Sankhyā : The Indian Journal of Statistics
2010, Volume 72-A, Part special, pp. 19-36

Ordinal Solution of Open Games and Analytic Sets

Ashok Maitra
Indian Statistical Institute, Kolkata, India

Abstract

We use the ordinal solution of open games to define constituents of analytic and coanalytic sets. Various properties of theses constituents are established and it is shown that they behave just as regularly as the classical constituents of Luzin and Sierpinski.

AMS (MOS) subject classification (1970). Primary 02K30, 04A15; Secondary 28A05.
Keywords and phrases. Analytic set, coanalytic set, constituents, prewellordering, open games.

1 Introduction

Exactly ten years ago David Blackwell (1967) had the remarkable insight that the reduction principle for coanalytic (Π^1_1) sets could be deduced from the determinacy of open games. This turned out to be an idea with far-reaching consequences. For, soon after the publication of Blackwell's paper, Addison and Moschovakis (1968) observed that the determinacy of open games could be made to yield a stronger property of coanalytic sets. They called it the prewellordering property. Furthermore, though it will not concern us here, they proved that the prewellordering property and hence the reduction principle hold at all Σ^1_{2k} and Π^1_{2k+1} ($k \geq 1$) levels of the projecive hierarchy under the hypothesis of the determinacy of projective games, thus settling an outstanding problem of descriptive theory.

Though the prewellordering property was first isolated by Addison and Moschovakis, the fact that it holds for coanalytic sets is implicit in the literature of classical descriptive set theory. Indeed it is an easy matter to check, using known results on constituents of a coanalytic set, that the constituents define a prewellordering (cf. Kuratowski, 1966, pp. 499-501).

Now in the approach of Blackwell-Addison-Moschovakis a prewellordering is defined directly by looking at a certain class of open games. Obviously this prewellordering determines what may be called constituents of the coanalytic set.

The main purpose of this article is to study these constituents. The motivation for this study arose from the question whether the covering theorem holds for these constituents and the question whether open games can be used to define constituents of analytic (Σ^1_1) sets.

We show that the answer to both questions is yes. We define constituents of analytic and coanalytic sets directly by using Blackwell's results on the ordinal solution of open games (see Blackwell, 1970). The basic idea here is to associate (Borel measurably) countable ordinals with subsets of the set of finite sequences of positive integers and Blackwell's analysis suggests how this should be done. Section 2 of this aticle is devoted to an exposition of Blackwell's results on open games. In Section 3 we present an example of an analytic non-Borel set due to Blackwell and also develop some machinery that is used in the sequel. We define constituents and show that they possess the same desirable measure and category theoretic properties as the 'classical' constituents in Section 4. The covering theorem is established for our constituents in Section 5. And finally we show in Section 6 that the constituents of a coanalytic set obtained through open games induce just the prewellordering that is obtained by the Blackwell-Addison-Moschovakis method.

2 Ordinal solution of open games

Let P be the set of finite sequences of positive integers, including the empty sequence e. Elements of P will be called *positions.* Let $S \subset P$. With S associate a two-person game $G(S)$ of complete information as follows. Two players, I and II, alternately choose positive integers, with I choosing first. A sequence $\omega = (n_1, n_2, \cdots)$ of such choices is called a *play.* A play $\omega = (n_1, n_2, \cdots)$ is a *win for I* just in case $e \notin S$ and $\langle n_1, n_2, \cdots, n_k\rangle \notin S$ for every $k \geq 1$; ω is a *win for II* otherwise. A *strategy* s for I is a function on the set of finite sequences of positive integers of even length into the set of positive integers. A play $\omega = (n_1, n_2, \cdots)$ is said to be *consistent* with a strategy s for I if $s(e) = n_1$ and $s(\langle n_1, \cdots, n_{2k}\rangle) = n_{2k+1}$ for all $k \geq 1$. A *winning strategy for I* is a strategy s for I such that all plays consistent with s are wins for I. One defines these notions analogously for player II.

Say that the game $G(S)$ is *determined* if either I has a winning strategy or II has a winning strategy.

The games $G(S)$ are just the special case of the games introduced by Gale and Stewart (1953) in which the winning set for player II is open. It is a well-known result of Gale and Stewart that such games are determined.

We now proceed to describe the ordinal solution of the games $G(S)$ due to Blackwell (1970).

Let then $p \in P$ and $B \subset P$. We say that *II can force the next position after p in B* if p is of even length and $pm \in B$ for every $m \geq 1$, or p is of odd length and $pm \in B$ for some $m \geq 1$. [Here pm denotes the concatenation of the two finite sequences p and $\langle m \rangle$.] We abbreviate the expression "II can force the next position after p in B" by $IICF_p(B)$. Analogously we define *I can force the next position after p in B* and abbreviate the expression by $ICF_p(B)$. Let us observe that,

$$\neg IICF_p(B) \Longrightarrow ICF_p(B^c). \tag{2.1}$$

We now define a map $\varphi : 2^P \to 2^P$ as follows:

$$\varphi(B) = B \cup \{p \in P : IICF_p(B)\}.$$

It is easy to see that φ is monotone, that is,

$$B_1 \subset B_2 \subset P \to \varphi(B_1) \subset \varphi(B_2).$$

Next we observe that for each $S \subset P$ there is a smallest fixed point of φ containing S. To see this, we define S_α by transfinite induction as follows:

$$S_0 = S; \qquad S_\alpha = \varphi(\cup_{\beta<\alpha} S_\beta), \quad 0 < \alpha.$$

Finally, set $W_S = \cup_{\alpha<\omega_1} S_\alpha$, where ω_1 is the first uncountable ordinal.

