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Indian Statistical Institute

Mid-semsetral examination : (2011-12)
M. Math I year
Advanced Functional Analysis

Date : 01.09.11 Maximum marks : 40 Duration : 2 hours.

Answer ANY THREE questions. Each question carries 15 marks.

(1)Let M, N be two bounded normal operators on a Hilbert space. Suppose that there is a bounded operator T which is invertible, i.e. has a bounded inverse, satisfying $M = TNT^{-1}$. Prove that there exists a unitary U such that $M = UNU^{-1}$ [Hint: Use the polar decomposition of T.] [15]

(2) Let $\mathcal{A}$ be a C^* algebra and a, b be positive elements of $\mathcal{A}$ satisfying $ab = ba$. Prove that $ab \geq 0$. [15]

(3) Prove that any finite dimensional C^* algebra is isometrically $*$-isomorphic with a $*$-subalgebra of $M_n(\mathbb{C})$ for some n. [15]

(4) (i) Let X be a locally convex topological vector space. Prove that the convex hull of every bounded subset is again bounded.

(ii) Give an example to show that local convexity is a necessary condition for the conclusion of (i). [8+7=15]

29·08·11

Probability Theory (M Math II) : Mid Semestral Exam

Answer All Questions in Two Hours
(maximum you can score 30)

1. When an estimator of θ is called maximum likelihood estimator (m.l.e.) for a family of probability functions $\{P_\theta\}_\theta$. Compute m.l.e. of Geo(p) based on $X \sim$ Geo(p). [7 pt]

2. Let $M, N \overset{iid}{\sim}$ Poi(λ). Compute $\mathbf{Ex}(M^N)$. [7 pt]

3. Suppose a coin is tossed (independently) infinitely many times. What is the probability that either number of Head or Tail is finite? [6 pt]

4. We define a sequence of random variables $X_1 = n$ (with probability one), $X_2, \ldots$ as follows: For each $i \geq 2$, $X_i | (X_1, \ldots, X_{i-1}) \sim \text{Unif}(\{1, 2, \ldots, X_{i-1}\})$. Compute $\mathbf{Ex}(X_m)$ for all $m \geq 2$. [7 pt]

5. Consider a fair random walk $\langle S_n \rangle_n$. Let $n = a + b$, $b < a$. Compute the following conditional probability $P(S_1 > 0, \ldots S_n > 0 \mid S_n = a - b)$. [7 pt]

6. Write down the probability function on the output of the following random experiment: Choose an element X at random from a set A. If $X \notin B$ ($\subseteq A$) then choose Y at random from B, otherwise define $Y = X$. [6 pt]

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Mid-Semestral Examination: 2011-12 (First Semester)

M. MATH. II YEAR
Commutative Algebra

Date : 05·09·11 Maximum Marks : 60 Duration : 3 Hours

Note: Answer any 7 questions from Groups A and any 2 from Group B.
Clearly state the results that you use.

GROUP A

Prove ANY SEVEN of the following statements.

1. Let R be an integral domain with field of fractions K. Then R is the intersection of the local rings $R_m(\subset K)$, as m varies over the set of maximal ideals of R. [8]

2. Let I be an ideal of an integral domain R. If both $I + (x)$ and $(I : x)$ are principal ideals, then I is a principal ideal. [8]

3. Let $I, P_1, P_2, \cdots, P_n$ be ideals of a ring R such that P_i is prime $\forall\, i \geq 3$. If I is not contained in any of the P_i, then there exists an element $x \in I$ that is not contained in any P_i. [8]

4. R is a reduced ring if and only if R_P is a reduced ring for every prime ideal P of R. (A ring is called reduced if 0 is the only nilpotent element of the ring.) [8]

5. Let M be an R-module. If P be an ideal of R that is maximal among all annihilators of non-zero elements of M then P is a prime ideal of R. [8]

6. If I is a finitely generated ideal of R satisfying $I^2 = I$ then there exists $f \in I$ for which $f^2 = f$ and $I = (f)$. [8]

7. Let $f = a_0 + a_1X + \cdots a_nX^n$ be an element of $R[X]$ $(a_i \in R\ \forall\, i)$. Then f is a unit in $R[X]$ if and only if a_0 is a unit and a_i is nilpotent for each $i \geq 1$. [8]

8. $\mathbb{C}[X, Y, Z, W]/(X^2 + Y^2 + Z^2 + W^2 - 1)$ is a UFD. ($\mathbb{C}$: the field of complex numbers.) [8]

GROUP B

Give an example each for ANY TWO of the following.

1. A ring R, an ideal I and a countably infinite collection of prime ideals $P_1, \cdots, P_n, \cdots$ of R such that $I \subseteq \bigcup_n P_n$ but $I \not\subseteq P_i$ for any i. [5]

2. A reduced ring R such that R_P is an integral domain for every prime ideal P of R but R is not an integral domain. [5]

3. A ring R containing a multiplicatively closed set S and two ideals I, J of R such that $S^{-1}(I : J) \neq (S^{-1}I : S^{-1}J)$. [5]

Indian Statistical Institute

Mid-Semestral Examination: 2011-2012
Programme: Master of Mathematics
Subject: Number Theory

Date: 07.09.2011
Duration: Two Hours and 30 Minutes
Maximum marks: 60

Answer all questions explaining and justifying each step.

p denotes a variable odd prime throughout.

1. If $p \mid (2^{2^n} + 1)$ for some positive integer n, then show that $p \equiv 1 (\text{mod } 2^{n+1})$. (6 marks)

2. Let $h(n)$ denote the number of distinct solutions modulo n to the congruence $x^2 + 1 \equiv 0(\text{mod } n)$. Evaluate $h(39)$ and $h(65)$. (6 marks)

3. Show that there is no integer solution to the equation $x^5 + y^5 = z^5$ with $1 \leq |xyz| \leq 10^5$. (6 marks)

4. Suppose a is a positive integer and $p \nmid a$. If $p \equiv \pm 1(\text{mod } 4a)$, then show that a is a quadratic residue of p. (12 marks)

5. Find all positive integers n such that $n \mid (10^n - 1)$. (12 marks)
Hint: infinite descent may be useful.

6. Let $S(X)$ denote the number of pairs of coprime integers m and n; $1 \leq m, n \leq X$. Derive an asymptotic formula (with a main term and an error term) for $S(X)$ as $X \longrightarrow \infty$. (12 marks)
Hint: capture the coprimality condition by μ.

7. Suppose p is an odd prime. Show that there is a primitive root modulo p^3. You can assume that $U(p)$, the group of units of $\mathbb{Z}/p\mathbb{Z}$, is cyclic. (12 marks)
Hint: if g is a generator of $U(p)$, consider g^{p-1} modulo p^2.

Fourier Analysis: M. Math II: Mid Semester Examination

September 9, 2011.

Maximum Marks 40 Maximum Time 2:30 hrs.

Answer all questions.

1. Give short answers to the following questions.

(a) Is $f(x) = e^{-\pi x^2} \sin(e^{\pi x^2})$ a Schwartz class function on $\mathbb{R}$? Justify.

(b) Show that if $f \in L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})$ then $\|f\|_p \leq \|f\|_{p_1}^{1-\theta} \|f\|_{p_2}^{\theta}$ where $\frac{1}{p} = \frac{(1-\theta)}{p_1} + \frac{\theta}{p_2}$.

