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Brownian Crossings via Regeneration Times

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Abstract

Let $\{B_t, t \geq 0\}$ be a standard one-dimensional Brownian motion. For each $t > 0$ let σ_t be the last entrance time before t into the interval (a, b), d_t the time of the first exit from (a, b) after t and $Y_t := B_t - B_{\sigma_t}$. In this paper we study i) the limit behaviour of the normalised occupation times of the process (Y_t), ii) the limiting joint distribution of $(t - \sigma_t, d_t - t)$ and $(d_t - t, B_{d_t} - B_t)$, conditioned on the event $\{B_t \in (a, b)\}$, as $t \to \infty$ and iii) derive renewal equations satisfied by the probabilities $\phi(t) := P_a\{0 < t - \sigma_t < u,\ 0 < B_t - B_{\sigma_t} < y\}$ and $\gamma(t) := P_a\{0 < d_t - t < u,\ 0 < B_{d_t} - B_t < y\}$.

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1 Introduction

Let $B \equiv (B_t)$ be a one-dimensional Brownian motion and $a < b$ real numbers. For $t \geq 0$, let σ_t be the last entrance time into the interval (a, b), before t, by B. Since $\mathrm{P(B_t \in (a, b) for\ some\ t > 0) = 1, \sigma_t}$ is well defined, for all t large, almost surely. Let N_t denote the number of crossings of (a, b) by time t (see precise definitions in Section 2). Let $Y_t := B_t - B_{\sigma_t}$, $t \geq 0$. In Section 3, Theorem 3.3, we derive a limit theorem for the occupation times of $\{Y_t : t \geq 0\}$. More precisely we show that

$$\frac{1}{\sqrt{t}} \int_0^t f(Y_t)\ dt \xrightarrow{d} \frac{\xi_{\frac{1}{2}}}{2\sqrt{2}(b-a)} \left(\int_S f\ d\pi \right) \tag{1.1}$$

where $\xrightarrow{d}$ stands for convergence in the sense of distributions; $\xi_{1/2}$ is the Mittag-Leffler distribution of index $1/2$ (see equation (2.1) below);

$S \equiv (-(b-a), b-a)$ and π is a finite measure on the Borel σ- field $\mathcal{S}$ (see Section 3 for the definition of π). When (Y_t) is a time homogeneous Markov process such results go back to Darling and Kac (1957) (see also Kallianpur and Robbins (1953); Papanicolau, Stroock and Varadhan (1977)). In Athreya (1986), a different proof of the results in Darling and Kac (1957) was given and the same results were also extended to regenerative processes. The process (Y_t) defined above and its semi-martingale structure was studied in Rajeev (1989b). Note that (Y_t) is not a Markov process and it is in general not a regenerative process as defined in Athreya (1986). However when $B_0 \equiv a$ (or $\equiv b$) then we can show that it is regenerative,with regeneration times having heavy tails. Let N_t denote the number of cycles (or renewals) completed by time t (see precise definition in Section 2). In Section 2, we develop limit theorems for N_t when the regeneration times have heavy tails. We apply this to the Brownian crossing process (Y_t) and derive (1.1) above in Theorem 3.3.

Let d_t be the first exit time from (a, b) by B after time $t \geq 0$. Analogous to renewal theory, the interval (σ_t, d_t) may be considered an 'excursion interval' for the excursions of B into (a, b). In Section (4), we derive the limiting joint distribution of $(t-\sigma_t, d_t-t)$ and $(d_t-t, B_{d_t}-B_t)$, conditioned on $\{B_t \in (a, b)\}$ as $t \to \infty$ (see Theorems 4.2 and 4.6 and Remark 4.7). Note that the conditioning event $\{B_t \in (a, b)\}$ has probability going to zero as $t \to \infty$. Thus the results of Section 4 are similar to those in branching processes for the critical and subcritical cases (see Athreya and Ney (2000)). In Section 5, as a consequence of the strong Markov property for Brownian motion, we derive (Theorem 5.1) a renewal equation for the probabilities

$$\phi(t) := P\{0 < t - \sigma_t < u,\ 0 < B_t - B_{\sigma_t} < y | B_0 = a\}$$

and

$$\gamma(t) := P\{0 < d_t - t < u,\ 0 < B_{d_t} - B_t < y | B_0 = a\}$$

where $0 < y < b - a$ and $u > 0$ are fixed numbers.