LEMMA 2.1. *For each $S \subset P$, W_S is the smallest fixed point of φ containing S.*

PROOF. Plainly, $W_S \subset \varphi(W_S)$. For the reverse inclusion, let $p \in \varphi(W_S)$. Then $p \in W_S$ or $IICF_p(W_S)$. If $p \in W_S$, we are done. So assume $IICF_p(W_S)$. If p is of odd length, then $pm \in W_S$ for some $m \geq 1$, so there is $\alpha < \omega_1$ such that $pm \in S_\alpha$ and hence $p \in \varphi(\cup_{\beta<\alpha+1} S_\beta) = S_{\alpha+1}$, so $p \in W_S$. If p is of even length, then $pm \in W_S$ for all $m \geq 1$. Choose $\alpha_m < \omega_1$ such that $pm \in S_{\alpha_m}$, $m \geq 1$. Let α be any ordinal less than ω_1

such that $\alpha_m \leq \alpha$ for all $m \geq 1$. Then $pm \in S_\alpha$ for all $m \geq 1$ and hence $p \in \varphi(\cup_{\beta<\alpha+1} S_\beta) = S_{\alpha+1}$, so $p \in W_S$. This proves that $\varphi(W_S) \subset W_S$. Hence $\varphi(W_S) = W_S$.

Let $B \subset P$ such that $S \subset B$ and $\varphi(B) = B$. It is easy to prove by induction on α that $S_\alpha \subset B$ for all α, which shows that $W_S \subset B$. This completes the proof. □

The above analysis enables us now to associate with each $S \subset P$ a function $\alpha_S : W_S \to \omega_1$ whose value at $p \in W_S$ is the smallest ordinal α such that $p \in S_\alpha$.

We are now in a position to state Blackwell's result on the games $G(S)$.

Theorem 2.1. *Let $S \subset P$. Then*

(a) $\alpha_S(p) = 0 \Longleftrightarrow p \in S$,

(b) $p \in W_S$ *and* $\alpha_S(p) > 0 \Longrightarrow IICF_p(\{q \in W_S : \alpha_S(q) < \alpha_S(p)\})$.

(c) $p \notin W_S \Longrightarrow ICF_p(W_S^c)$.

Proof. (a) is obvious. To check (b), assume that $p \in W_S$ and $\alpha_S(p) = \alpha > 0$. Then $p \in S_\alpha$ and $p \notin S_\beta$ for $\beta < \alpha$. It now follows from the definition of S_α that $IICF_p(\cup_{\beta<\alpha} S_\beta)$. But $\cup_{\beta<\alpha} S_\beta = \{q \in W_S : \ \alpha_S(q) < \alpha_S(p)\}$. This proves (b).

To prove (c), let $p \notin W_S$. It now follows from Lemma 2.1 that $p \notin \varphi(W_S)$. Hence $\neg IICF_p(W_S)$, so from (2.1) we have: $ICF_p(W_S^c)$. This terminates the proof. □

The game theoretic content of Theorem 2.1 is set forth in the following

Corollary 2.1. *If $p \in W_S$, then II has a winning strategy in the game $G(S)$ starting at p. If $p \notin W_S$, then I has a winning strategy in the game $G(S)$ starting at p.*

This is an immediate consequence of Theorem 2.1 and we omit the easy proof. The result of Gale and Stewart on the determinacy of the game $G(S)$ falls out of Corollary 2.1. Indeed, if $e \in W_S$ then II has a winning strategy and if $e \notin W_S$ then I has a winning strategy. So $G(S)$ is determined.

3 Example of an analytic non-Borel set

Here and in the sequel a subset of a Polish space is said to be *analytic* if it is empty or a continuous image of the space Σ, where $\Sigma = N^N$, N being the set of positive integers and the topology on Σ is the product of discrete topologies.

Topologize the set 2^P by giving it the product of discrete topologies on $\{0, 1\}$, so it becomes a homeomorph of the Cantor set. We shall use the same symbol to denote a subset of P as well as its indicator function. Set

$$E = \{S \in 2^P : I \text{ has a winning strategy in } G(S)\}.$$

We shall prove in this section that E is an analytic non-Borel set of 2^P. This fact was stated without proof by Blackwell (1970).

To establish the non-Borel nature of E, we need some preliminaries which will also be useful in the sequel. Let X be a Polish space and let f be a continuous function on Σ into X. With each $x \in X$ associate a game $G'(x)$ of complete information between players I and II as follows. Players alternately choose positive integers $m_1, n_1, m_2, n_2, \cdots$ with I making the first move. Player II wins just in case $x \notin cl(f(\Sigma(m_1)))$, or there is $i \geq 2$ such that $x \in cl(f(\Sigma(n_1, \cdots, n_{i-1})))$ and $x \notin cl(f(\Sigma(m_1, \cdots, m_i)))$, where cl denotes the closure operator on X and for $p \in P$, $\Sigma(p)$ is the set of all infnite sequences of positive integers for which p is an initial segment.

The game $G'(x)$ is just one of the games considered in Section 2. To formalize this, define a map $\psi : X \to 2^P$ by:

$$\langle n_1, n_2, \cdots n_k\rangle \in \psi(x) \iff (x \notin cl(f(\Sigma(n_1)))) \text{ or}$$

$$(\exists i \geq 1)(2i + 1 \leq k) \text{ and } x \in cl(f(\Sigma(n_2, n_4, \cdots, n_{2i})))- \\ cl(f(\Sigma(n_1, n_3, \cdots, n_{2i+1}))).$$

It is easy to see that the game $G'(x)$ is precisely the game $G(\psi(x))$. The next lemma summarizes the main facts about the function ψ and the games $G(\psi(x))$.

Lemma 3.1. (a) *The function $\psi : X \to 2^P$ is Borel measurable.*

(b) $x \in f(\Sigma) \Longrightarrow I$ has a winning strategy in $G(\psi(x))$.

(c) $x \notin f(\Sigma) \Longrightarrow II$ has a winning strategy in $G(\psi(x))$.

Proof. We omit the easy proof of (a). To prove (b) let $x \in f(\Sigma)$. So there is $m_1, m_2, \cdots$ such that $x = f(\{m_k\})$. To win $G(\psi(x))$, I has only to play $m_1, m_2, \cdots$. Suppose next that $x \notin f(\Sigma)$. It is easy to see that II can win $G(\psi(x))$ by imitating I's move. This completes the proof. □

The function ψ defined above will be called the *canonical map* on X to 2^P induced by the function f. The other fact we need is that the map $\varphi : 2^P \to 2^P$ introduced in Section 2 is Borel measurable.

