $\mathbb{C}[X,Y,Z]/(X^2 + Y^2 + Z^2 - 1)$ is not a Euclidean domain.

   (iv) Any submodule of a finitely generated module is finitely generated.

   (v) If every ideal of a non-zero commutative ring R is a free R-module, then R is either a field or a PID. [$5 \times 3 = 15$]

Indian Statistical Institute

Backpaper examination : (2010-11)

M. Math I year

Analysis of Several Variables

Date :12 . 1 . 11 Maximum marks : ~~100~~ 45 Duration : 3 hours.

Answer ALL questions. Marks are indicated in brackets.

(1) Prove that a continuously differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ cannot be one-to-one.
[15]

(2) Let $E = \{x \in \mathbb{R}^3 : x \neq 0\}$. and ω be the 2-form in E given by

$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}.$$

(i) Prove that ω is closed, i.e. $d\omega = 0$ in E.

(ii) Is ω exact in E ? Justify your answer.

[10+20=30]

(3) Let E be as in Problem 2 above. Prove or disprove (with arguments) the following: Every closed one-form in E is exact.

[25]

(4) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable function such that $F(x, y, z) = F(x', y', z')$ whenever $x - y = x' - y'$ and $y - z = y' - z'$. Prove that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \equiv 0.$$

[15]

(5) Let γ be the curve obtained by intersecting the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ with the plane $\{(x, y, z) : x + y + z = 1\}$, and orientation of γ is chosen in such a way that the curve starts at $(1, 0, 0)$ and then moves into the positive octant first. Evaluate the following integral:

$$\int_\gamma (y dx + z dy + x dz).$$

[15]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination – Semester I : 2010-2011

M. Math. I Year

Measure Theoretic Probability

Date: 18. 1. 11 **Maximum Score : 45** Time : 3 Hours

Note : This paper carries **FOUR QUESTIONS**, each carrying **25 MARKS**. Answer as much as you can. The **MAXIMUM** you can score is **45**.

RESULTS USED MUST BE CLEARLY STATED.

1. Let G be a non-decreasing real-valued function on $\mathbb{R}$, which is continuously differentiable everywhere. Show that the Radon measure μ induced by G satisfies $\mu(B) = \int_B G'(x)\,dx$, for every Borel set $B \subset \mathbb{R}$ and that $\int f\,d\mu = \int f(x)G'(x)\,dx$, for any measurable function $f : \mathbb{R} \to \mathbb{R}$.

2. Let λ denote the Lebesgue measure on the real line. Show that for any Borel set $B \subset \mathbb{R}$ with $\lambda(B) < \infty$ and for any $\epsilon > 0$, there is a finite disjoint union F of intervals such that $\lambda(B \Delta F) < \epsilon$. Use this to show that continuous functions with compact support are dense in $L_1(\mathbb{R}, \mathcal{B}, \lambda)$.

3. (a) State and prove Borel-Cantelli Lemma.

   (b) Show that if $\{X_n\}$ is a sequence of real random variables converging in probability to a random variable X, then there is a subsequence converging almost surely.

4. (a) State and prove Fubini's Theorem.

   (b) Using Fubini's Theorem, show that for any non-negative random variable X, $E[\tan^{-1}(X)] = \int_0^\infty (1+u^2)^{-1} P(X > u)\,du$.

INDIAN STATISTICAL INSTITUTE
Semester(Back Paper) Examination : 2010-2011
M.Math.-I & M.Stat-II
Topology

Date : 20 · 1 · 11 | Maximum Score : 45 | Time : 3 Hours

Answer all questions.
Any result that you use should be stated clearly.

1.
- Define the notion of a topology on a non-empty set X.
- Let (X, d) be a metric space. Prove that the metric d on X defines a topology on X.

[2+8=10]

2. Let $f : X \longrightarrow Y$ be a continuous surjection and X be connected. Prove that Y is also connected.

[6]

3.
- What is a normal topological space?
- Prove that a metric space is always a normal space.
- Prove that a compact T_2 space is normal.

[2+4+6=12]

4.
- Explain the notion of local connectedness of a topological space X.
- Prove that a space X is locally connected if and only if for every open set U of X, each path component of U is open in X.

[2+10=12]

5.
- Explain the notion of sequential compactness.
- Prove that a sequentially compact metric space is compact.

[2+7=9]

P.T.O

6. Let X be a Hausdorff space and A be a compact subset in X and x be a point not belonging to A. Prove that there exist disjoint open subsets U and V in X such that $x \in U$ and $A \subset V$.