2 Limiting distribution of a renewal process with heavy tailed regeneration times

Let $(X_i)_{i \geq 1}$ be i.i.d. positive non lattice random variables. such that $P(X_1 > t) \sim c/\sqrt{t}$ as $t \to \infty$ for some $0 < c < \infty$.

Let $\eta_0 = 0, \eta_j = X_1 + X_2 + \ldots + X_j, j \geq 1$. For $0 < t < \infty$, let $N_t := \sup\{k : \eta_k \leq t\}$. Note that $N_t = k$ if $\eta_k \leq t < \eta_{k+1}, k = 0, 1, 2, \ldots$.

Let $F(x) = P(X_1 \le x), 0 < x < \infty$. Let ξ_α denote a random variable with the Mittag–Leffler distribution with parameter $\alpha, 0 < \alpha < 1$, i.e. it is a random variable with values in $[0, \infty)$ and moments

$$E\xi_\alpha^k = k!(\Gamma(\alpha k + 1))^{-1} \quad k = 0, 1, 2, \cdots \tag{2.1}$$

It maybe noted that the moments of ξ_α satisfy the Carleman condition viz.

$$\sum_1^\infty \frac{1}{\mu_{2k}^{\frac{1}{2k}}} = \infty$$

where $\mu_k := E\xi_\alpha^k$; hence the moments determine the distribution of ξ_α uniquely (see Feller, 1986, VII.4, p. 224). In what follows, for two non-negative functions $a(t), b(t)$ and $0 \le t_0 \le \infty$, we write $a(t) \sim b(t)$ as $t \to t_0$ if $\lim_{t \to t_0} a(t)/b(t) = 1$. The following result and (Theorem 2.3 below) were mentioned in Athreya (1986), but we give a proof for the sake of completeness.

THEOREM 2.1. *Let $(N_t), (X_i), (\eta_i)$ be as above. Suppose that $P(X_1 > t) \sim C/\sqrt{t}$ as $t \to \infty$ for some $0 < C < \infty$. Then*

a) *as $t \to \infty, EN(t)/\sqrt{t} \to \frac{2}{C\pi}$,*

b) *as $t \to \infty, N(t)/\sqrt{t} \xrightarrow{d} \frac{1}{C\sqrt{\pi}}\xi_{1/2}$*

To prove the theorem, we need the following Tauberian theorem. The Gamma function is denoted as $\Gamma(.)$.

THEOREM 2.2. (Karamata's Tauberian Theorem) *Let $U(\cdot)$ be a nondecreasing non-negative function on $(0, \infty)$. Let $W(\tau) \equiv \int_0^\infty e^{-\tau t} U\ (dt)$ for $\tau > 0$. Let $L(\cdot)$ be slowly varying at ∞.*
Then

$$U(t) \sim \frac{t^\rho L(t)}{\Gamma(\rho + 1)}, \quad \textit{for some } 0 < \rho < \infty, \text{as t} \uparrow \infty$$

iff

$$W(s) \sim s^{-\rho} L\left(\frac{1}{s}\right) \quad \textit{as } s \downarrow 0.$$

For a proof, we refer to Feller Vol. II, p. 421, Theorem 2.

PROOF OF THEOREM 2.1. a). We note that $E(N(t))^k < \infty, k = 1, 2, \cdots$ (see Athreya and Lahiri (2006), Cor. 3.5.2.) and further that

$$N(t) = \begin{cases} 0 \text{ if } \eta_1 > t \\ 1 + \tilde{N}(t - \eta_1), \eta_1 \le t, \end{cases} \tag{2.2}$$

$\{\tilde{N}(u) : u \geq 0\}$ being the same process as $\{N(u) : u \geq 0\}$ but defined in terms of $\{\eta_j : j \geq 2\}$, i.e. $\tilde{N}_u := \sup\{k : \tilde{\eta}_k \leq u\}$ where $\tilde{\eta}_0 := 0, \tilde{\eta}_j := \sum_{i=1}^{j} X_{i+1}, j = 1, 2 \cdots$.