Lemma 3.2. *The map φ is Borel measurable.*

Proof. It suffices to prove that for fixed $p \in P$, the map $S \mapsto \varphi(S)(p)$ is Borel measurable. We distinguish two cases.

Case 1. The length of p is odd. Then

$$\{B : \varphi(B)(p) = 1\} = \{B \in 2^P : B(p) = 1\} \bigcup \bigcup_{m=1}^{\infty} \{B \in 2^P : B(pm) = 1\}.$$

Case 2. The length of p is even. Then

$$\{B : \varphi(B)(p) = 1\} = \{B \in 2^P : B(p) = 1\} \bigcup \bigcap_{m=1}^{\infty} \{B \in 2^P : B(pm) = 1\}.$$

In either case, the set $\{B \in 2^P : \varphi(B)(p) = 1\}$ is seen to be Borel, which completes the proof. □

Theorem 3.1. *The set E is an analytic non-Borel subset of 2^P.*

Proof. First we prove that E is analytic. For this observe that for $S \in 2^P$

$$S \in E \iff (\exists B \in 2^P)(S \subset B, \varphi(B) = B \text{ and } e \notin B).$$

The above equivalence follows from Lemma 2.1 and the fact noted in Section 2 that I has a winning strategy in $G(S)$ just in case $e \notin W_S$. Consequently E is the projection to the first coordinate of the set

$$\{(S, B) \in 2^P \times 2^P : S \subset B, \varphi(B) = B \text{ and } e \notin B\}.$$

But as is easily seen by using Lemma 3.2 the above set is Borel in $2^P \times 2^P$. Hence E is analytic.

To show that E is not Borel in 2^P, let X be an uncountable Polish space. As is well-known X contains an analytic non-Borel set A. Let f be a continuous function on Σ onto A and let ψ be the canonical map on X to 2^P induced by f. It is immediate from Lemma 3.1 that $\psi^{-1}(E) = A$. As ψ is Borel measurable and A is not Borel it follows that E is not Borel, which terminates the proof. □

4 Constituents of analytic and coanalytic sets

The ordinal analysis in Section 2 of the games $G(S)$ provides us with a method of associating (Borel measurably) ordinals with subsets of P. This in turn enables us to define constituents of an analytic set as well as constituents of its complement. The constituents defined here have points of similarity with but are different from the "classical" constituents of Luzin and Sierpinski (1923), Sierpinski (1926, 1933) and Selivanowski (1933). However, as will be shown, our constituents have the same desirable properties as the "classical" ones and are in fact somewhat simpler.

To begin with define maps $\varphi_\alpha : 2^P \to 2^P$ for each ordinal $\alpha < \omega_1$ as follows:

$$\varphi_\alpha(S) = S_\alpha.$$

LEMMA 4.1. *For each $\alpha < \omega_1$, the maps φ_α are Borel measurable.*

PROOF. The proof is by induction on α. For $\alpha = 0$, the map φ_α is the identity function and so Borel measurable. Suppose that $0 < \alpha < \omega_1$ and φ_β is Borel measurable for $\beta < \alpha$. Define ψ_α by :

$$\psi_\alpha(S) = \cup_{\beta<\alpha} S_\beta = \cup_{\beta<\alpha} \varphi_\beta(S), \quad S \in 2^P.$$

Plainly ψ_α is Borel measurable and $\varphi_\alpha = \varphi \circ \psi_\alpha$. It now follows from Lemma 3.2 that φ_α is Borel measurable. This completes the proof. □

Next we define two ordinal valued functions σ and τ. To define σ let $S \subset P$. Since W_S is countable and the values of α_S are countable ordinals, there is $\alpha < \omega_1$ such that $\alpha_S(p) \leq \alpha$ for all $p \in W_S$. It follows from the definition of α_S that $W_S \subset S_\alpha$ and hence $W_S = S_\alpha$. We define $\sigma(S)$ to be the smallest ordinal β such that $W_S = S_\beta$. Thus σ is a function on 2^P into ω_1. The function τ has for domain the set $2^P - E$ and its value at $S \in 2^P - E$ is defined to be $\alpha_S(e)$ (note that $\alpha_S(e)$ is defined because $e \in W_S$). The values of τ are again countable ordinals.

LEMMA 4.2. *For each $\alpha < \omega_1$, the sets $\{S \in 2^P : \sigma(S) \leq \alpha\}$ and $\{S \in 2^P - E : \tau(S) \leq \alpha\}$ are Borel in 2^P.*

Proof. Use Lemma 2.1 to see that for any $S \in 2^P$

$$\sigma(S) \leq \alpha \Longleftrightarrow \varphi \circ \varphi_\alpha(S) = \varphi_\alpha(S).$$

Hence

$$\{S \in 2^P : \sigma(S) \leq \alpha\} = \{S \in 2^P : \varphi \circ \varphi_\alpha(S) = \varphi_\alpha(S)\}.$$

Since φ and φ_α are Borel measurable it follows that $\{S \in 2^P : \sigma(S) \leq \alpha\}$ is a Borel subset of 2^P.

Next note that for any $S \in 2^P$

$$\begin{aligned} e \in S_\alpha &\Longleftrightarrow e \in W_S \quad \text{and} \quad \tau(S) \leq \alpha \\ &\Longleftrightarrow S \in (2^P - E) \quad \text{and} \quad \tau(S) \leq \alpha. \end{aligned}$$

Consequently,

$$\{S \in 2^P - E : \tau(S) \leq \alpha\} = \{S \in 2^P : \varphi_\alpha(S)(e) = 1\}$$

so that $\{S \in 2^P - E : \tau(S) \leq \alpha\}$ is a Borel set in 2^P, which completes the proof. □

We are now ready to define constituents. Let then A be a non-empty analytic subset of a Polish space. Let f be a continuous function on Σ onto A and let ψ be the canonical map on X to 2^P induced by f. For $\alpha < \omega_1$ set

$$A_\alpha = \{x \in X : e \notin \varphi_\alpha \circ \psi(x)\},$$

$$B_\alpha = \{x \in X : \sigma(\psi(x)) > \alpha\}.$$

Theorem 4.1. *(a) The sets A_α, B_α, $\alpha < \omega_1$, are Borel in X.*

(b) $\alpha < \beta < \omega_1 \Longrightarrow A_\alpha \supset A_\beta$.