[6]

7. Suppose X and Y are compact spaces. Prove that $X \times Y$ with the product topology is compact.

[8]

8.
- Explain the notion of a filter on a set X.
- Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a function. Prove that f is continuous if and only if for every filter $\mathcal{F}$ on X converging to a point $x \in X$, the filter $f_*(\mathcal{F})$ converges to $f(x)$.

[4+10=14]

9. Prove that a contractible space is simply connected.

[10]

10. Compute fundamental group of S^1 and hence show that there does not exist a continuous function $f : D^2 \longrightarrow S^1$ whose restriction to S^1 is the identity map $id : S^1 \longrightarrow S^1$.

[8+5=13]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination
Second semester 2010–2011
M.Math (First year)
Differential Geometry

21

Date: February, 2011

Maximum Marks: 60
Duration: 2 hours 30 minutes

Answer all questions.
State clearly any result that you use in your answer.

(1) Let $\mathbf{X}$ be a vector field on $\mathbb{R}^2$ defined by $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_1, -x_2)$. Determine the maximal integral curve of this vector field through a point (a, b). What happens when $(a, b) = (0, 0)$? 6

(2) Let $\alpha : I \to \mathbb{R}^n$ be a maximal integral curve of a smooth vector field Y satisfying $\alpha(0) = \alpha(t_0)$, where $0, t_0 \in I$. Prove that $I = \mathbb{R}$ and α is periodic. 8

(3) Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $p_0 \notin S$. Suppose p is a point of S such that $\|p - p_0\| \leq \|q - p_0\|$ for all $q \in S$. Then show that $(p, p - p_0)$ is orthogonal to S_p. 8

(4) Consider the right circular cylinder $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$. If α is a geodesic in S then prove that it is of the following form:

$$\alpha(t) = (\cos(at + b), \sin(at + b), ct + d) \text{ for } t \in \mathbb{R},$$

where a, b, c, d are constants. 8

(5) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function defined by $f(x, y, z) = x^2 + y^2 - z^2$. Let $S = f^{-1}(1)$ and $p = (1, 0, 0)$. Note that $p \in S$.

(a) Prove that S is a 2-surface in $\mathbb{R}^3$. Show that $\mathbf{v} = (p, 0, 1, 1)$ and $\mathbf{w} = (p, 0, -1, 1)$ are in the tangent space S_p.

(b) Choose an orientation for S. Determine the matrix of the Weingarten map L_p relative to the basis $\mathbf{v}, \mathbf{w}$. Hence find the Gauss-Kronecker curvature of S at p. 4+8

(6) Let $\alpha : I \to S$ be a smooth curve in an n-surface S in $\mathbb{R}^{n+1}$.

(a) Explain the notion of a parallel vector field on S along α.

(b) Justify the following statement:
There is a unique parallel vector field $\mathbf{X}$ *along* α *such that* $\mathbf{X}(t_0) = (p, v)$, *where* $t_0 \in I$, $\alpha(t_0) = p$ *and* $(p, v) \in S_p$.

(c) Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are two parallel vector fields on S along α. Then show that the dot product of X and Y is a constant function along α.

(d) Show by an example that if α and β are two curves in S from p to q then the parallel transport of $(p, v) \in S_p$ along α and β need not give the same vector at q. 4+4+4+6

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2010-2011

M. Math I Year

Algebra II

Date: February 22, 2011 **Maximum Marks: 35** **Duration:** 3 hours

1. Indicate TRUE/FALSE with justification. [8]

   (a) A finite field cannot be algebraically closed.

   (b) Let $k \subset F \subset L$ be a tower of fields where F/k and L/F are both finite Galois. Then L/k is also finite Galois.

   (c) If L/k is a field extension of prime degree, then $L = k(\alpha)$ for every $\alpha \in L \setminus k$.

   (d) i. Any degree 2 extension of a field is normal.

      ii. Any degree 2 extension of a field is separable.

2. Let $f(X)$ be an irreducible polynomial in $k[X]$ and let L be a finite normal field extension of k. Suppose that $f(X) = g(X)h(X)$, where $g(X), h(X) \in L[X]$ and $g(X), h(X)$ are irreducible in $L[X]$. Show that there exists a k-automorphism σ of L such that when σ is extended to $L[X]$ by defining $\sigma(X) = X$, we have $\sigma(g(X)) = h(X)$. [8]

3. Let k be a field and $f(X), g(X)$ be coprime in $k[X]$. Prove that

$$[k(t) : k(\frac{f(t)}{g(t)})] = \max\{\deg f, \deg g\}$$

[8]

4. (a) Let n be a positive integer and $\Phi_n(X)$ denote the nth cyclotomic polynomial over $\mathbb{Q}$. Prove that $\Phi_n(X) \in \mathbb{Z}[X]$. [3]

   (b) Prove that the degree of the cyclotomic field extension $\mathbb{Q}(\zeta_n)$ is $\varphi(n)$, where φ is the Euler φ-function. [5]

5. (a) Let $X^3 + bX + c \in \mathbb{Q}[X]$ be irreducible and let K be its splitting field. Prove that $[K : \mathbb{Q}] = 3$ or 6 according as whether $-4b^3 - 27c^2$ is a square or not. [5]

   (b) Find the Galois group of $X^3 - 4X + 1$ over $\mathbb{Q}$. [3]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination (2010–2011)

M MATH I

Functional Analysis

Date : 24.02.2011 | Maximum Marks : 60 | Time : 2 hrs.