So taking expectations in (2.2), we get

$$EN(t) = F(t) + E\{\tilde{N}(t - \eta_1) : \eta_1 \leq t\}, t \geq 0. \tag{2.3}$$

Let $M(t) := EN(t), t \geq 0$. Then from (2.3), using the independence of $(\tilde{N}_u)$ and η_1 we get

$$M(t) = F(t) + \int_{[0,t]} M(t-u)\, dF(u) \tag{2.4}$$

$$= F(t) + M * F(t) \tag{2.5}$$

Let

$$\tilde{M}(s) := \int_{[0,\infty)} e^{-su}\, dM(u) \text{ and } \tilde{F}(s) \equiv \int_{[0,\infty)} e^{-su} dF(u), s \geq 0.$$

Then, taking Laplace transforms in eqn (2.4), we get

$$\tilde{M}(s) = \frac{\tilde{F}(s)}{1 - \tilde{F}(s)}.$$

Next,

$$1 - \tilde{F}(s) = \int_{[0,\infty)} (1 - e^{-st})\, dF(t) = \int_{[0,\infty)} \left(s \int_0^t e^{-sx} dx \right) dF(t)$$

$$= s \int_0^\infty e^{-sx} \left(\int_{[x,\infty)} dF(t) \right) dx = s \int_0^\infty e^{-sx} (1 - F(x))\, dx.$$

By hypothesis $1 - F(x) \sim C/\sqrt{x}$ as $x \to \infty$ and hence $\int_0^t (1 - F(x))\, dx \sim 2C\sqrt{t}$.

So by Karamata's Tauberian Theorem 2.2, as $s \downarrow 0$, we get

$$\frac{1 - \tilde{F}(s)}{s} \sim s^{-\frac{1}{2}} 2C\Gamma\left(\frac{3}{2}\right).$$

This implies that as $s \downarrow 0$, $1 - \tilde{F}(s) \sim \sqrt{s} C\Gamma(1/2)$. Also as $s \downarrow 0, \tilde{F}(s) \uparrow \int_{[0,\infty)} dF(t) = 1$. So as $s \downarrow 0$,

$$\tilde{M}(s) \sim \frac{1}{\sqrt{s} C\Gamma\left(\frac{1}{2}\right)}.$$

By Karamata's Theorem 2.2 above, this yields, as $t \to \infty$,

$$M(t) \sim \frac{\sqrt{t}}{\Gamma\left(\frac{3}{2}\right)} \frac{1}{C\Gamma\left(\frac{1}{2}\right)} = \frac{2\sqrt{t}}{C\pi}$$

This proves part a) of the theorem.

b). Let $M_k(t) = E(N(t))^k, k \geq 1$.

Then,

$$\begin{aligned} M_k(t) &= E((N(t))^k : \eta_1 > t) + E((1 + \tilde{N}(t - \eta_1))^k : \eta_1 \leq t) \\ &= E\left(\sum_{j=0}^{k-1} \binom{k}{j} (\tilde{N}(t - \eta_1)^j : \eta_1 \leq t)\right) + E((\tilde{N}(t - \eta_1)^k : \eta_1 \leq t)) \\ &= \sum_{j=0}^{k-1} \binom{k}{j} \int_{[0,t]} M_j(t - u)\ dF(u) + \int_{[0,t]} M_k(t - u)\ dF(u). \end{aligned}$$

Thus, we have,

$$M_k(t) = \sum_{j=0}^{k-1} \binom{k}{j} (M_j * F)(t) + M_k * F(t). \tag{2.6}$$

where $*$ stands for convolution as in eqn (2.4). Let

$$\tilde{M}_j(s) = \int_{[0,\infty)} \bar{e}^{su}\ dM_j(u), j = 1, 2, \ldots, \text{and} \quad \tilde{F}(s) = \int_{[0,\infty)} \bar{e}^{su}\ dF(u).$$

Then taking Laplace transforms in eqn (2.5) we get,

$$\tilde{M}_k(s)(1 - \tilde{F}(s)) = \left(\sum_{j=0}^{k-1} \binom{k}{j}\ \tilde{M}_j(s)\right)\ \tilde{F}(s).$$