(c) Show that for any $f \in C_c^{\infty}(\mathbb{R}^n)$ and any $p \in [1, \infty)$, $\|f\|_{p,\infty} \leq \|f\|_p$.

(d) For $1 < p < 2$, let $f \in L^p(\mathbb{R}^n)$ be once differentiable and $f' \in L^p(\mathbb{R}^n)$. Then show that f vanishes at infinity.

3+3+3+5=14

2. (a) Let $y \in \mathbb{R}^n$ be fixed. Let $B(y, r)$ be the ball in $\mathbb{R}^n$ of radius $r > 0$ with centre at y. Define $f(x) = |B(y, |x|)|^{-\delta}$ for some $\delta \in \mathbb{R}$. Give argument to show that for any δ, f is not an L^p-function for any $p \geq 1$.

(b)Find the range of $\delta \in \mathbb{R}$ so that the Fourier transform of the function f exists (as a measurable function) and determine its Fourier transform.

(c) Let ϕ be a Schwartz class function on $\mathbb{R}^n$. Find the range of δ for which $g(x) = \phi(x)|B(y, |x|)|^{-\delta}$ is in $L^1(\mathbb{R}^n)$ 7+5+5=17

3. For a point $x \in \mathbb{R}^n$ and for $r > 0$ let $B(x, r)$ be the ball of radius r with centre at x. For a locally integrable function f define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

(The supremum is taken over all balls with centre at x.)

If $\|Mf\|_q \leq C\|f\|_p$, the show that $p = q$.

What is your conclusion if $\|Mf\|_{q,\infty} \leq C\|f\|_{p,\infty}$? 10+3=13

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination 2011–2012

M.Math (Second year)

Differential Topology

Maximum Marks: 60 Date: [redacted] 12.09.11

Duration: 2 hours 30 minutes

Answer all questions.

State clearly any result that you use in your answer.

A manifold is assumed to be a subset of an Euclidean space.

(1) Let M and N be manifolds such that $N \subset M \subset \mathbb{R}^q$. Prove the following:

(a) The inclusion map i of N into M is a smooth map.

(b) The tangent space of N at a point x is a subspace of the tangent space of M at x. 3+3

(2) Show by an example that the image of a one-to-one immersion need not be a submanifold. 7

(3) Show that a non-degenerate critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ is an isolated critical point. 9

(4) (a) Let S^n denote the n-sphere in $\mathbb{R}^{n+1}$. Prove that any smooth map $f : M \to S^n$ is homotopic to a constant map if $n > \dim M$.

(b) Give an example to show that this need not be true when $n \leq \dim M$. 6+6

(5) For each $a = (a_1, a_2) \in \mathbb{R}^2$, define a function $f_a : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = (t, t^2) + (a_1, a_2), \quad t \in \mathbb{R}.$$

Show that there exists an $\varepsilon > 0$ such that f_a is transversal to the submanifold $\Delta = \{(x, x) | x \in \mathbb{R}\}$ of $\mathbb{R}^2$ whenever $\|(a_1, a_2)\| < \varepsilon$. 12

(6) Let $f : M \to \mathbb{R}^n$ be an immersion. Suppose that $\cup_{x \in M} df_x(T_x M)$ is a proper subset of $\mathbb{R}^n$.

(a) Prove that there exists an immersion $g : M \to \mathbb{R}^{n-1}$.

(b) Apply (a) to show that the open upper hemisphere of a sphere S^n immerses in $\mathbb{R}^n$. 9+5

INDIAN STATISTICAL INSTITUTE

Semestral Examination 2011–2012

M.Math (Second year)

Differential Topology

Maximum Marks: 60 | Date: 16 November, 2011

Duration: 2 hours 30 minutes

Answer all questions.
State clearly any result that you use in your answer.

(1) (a) Let V be a vector space of dimension k and $\phi_1, \phi_2, \ldots, \phi_k$ belong to V^*. If $A : V \to V$ a linear map then show that

$$A^*\phi_1 \wedge \cdots \wedge A^*\phi_k = (\det A)\phi_1 \wedge \cdots \wedge \phi_k.$$

(b) *Recall that a manifold M is oriented if each tangent space T_xM is oriented and M admits local parametrizations by orientation preserving diffeomorphisms.* Prove that if M is an oriented manifold of dimension k in $\mathbb{R}^N$ then there is a nowhere vanishing k-form on M. 6+6

(2) Let $h : \mathbb{R} \to S^1$ be defined by $h(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$.

(a) Show that if ω is any 1-form on S^1, then

$$\int_{S^1} \omega = \int_0^{2\pi} h^*\omega.$$

(b) Let $\int_{S^1} \omega = 0$ for some 1-form ω. Prove that ω is exact. *Hint: Define a function g on $\mathbb{R}$ by $\int_0^t h^*\omega$. Use this g to define a primitive of ω.*

(c) Prove that the first de Rham cohomology group $H^1(S^1)$ is isomorphic to $\mathbb{R}$. 8+6+4

(3) (a) Let M be an n-dimensional manifold in $\mathbb{R}^{n+1}$. Suppose that there is a smooth normal vector field on M. Show that M is orientable.

(b) Note that an n-sphere S^n is orientable. Consider the antipodal map $a : S^n \to S^n$. Given an n, determine whether a is orientation preserving or orientation reversing.

(c) Which of the real projective spaces $\mathbb{R}P^n$ are non-orientable? Justify your answer. 6+6+6

(4) Let M be an oriented manifold of dimension n without boundary.

(a) Consider the product manifold $M \times [0, 1]$ with the product orientation. Describe the boundary orientation on $M \times \{0, 1\}$.

P.T.O

INDIAN STATISTICAL INSTITUTE

Semestral Examination 2011–2012

M.Math (Second year)

Differential Topology

Maximum Marks: 60 Date: 16 November, 2011

Duration: 2 hours 30 minutes

Answer all questions.

State clearly any result that you use in your answer.

(1) (a) Let V be a vector space of dimension k and $\phi_1, \phi_2, \ldots, \phi_k$ belong to V^*. If $A : V \to V$ a linear map then show that

$$A^*\phi_1 \wedge \cdots \wedge A^*\phi_k = (\det A)\phi_1 \wedge \cdots \wedge \phi_k.$$

(b) *Recall that a manifold M is oriented if each tangent space T_xM is oriented and M admits local parametrizations by orientation preserving diffeomorphisms.* Prove that if M is an oriented manifold of dimension k in $\mathbb{R}^N$ then there is a nowhere vanishing k-form on M. 6+6

(2) Let $h : \mathbb{R} \to S^1$ be defined by $h(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$.

(a) Show that if ω is any 1-form on S^1, then

$$\int_{S^1} \omega = \int_0^{2\pi} h^*\omega.$$

(b) Let $\int_{S^1} \omega = 0$ for some 1-form ω. Prove that ω is exact. *Hint: Define a function g on $\mathbb{R}$ by $\int_0^t h^*\omega$. Use this g to define a primitive of ω.*

(c) Prove that the first de Rham cohomology group $H^1(S^1)$ is isomorphic to $\mathbb{R}$. 8+6+4

(3) (a) Let M be an n-dimensional manifold in $\mathbb{R}^{n+1}$. Suppose that there is a smooth normal vector field on M. Show that M is orientable.