This paper carries 65 marks. Maximum you can score is 60. Precisely justify all your steps. Carefully state all the results you are using.

1. Let X be a normed linear space over $\mathbb{K}$ and $f : X \to \mathbb{K}$ be a linear map. Show that $\ker f = \{x \in X : f(x) = 0\}$ is either closed or dense in X. [15]

2. Let $\{e_n\}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Suppose $\{f_n\} \subseteq \mathcal{H}$ is an orthonormal set such that
$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1.$$
Show that $\{f_n\}$ is also an orthonormal basis for $\mathcal{H}$. [15]

3. Let Y be a linear subspace of $C[0,1]$. Show that Y is closed in $L^2[0,1]$ if and only if Y is finite dimensional. [5+(5+10+10+5)=35]

   To prove necessity, argue as follows :

   (a) Show that Y is also closed in $C[0,1]$.

   (b) Show that there is a constant $M > 0$ such that for all $f \in Y$,
   $$\|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2.$$

   (c) If $\{f_1, f_2, \ldots, f_n\}$ is a finite orthonormal set in $Y \subseteq L^2[0,1]$, then show that
   $$\sum_{k=1}^{n} |f_k(x)|^2 \leq M^2 \text{ for all } x \in [0,1].$$

   [Hint : Consider $f = \sum_{k=1}^{n} \alpha_k f_k$ in (b) for a suitable choice of α_k's.]

   (d) Show that $\dim(Y) \leq M^2$.

   [Note : The constant in (c) and (d) is same as that in (b).]

Indian Statistical Institute
Mid-semester Examination
2010-11 (Second Semester)
M. Math. 1st. Year
Complex Analysis

Date and Time: Friday 25.2.11, 2:30 - 4:30 pm

Total Points: 3

Answers must be justified with clear and precise arguments. If you use any theorem/propositic proved in class state it explicitly.

1. Suppose that f is holomorphic in a region Ω and γ is a closed curve in Ω. Show that
$$\int_\gamma \overline{f(z)} f'(z) dz$$
is purely imaginary. (Hint: write $w = f(z)$ and do the calculation in the image space.) 5 pts

2. Using contour integration show that
$$\int_0^\infty \frac{1}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}, a > 0.$$
5 pts

3. Suppose that f and g are holomorphic in a region containing the disc $\{z : |z| \leq 1\}$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $\{z; |z| \leq 1\}$. Let
$$f_\epsilon(z) = f(z) + \epsilon g(z).$$
Show that if ϵ is sufficiently small then 5 + 5 = 10 pts
(a) $f_\epsilon(z)$ has a unique zero in $\{z : |z| \leq 1\}$, and
(b) if z_ϵ is this zero then the mapping $\epsilon \mapsto z_\epsilon$ is continuous.

4. Suppose A is an $n \times n$ matrix. For ζ such that $|\zeta|$ is sufficiently large, the expansion
$$(\zeta I - A)^{-1} = \sum_{k=0}^{\infty} \frac{A^k}{\zeta^{k+1}}$$
is valid (e.g. when $|\zeta| > ||A||$, a suitable matrix norm of A). Assuming the above expansion,
5 + 5 = 10 pts
(a) for $|\zeta| = r$, with r sufficiently large, show that for any polynomial of degree m in A, one has
$$p(A) = \frac{1}{2\pi i} \int_{|\zeta|=r} p(\zeta)(\zeta I - A)^{-1} d\zeta.$$
(Note that the integrand on the right hand side contains matrices. You can first prove it for $p(\zeta) = \zeta^k, k \in \mathbb{N}$.)
(b) Call the above the Cauchy integral formula for polynomials with matrix argument. Now prove the Cayley-Hamilton theorem: if $p(\zeta)$ is the characteristic polynomial of A, then $p(A) = 0$, by completing the argument below,
The characteristic polynomial of A is $p(\zeta) = det(\zeta I - A)$. Note that the ijth entry of $(\zeta I - A)^{-1}$ is
$$((\zeta I - A)^{-1})_{ij} = \frac{M_{ij}(\zeta)}{[det(\zeta I - A)]},$$
where $M_{ij}(\zeta)$ is the ijth cofactor, a polynomial of degree $n-1$ in ζ. **You have to prove that for this particular $p(\zeta) = det(\zeta I - A)$, the Cauchy integral formula proved in (a) gives the zero matrix.**

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination : 2010-2011
M.Math.-I & M.Stat-II
Topology-II

Date :28.02.11 | Maximum Score : 40 | Time : 2 Hours

Answer all questions.
Any result that you use should be stated clearly.