For $k = 2$, using the fact that $\tilde{M}_0(s) \equiv 0$, we get

$$\tilde{M}_2(s)(1 - \tilde{F}(s)) = 2\tilde{M}_1(s)\ \tilde{F}(s).$$

Since $\tilde{M}_1(s) = \tilde{F}(s)/1 - \tilde{F}(s)$, we get

$$\tilde{M}_2(s) = 2\frac{(\tilde{F}(s))^2}{(1-\tilde{F}(s))^2}.$$

Similarly for $k = 3$, we get

$$\tilde{M}_3(s) \quad = \quad \frac{3 \times 2(\tilde{F}(s))^3}{(1-\tilde{F}(s))^3} + 3\frac{(\tilde{F}(s))^2}{(1-\tilde{F}(s))^2}.$$

By induction

$$\tilde{M}_k(s) = k!\frac{(\tilde{F}(s))^k}{(1-\tilde{F}(s))^k} + \text{ lower order terms.}$$

Since $(1 - \tilde{F}(s)) \sim \sqrt{s} C\Gamma(1/2)$ as $s \downarrow 0$, and $\tilde{F}(s) \uparrow 1$ as $s \downarrow 0$ it follows from (2.6) that as $s \downarrow 0$,

$$\tilde{M}_k(s) \sim \frac{k! s^{-k/2}}{(C\sqrt{\pi})^k}, k = 1, 2, \ldots$$

By Karamata's Theorem 2.2 above, we get as $t \uparrow \infty$

$$E(N(t))^k \sim \frac{k! t^{k/2}}{(C\sqrt{\pi})^k} \; \frac{1}{(\Gamma(\frac{k}{2}+1))}, \; \forall \; k = 1, 2, \ldots$$

i.e.

$$E\left(\frac{N(t) \; C\sqrt{\pi}}{\sqrt{t}}\right)^k \to \frac{k!}{\Gamma(\frac{k}{2}+1)} = E(\xi_{1/2})^k, k = 1, 2, \ldots$$

Then part b) follows from the method of moments.

A more general result holds. Let $L(t)$ be a slowly varying function at ∞.

Theorem 2.3. *Suppose for some* $0 < \alpha < 1$, $P(X_1 > t) \sim t^{-\alpha} L(t)$. *Then* $\forall \; k = 1, 2, \ldots$

$$E\left(\frac{N(t) \; C_\alpha \; L(t)}{t^\alpha}\right)^k \to k! \; (\Gamma(k\alpha+1))^{-1} \tag{2.7}$$

where $C_\alpha = \Gamma(1+\alpha)/(1-\alpha)$ *and hence*

$$\frac{N(t) \; L(t)}{t^\alpha} \overset{d}{\to} \frac{\xi_\alpha}{C_\alpha}$$

PROOF. Since $P(X_1 > t) \sim t^{-\alpha} L(t)$, as $t \to \infty$, we have

$$\int_0^t P(X_1 > x)dx \sim \frac{t^{1-\alpha} L(t)}{(1-\alpha)} = \frac{t^{1-\alpha}\ L(t)\ \Gamma(1+\alpha)}{\Gamma(1+\alpha)(1-\alpha)}$$

as $t \to \infty$, and hence by Karamata's Tauberian Theorem 2.2

$$\int_0^\infty e^{-st}\ P(X_1 > t)\ dt \sim \bar{s}^{(1-\alpha)}\ L\left(\frac{1}{s}\right)\ \frac{\Gamma(1+\alpha)}{(1-\alpha)} \text{ as } s \downarrow 0.$$

But

$$\begin{aligned} \int_0^\infty e^{-st}\ P(X_1 > t)\ dt \ &= \ \int_0^\infty e^{-st} \left(\int_t^\infty dF(x) \right)\ dt \\ = \int_0^\infty \left(\int_0^x e^{-st} dt \right)\ dF(x) \ &= \ \int_0^\infty \frac{(1 - e^{-sx})}{s}\ dF(x) = \frac{1 - \tilde{F}(s)}{s} \end{aligned}$$

Thus

$$(1 - \tilde{F}(s)) \sim s^\alpha L\left(\frac{1}{s}\right)\ \frac{\Gamma(1+\alpha)}{(1-\alpha)}.$$