(b) Note that an n-sphere S^n is orientable. Consider the antipodal map $a : S^n \to S^n$. Given an n, determine whether a is orientation preserving or orientation reversing.

(c) Which of the real projective spaces $\mathbb{R}P^n$ are non-orientable? Justify your answer. 6+6+6

(4) Let M be an oriented manifold of dimension n without boundary.

(a) Consider the product manifold $M \times [0, 1]$ with the product orientation. Describe the boundary orientation on $M \times \{0, 1\}$.

P.T.O

(b) Suppose that M is compact and without boundary. Let $f_0, f_1 : M \to N$ be homotopic maps. Prove that

$$\int_M f_0^*\omega = \int_M f_1^*\omega$$

for any closed n-form ω on N. 6+6

Indian Statistical Institute

Semsetral examination : (2011-12)
M. Math II year
Advanced Functional Analysis

Date : 21·11·11 Maximum marks : 60 Duration : 3 hours.

Answer ANY TWO questions from Group A and ANY ONE from Grup B. Marks are indicated in bracket.

Group A (Answer ANY TWO questions).

(1) Let T be a bounded normal operator which has a one-sided inverse S such that $ST = I$. prove that T is invertible. [15]

(2) Let $\mathcal{A}$ be a unital separable C^* algebra such that any irreducible representation of $\mathcal{A}$ is finite dimensional. Prove that there is a faithful tracial state τ on $\mathcal{A}$ (tracial means $\tau(ab) = \tau(ba)$ for all a, b).
(Hint: First argue that there is a countable family of pure states $\{\phi_n\}$ of $\mathcal{A}$ which is separating for $\mathcal{A}$, i.e. for any nonzero a there is some ϕ_n with $\phi_n(a) \neq 0$.)
[15]

(3) Let X be a topological vector space with a dense subspace X_0, Y Banach space and $T : X_0 \to Y$ be a continuous linear map. prove that T extends to a continuous linear map from X to Y. [15]

Group B (Answer ANY ONE question)

(4)(i) Let G be a finite abelian group with n elements and let α be the canonical G-action on the finite dimensional C^* algebra $C(G)$ given by $\alpha_g(f)(h) = f(g^{-1}h)$. Prove that the crossed product C^* algebra $C(G) \times_\alpha G$ is isomorphic with $M_n(\mathbb{C})$.
(Hint: Identify $M_n(\mathbb{C})$ with $\mathcal{B}(l^2(G))$ and try to construct a $*$-homomorphism π from $M_n(\mathbb{C})$ to the crossed product algebra by defining $\pi(|\chi_g >< \chi_h|)$, where χ_g denotes the characteristic function of the singleton $\{g\}$.)
(ii) Using (i) or otherwise, prove that any irreducible representation of the universal C^* algebra $\mathcal{A}_{\frac{1}{n}}$ generated by two unitaries U, V satisfying $UV = \exp(\frac{2\pi i}{n})VU$ must be n-dimensional. [15+15=30]

OR

(5) (i) Let $\mathcal{H}$ be any separable infinite dimensional Hilbert space and let $\mathcal{A}$ denote the C^*-subalgebra of $C([0,1], \mathcal{B}(\mathcal{H}))$ (with $\|F\| := \sup_{x\in[0,1]} \|F(x)\|$) consisting of those functions $F : [0,1] \to \mathcal{B}(\mathcal{H})$ for which $F(0)$ is a scalar multiple of I. prove that $\mathcal{A}$ does not have any projections except the trivial ones, i.e. 0 and 1.
(ii) Prove that there is an injective C^*-homomorphism from $C^*(\mathbb{F}_2)$ to $\mathcal{A}$, and hence $C^*(\mathbb{F}_2)$ does not have any nontrivial projection.
(Hint: Observe that $C^*(\mathbb{F}_2)$ admits a faithful representation in some separable Hilbert space.) [15+15=30]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2011-12 (First Semester)

M. MATH. II YEAR
Commutative Algebra

Date : 23.11.2011 | Maximum Marks : 70 | Duration : $3\frac{1}{2}$ Hours

ANSWER ANY SIX QUESTIONS.
Clearly state the results that you use.

1. Let $A = \mathbb{C}[X,Y]/(X^2+Y^2-1)$.
   (i) Prove that $A \cong \mathbb{C}[T,T^{-1}]$.
   (ii) Deduce that A is a PID.
   (iii) Explain why the relation $\bar{X}\bar{X} = (1-\bar{Y})(1+\bar{Y})$ does not contradict the fact that A is a UFD. [5+2+5=12]

2. (i) Prove that any finitely generated projective module over a local ring is free.
   (ii) Let $B = \mathbb{R}[X,Y]/(X^2+Y^2-1)$ and M the ideal $(x, y-1)B$ where x and y denote respectively the images of X and Y in B. Prove that M is a projective B-module. [6+6=12]

3. (i) Compute $\mathbb{C}[X]/(X^5) \otimes_{\mathbb{C}} \mathbb{C}[X]/(X^7)$.
   (ii) Let M and N be finitely generated R-modules such that $M \otimes_R N = 0$. Prove that $Ann_R M + Ann_R N = R$.
   (iii) Prove that any flat module over an integral domain is torsion-free. [3+5+4=12]

4. (i) Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$ be a graded ring. Show that if R is Noetherian, then R is a finitely generated algebra over R_0.
   (ii) Let $R \subset A \subset B$ be rings such that R is Noetherian and B is a finitely generated R-algebra. Suppose that B is also finitely generated as an A-module. Show that B is integral over A and that A is finitely generated as an R-algebra. [6+6=12]

5. (i) Suppose that R is a subring of A and α is a unit in A. Show that $R[\alpha] \cap R[\alpha^{-1}]$ is integral over R.
   (ii) Let B be a Noetherian local domain with field of fractions $K (\neq B)$ and maximal ideal m. Show that if $xm \subseteq m$ for some x in K, then x is integral over B. [7+5=12]

6. Let $R = \mathbb{C}[X,Y]/(Y^2 - X^2 - X^3)$. Show that
   (i) R is not normal.
   (ii) The normalisation of R is of the form $\mathbb{C}[t]$, a polynomial ring in one variable over $\mathbb{C}$.
   (iii) Display an explicit integral equation over R satisfied by t.
   (iv) Prove that $t \notin R$. [12]

[P.T.O.]

7. (i) Let k be an algebraically closed field and I a proper ideal in $k[X_1, \cdots, X_n]$. Prove that $\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$. [You may assume the Weak Nullstellensatz.]

(ii) Let k be a field, A an integral domain which is finitely generated as a k-algebra and L the field of fractions of A. Show that if $A \neq L$, then $A[1/f] \neq L$ for any $f \in A$. [6+6=12]

8. (i) Let V be an affine algebraic set in $\mathbb{A}_k^n$ and $p \in \mathbb{A}_k^n \setminus V$. Show that there exists g in $k[X_1, \cdots, X_n]$ such that $g(x) = 0 \ \forall \ x \in V$ and $g(p) = 1$.