1.
- Define a Contractible space.
- Suppose X is a contractible space and $f : X \longrightarrow Y$ is a surjective continuous function. Should Y be also contractible? Justify your answer.
- Prove that retract of a contractible space is contractible.

[2+3+5=10]

2.
- Prove that $\mathbb{R}^{n+1} - \{(0, 0, \cdots, 0)\}$ has the same homotopy type as the space S^n.
- Suppose $f : X \longrightarrow S^n$ is a continuous function which is not surjective. Prove that f is nullhomotopic.

[5+5 =10]

3.
- Define degree $deg([\sigma])$ of an element $[\sigma] \in \pi_1(S^1, 1)$. Prove that the degree of the homotopy class represented by the loop $\sigma : [0, 1] \longrightarrow S^1$ given by $\sigma(t) = e^{2\pi imt}$ is $m \in \mathbb{Z}$.
- Prove that the function $deg : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ is an isomorphism.

[[illegible]+6=10]

P.T.O

40

4. • Let X, Y be path connected spaces and $X \simeq Y$. Prove that for any $x_0 \in X$ and $y_0 \in Y$,

$$\pi_1(x, x_0) \cong \pi_1(Y, y_0).$$

• Is S^1 homotopically equivalent to S^2? Justify your answer.

[8+2=10]

5. • Prove that for any two pointed spaces (X, x_0) and (Y, y_0).

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(x, x_0) \times \pi_1(Y, y_0).$$

• Compute fundamental group of $T^2 \vee S^1$, where $T^2 = S^1 \times S^1$.

[5+5=10]

INDIAN STATISTICAL INSTITUTE

Semester Examination

Second semester 2010–2011

M.Math (First year)

Differential Geometry

Date: 29 April, 2011

Maximum Marks: 60

Duration: 3 hours

Answer all questions. State clearly any result that you use in your answer.

Unless stated otherwise all maps and vector fields are assumed to be smooth.

For any n-surface S and a point $p \in S$, S_p will denote the tangent space of S at p.

(1) Let $\phi : U \to \mathbb{R}^{n+1}$ be a smooth parametrized n-surface in $\mathbb{R}^{n+1}$. Let $\mathbf{e}_i$, $i = 1, 2, \ldots, n$, denote the coordinate vector fields on $\mathbb{R}^n$ and $\mathbf{E}_i = d\phi(\mathbf{e}_i)$ the coordinate vector fields along ϕ.

(a) Show that $\nabla_{\mathbf{e}_i}\mathbf{E}_j = \nabla_{\mathbf{e}_j}\mathbf{E}_i$.

(b) Suppose that ϕ is one-to-one. Define vector fields $\mathbf{V}_i$ on $\phi(U)$ by $\mathbf{V}_i = \mathbf{E}_i \circ \phi^{-1}$. Prove that the Lie bracket $[\mathbf{V}_i, \mathbf{V}_j] = 0$. 6+7

(2) Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices over reals and $S_n(\mathbb{R})$ the subset of $M_n(\mathbb{R})$ consisting of all symmetric matrices. Consider the map $\Phi : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ defined by $\Phi(A) = AA^T$. Identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$.

(a) Prove that $S = \Phi^{-1}(I)$ is a surface in $\mathbb{R}^{n^2}$. Find the dimension of S.

(b) Let $A \in S$. Show that the tangent space of S at A consists of all $n \times n$ matrices H which satisfy the relation $AH^TA = -H$, where H^T denotes the transpose of the matrix H. 8+5

(3) (a) Let ω_i, $i = 1, 2, 3$ be 1-forms on $\mathbb{R}^3$. Define $\omega_1 \wedge \omega_2 \wedge \omega_3$ by

$$\omega_1 \wedge \omega_2 \wedge \omega_3(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \textstyle\sum_\sigma \operatorname{sgn} \sigma \ \omega_1(\mathbf{v}_{\sigma 1})\omega_2(\mathbf{v}_{\sigma 2})\omega_3(\mathbf{v}_{\sigma 3}),$$

where the sum is taken over all permutations σ of $\{1, 2, 3\}$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3_p$, $p \in \mathbb{R}^3$. Show that $\omega_1 \wedge \omega_2 \wedge \omega_3$ is a 3-form on $\mathbb{R}^3$.