Proceeding as in the proof of Theorem 2.1, we get as $s \downarrow 0$ and $k = 1, 2, \cdots$,

$$\tilde{M}_k(s) \ \sim \ s^{-\alpha k}\ \frac{k!}{(L(\frac{1}{s})C_\alpha)}.$$

By Karamata's Tauberian Theorem 2.2 we get as $t \to \infty$ and $k = 1, 2, \cdots$

$$M_k(t) \ \sim \ \frac{t^{\alpha k}\ \ k!}{\Gamma(1+\alpha k)\ \ (L(t)C_\alpha)^k}$$

and hence eqn (2.7) follows. So by the method of moments,

$$\frac{N(t)\ \ L(t)}{t^\alpha} \xrightarrow{d} \frac{\xi_\alpha}{C_\alpha}.$$

3 Limiting distribution of occupation times of $\{Y_t : t \geq 0.\}$

Let a and b are real numbers with $a < b$. Let (B_t) be a one-dimensional Brownian motion with $B_0 \equiv a$. Define a sequence of stopping times $\{\tau_k\}_{k \geq 0}$ as follows: Let $\tau_0 \equiv 0$. Let

$$\begin{aligned} \tau_{2k+1} &= \inf\{s > \tau_{2k} : B_s = b\}, k = 0, 1, 2, \cdots \\ \tau_{2k} &= \inf\{s > \tau_{2k-1} : B_s = a\}, k = 1, 2, \cdots \end{aligned}$$

We will denote by T_c the first passage time to level $c \in \mathbb{R}$ by an independent real valued standard Brownian motion (W_t) viz. $T_c = \inf\{s > 0 : W_s = c\}$. Let $T_0 \equiv 0$ and let $T_k := \tau_{2k} - \tau_{2(k-1)}, k = 1, 2, \cdots$. Note that by the strong Markov property, $T_k \stackrel{d}{=} T_{2(b-a)}$ where $\stackrel{d}{=}$ denotes equality in law. It follows again from the strong Markov property and the symmetry of the standard Brownian motion that $(T_k)_{k \geq 1}$ are independent and identically distributed. The T_k's thus generate a renewal process and we continue to denote by N_t the number of renewals, i.e. $N_t = \sup \{k : \tau_{2k} \leq t\}$, $t \geq 0$. Note that $\tau_{2i} := \sum_{j=1}^{i} T_j, \quad i = 1, \cdots$. Let $\sigma_t := \sup \{s \leq t : B_s \in (a, b)^c\}$ be the last entrance times of B into the interval (a, b) before time t with the convention that $\sup\{\phi\} = 0$. Let $Y_t := B_t - B_{\sigma_t}$ be the associated crossing process (see Rajeev (1989b)).

Define the measure π on the Borel σ-field $\mathcal{S}$ of $S := (-(b-a), b-a)$ as follows: For $A \in \mathcal{S}$ define $\pi(A) := 1/2\{\pi^+(A) + \pi^-(A)\}$, where

$$\pi^+(A) \quad := \quad E \int_0^{T_{b-a}} I_A((W_s)^+) \, ds$$

and

$$\pi^-(A) \quad := \quad E \int_0^{T_{-(b-a)}} I_A(-(W_s)^-) \, ds$$

Note that $\pi^+(A), \pi^-(A)$ are respectively the expected time spent in the portion of A above (respectively, below) zero up to the time (W_t) reaches $b-a$ (respectively $-(b-a)$). Further $\pi^+(A), \pi^-(A)$ are supported on $(0, b-a)$ and $(-(b-a), 0)$. Since $\pi^+\{0\}$ is the expected time spent by a standard Brownian motion below 0 during $[0, T_{b-a})$ it is necessarily infinity. Similarly $\pi^-\{0\} = \infty$. However the following holds:

PROPOSITION 3.1.

$$\pi^+((0, b-a)) = \pi^-(-(b-a), 0) = (b-a)^2.$$

PROOF. Let $(L(t,x)) \equiv (L^W(t,x))$ denote the local time process of (W_t) at $x \in \mathbb{R}$. Since $W_0 \equiv 0$, we have using Tanaka's formula (see Revuz and Yor (1991), Chap. VI, Sec. 1), for $0 \le x \le c$,

$$\begin{aligned} EL(T_c, x) = \lim_{t\uparrow\infty} E(L(T_c \wedge t, x)) &= \lim_{t\uparrow\infty} 2E(W_{T_c\wedge t} - x)^+ \\ &= 2(c-x) \end{aligned}$$