(c) $\alpha < \beta < \omega_1 \Longrightarrow A_\alpha - B_\alpha \subset A_\beta - B_\beta$.

(d) $A = \bigcap_{\alpha<\omega_1} A_\alpha = \bigcup_{\alpha<\omega_1}(A_\alpha - B_\alpha)$.

Proof. The Borel measurability of φ_α and ψ imply that A_α is Borel in X, while Lemma 4.2 implies that B_α is Borel in X. To prove (b) note that if $\alpha < \beta < \omega_1$ then $\varphi_\alpha \circ \psi(x) \subset \varphi_\beta \circ \psi(x)$ for each $x \in X$, so that $A_\alpha \supset A_\beta$. For (c) let $x \in A_\alpha - B_\alpha$, so $e \notin \varphi_\alpha \circ \psi(x)$ and $\sigma(\psi(x)) \leq \alpha$. If $\alpha < \beta$ then $\sigma(\psi(x)) \leq \beta$ so $x \notin B_\beta$. Since $\sigma(\psi(x)) \leq \alpha$ it follows that $\varphi_\alpha \circ \psi(x) = \varphi_\beta \circ \psi(x)$ and so $e \notin \varphi_\beta \circ \psi(x)$ and hence $x \in A_\beta$. Thus $x \in A_\beta - B_\beta$, which proves (c).

Next observe that for any $x \in X$,

$$\begin{aligned}
&x \in A \\
\iff &\psi(x) \in E \\
\iff &I \ \text{ has a winning strategy in } \ G(\psi(x)) \\
\iff &e \notin W_{\psi(x)} \\
\iff &(\forall \alpha < \omega_1)\ (e \notin \varphi_\alpha \circ \psi(x)) \\
\iff &(\forall \alpha < \omega_1)\ (x \in A_\alpha).
\end{aligned}$$

Hence $A = \bigcap_{\alpha<\omega_1} A_\alpha$. Finally note that for any $x \in X$,

$$\begin{aligned}
x \in A &\iff e \notin W_{\psi(x)} \\
&\iff (\exists \alpha < \omega_1)\ (W_{\psi(x)} = \varphi_\alpha \circ \psi(x) \ \text{ and } \ e \notin \varphi_\alpha \circ \psi(x)) \\
&\iff (\exists \alpha < \omega_1)\ (\sigma(\psi(x)) \leq \alpha \ \text{ and } \ e \notin \varphi_\alpha \circ \psi(x)) \\
&\iff (\exists \alpha < \omega_1)\ (x \in A_\alpha - B_\alpha),
\end{aligned}$$

so that $A = \bigcup_{\alpha<\omega_1}(A_\alpha - B_\alpha)$. This proves (d). $\square$

We shall call the sets $A_\alpha - B_\alpha$, $\alpha < \omega_1$, the *constituents of the analytic set A determined by the function f*, while the sets $A^c_\alpha = \{x \in X : \tau(\psi(x)) \leq \alpha, \alpha < \omega_1\}$ will be called the *constituents of the coanalytic set $X - A$ determined by f*. We have given above incidentally a new proof of the fact that an analytic set can be expressed as the intersection as well as union of $\aleph_1$ Borel sets.

Next we prove that our constituents have all the desirable properties of the "classical" constituents.

Theorem 4.2. *Let μ be a finite measure on the Borel subsets of X. Then A is μ-measurable and there is an $\alpha_0 < \omega_1$ such that $\overline{\mu}(A) = \mu(A_{\alpha_0} - B_{\alpha_0})$ and $\overline{\mu}(X - A) = \mu(A^c_{\alpha_0})$, where $\overline{\mu}$ is the completion of μ.*

Proof. For each $p \in P$ and $\alpha < \omega_1$, define

$$L_\alpha(p) = \{x \in X : p \notin \varphi_\alpha \circ \psi(x) \quad \text{and} \quad \sigma(\psi(x)) > \alpha\}.$$

Then $L_\alpha(p)$ is Borel in X and $\beta < \alpha < \omega_1$ implies $L_\beta(p) \supset L_\alpha(p)$. Fix $p \in P$. The sets $L_\alpha(p)$, $\alpha < \omega_1$, form a transfinite sequence of non-increasing Borel sets. Since the measure algebra $\mathcal{B}(\mu)$ ($\mathcal{B}$ = Borel σ-field on X) satisfies the countable chain condition, it follows that there exists $\beta(p) < \omega_1$ such that $\mu(L_\alpha(p)) = \mu(L_{\alpha+1}(p))$ for all $\alpha \geq \beta(p)$. Now let α_0 be a countable ordinal such that $\beta(p) \leq \alpha_0$ for all $p \in P$. Then we have: $\mu(L_\alpha(p)) = \mu(L_{\alpha+1}(p))$ for all $\alpha \geq \alpha_0$ and for $p \in P$. In particular $\mu(L_{\alpha_0}(p) - L_{\alpha_0+1}(p)) = 0$ for all $p \in P$. Put $L = \bigcup_{p \in P} (L_{\alpha_0}(p) - L_{\alpha_0+1}(p))$, so L is Borel and $\mu(L) = 0$. We now claim that

$$A_{\alpha_0} - B_{\alpha_0} \subset A \subset (A_{\alpha_0} - B_{\alpha_0}) \cup L \tag{4.1}$$

The inclusion on the left follows from Theorem 4.1. To prove the inclusion on the right, let $x \in A$ and $x \notin A_{\alpha_0} - B_{\alpha_0}$. Since $A \subset A_{\alpha_0}$, we have: $x \in A_{\alpha_0}$. Consequently, as $x \notin A_{\alpha_0} - B_{\alpha_0}$, x must be in B_{α_0}, so that $\sigma(\psi(x)) > \alpha_0$. This implies that $\varphi_{\alpha_0} \circ \psi(x) \neq \varphi_{\alpha_0+1} \circ \psi(x)$.