(b) Let x_1, x_2, x_3 denote the coordinate functions on $\mathbb{R}^3$. Consider the 3-form $\omega = dx_1 \wedge dx_2 \wedge dx_3$. Determine the value of $\omega(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ in terms of the coordinates of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(c) Let $\mathbf{N}$ be a vector field on $\mathbb{R}^3$ defined by $\mathbf{N}(x_1, x_2, x_3) = (x_1, x_2, x_3, x_1, x_2, 0)$. Define a 2-form ζ on $\mathbb{R}^3$ by $\zeta(\mathbf{v}_1, \mathbf{v}_2) = \omega(\mathbf{v}_1, \mathbf{v}_2, \mathbf{N}(p))$, where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3_p$ for $p \in \mathbb{R}^3$. Show that $\zeta = (x_1\, dx_2 - x_2\, dx_1) \wedge dx_3$. 5+4+7

(4) Let S denote the subset of $\mathbb{R}^3$ defined by $x_1^2 + x_2^2 = 1$, $-1 \leq x_3 \leq 1$.

(a) Show that S is a surface with boundary.

(b) Let $i : S \to \mathbb{R}^3$ denote the inclusion map. Show that $i^*\zeta$ is of the form $d\eta$ for some 1-form η on S, where ζ is as in 3(c).

(c) Prove that $\int_S i^*\zeta = 4\pi$. 6+5+7

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2010-2011

M. Math I Year

Algebra II

Date : May 2, 2011 **Maximum Marks: 65** **Duration: $3\frac{1}{2}$ hours**

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1. Indicate True / False with justification for the following statements. [$5 \times 3 = 15$]
   (a) There are finite extensions of $\mathbb{R}$ of odd degree > 1.
   (b) Let F be a field of characteristic $p > 0$ and $K = F(a_1, \cdots, a_n)$ where $a_i^p \in F$ for all i. The only F-automorphism of K is the identity.
   (c) $[\mathbb{Q}(\sqrt[4]{2}, \sqrt[4]{18}) : \mathbb{Q}] = 16$.
   (d) Let $K = \mathbb{F}_p(x, y)$ be the rational function field in two variables over $\mathbb{F}_p$. Let $F = \mathbb{F}_p(x^p, y^p)$. Then K is a simple extension of F.
   (e) There may exist $f(X) \in \mathbb{Q}[X]$ of degree 6 with distinct non-rational roots such that the Galois group of the splitting field of f is $\mathbb{Z}/2\mathbb{Z}$.

2. (a) A *Fermat prime* is a number of the form $2^{2^r} + 1$ for some r. Let p be an odd prime such that a regular p-gon is constructible. Show that p is a Fermat prime. [8]
   (b) Let $c \in \mathbb{R}$ be algebraic over $\mathbb{Q}$ and N be the normal closure of $\mathbb{Q}(c)|\mathbb{Q}$. Assume that $[N : \mathbb{Q}] = 2^r$ for some positive integer r. Prove that c is constructible. [7]

3. (a) Let $f(X) \in \mathbb{Q}[X]$ and K be the splitting field of f over $\mathbb{Q}$. Suppose that f is solvable by radicals. Prove that $\mathrm{Gal}(K|\mathbb{Q})$ is a solvable group. [8]
   (b) Consider $f(X) = X^5 - 10X + 5 \in \mathbb{Q}[X]$. Show that f is not solvable by radicals. [7]

4. (a) Let $f(X) = X^4 + aX^3 + bX^2 + cX + d \in \mathbb{Q}[X]$ be irreducible. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of f in a splitting field K of f. Consider the resolvent cubic $r(X) = X^3 - bX^2 + (ac - 4d)X + 4bd - a^2d - c^2$, whose roots in K are $\alpha_1\alpha_2 + \alpha_3\alpha_4$, $\alpha_1\alpha_3 + \alpha_2\alpha_4$, $\alpha_1\alpha_4 + \alpha_2\alpha_3$. Let L be a splitting field of $r(X)$ over $\mathbb{Q}$. Now prove the following : [8]
      (i) $\mathrm{Gal}(K|\mathbb{Q}) \simeq S_4 \Longleftrightarrow [L : \mathbb{Q}] = 6 \Longleftrightarrow r(X)$ is irreducible over $\mathbb{Q}$ and $\mathrm{disc}(f) \notin \mathbb{Q}^2$.
      (ii) $\mathrm{Gal}(K|\mathbb{Q}) \simeq A_4 \Longleftrightarrow [L : \mathbb{Q}] = 3 \Longleftrightarrow r(X)$ is irreducible over $\mathbb{Q}$ and $\mathrm{disc}(f) \in \mathbb{Q}^2$.
   (b) Let $f(X) = X^4 - 4X^3 + 4X^2 + 6 \in \mathbb{Q}[X]$. Determine the Galois group of f. [7]