We then have, from the occupation density formula (see Revuz and Yor (1991), Chap. VI, Sec. 1),

$$\begin{aligned} \pi^+((0,(b-a))) &:= E\int_0^{T_{b-a}} I_{(0,(b-a))}((W_s)^+)\ ds \\ &= E\int_0^{b-a} L(T_{b-a}, x)\ dx \\ &= \int_0^{b-a} EL(T_{b-a}, x)\ dx \\ &= \int_0^{b-a} 2((b-a)-x)\ dx \\ &= (b-a)^2. \end{aligned}$$

By symmetry, $\pi^+((0, b-a)) = \pi^-(-(b-a), 0)$ and the proof is complete.

REMARK 3.1. It can be checked that the random variable $\xi_{1/2}$ defined via equation (2.1) with $\alpha = 1/2$, has the same distribution as $\sqrt{2}|X|$, where $X \sim N(0,1)$. It is also known that $|X|$ has the same distribution as the random variable $\frac{1}{\sqrt{t}} \sup_{s\le t} W_s$. Further as a consequence of Skorokhod's lemma (see Revuz and Yor (1991), Chap. VI, Sec. 2) the latter has the same distribution as $\frac{L(t,0)}{\sqrt{t}}$ for each $t > 0$.

We then have the following result.

THEOREM 3.3. *Let* $\{N_t, t \ge 0\}, \{Y_t : t \ge 0.\}, \pi,$ and $\xi_{1/2}$ *be as above. Then*

1. *The process* (Y_t) *is regenerative with regenerative times* $\tau_{2k}, k = 1, 2, \cdots$.

2. *Further as* $t \to \infty$ *we have* $N_t/\sqrt{t} \xrightarrow{d} \frac{\xi_{1/2}}{2\sqrt{2}(b-a)}$

3. *Let* $f : S \to \mathbb{R}, \int |f|\ d\pi < \infty$. *Then as* $t \to \infty$,

$$\frac{1}{\sqrt{t}} \int_0^t f(Y_t)\ dt \xrightarrow{d} \frac{\xi_{\frac{1}{2}}}{2\sqrt{2}(b-a)} \left(\int_S f\ d\pi \right).$$

PROOF. 1). It suffices to show following the definition of a regenerative process (see Section 2; Athreya (1986)) that $\{Y_t, \ \ \tau_{2i} \le t < \tau_{2(i+1)}\}$ for $i = 1, 2, \cdots$ are independent and identically distributed. We first note that during a down crossing interval

$$\{Y_t, \ \ \tau_{2i+1} \le t < \tau_{2(i+1)}\} \stackrel{d}{=} \{-(W_t)^-, \ \ 0 \le t < T_{-(b-a)}\}$$

for $i = 1, 2, \cdots$ and that during an up crossing interval

$$\{Y_t, \ \ \tau_{2i} \le t < \tau_{2i+1}\} \stackrel{d}{=} \{(W_t)^+, \ \ 0 \le t < T_{(b-a)}\}$$

for $i = 0, 1, \cdots$. By the strong Markov property, the processes $\{Y_t, \ \ \tau_{2i} \le t < \tau_{2(i+1)}\}$ for $i = 1, 2, \cdots$ are independent. Moreover, for each fixed i, the processes $\{Y_t, \ \ \tau_{2i} \le t < \tau_{2i+1}\}$ and $\{Y_t, \ \ \tau_{2i+1} \le t < \tau_{2(i+1)}\}$ are independent. Since

$$\{Y_t, \ \ \tau_{2i} \le t < \tau_{2(i+1)}\} = \{Y_t, \ \ \tau_{2i} \le t < \tau_{2i+1}\} \bigcup \{Y_t, \ \ \tau_{2i+1} \le t < \tau_{2(i+1)}\}$$

it follows from the above observations that the law of $\{Y_t, \ \ \tau_{2i} \le t < \tau_{2(i+1)}\}$ is determined as a functional of $\{(W_t)^+, \ 0 \le t < T_{(b-a)}\}$ and $\{-(W'_t)^-, \ 0 \le t < T_{-(b-a)}\}$, where (W_t) and (W'_t) are two independent standard Brownian motions. Hence $\{Y_t, \ \ \tau_{2i} \le t < \tau_{2(i+1)}\}$ for $i = 1, 2, \cdots$ are identically distributed. It follows that (Y_t) is regenerative.