(ii) Suppose that V is an affine algebraic set in $\mathbb{C}^n$ such that $(\mathbb{C}[V])^\star = \mathbb{C}^\star$. Show that any non-constant polynomial function $f : V \to \mathbb{C}$ must be surjective.

(iii) Give an example of a non-constant polynomial function $f : V \to \mathbb{C}$ from an affine algebraic set V in $\mathbb{C}^2$ such that f is not surjective. [4+6+2=12]

Fourier Analysis: M. Math II: Semester Examination

November 25, 2011.

Maximum Marks 60 Maximum Time 3 hrs.

Answer all questions.

(1) Let $f \in L^p(\mathbb{R}), 1 < p < 2$. Find the Fourier transform of its dilation $\delta_r f$ at $\xi \in \mathbb{R}$. 9

(2) Define the *radial part* of a function $f \in L^1(\mathbb{R}^n)$. Show that if its radial part is zero then the ideal generated by f is not dense in $L^1(\mathbb{R}^n)$. 5

(3) Let $u, v \in L^1(\mathbb{R})$ and $\|v\|_1 < 1$. Show that $\widehat{u}/(1+\widehat{v}) = \widehat{f}$ for some $f \in L^1(\mathbb{R})$. 5

(4) Let (X, μ) and (Y, ν) be two measure spaces and $K(x, y)$ be a measurable function on the product space $X \times Y$. Assume that for some $M > 0$

$$\int_Y |K(x,y)|d\nu(y) \leq M \text{ for a. e. } x \in X \text{ and } \int_X |K(x,y)|d\mu(x) \leq M \text{ for a. e. } y \in Y.$$

Show that if $T(f) = \int_Y f(y)K(x,y)d\nu(y)$ then $\|Tf\|_p \leq M\|f\|_p$ for $1 \leq p \leq \infty$. 9

(5) Define central and noncentral Hardy-Littlewood maximal functions of $f \in L^1_{loc}(\mathbb{R}^n)$ using balls and denote them by Mf and $M_1 f$ respectively. Show that $M_1 f \leq CMf$. What property of the Lebesgue measure is crucial for this? 3+4+2=9

(6) On $\mathbb{R}^2$ define a multiplier operator $mf(x_1, x_2) = \int_{\mathbb{R}} \int_0^\infty \widehat{f}(\xi_1, \xi_2) e^{i(x_1\xi_1 + x_2\xi_2)} d\xi_1 d\xi_2$. Show that m is strong type (p, p) for $p > 1$. 10

(7) Suppose E is a measurable subset of $\mathbb{R}^n$. Let B denotes the balls in $\mathbb{R}^n$. Show that

$$\lim_{|B|\to 0, x\in B} \frac{|B \cap E|}{|B|} = 1 \text{ for a.e. } x \in E \text{ and } \lim_{|B|\to 0, x\in B} \frac{|B \cap E|}{|B|} = 0 \text{ for a.e. } x \notin E.$$

(Use Lebesgue Differentiation theorem for locally integrable functions) 10

(8) On $\mathbb{R}$ we define p. v. $\frac{1}{x}$ by

$$\text{p. v. } \frac{1}{x}(\phi) = \lim_{t\to 0} \int_{|x|>t} \frac{\phi(x)}{x} dx.$$

Show that p. v. $\frac{1}{x}$ is a tempered distribution. 7

INDIAN STATISTICAL INSTITUTE

First Semestral Examination: 2011-12

Course Name : M Math II   Subject Name : **Basic Probability**

Date : 28th November 2011   Maximum Marks: 60   Duration: 3 Hours

Note: Attempt all questions. Marks are given in brackets. State the results you want to use.

Problem 1 (8). Let Ω be a sample space of a probability function P such that Ω is disjoint union of uncountable events A_α, $\alpha \in \mathcal{A}$ (uncountable). Then prove that at most countable many A_α's have positive probability.

Problem 2 (8). Let A be a finite set and $X_1, \ldots, X_n \overset{iid}{\sim} \text{Unif}(A)$. Prove that

$$\Pr[X_i = X_j \text{ for some } i \neq j] \geq \frac{\binom{n}{2}}{|A|} - \frac{\binom{n}{3}}{|A|^2}.$$

Problem 3 (8). Let $X_i \sim N(\mu_i, \sigma^2)$, $i = 0, 1$ and $B \sim \text{Unif}(\{0, 1\})$. Moreover, X_0, X_1 and B are mutually independent. Find the probability density function of X_B.

Problem 4 (10). Let X_0, X_1, B and B are mutually independent random variables where $B \sim \text{Unif}(\{0, 1\})$. Prove that for any function f, ,

$$|\Pr[f(X_B) = B] - 1/2| \leq \text{dist}_{\text{TV}}(X_0, X_1)$$

where dist_{TV} is the total variation function.

Problem 5 (8). Let X be a random variable taking values from $\{0, 1, 2, \ldots\}$ such that for all non-negative integers s and t, $P(X \geq s + t | X \geq s) = P(X \geq t)$. What can you say about the random variable X?

Problem 6 (8). Define and compute the (differential) entropy of $\text{Exp}(\lambda)$: $\lambda e^{-\lambda x}, x > 0$.

Problem 7 (8). Construct three pairwise independent random variables X_1, X_2 and X_3 which are not mutually independent.

Problem 8 (6 + 6 = 12).

1. Prove that
$$\Pr[X - \mu \geq k\sigma] \leq \frac{1}{1 + k^2}$$
where μ and σ^2 are mean and variance of X. (Hint: use Cauchy-Schwarz inequality: $\mathbf{E}(Z^2)\mathbf{E}(W^2) \geq \mathbf{E}^2(ZW)$.
2. Prove that, if median m (i.e. $\Pr[X < m] = 1/2$) exists, then $|\mu - m| \leq \sigma$.

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2011-12 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date : 20.2.2012 | Maximum Marks : 30 | Duration : $2\frac{1}{2}$ Hours

Answer ANY FOUR questions

1. Let k be a field, $A = k[X_1, \cdots, X_n]$ and $m = (X_1 - a_1, \cdots, X_n - a_n)$. Let $I = (f_1, \cdots, f_m)$ be an ideal of A contained in m and $R = A_m/IA_m$. Let r be the rank of the corresponding Jacobian matrix :— the $m \times n$-matrix whose (i,j)th entry is $(\frac{\partial f_i}{\partial X_j}|_{(a_1, \cdots, a_n)})$.

   (i) Show that R is a regular local ring if and only if dim $R = n - r$.

   (ii) Let $B = \mathbb{C}[X, Y, Z]/(XY - Z^2)$. Describe all maximal ideals m of B for which B_m is a regular local ring. [6+2=8]

2. Let $B = \mathbb{C}[X, Y, Z]/(XY - Z^2)$. Let $x = \overline{X}$, $y = \overline{Y}$, $z = \overline{Z}$.
   Let $P = (x, z)B$, $I_1 = P^2 + yB$ and $I_2 = (x, z^2)B$.

   (i) Show that xB is P-primary.

   (ii) Verify that $P^2 = I_1 \cap I_2$ is an irredundant primary decomposition of P^2. Mention the associated prime ideals and identify the isolated and embedded components.