5. Let F be field containing a primitive nth root of unity and let $K|F$ be an n-Kummer extension. Let $\mu(F)$ denote the set of all nth roots of unity in F. Define the *Kummer pairing* $B : \mathrm{Gal}(K|F) \times \mathrm{kum}(K|F) \longrightarrow \mu(F)$ by $B(\sigma, \alpha F^*) = \sigma(\alpha)/\alpha$. Prove that :
   (a) B is well defined.
   (b) B is bilinear.
   (c) B is non-degenerate.
   (d) $\mathrm{kum}(K|F) \simeq \mathrm{Gal}(K|F)$ (clearly state the results you used).
   (e) There is an injective group homomorphism $f : \mathrm{kum}(K|F) \longrightarrow F^*/F^{*n}$. [1+4+3+3+4]

INDIAN STATISTICAL INSTITUTE

Second Semester Examination : 2010-11

M. MATH. I

Functional Analysis

Date : 06.05.2011 | Maximum Marks : 100 | Duration : 3 hrs.

The question carries 110 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. Given a sequence $\{\alpha_n\}$ of complex numbers, show that there exists a complex measure μ on $[0,1]$ such that $|\mu|([0,1]) \leq 1$ and $\int_0^1 x^n d\mu = \alpha_n$ for all $n \geq 0$ if and only if for any scalars $a_0, a_1, \ldots, a_n$,
$$\left|\sum_{k=0}^{n} a_k\alpha_k\right| \leq \sup\{|a_0 + a_1x + \ldots + a_nx^n| : x \in [0,1]\}.$$ [10]

2. Let c_0 be the space of all sequences of real numbers converging to zero equipped with the sup norm. Let A be the subset of c_0 consisting of all *rational* sequences. Show that A cannot be a G_δ set in c_0. [15]

3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a linear map. Define
$$\|x\|_1 = \|T(x)\|_Y, \quad x \in X.$$
   (a) Is $\|\cdot\|_1$ a norm on X? If yes, prove it. If not, can you find a necessary and sufficient condition on T that will make it a norm on X? [7]
   (b) Given that $\|\cdot\|_1$ is a norm on X, find a necessary and sufficient condition on the range of T that will make $\|\cdot\|_1$ a complete norm. [8]
   (c) Given that $(X, \|\cdot\|_1)$ is a Banach space, prove that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_X$ if and only if $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is continuous. [10]

4. (a) Let X^* be a dual Banach space. Show that the norm on X^* is w*-lower semicontinuous. [5]

   (b) Let Y be a w*-closed subspace of X^*. Let $x^* \in X^*$. Show that there exists $y^* \in Y$ such that $\|x^* - y^*\| = \inf\{\|x^* - z^*\| : z^* \in Y\}$. [10]

5. Given a sequence $\{\alpha_n\}$ of scalars, define $T : \ell_2 \to \ell_2$ by

$$T(\{x_n\}) = \{\alpha_n x_n\}, \qquad \{x_n\} \in \ell_2.$$

   (a) Show that T is bounded if and only if $\{\alpha_n\} \in \ell^\infty$. And in that case, T is normal with $\|T\| = \|\{\alpha_n\}\|_\infty$.

   Suppose now that $\{\alpha_n\} \in \ell^\infty$. Show that

   (b) T is invertible if and only if $\{\alpha_n\}$ is bounded away from 0.

   (c) λ is an eigenvalue of T if and only if $\lambda = \alpha_n$ for some $n \geq 1$, and, $\lambda \in \sigma(T)$ if and only if $\lambda \in \overline{\{\alpha_n : n \geq 1\}}$.

   (d) T is compact if and only if $\{\alpha_n\} \in c_0$. [$15 + 8 + 7 + 15 = 45$]

Indian Statistical Institute
Semestral Examination
2010-11 (Second Semester)
M. Math. 1st. Year
Complex Analysis

Date and Time: Wednesday, 11.5.11, 10:30 am - 1:30 pm   Total Points: $5 \times 14 = 70$

Answers must be justified with clear and precise arguments. If you use any theorem/proposition proved in class state it explicitly.

1. (a) Evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx, \ t \in \mathbb{R},$$

starting from the contour $[-R, R] \cup \{Re^{i\theta}, 0 \leq \theta \leq \pi\}$. You may first assume $t > 0$, but the final result for $t \in \mathbb{R}$ should be stated clearly.

(b) Let f be an entire function (i.e. holomorphic on $\mathbb{C}$) and suppose there is a constant M, an $R > 0$ and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree n.