Choose $p \in \varphi_{\alpha_0+1} \circ \psi(x) - \varphi_{\alpha_0} \circ \psi(x)$, so that $x \in L_{\alpha_0}(p)$. But as $p \in \varphi_{\alpha_0+1} \circ \psi(x)$ it follows that $x \notin L_{\alpha_0+1}(p)$. Hence $x \in L_{\alpha_0}(p) - L_{\alpha_0+1}(p)$, so $x \in L$. This proves the claim. Since $A_{\alpha_0} - B_{\alpha_0}$ is a Borel set and $\mu(L) = 0$, it follows from (4.1) that A is μ-measurable and that $\overline{\mu}(A) = \mu(A_{\alpha_0} - B_{\alpha_0})$.

Next note that the argument used to establish the inclusion on the right of (4.1) actually proves that $B_{\alpha_0} \subset L$. Hence $\mu(A_{\alpha_0} \cap B_{\alpha_0}) = 0$ and so $\overline{\mu}(A) = \mu(A_{\alpha_0})$, from which we conclude that $\overline{\mu}(X - A) = \mu(A^c_{\alpha_0})$. This completes the proof of Theorem 4.2. □

We have thus reproved Luzin's theorem that an analytic set is universally measurable. The next result gives the category analogue of Theorem 4.2.

Theorem 4.3. *The set A possesses the Baire property and there exists $\alpha_0 < \omega_1$ such that $A - (A_{\alpha_0} - B_{\alpha_0})$ and $(X - A) - A^c_{\alpha_0}$ are meagre.*

Proof. Use the fact that the quotient algebra $\mathcal{B}/\mathcal{N}$ ($\mathcal{N} =$ the σ-ideal of meagre Borel sets) satisfies the countable chain condition to prove that there is $\alpha_0 < \omega_1$ such that $L_{\alpha_0}(p) - L_{\alpha_0+1}(p)$ is meagre for every $p \in P$. Define L as in the proof of Theorem 4.2. Then L is meagre. Since $A_{\alpha_0} - B_{\alpha_0}$ is Borel in X and hence possesses the Baire property it follows from (4.1) that $A - (A_{\alpha_0} - B_{\alpha_0})$ is meagre and that A possesses the Baire property. Furthermore $(X - A) - A^c_{\alpha_0} \subset A_{\alpha_0} \cap B_{\alpha_0} \subset L$, so that $(X - A) - A^c_{\alpha_0}$ is meagre. This completes the proof. □

5 Covering theorem and the first principle of seperation for analytic sets

An extremely important property of the "classical" constituents is the Covering Theorem of Luzin. We prove below the Covering Theorem for our constituents.

Let A be a non-empty analytic subset of a Polish space X and let f be a continuous function on Σ onto A. Denote by ψ the canonical map on X to 2^P induced by f. Finally let A_α, B_α be as in the previous section.

THEOREM 5.1. *If A' is an analytic subset of X such that $A' \cap A_\alpha \neq \emptyset$ for every $\alpha < \omega_1$, then $A \cap A' \neq \emptyset$.*

PROOF. We shall define two sequences $m_1^0, m_2^0, \cdots$ and $n_1^0, n_2^0, \cdots$ of positive integers inductively such that

$$cl\left(f\left(\Sigma(m_1^0, m_2^0, \cdots, m_k^0)\right)\right) \cap cl\left(g\left(\Sigma(n_1^0, n_2^0, \cdots, n_k^0)\right)\right) \neq \emptyset, \tag{5.1}$$

for every $k \geq 1$, where g is a continuous function on Σ onto A'. Since f and g are continuous, the diameters of sets in (5.1) tend to 0 as $k \to \infty$. The completeness of X now implies that the intersection over all k of the sets in (5.1) reduces to a singleton, say, $\{x_0\}$. Plainly $x_0 \in A \cap A'$.

We first define m_1^0. Fix $\alpha < \omega_1$. We then have:

$$A' \cap A_\alpha = A' \cap \{x \in X : e \notin \varphi_\alpha \circ \psi(x)\}$$

$$\subset A' \bigcap \left(\bigcup_{m_1=1}^{\infty} \{x : \langle m_1 \rangle \notin \bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)\} \right).$$

Since $A' \cap A_\alpha \neq \emptyset$, it follows that there is $m_1(\alpha)$ such that

$$A' \bigcap \{x : \langle m_1(\alpha) \rangle \notin \bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)\} \neq \emptyset.$$

This sets up a map $\alpha \mapsto m_1(\alpha)$ from ω_1 to N, so there exists m_1^0 such that $m_1(\alpha) = m_1^0$ for uncountably many α. Hence

$$A' \cap \{x : \langle m_1^0 \rangle \notin \bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)\} \neq \emptyset$$

for uncountably many α. Now for fixed x the sets $\bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)$ are obviously non-decreasing in α and so

$$A' \cap \{x : \langle m_1^0 \rangle \notin \bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)\} \neq \emptyset \tag{5.2}$$

for all $\alpha < \omega_1$. Replacing α by $\alpha + 1$ in (5.2) we get

$$A' \cap \{x : \langle m_1^0 \rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset \tag{5.3}$$

for all $\alpha < \omega_1$. Now

$$\{x : \langle m_1^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \subset \{x : \langle m_1^0, m_1^0\rangle \notin \bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)\}.$$

So (5.3) yields

$$A' \cap \{x : \langle m_1^0, m_1^0\rangle \notin \bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)\} \neq \emptyset \tag{5.4}$$

for all $\alpha < \omega_1$. Replacing α by $\alpha + 1$ in (5.4) yields

$$A' \cap \{x : \langle m_1^0, m_1^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset$$

for all $\alpha < \omega_1$. Now $A' = \bigcup_{n_1=1}^\infty g(\Sigma(n_1))$, so for every $\alpha < \omega_1$ there is $n_1(\alpha)$ such that

$$g(\Sigma(n_1(\alpha))) \cap \{x : \langle m_1^0, m_1^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset.$$

Hence there is n_1^0 such that

$$g(\Sigma(n_1^0)) \cap \{x : \langle m_1^0, m_1^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset$$

for uncountably many α and hence for all $\alpha < \omega_1$.