   [3+5=8]

3. (i) Let K be a field. Compute the Krull dimension of the ring $K[X^2, Y^2, XY^2, X^3]$.

   (ii) Let M be a nonzero finitely generated module over a Noetherian ring R and let I be an ideal of R. Show that either I contains a nonzerodivisor on M or I annihilates an element of M. [3+5=8]

4. Let R be a Noetherian domain.

   (i) Show that, for every prime ideal P of R, the symbolic power $P^{(n)} (:= P^n R_P \cap R)$ is a P-primary ideal of R.

   (ii) Prove that if x is a nonzero non-unit in R, then the height of xR is one. [2+6=8]

5. (i) Let R be a Noetherian domain. Prove that R is a UFD if and only if every prime ideal minimal over a principal ideal is itself principal.

   (ii) Let $P \subsetneqq Q$ be prime ideals in a Noetherian ring R. Show that if there exists one prime ideal P_1 in R with $P \subsetneqq P_1 \subsetneqq Q$, then there exist infinitely many prime ideals P_i in R such that $P \subsetneqq P_i \subsetneqq Q$. [3+5=8]

6. Let R be a Noetherian domain of dimension one. Prove that the normalisation of R is Noetherian. [8]

Indian Statistical Institute

Mid-semsetral examination : (2011-12)

M. Math II year

Special Topics (K theory of C^* algebras)

Date : 24.2.12 Maximum marks : 40 Duration : 2 hours.

Answer ANY TWO questions. Each question carries 20 marks.

(1) Prove that $K_0(\mathcal{A})$ is countable for a separable C^* algebra $\mathcal{A}$.

(2) Let $\mathcal{A}$ be a unital C^* algebra and a be an element of $\mathcal{A}$ which is positive and $\|a\| \leq 1$. Consider

$$p = \begin{pmatrix} a & (a-a^2)^{\frac{1}{2}} \\ (a-a^2)^{\frac{1}{2}} & 1-a \end{pmatrix}.$$

Prove that p is a projection and $p \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathcal{A})$.

(3) Let $X \subseteq [0,1]$ denote the Cantor set. Prove that $K_0(C(X)) \cong Z[\frac{1}{2}]$ as abelian group, where Z denotes the set of integers.

INDIAN STATISTICAL INSTITUTE

MID-SEMESTER EXAMINATION : (2011-2012)
M. MATH II
ALGEBRAIC NUMBER THEORY
FEBRUARY 24, 2012 MAXIMUM MARKS : 35 DURATION : $2\frac{1}{2}$ HOURS

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(1) Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\mathbb{Z}[\omega] = \{a + b\omega | a, b \in \mathbb{Z}\}$.

(a) Find all the units in $\mathbb{Z}[\omega]$. [4]

(b) Prove that $1 - \omega$ is irreducible in $\mathbb{Z}[\omega]$ and that $3 = u(1-\omega)^2$ for some unit u in $\mathbb{Z}[\omega]$. Find the order of $\frac{\mathbb{Z}[\omega]}{(1-\omega)}$. [3]

(c) Let $K = \mathbb{Q}(\omega)$. Is $\mathcal{O}_K = \mathbb{Z}[\omega]$? Justify your answer. [3]

(2) Let A be an integral domain which is integrally closed in its field of fractions K. Let L be a separable extension of K of degree n. Let B be the integral closure of A in L. Now answer the following questions.

(a) Prove that $\alpha \in B^*$ if and only if $N_{L|K}(\alpha) \in A^*$. [2]

(b) If $\alpha_1, \cdots, \alpha_n \in B$ form a basis of the K-vector space L, and if $d = \text{disc}(\alpha_1, \cdots, \alpha_n)$, then prove that $dB \subseteq A\alpha_1 + \cdots + A\alpha_n$. [4]

(c) Assume further that A is a PID. Then prove that any finitely generated non-zero B-submodule M of L is a free A-module of rank n. [4]

(d) Consider the case when $A = \mathbb{Z}$, $L = \mathbb{Q}(\sqrt{5})$. Find a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module B. [5]

(3) Let $K = \mathbb{Q}(\sqrt{-5})$ and $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$ the ring of algebraic integers in K.

(a) Prove that $\mathcal{P} = (2, 1+\sqrt{-5})$ is a prime ideal of $\mathcal{O}_K$ with norm 2. [3]

(b) Show that $\mathcal{P}$ is the only ideal with norm 2. Deduce that the class group Cl_K is of order 2 (clearly state the result(s) you used). [4]

(c) How many ideals are there of norm 3 in $\mathbb{Z}[\sqrt{-5}]$? [3]

(4) (a) Let K be a number field and $\mathcal{O}_K$ be its ring of integers. Prove that the group of units $\mathcal{O}_K^*$ is finite if and only if either $K = \mathbb{Q}$ or K is an imaginary quadratic extension of $\mathbb{Q}$. (State the results you used). [2]

(b) Let $K = \mathbb{Q}(\sqrt{-d})$, where d is a square-free, positive integer. Give a complete description of the group of units $\mathcal{O}_K^*$, with justification. [4]

(c) Find the fundamental units for the following number fields : (i) $\mathbb{Q}(\sqrt{3})$, (ii) $\mathbb{Q}(\sqrt{5})$. [4]

INDIAN STATISTICAL INSTITUTE
Mid-Semesteral Examination : 2011-12
M. Math. - Second Year
Mathematical Logic

Date : 27. 02. 2012 Maximum Score : 100 Time :3 Hours

1. The paper carries 120 marks. You are free to answer all the questions. Maximum Score: 100

2. You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) Show that if a set $\mathcal{A}$ of formulas of a language for propositional logic is finitely satisfiable, it is satisfiable. [15]

(2) Show that any two countably infinite models of DLO are isomorphic. [15]

(3) Show that two uncountable models of the theory of divisible, torsion-free, abelian groups are isomorphic if and only if they have the same cardinality. [15]

(4) Show that every countable consistent theory has a countable model. [15]

(5) Let M be the structure of a first order language L, α an automorphism of M and $A \subset M$ is such that for every $a \in A$, $\alpha(a) = a$. Show that if $B \subset M^n$ is A-definable, $\alpha(B) \subset B$. Use this to show that the set of all real numbers $\mathbb{R}$ is not a definable subset of the ring $\mathbb{C}$ of complex numbers. [20]

(6) Let T be a first order theory and A a sentence undecidable in T. Show that $T[A]$ is consistent. [10]

(7) Show that every consistent theory has a complete simple extension. [10]

(8) Show that if T is a complete, Henkin theory, the canonical structure of the language of T is a model of T. [20]

Indian Statistical Institute, Kolkata

Midsemestral Examinations : M.Math.II year & M.Stat.II year

Ergodic Theory

Maximum marks : 30 February 29, 2012 Time : 3 hours

Answer all questions

1. Let T be a measure preserving invertible transformation of the probability space $(X, \mathcal{B}, m)$. Say that T has a countable Lebesgue spectrum if there exists $f_0, f_1, f_2, \cdots \in L^2(X, \mathcal{B}, m)$ such that f_0 is the constant function 1 and the family $\{f_0, U_T^k f_j : k = 0, \mp 1, \mp 2, \cdots, j = 1, 2, 3, \cdots\}$ is an orthonormal basis of $L^2(X, \mathcal{B}, m)$.