$7 + 7 = 14$ pts.

2. (a) Consider the following map from $\mathbb{D}$ to $\mathbb{C}$ given by

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, z \in \mathbb{D},$$

where $\alpha \in \mathbb{D}$. Prove that the map is $\mathbb{D}$ valued.

(b) Show that there is no one-one analytic function f which maps $\{z : 0 < |z| < 1\}$ onto an annulus $\Omega = \{z : r < |z| < R\}$ where $r > 0$. (Hint: if there is, then show that 0 is a removable singularity of f, hence if $\tilde{f}$ is the analytic extension of f then by the open mapping theorem $\tilde{f}(0) \in \Omega$. But for points w near $\tilde{f}(0)$ how many solutions are there for $\tilde{f}(z) = f(z) = w$?) $7 + 7 = 14$ pts.

3. (a) Prove Rouche's theorem: Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If

$$|f(z)| > |g(z)| \text{ for all } z \in C,$$

then f and $f + g$ have the same number of zeros inside the circle C.

(b) Consider $f(z) = z^5 + 5z^3 + z - 2$. Show that f has 3 roots in $B(0, 1)$. Does f have all its 5 roots in $B(0, 5/2)$? $7 + (3 + 4) = 14$ pts.

4. (a) Use the product

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} (1 + \frac{z}{n})^{-1} e^{z/n}$$

that was constructed in an ad hoc manner and was shown to be an analytic continuation of $\int_0^\infty e^{-t} t^{s-1} dt, s > 1$, to show that $\Gamma(z)\Gamma(1 - z) = \pi / \sin \pi z$.

(b) From the above deduce the value of $\int_0^\infty e^{-t} t^{-(1/2)} dt$. $7 + 7 = 14$ pts.

5. Let f be analytic on $G = \{z : Re(z) > 0\}$, one-one, with $Ref(z) > 0$ for all $z \in G$, and $f(a) = a$ for some real number $a \in G$. Show that $|f'(a)| \leq 1$. As an intermediate step you may consider a conformal map taking G to $\mathbb{D}$ such that $g(a) = 0$. 14 pts.

INDIAN STATISTICAL INSTITUTE

Second Semestral Backpaper Examination: 2010-2011

M. Math. I Year

Algebra II

Date: 28.06.11 **Maximum Marks: 100** **Duration: 3 hou**

1. Let L be an algebraic extension of a field k. Show that any k-endomorphism of L is an autom phism. Is the result true if L is not algebraic over k ? [

2. Let G be a finite group. Show that G can be realised as the Galois group of a suitable finite Gal extension. [

3. Prove that every field has an algebraic closure. [

4. Let F be a field of characteristic p. Answer the following :

   (a) Let $K|F$ be a cyclic Galois extension of degree p. Then prove that $K = F(\beta)$ with $\beta^p - \beta - a$ 0 for some $a \in F$.

   (b) Let $a \in F$ be such that there is no $b \in F$ with $b^p - b = a$. Prove that $f(X) = X^p - X$ - is irreducible over F and the splitting field of f over F is a cyclic Galois extension of F degree p.

   [

5. Let k be a field and E be a finite Galois extension of k. Show that an intermediary field ($k \subset F \subset E$) is a normal extension of k if and only if $\mathrm{Gal}(E/F)$ is a normal subgroup $\mathrm{Gal}(E/k)$. In this case, prove that

$$\mathrm{Gal}(E/k)/\mathrm{Gal}(E/F) \simeq \mathrm{Gal}(F/k)$$

[

6. Let $f(X) = X^4 - 2 \in \mathbb{Q}[X]$. Find the Galois group of the splitting field E of f over $\mathbb{Q}$. Find intermediate fields of $E/\mathbb{Q}$.

7. Let k be a field and $f \in k[X]$ be an irreducible and separable polynomial. Assume that the Ga group of f over k is *abelian*. Let E be a splitting field of f over k and let $\alpha_1, \cdots, \alpha_n$ be the ro of f in E. Show that $E = k(\alpha_i)$ for any i.

INDIAN STATISTICAL INSTITUTE
Semester(Back Paper) Examination : 2010-2011
M.Math.-I & M.Stat-II
Topology-II

Date : 30.6.11 Maximum Score : 100 Time : 3 Hours

Answer all questions.
Any result that you use should be stated clearly.

1.
   - Compute singular homology groups of a one point space.
   - State Excision axiom of singular homology theory.
   - Prove that $H_p(S^n, D^n_+) \cong H_p(D^n, S^{n-1})$ for all p. Here S^n is the n-sphere, D^n_+ is the closed upper hemisphere in S^n and D^n is the closed n-ball in $\mathbb{R}^n$.
   - Prove that two chain maps which are chain homotopic induces the same map in homology.