For the inductive step assume that $m_1^0, m_2^0, \cdots, m_k^0, n_1^0, n_2^0 \cdots, n_k^0$ have been chosen so that

$$g(\Sigma(n_1^0, n_2^0, \cdots, n_k^0)) \cap \{x : \langle m_1^0, m_1^0, m_2^0, m_2^0, \cdots, m_k^0, m_k^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset \tag{5.5}$$

for all $\alpha < \omega_1$. Now

$$\begin{aligned}\{x : \langle m_1^0, m_1^0, \cdots, m_k^0, m_k^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \subset \\ \bigcup_{m_{k+1}=1}^\infty \{x : \langle m_1^0, m_1^0, \cdots, m_k^0, m_k^0, m_{k+1}\rangle \notin \cup_{\beta<\alpha}\varphi_\beta \circ \psi(x)\}.\end{aligned} \tag{5.6}$$

Consequently by arguing as above and using (5.5) and (5.6) one deduces that there is m_{k+1}^0 such that

$$g(\Sigma(n_1^0, \cdots, n_k^0)) \cap \{x : \langle m_1^0, m_1^0, \cdots, m_k^0, m_k^0, m_{k+1}^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset \tag{5.7}$$

for all $\alpha < \omega_1$. Since

$$\{x : \langle m_1^0, m_1^0, \cdots m_k^0, m_k^0, m_{k+1}^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \subset$$

$$\{x : \langle m_1^0, m_1^0, \cdots m_k^0, m_k^0, m_{k+1}^0, m_{k+1}^0\rangle \notin \bigcup_{\beta<\alpha} \varphi_\beta \circ \psi(x)\}$$

it follows from (5.7) that

$$g(\Sigma(n_1^0, \cdots, n_k^0)) \cap \{x : \langle m_1^0, m_1^0, \cdots, m_{k+1}^0, m_{k+1}^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset \quad (5.8)$$

for all $\alpha < \omega_1$. Since

$$g(\Sigma(n_1^0, \cdots, n_k^0)) = \bigcup_{n_{k+1}=1}^{\infty} g(\Sigma(n_1^0, \cdots, n_k^0, n_{k+1}))$$

it follows from (5.8) that there is n_{k+1}^0 such that

$$g(\Sigma(n_1^0, \cdots, n_k^0, n_{k+1}^0)) \cap \{x : \langle m_1^0, m_1^0, \cdots, m_k^0, m_k^0, m_{k+1}^0, m_{k+1}^0\rangle \notin \varphi_\alpha \circ \psi(x)\} \neq \emptyset$$

for all $\alpha < \omega_1$, which implies the proof of the inductive step.

Putting $\alpha = 0$ in (5.5) we get:

$$g(\Sigma(n_1^0, \cdots, n_k^0)) \cap \{x : \langle m_1^0, m_1^0, \cdots, m_k^0, m_k^0\rangle \notin \psi(x)\} \neq \emptyset$$

for each $k \geq 1$. Thus

$$cl\ (g(\Sigma(n_1^0, \cdots, n_k^0))) \cap cl\ (f(\Sigma(m_1^0, \cdots, m_k^0))) \neq \emptyset$$

for all $k \geq 1$ and the proof is complete. □

The Covering Theorem can now be stated as

THEOREM 5.1′. *If A' is an analytic subset of X such that $A' \subset X - A$, then there is $\alpha < \omega_1$ such that $A' \subset A_\alpha^c$.*

An immediate consequence of Theorem 5.1′ and the fact that a Borel subset of X is analytic is:

COROLLARY 5.1. *The following conditions on the coanalytic set $X - A$ are equivalent.*

(a) $X - A$ is analytic.

(b)$X - A = A_\alpha^c$ for some $\alpha < \omega_1$

(c) The function $\tau o \psi$ is bounded on $X - A$.

The first principle of Seperation for analytic sets, viz., any pair of disjoint analytic sets can be seperated by a Borel set, follows from Theorem 5.1′, while Souslin's theorem that a set which is both analytic and coanalytic is Borel falls out of Corollary 5.1.

6 The prewellordering property of coanalytic sets

The purpose of this section is to show that our constituents, just like the "classical" ones, can be used to establish the prewellordering property of coanalytic sets.

Let C be a coanalytic subset of a Polish space X. Following Kechris and Moschovakis (1971) we say that C possesses the prewellordering property if there exist a function ρ on C into an ordinal and relations R, R' on X such that R is a coanalytic subset of $X \times X$ and R' is an analytic subset of $X \times X$ and for every $y \in C$ the following condition holds.

$$(\forall\, x \in X)\ (x \in C \ \text{ and } \ \rho(x) \leq \rho(y) \Longleftrightarrow xRy \Longleftrightarrow xR'y) \tag{6.1}$$

Suppose that C is a coanalytic subset of a Polish space X. Assume without loss of generality that $C \neq X$. We shall show that the coanalytic set C possesses the prewellordering property. Put $A = X - C$. Hence A is analytic. Let f be a continuous function on Σ onto A and let ψ be the canonical map on X to 2^P induced by f. For ρ take the function $\tau \circ \psi$ whose domain is C (recall that the domain of τ is $2^P - E$).

In order to define the relations R and R', we associate with $x, y \in X$ a two person game $G(x,y)$ of complete information between players I and II as follows. Players I and II alternately choose positive integers $m_1, n_1, m_2, n_2, \cdots$ with I making the first move. Player II wins just in case $x \notin cl\ (f(\Sigma(m_1)))$ or there is $i \geq 2$ such that $y \in cl\ (f(\Sigma(n_1, \cdots, n_{i-1})))$ and $x \notin cl\ (f(\Sigma(m_1, \cdots, m_i)))$. Now define

$$R = \{(x,y) \in X \times X : II \ \text{ has a winning strategy in } \ G(x,y)\}$$

Lemma 6.1. *R is coanalytic in $X \times X$.*

Proof. Define $F : X \times X \to 2^P$ as follows.

$$\langle n_1, n_2, \cdots, n_k\rangle \in F(x,y) \Longleftrightarrow (x \notin cl(f(\Sigma(n_1)))) \text{ or}$$

$$(\exists i \geq 1)(2i+1 \leq k \text{ and } y \in cl(f(\Sigma(n_2, n_4, \cdots, n_{2i})))$$
$$\text{and } x \notin cl(f(\Sigma(n_1, n_3, \cdots, n_{2i+1})))).$$

It is easy to check that F is Borel measurable. Moreover it is fairly obvious that the game $G(x,y)$ is identical with the game $G(F(x,y))$. Consequently, $R = F^{-1}(2^P - E)$. The Borel measurability of F and the fact that $2^P - E$ is coanalytic now imply that R is coanalytic. This completes the proof. □

To define the relation R' observe that if $y \in C$ player II can win $G(x, y)$ with a strategy t that ensures every $k \geq 1$ that $y \in cl(f(\Sigma(n_1, n_2, \cdots, n_k)))$ whenever $x \in cl(f(\Sigma(m_1, m_2, \cdots, m_k)))$, where $(m_1, n_1, m_2, n_2, \cdots)$ is a play consistent with t. More formally we proceed as follows.