2). From the distribution of the hitting time of Brownian motion, $\sqrt{t}P(T_{2(b-a)} > t) \to C$, where $C > 0$ is a constant. Indeed for $x > 0$,we have as $t \to \infty$,

$$P(T_x > t) = 2\Phi\left(\frac{x}{\sqrt{t}}\right)\Big) - 1 \sim \frac{\sqrt{2}x}{\sqrt{\pi t}}$$

Taking $X_1 := T_1 \stackrel{d}{=} T_{2(b-a)}$, the assertion 2) follows from Theorem 2.1 with the constant C in that theorem now evaluated as $C = 2(b-a)\sqrt{2/\pi}$.

3). We have

$$\frac{1}{\sqrt{t}} \int_0^t f(Y_t) \ dt \ \ = \ \ \frac{N_t}{\sqrt{t}} \frac{1}{N_t} \int_0^t f(Y_t) \ dt \tag{3.1}$$

Since (Y_t) is regenerative, we have as in Proposition 2 of Athreya (1986), for f satisfying the given hypothesis that $\int\limits_S |f| \ d\pi < \infty$ (which, since $\pi\{0\} = \infty$ implies $f(0) = 0$),

$$\frac{1}{N_t} \int_0^t f(Y_t) \ dt \ \to \ E \int_0^{\tau_2} f(Y_s) \ ds = \int\limits_S f \ d\pi$$

almost surely, as $t \to \infty$,. It follows, using part 2) of the theorem that the RHS in eqn (3.1) converges in distribution, as $t \to \infty$ to $\frac{\xi_{1/2}}{2\sqrt{2}(b-a)}\left(\int_S f\,d\pi\right)$. This completes the proof of the theorem.

REMARK 3.4. We remark that in the case of crossings of an interval (a,b) by a Brownian motion, Theorem 2.1a) maybe directly proved as follows: from Rajeev (1989a) we have the following relationship between the expected number of crossings N'_t of (a,b) by the Brownian motion (B_t) during $[0,t]$ and the expected time spent in (a,b) by (B_t) during $[0,t]$:

$$E \int_0^t I_{(a,b)}(B_s)\, ds = (b-a)^2 EN'_t + E(B_t - B_{\sigma_t})^2.$$

Since $|B_t - B_{\sigma_t}| \leq b-a$, it follows that

$$\begin{aligned}\lim_{t\to\infty} \frac{EN'_t}{\sqrt{t}} &= \lim_{t\to\infty} \frac{1}{(b-a)^2}\frac{1}{\sqrt{t}}\int_0^t \frac{1}{\sqrt{2\pi s}}\int_a^b e^{-\frac{x^2}{2s}} dx\ ds \\ &= \sqrt{\frac{2}{\pi}}\frac{1}{b-a}\end{aligned}$$

It follows that

$$\lim_{t\to\infty} \frac{EN_t}{\sqrt{t}} = \frac{1}{2}\lim_{t\to\infty}\frac{EN'_t}{\sqrt{t}} = \frac{1}{\sqrt{2\pi}(b-a)}$$

REMARK 3.5. By the symmetry of Brownian motion around zero, $\{|Y_t|, \tau_j \leq t < \tau_{j+1}\} \stackrel{d}{=} \{(W_t)^+,\ \ 0 \leq t < T_{b-a}\}$ for $j = 0, 1, \cdots$. In particular, $(|Y_t|)$ is a regenerative process with regeneration times $\tau_i,\ \ i = 0, 1, \cdots$. Now it follows from Theorem 1 of Athreya (1986), with $S = [0, b-a)$ and $f : S \to \mathbb{R}$, $\int |f|\, d\pi < \infty$, that as $t \to \infty$,

$$\frac{1}{\sqrt{t}}\int_0^t f(|Y_t|)\ dt \xrightarrow{d} \frac{\xi_{\frac{1}{2}}}{\sqrt{2}(b-a)}\left(\int f\ d\pi^+\right).$$