[5+2+3+5=15]

2. Prove that for any $n > 0$, $\tilde{H}_i(S^n) \cong \mathbb{Z}$ if $i = n$ and 0, otherwise, where $\tilde{H}_i$ denotes ith reduced homology.

[15]

3. Let X be a path connected, locally path connected space and G be a finite group acting freely on X. Prove that the quotient map $q : X \longrightarrow X/G$ is a covering map.

[15]

P.T.O

4. Prove that for a regular covering space $p : E \longrightarrow B$, the deck transformation group is isomorphic to

$$\pi_1(B, b)/p_*\pi_1(E, e),$$

where $b \in B$ and $e \in p^{-1}(b)$.

[15]

5.
- Define a finite cell complex.
- Describe a cell complex structure of $\mathbb{R}P^2$.
- Compute homology groups of $\mathbb{R}P^2$.

[5+5+15=20]

6.
- Define Hurewicz homomorphism from the fundamental group of a path connected space X to $H_1(X)$.
- Prove that for a path connected space X whose fundamental group $\pi_1(X)$ is abelian, $\pi_1(X) \cong H_1(X)$.
- Compute $H_1(T^2)$.

[5+10+5=20]

INDIAN STATISTICAL INSTITUTE

Back-paper Examination

Second semester 2010–2011

M.Math (First year)

Differential Geometry

Date: 01, July, 2011

Maximum Marks: 100

Duration: 3 hours 30 minutes

Answer all questions.

State clearly any result that you use in your answer.

Unless stated otherwise all maps and vector fields are assumed to be smooth.

For any n-surface S and a point $p \in S$, S_p will denote the tangent space of S at p.

(1) Consider the vector field $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$ on $\mathbb{R}^2$. For $p \in \mathbb{R}^2$, let α_p denote the maximal integral curve of $\mathbf{X}$ such that $\alpha_p(0) = p$. Define for each $t \in \mathbb{R}$ a transformation $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi_t(p) = \alpha_p(t)$. Show that

(a) ϕ_t is a homeomorphism of $\mathbb{R}^2$;

(b) $t \mapsto \phi_t$ is a homomorphism from the additive group of real numbers to the group of all homeomorphisms of $\mathbb{R}^2$ under composition. 15

(2) Let $\mathbf{X}$ and $\mathbf{Y}$ be two tangent vector fields on an n-surface S in $\mathbb{R}^{n+1}$. Prove that the vector field $[\mathbf{X}, \mathbf{Y}] = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X}$ is also tangent to S. 12

(3) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$. Prove that

$$\tilde{S} = \{(p, v) \in S \times \mathbb{R}^{n+1} : v \in S_p, \|v\| = 1\}$$

is a $(2n-1)$-surface in $\mathbb{R}^{2n+2}$. 12

(4) Let S_1 and S_2 be two n-surfaces in $\mathbb{R}^{n+1}$ and $\alpha : I \to \mathbb{R}^{n+1}$ be a parametrized curve which lies in $S_1 \cap S_2$. Show by an example that a vector field $\mathbf{X}$ parallel along α in S_1 need not be parallel along α in S_2. 10

(5) Show that a parametrized curve α in the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$ is a geodesic if and only if it is of the form

$$\alpha(t) = (\cos at)\mathbf{e} + (\sin at)\mathbf{e}',$$

where $\{\mathbf{e}, \mathbf{e}'\}$ is an orthogonal pair of unit vectors in $\mathbb{R}^3$ and $a \in \mathbb{R}$. 12

(6) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function such that $\nabla f(p) \neq 0$ for all $p \in \mathbb{R}^{n+1}$ and let $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by $g(p) = f(ap)$, where $a \in \mathbb{R}$, $a \neq 0$. Let $S = f^{-1}(c)$ and $\tilde{S} = g^{-1}(c)$ for some real number c. Show that the Gauss-Kronecker curvatures K and $\tilde{K}$ of the n-surfaces of S and $\tilde{S}$ respectively are related by $\tilde{K}(p) = a^n K(ap)$ for all $p \in \tilde{S}$. 15

(7) Consider the 2-sphere S^2 in $\mathbb{R}^3$ defined by the equation $x_1^2 + x_2^2 + x_3^2 = 1$. Orient S^2 by the outward unit normal vector field. Show that the volume form of S^2 oriented as above is $\zeta = x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2$. 12

(8) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function defined by $f(x_1, x_2, x_3) = (x_2x_3, x_1x_3, x_1x_2)$. Let $\omega = x_1\, dx_1 + x_2\, dx_2 + x_3\, dx_3$. Obtain the expression for $f^*\omega$ in the form $f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3$.

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