   (a) Let $\mathcal{P}$ be the spectral measure corresponding to U_T. Show that if T has a countable Lebesgue spectrum then for any $f \in L^2(X, \mathcal{B}, m), < f, 1 >= 0, < f, f >= 1$, the measure $< \mathcal{P}(.)f, f >$ is the Lebesgue measure on $\mathbb{T}$. [4]

   (b) If T has countable Lebesgue spectrum, show that T is mixing. [4]

2. Let G be a compact, connected metric abelian group. Let $Tx = aAx$ be an affine transformation of G (i.e. $a \in G$ and A is a continuous homomorphism of G onto itself.) Suppose that for $x_0 \in X$, the orbit $\{T^n x_0, n \geq 0\}$ is dense in G. Show that if for some character γ of G, and positive integer k, $\gamma \circ A^k = \gamma$, then $\gamma \circ A = \gamma$. [5]

3. Let T be a measurable transformation of the measurable space $(X, \mathcal{B})$ and $\mathcal{M}_T$ the space of all probability measures μ on $\mathcal{B}$ such that μ is $T-$invariant. $\mathcal{M}_T$ is a convex subset of the space of probability measures on $\mathcal{B}$.

   Show that

   (a) if $\mu, \nu \in \mathcal{M}_T, \nu << \mu$, then the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is a $T-$invariant function *a.e.* [2]

   (b) if, in addition to the hypothesis in (a), μ is given to be ergodic, then $\nu = \mu$. [2]

   (c) $\mu \in \mathcal{M}_T$ is ergodic if and only if μ is an extreme point of $\mathcal{M}_T$. [3]

4. Let T be the transformation on $[0, 1)$ defined by $Tx =< \frac{1}{x} >$, if $x \neq 0$ and $T0 = 0$, called the continued fraction map and μ the measure given by $\mu(a, b) = \int_0^1 \frac{1}{1+x} dx$, called the Gauss measure.

   (a) Show that T preserves μ. [5]

   (b) Assume the facts (i) and (ii):

   (i) For $x \in (0, 1)$ irrational, $x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$ is the (simple) continued fraction representation of x where $a_1 = [\frac{1}{x}], a_2 = [\frac{1}{Tx}], a_3 = [\frac{1}{T^2 x}], \cdots$

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2011-2012
M. Math. - II Year
Topology-III

Date : 02. 03. 2012 Maximum Score : 40 Time :3 Hours

Any result that you use should be stated clearly.

(1) **(a)**: State Eilenberg-Steenrod axioms for Singular Cohomolgy.
(b): Compute $H^q(S^n)$ for $q \geq 1,\ n \geq 1$.
[10+15=25]

(2) **(a)**: Define the notion of cochain homotopy between two cochain maps.
(b): Prove that if $f^* : C^* \longrightarrow D^*$ is a cochain homotopy equivalence, then the induced maps in cohomology groups are isomorphisms.
[5+10=15]

(3) **(a)**: Define reduced cohomology groups $\widetilde{H}^p(X)$, $p \geq 0$, of a topological space X.
(b): Prove that $H^p(X) \cong \widetilde{H}^p(X) \oplus \mathbb{Z}$.
(c): Compute Cohomolgy groups of $S^1 \bigvee S^3$.
[5+5+5=15]

INDIAN STATISTICAL INSTITUTE
Semesteral Examination : 2011-12
M. Math. - Second Year
Mathematical Logic

Date : 23. 04. 2012 Maximum Score : 100 Time :4 Hours

1. *Answer all the questions.*
2. *You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.*

(1) Answer the following questions giving only a brief justification.
   (a) Is the class of all finite sets elementary?
   (b) Let M be a model of Peano arithmetic. Is it true that M must be unbounded, i.e., for every $x \in M$ there is a $y \in M$ such that $x < y$?
   (c) Let G_1 and G_2 be two ordered, divisible, torsion-free, abelian groups. Are G_1 and G_2 elementarily equivalent?
   (d) Let R_1, R_2 be real closed fields with R_1 a subfield of R_2. Is R_1 elementarily embedded in R_2?
   (e) Let $\mathbb{F}$ be an algebraically closed field, $C \subset \mathbb{F}^n$ constructible and $f_i \in \mathbb{F}[X_1, \cdots, X_n]$, $1 \leq i \leq m$. Is $f(C) \subset \mathbb{F}^m$ constructible, where $f = (f_1, \cdots, f_m)$?

[5 × 5 = 25]

(2) Let $\varphi[x, x_0, \cdots, x_{n-1}]$ be an open formula of a theory T whose language contains a constant symbol, say c. Show that the following two statements are equivalent.
   (a) There is an open formula $\psi[x_0, \cdots, x_{n-1}]$ such that

$$T \vdash \forall x_0 \cdots \forall x_{n-1}(\exists x\varphi \leftrightarrow \psi).$$

   (b) For any two models $M, N \models T$, for any common substructure $A \subset M, N$, for any $\overline{a} = (a_0, \cdots, a_{n-1}) \in A^n$,

$$M \models \exists x\varphi[x, \overline{a}] \Leftrightarrow N \models \exists x\varphi[x, \overline{a}].$$

[20]

(3) Let κ be an infinite cardinal and T a consistent κ-theory.
   (a) Assuming that T has an infinite model, show that T has a model of cardinality κ.
   (b) If all models of T are infinite and if T is κ-categorical, show that T is complete.

[10 + 10 = 20]

(4) Let R be a real field.

(a) Show that the field of rational functionals $R(X_1, \cdots, X_n)$ over R is real.
(b) Assume, moreover, that R is real closed. Let $f \in R(X_1, \cdots, X_n)$ be such that for no $\overline{a} \in R^n$, $f(\overline{a}) < 0$. Show that f is a sum of squares in $R(X_1, \cdots, X_n)$.

[5 + 10 = 15]

(5) (a) Show that there is no recursive set $U \subset \mathbb{N} \times \mathbb{N}$ universal for all recursive subsets of $\mathbb{N}$.
(b) Let $R \subset \mathbb{N}^k$ as well as $\mathbb{N}^k \setminus R$ be semi-recursive. Show that R is recursive.

[10 + 10 = 20]

(6) (a) Show that every complete axiomatized theory is decidable.
(b) Show that no axiomatized consistent extension of the theory N is complete.

[10 + 10 = 20]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2011-12 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date : 27.4.2012 | Maximum Marks : 70 | Duration : 4 Hours

Note: Answer two questions from Group A, four from Group B and four from Group C.

GROUP A

Answer any TWO questions.
Each question carries 14 marks.

1. Let $A = \mathbb{C}[X, Y, Z]/(XY - Z^2)$ and $P = (x, z)$, the ideal of A generated by x and z, the images of X and Z respectively. Show that

   (i) A is a Noetherian integral domain.

   (ii) A_P is a discrete valuation ring with $PA_P = zA_P$.

   (iii) xA is a P-primary ideal.