Denote by P_o the set of finite sequences of odd length and by P_e the set of finite sequences of positive even length. If $p = \langle n_1, n_2, \cdots, n_{2k}\rangle \in P_e$, denote the sequence $\langle n_1, n_3, \cdots, n_{2k-1}\rangle$ by p_o and the sequence $\langle n_2, n_4, \cdots, n_{2k}\rangle$ by p_e. The set of strategies of player II is N^{P_o} which is topologized by giving it the product of discrete topologies on N. If $p = \langle n_1, n_2, \cdots, n_k\rangle \in P$ and $t \in N^{P_o}$, we say that p is *consistent* with t if $t(\langle n_1\rangle) = n_2$, $t(\langle n_1, n_2, n_3\rangle) = n_4$, $\cdots$. Denote by C_p the set of $t \in N^{P_o}$ such that p is consistent with t. Plainly C_p is clopen in N^{P_o}. Define R' by

$$xR'y \iff (\exists t \in N^{P_o})(\forall p \in P_e)$$

$$\{t \in C_p \text{ and } x \in cl(f(\Sigma(p_o))) \Longrightarrow y \in cl(f(\Sigma(p_e)))\}$$

so that

$$R' = \Pi\Bigg(\bigcap_{p\in P_e}\Big[\Big(X \times X \times \big(N^{P_0} - C_p\big)\Big) \bigcup \Big(\big(X - cl\big(f(\Sigma(p_o))\big)\big) \times X \times N^{P_o}\Big) \bigcup \Big(C_p \times cl\big(f(\Sigma(p_o))\big) \times cl\big(f(\Sigma(p_e))\big)\Big)\Big]\Bigg),$$

where Π is projection to $X \times X$. As the set within the square brackets is Borel in $X \times X \times N^{P_o}$ and the intersection preceeding it is countable, R' is analytic. Thus we have

Lemma 6.2. *R' is analytic.*

The next two lemmmas are obvious.

Lemma 6.3. *If $y \in C$, then $xRy \iff xR'y$.*

Lemma 6.4. *If $y \in C$, then $xRy \Longrightarrow x \in C$.*

Lemma 6.5. *Let $x, y \in C$. Then $\tau \circ \psi(x) \leq \tau \circ \psi(y) \iff xRy$.*

Proof. We sketch the proof. Let $x, y \in C$ and assume that $\tau \circ \psi(x) \leq \tau \circ \psi(y)$. Put $S = \psi(x)$ and $T = \psi(y)$ and $\alpha = \tau(S)$. We shall describe a winning strategy for player II in the game $G(x, y)$. Suppose that I plays $n_1, n_2, \cdots$ in $G(x, y)$. Let k be the smallest positive integer such that $x \notin cl(f(\Sigma(n_1, n_2, \cdots, n_k)))$. If $k = 1$ then any sequence of moves will win

$G(x,y)$ for II. Assume $k > 1$. By Theorem 2.1 $\alpha_S(\langle n_1\rangle) = \alpha_1 < \alpha$. Since $\alpha_T(e) \geq \alpha_S(e) > \alpha_1$ there is m_1 such that $\alpha_T(\langle m_1\rangle) \geq \alpha_1$. Then this m_1 is II's reply to I's first move in $G(x,y)$. To determine II's response to I's second move n_2 proceed as follows. Use Theorem 2.1 to obtain l_1 such that $\alpha_S(\langle n_1, l_1\rangle) = \alpha_2 < \alpha_1$ so that by Theorem 2.1 again $\alpha_S(n_1, l_1, n_2\rangle) = \alpha_3 < \alpha_2$. Now $\alpha_T(\langle m_1\rangle) > \alpha_2$, so $\alpha_T(\langle m_1, m_1\rangle) \geq \alpha_2$ and hence there is m_2 such that $\alpha_T(\langle m_1, m_1, m_2\rangle) \geq \alpha_3$. II now plays m_2 in $G(x,y)$. Continuing in this manner, II's moves $m_1, m_2, \cdots, m_{k-1}$ can be determined so that $y \in cl(f(\Sigma(m_1, m_2, \cdots, m_{k-1})))$. We have thus described a winning strategy for II in $G(x,y)$. Hence xRy.

For the converse assume that $x, y \in C$ and $\tau \circ \psi(y) < \tau \circ \psi(x)$. Let S and T be as above and $\beta = \tau(T)$. We shall describe a winning strategy for player I in $G(x,y)$. Since $\alpha_S(e) > \beta$ there is n_1 such that $\alpha_S(\langle n_1\rangle) \geq \beta$. I's first move in $G(x,y)$ is n_1. Suppose that m_1 is II's response to n_1 in $G(x,y)$. By Theorem 2.1 $\alpha_T(\langle m_1\rangle) = \beta_1 < \beta$. If $\beta_1 = 0$ then player I wins $G(x,y)$. Assume that $\beta_1 > 0$. Then by Theorem 2.1 there is l_1 such that $\alpha_T(\langle m_1, l_1\rangle) = \beta_2 < \beta_1$. Now $\alpha_S(\langle n_1\rangle) > \beta_1$ so $\alpha_S(\langle n_1, n_1\rangle) \geq \beta_1 > \beta_2$. hence there is n_2 such that $\alpha_S(\langle n_1, n_1, n_2\rangle) \geq \beta_2$. Then n_2 is I's second move in $G(x,y)$. Continuing in this manner I's moves $n_1, n_2, \cdots$ against II's $m_1, m_2, \cdots$ can be specified so that $x \in cl(f(\Sigma(n_1, n_2, \cdots, n_k)))$ whenever $y \in cl(f(\Sigma(m_1, m_2, \cdots, m_{k-1})))$. Clearly the strategy thus described is a winning strategy for I in $G(x,y)$ and hence $\neg(xRy)$. This completes the proof of Lemma 6.5. $\square$

Lemmas 6.4 and 6.5 now establish (6.1). We have thus proved

THEOREM 6.1. *If C is a coanalytic subset of a Polish space X, then C possesses the prewellordering property.*

Addison and Moschovakis (1968) have shown that the reduction principle is a consequence of the prewellordering property. We now prove the reduction principle for countably many coanalytic sets, first established by Kuratowski (1936).