   (iv) A_m is a regular local ring for all but one maximal ideals m of A. [2+4+4+4=14]

2. Let R be a Noetherian ring.

   (i) Suppose that x is an element in R which is neither a unit nor a zerodivisor. Prove that $R/x^{n-1}R \cong xR/x^nR$ as R-modules for each $n \geq 1$; and hence construct a short exact sequence
$$0 \to R/x^{n-1}R \to R/x^nR \to R/xR \to 0.$$
   Deduce that $Ass_R\ (R/x^nR) = Ass_R\ (R/xR)\ \forall n \geq 1$.

   (ii) Compute $Ass_R\ (R/xR)$ and $Ass_R\ (R/x^2R)$ when $R = \mathbb{C}[X, Y]/(XY)$ and x is the image of X in R. [8+6=14]

3. (i) Let P be a prime ideal of a Noetherian ring R of height r. Show that there exist $a_1, \cdots, a_r \in P$ such that $\text{ht}(a_1, \cdots, a_r) = r$.

   (ii) Let $R = \mathbb{C}[X^2, Y^2, XY^2, X^3]$. Show that the principal ideal X^3R has an associated prime ideal of height 2. [9+5=14]

P.T.O.

GROUP B

Answer ANY FOUR questions.

Each question carries 4 marks.

1. Let t be a nonzero non-unit element of a Noetherian domain R. Prove that $\bigcap_n t^n R = (0)$.

2. Examine whether $\mathbb{C}[[X]][Y]_{(Y)}$ is a discrete valuation ring.

3. Show that every radical ideal of a valuation ring is a prime ideal.

4. If R is a valuation ring of dimension one with field of fractions K, then show that there does not exist any ring A with $R \subsetneqq A \subsetneqq K$.

5. Examine whether every primary ideal of a Dedekind domain is irreducible.

6. Show that any nonzero ideal of any Dedekind domain R is a finitely presented projective R-module of rank one. $[4 \times 4 = 16]$

GROUP C

Answer ANY FOUR questions.

Each question carries 8 marks.

1. Define the concepts of "a (additive) valuation" and "a place" of a field K. Given a discrete valuation ring (R, t) with field of fractions K, describe (without proof) the valuation and the place of K defined by R.

   If $\mathcal{P}$ is the place and v the valuation of $\mathbb{Q}(X)$ defined by the valuation ring $\mathbb{Q}[\frac{1}{X}]_{(\frac{1}{X})}$, compute $\mathcal{P}(\frac{1}{X-1})$, $v(\frac{1}{X-1})$, $\mathcal{P}(\frac{X}{2X^2+1})$ and $v(\frac{X}{2X^2+1})$. [8]

2. Let $A = \mathbb{C}[X, Y, Z, W]/(XY - ZW) = \mathbb{C}[x, y, z, w]$ (where x, y, z, w denote the images in A of X, Y, Z, W respectively). Find three elements a, b, c in A such that $B = \mathbb{C}[a, b, c]$ is isomorphic to the polynomial ring in three variables over $\mathbb{C}$ and A is integral over B. Write down explicit integral equations satisfied by x, y, z, w over B. [8]

3. Give an example of a ring R containing a maximal ideal M and an M-primary ideal I such that $M \neq (I : x)$ for any $x \in R$. [8]

4. Show that if a Noetherian local ring A contains at least one principal prime ideal P of height one, then A must be an integral domain. [8]

5. Let R be a discrete valuation ring and t a uniformising parameter of R. Let A be an R-algebra such that $R \subset A \subset R[X]$. Show that $A[1/t]$ is a Noetherian ring. [8]

6. Let R be a Noetherian normal domain and P a prime ideal of R such that the height of P is at least 2. Let $a \in P$. Show that P/aR contains a nonzerodivisor of R/aR. [8]

Indian Statistical Institute

semsetral examination : (2011-12)

M. Math II year

Special Topics (K theory of C^* algebras)

Date : 02.05.12 Maximum marks : 60 Duration : 3 hours.

Answer all the questions. Marks are indicated in brackets. The maximum you can score is 60.

(1) Let $\mathcal{O}_2$ be the universal unital C^* algebra generated by two isometries S_1, S_2 such that $S_1S_1^* + S_2S_2^* = 1$. Prove that $K_0(\mathcal{O}_2) = 0$. [25]

(2) Let $\phi : M_6(\mathbb{C}) \to M_{12}(\mathbb{C})$, $\psi : M_4(\mathbb{C}) \to M_{12}(\mathbb{C})$ be the $*$-homomorphisms given by $\phi(a) := \mathrm{diag}(a, a)$, $\psi(b) := \mathrm{diag}(b, b, b)$. Consider the C^* algebra $\mathcal{A}$ defined below:

$$\mathcal{A} := \{F \in C([0,1], M_{12}(\mathbb{C})) : \ \exists\, a \in M_6(\mathbb{C}),\ b \in M_4(\mathbb{C}) \text{ s.t. } F(0) = \phi(a),\ F(1) = \psi(b)\}.$$

Compute $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$ using the six-term exact sequence corresponding to a short exact sequence of the following form:

$$0 \to S(M_{12}(\mathbb{C})) \to \mathcal{A} \to M_6(\mathbb{C}) \oplus M_4(\mathbb{C}) \to 0.$$

[25]

(3) Let D be the open unit disc of $\mathbb{R}^2$, $\overline{D}$ be its closure (i.e. the closed unit disc) and S^1 be the unit circle. Consider the following short exact sequence:

$$0 \to C_0(D) \to C(\overline{D}) \to C(S^1) \to 0,$$

where the homomorphism from $C_0(D)$ to $C(\overline{D})$ is the natural inclusion, and the homomorphism from $C(D)$ to $C(S^1)$ is obtained by restriction, i.e. $f \mapsto f|_{S^1}$. Denote by $\delta_1 : K_1(C(S^1)) \to K_0(C_0(D))$ the index map corresponding to the above short exact sequence. Let $u \in C(S^1)$ be given by $u(z) = z$. Prove that $\delta_1([u]_1) = [e]_0 - [f]_0$, where

$$e(z) = \begin{pmatrix} |z|^2 & z(1-|z|^2)^{\frac{1}{2}} \\ \overline{z}(1-|z|^2)^{\frac{1}{2}} & 1-|z|^2 \end{pmatrix}, \quad f(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

[15]

INDIAN STATISTICAL INSTITUTE

SECOND SEMESTRAL EXAMINATION : (2011-2012)
M. MATH II
ALGEBRAIC NUMBER THEORY

DATE : MAY 2, 2012. MAXIMUM MARKS : 65 DURATION : 4 HOURS

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(1) Let $l \in \mathbb{Z}$ be a prime and $n = l^m$, $m \geq 1$. Let ζ be a primitive nth root of unity and write $K = \mathbb{Q}(\zeta)$. Let $\mathcal{O}_K$ be the ring of integers of K. Prove that
   (a) $l\mathcal{O}_K = (1-\zeta)^d \mathcal{O}_K$ where $d = \varphi(l^m) = [K : \mathbb{Q}]$; [6]
   (b) $(1-\zeta)\mathcal{O}_K \in \mathrm{Spec}(\mathcal{O}_K)$ and inertia degree of $(1-\zeta)\mathcal{O}_K$ is 1; [3]
   (c) the basis $1, \zeta, \cdots, \zeta^{d-1}$ of the $\mathbb{Q}$−vector space K has discriminant $\pm l^s$, where $s = l^{m-1}(ml - m - 1)$. [5]
   (d) $1, \zeta, \cdots, \zeta^{d-1}$ is an integral basis of $\mathcal{O}_K$. [6]

(2) Let R be a Dedekind domain and K its quotient field. Let L be a *Galois* extension of K of degree n and S be the integral closure of R in L.
   (a) Prove that S is a Dedekind domain. [6]
   (b) Let $\mathfrak{p} \in \mathrm{Spec}(R) \setminus (0)$. Prove that $P \mapsto \sigma(P)$ defines an action of $\mathrm{Gal}(L|K)$ on the set $T = \{P \in \mathrm{Spec}(S) | P \cap R = \mathfrak{p}\}$. Show that this action is transitive. [4]
   (c) Let $P \in \mathrm{Spec}(S) \setminus (0)$. Define the *decomposition group* G_P of P over K. For $\sigma \in \mathrm{Gal}(L|K)$, show that $G_{\sigma(P)} = \sigma G_P \sigma^{-1}$. [4]
   (d) Let $\mathfrak{p} \in \mathrm{Spec}(R) \setminus (0)$. Prove that

$$\mathfrak{p}S = (\prod_{\sigma} \sigma(P))^e,$$

   where $P \in \mathrm{Spec}(S)$ such that $P \cap R = \mathfrak{p}$ and σ varies over a system of representatives of $\mathrm{Gal}(L|K)/G_P$. [6]

[P.T.O.]

(3) (a) Consider the number field $K = \mathbb{Q}(2^{\frac{1}{3}})$. Assuming the fact that $\mathcal{O}_K = \mathbb{Z}[2^{\frac{1}{3}}]$, determine the prime ideal factorization of $7\mathcal{O}_K$ and $31\mathcal{O}_K$. [6]

(b) Let K be a quadratic number field with discriminant d and let p be an odd prime. Prove that $p\mathcal{O}_K = P^2$ for some $P \in \mathrm{Spec}\mathcal{O}_K$ if and only if p divides d. [5]

(c) Let ω be a complex cube root of unity and consider the ring $\mathbb{Z}[\omega]$. Let $p \in \mathbb{Z}$ be a prime. If $p \equiv 2 \pmod 3$, then prove that p remains a prime in $\mathbb{Z}[\omega]$. [4]

(4) (a) Let α be a unit in $\mathbb{Z}_2$. Show that α is a square in $\mathbb{Q}_2$ if and only if $\alpha \equiv 1 \pmod 8$.

(b) Prove directly that any sequence of $\mathbb{Z}_p$ has a convergent subsequence.

(c) If p, q are distinct odd primes, prove that $\mathbb{Q}_p$ is not isomorphic to $\mathbb{Q}_q$ (as fields).

(d) Prove that a p-adic integer $\alpha = (x_n)_{n \geq 0}$ is a unit if and only if $x_0 \not\equiv 0 \pmod p$. [4 × 4]

INDIAN STATISTICAL INSTITUTE
Semestral Examination : 2011-2012
M. Math. - II Year
Topology-III

Date :04-05-2012 | Maximum Score : 60 | Time :3 Hours

Any result that you use should be stated clearly.

(1) **a:** Define the notion of a CW-complex.
b: Describe a CW-complex structure for $\mathbb{R}P^n$, $n > 0$, indicating the cells and their characteristic maps explicitly.
c: Prove that for a CW-complex X, the inclusion $i : X^{p+1} \hookrightarrow X$ induces isomorphism $i_* : H_p(X^{p+1}) \longrightarrow H_p(X)$.
d: Prove that if A is a compact subspace of a CW-complex X, then A is contained in a finite subcomplex.

[6+5+8+6=25]

(2) **a:** State Universal coefficient theorem for singular cohomology.
b: Suppose for a space X, $H_{n-1}(X)$ is free abelian. Prove that the nth cohomology of X is dual to its nth homology.

[4+6=10]

(3) **a:** Let M be a topological manifold of dimension n and $p \in M$. Prove that

$$H_n(M, M - p) \cong \mathbb{Z}.$$

b: Define orientation on a manifold.

[6+4=10]

(4) **a:** State Mayer-Vietoris long exact sequence for singular homology.
b: Use Mayer-Vietoris long exact sequence to compute homology groups of $T^2 = S^1 \times S^1$.

[5+10=15]

(5) **a:** Define Complex projective space $\mathbb{C}P^n$ of dimension n.
b: Use cellular chain complex to compute homology groups of $\mathbb{C}P^n$.

[4+6=10]

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2011-2012, Second Semester
M-Stat II (MSP)and M-Math II
Ergodic Theory

Date: 07·05·12 **Max. Marks 70** **Duration: 3 Hours**

Note: Answer all questions.
All the measures considered are probability measures unless otherwise stated.

1. a) Prove Poincaré's recurrence theorem.

   b) Let $(X, \mathcal{B}, m, T)$ be a measure-preserving dynamical system. Show that T is ergodic if and only if $m(\cup_{n=1}^{\infty} T^{-n}A) = 1$ for all $A \in \mathcal{B}$ with $m(A) > 0$.

   c) Let T be measure-preserving ergodic and invertible on $(X, \mathcal{B}, m)$. Let $A \in \mathcal{B}$ and $m(A) > 0$. Prove that $\int_A R_A dm = 1$ where

   $$R_A(x) = \inf\{n \geq 1 : T^n(x) \in A\}.$$

   [8+7+8]

2. Let $K = \{z \in \mathcal{C} : |z| = 1\}$ with Borel σ-field and Lebesgue measure. Let $T : K \to K$ be defined as $Tz = az$ where a is not a root of unity. Show that
   a) T is ergodic
   b) $T \times T$ on $K \times K$ is not ergodic. [5+5]

3. a) Let $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ be measure-preserving and ergodic. Suppose that f and g are eigenfunctions of T corresponding to the eigenvalue λ. Show that $f = cg$ almost everywhere for some constant c.

   b) If T_1 and T_2 are invertible measure-preserving transformation on $(X, \mathcal{B}, m)$ such that $T_1 T_2 = T_2 T_1$, T_1 is ergodic, T_2 is weak mixing. Then show that T_1 is weak mixing.

   [6+6]

4. a) If T is an invertible ergodic measure-preserving transformation with discrete spectrum then show that T and T^{-1} are conjugate.

   b) Let $K = \{z \in C : |z| = 1\}$ with Lebesgue measure. Let $Tz = z^2$. Does T have discrete spectrum? Justify your answer.

[6 + 6]

5. a) Let $(K, \mathcal{B}, \lambda, T)$ be the dynamical system as given in 4.(b). Find the value of $h(T)$, the entropy of the system.

   b) Let $(X, \mathcal{B}, m, T)$ be a measure-preserving dynamical system. Let $\mathcal{A}$ be a countable measurable partition with finite entropy. Let $I^*(x) = \sup_{n\geq 1} I_{\mathcal{A}|\bigvee_{i=1}^{n} T^{-i}\mathcal{A}}(x)$ where $I_{\cdot|\cdot}$ denotes the conditional information function. Show that for all $A \in \mathcal{A}$, $m(x \in A : I^*(x) > \gamma)$ converges to zero at exponential rate as $\gamma \to \infty$.

[5+10]