THEOREM 6.2. *Let $C_n, n \geq 1$ be coanalytic subsets of a Polish space X. Then there exist coanalytic subsets $B_n, n \geq 1$ of X such that (i) $B_n \subset C_n, n \geq 1$, (ii) $B_n \cap B_m = \emptyset$ for $n \neq m$ and (iii) $\cup_{n\geq 1} B_n = \cup_{n\geq 1} C_n$.*

PROOF. Let N be the set of positive integers. Set $C = \cup_{n\geq 1}(C_n \times \{n\})$. Then C is a coanalytic subset of the Polish space $X \times N$. By Theorem 6.1 there exists $\rho : C \to \omega_1$ and relations R and R' on $X \times N$ such that R is a coanalytic subset of $(X \times N) \times (X \times N)$ and R' is an analytic subset of $(X \times N) \times (X \times N)$ and such that (6.1) is satisfied.

Define

$$B_1 = \{x \in C_1 : (\forall n \geq 1)\ (x \in C_1 \Longleftrightarrow \rho(x,1) \leq \rho(x,n))\}$$

and for $m \geq 2$,

$$B_m = \{x \in C_m : (\forall i < m)(x \in C_i \Longrightarrow \rho(x,m) < \rho(x,i) \text{ and}$$
$$(\forall n \geq m)(x \in C_n \Longrightarrow \rho(x,m) \leq \rho(x,n)\}.$$

It is easy to verify that the sets B_n satisfy conditions (i) - (iii) in the statement of the theorem. It remains to verify that the sets B_n are coanalytic. Observe that

$$x \in B_1 \Longleftrightarrow (x \in C_1) \text{ and } (\forall n \geq 1)((x,1)R(x,n) \text{ or } \neg((x,n)R'(x,1)))$$

and for $m > 1$

$$x \in B_m \Longleftrightarrow (x \in C_m) \text{ and } (\forall i < m)(((x,m)R(x,i)) \text{ and } \neg((x,i)R'(x,m)))$$
$$\text{and } (\forall n \geq m)(((x,m)R(x,n)) \text{ or } \neg((x,n)R'(x,m))).$$

It follows immediately that the sets B_n are coanalytic. This completes the proof. □

Acknowledgement. Discussions with B. V. Rao and K. P. S. Bhaskara Rao are gratefully acknowledged.

References

Addison, J.W. and Moschovakis, Y.N. (1968) Some consequences of the axiom of definable determinateness. *Proc. Natl. Acad. Sci. USA*, **59**, 708–712.

Blackwell, D. (1967). Infinite games and analytic sets. *Proc. Natl. Acad. Sci. USA*, **58**, 1836–1837.

Blackwell, D. (1970). Ordinal solution of G_δ games. In *Lecture presented at A.M.S. meeting in Davis, Calif.*, Audio recordings of mathematical lectures **27**. AMS, Providence.

Gale, D. and Stewart, F.M. (1953). Infinite games with perfect information. In *Contributions to the theory of games, Vol. 2*, (H.W. Kuhn and A.W. Tucker, eds.). Ann. Math. Studies, **28**. Princeton University Press, Princeton, NJ, 245–260.

Kechris, A.S. and Moschovakis, Y.N. (1971). Notes on the theory of scales. Unpublished manuscript.

Kuratowski, K. (1936). Sur les theorems de separation dans la theorie des ensembles. *Fund. Math.*, **26**, 183–191.

Kuratowski, K. (1966). *Topology*, Vol. 1. PWN, Warsaw, Academic Press, New York.

Lusin, N. and Sierpinski, W. (1923). Sur un ensemble non measurable (B). *Journal de Mathematiques*, 7e serie, t. II , 53–72.

Selivanowski, E. (1933). Sur les proprieties des constituantes des ensembles analytiques. *Fund. Math.*, **21**, 20–28.

Sierpinski, W. (1926). Sur une propriete des ensembles (A). *Fund. Math.*, **8**, 362–369.

Sierpinski, W. (1933). Sur les constituantes des ensembles analytiques. *Fund. Math.*, **12**, 29–34.

Ashok Maitra
Dean's Office
Indian Statistical Institute
203 B.T. Road
Kolkata 700108, India

Editors' notes: This unpublished notes written during 1977–78 was the first exposition of Blackwell's ideas for the benefit of those working in Descriptive Set Theory but not familiar with Effective Set Theory. Along with Addison and Moschovakis (1968) referred to in the original article, Martin (1968) should also be mentioned. The unpublished manuscript of Kechris and Moschovakis (1971) referred to in the original article has since appeared as Kechris and Moschovakis (1978). Standard reference for this article now is the book by Moschovakis (1980).

Additional References

Kechris, A.S. and Moschovakis, Y.N. (1978) Notes on the theory of scales. *Cabal Seminar 76–77 (Proc. Caltech-UCLA Logic Sem., 1976–77)*, (A.S. Kechris and Y.N. Moschovakis, eds.). Lecture Notes in Math., **689**. Springer, Berlin-New York, 1–53.

Martin, D.A. (1968) The axiom of determinateness and reduction principles in the analytical hierarchy. *Bull. Amer. Math. Soc.*, **74**, 687–689.

Moschovakis, Y.N. (1980) *Descriptive set theory.* Studies in Logic and the Foundations of Mathematics, **100**. North-Holland Publishing Co., Amsterdam.