4 Limiting distributions related to excursion intervals

Let (B_t) be a Brownian Motion as before but with B_0 arbitrary, $a < b$ and (σ_t) be as in the previous section. Let for $t \geq 0$,

$$d_t := \inf\{s > t : B_s \notin (a,b)\}.$$

From the properties of Brownian motion, it follows that $d_t < \infty$ almost surely for all $t \geq 0$. For $x \in (a,b)$ and $u > 0$ define

$$\psi(x,u,a,b) \quad := \quad P\left(a < \min_{0\leq s\leq u} B_s < \max_{0\leq s\leq u} B_s < b \mid B(0) = x\right)$$

It is known (see Revuz and Yor (1991), Chap. 3, Sec. 3, ex. 3.15 for the case $a < 0 < b$) that for $a < b$ real numbers, $\psi(x,u) \equiv \psi(x,u,a,b) := \int_a^b K(u,x,a,b,y)\, dy$ where

$$K(u,x,a,b,y) = \frac{1}{\sqrt{2\pi u}} \sum_{n=-\infty}^{n=\infty} \left[e^{-\frac{(y+2n(b-a))^2}{2u}} - e^{-\frac{(y-2(b-x)+2n(b-a))^2}{2u}} \right] \quad (4.1)$$

Thus $0 \leq \psi(x,u) \leq 1$, $\psi(x,u) \to 1$ as $u \to 0$ and $\psi(x,u) \to 0$ as $u \to \infty$, and is jointly continuous in x and u.

PROPOSITION 4.1. *$t - \sigma_t$ and $d_t - t$ converge to zero in probability as $t \to \infty$.*

PROOF. Enough to note that for $-\infty < a < b < \infty$, $P(B_t \in (a,b)) \to 0$ as $t \to \infty$ and that for $\epsilon > 0$, $\{t - \sigma_t > \epsilon\}$ and $\{d_t - t > \epsilon\}$ are both contained in $\{B_t \in (a,b)\}$.

THEOREM 4.2. *Let $u_1 > 0$, $u_2 > 0$ and $\psi(.,.)$ be as above. Then*

$$\lim_{t\to\infty} P(t - \sigma_t > u_1,\ d_t - t > u_2 \mid B_t \in (a,b)) = \frac{1}{b-a} \int_a^b \psi(x, u_1 + u_2)\, dx.$$

PROOF. Let $u_1 > 0,\ u_2 > 0$. We note that,

$$\{t - \sigma_t > u_1,\ d_t - t > u_2\} = \left\{a < \inf_{t-u_1\leq s\leq t+u_2} B_s < \sup_{t-u_1\leq s\leq t+u_2} B_s < b\right\}.$$

Using the Markov property at time $t - u_1$ we get

$$\begin{aligned} &P\{t - \sigma_t > u_1,\ d_t - t > u_2 \mid B_t \in (a,b)\} \\ &\quad = \frac{P\left\{a < \inf_{t-u_1\leq s\leq t+u_2} B_s < \sup_{t-u_1\leq s\leq t+u_2} B_s < b\right\}}{P(B_t \in (a,b))} \\ &\quad = \frac{\int_a^b P\left\{a < \inf_{0\leq s\leq u_1+u_2} B_s < \sup_{0\leq s\leq u_1+u_2} B_s < b \mid B_0 = x\right\} P(B_{t-u_1} \in dx)}{\int_a^b P(B_t \in dx)} \end{aligned}$$

$$= \frac{\int_a^b \psi(x, u_1 + u_2) \frac{1}{\sqrt{2\pi(t-u_1)}} \varphi\left(\frac{x}{\sqrt{t-u_1}}\right) dx}{\frac{1}{\sqrt{2\pi t}} \int_a^b \varphi\left(\frac{x}{\sqrt{t}}\right) dx}$$

where $\varphi(x) = e^{-x^2/2}$. The result follows on letting $t \to \infty$ in the RHS above and using the dominated convergence theorem.

COROLLARY 4.3.

$$\begin{aligned} \lim_{t\to\infty} P\{t - \sigma_t \le u \mid B_t \in (a,b)\} &= \lim_{t\to\infty} P\{d_t - t \le u \mid B_t \in (a,b)\} \\ &= \int_a^b \frac{1 - \psi(x,u)}{b-a}\, dx. \end{aligned}$$

PROOF. Let $u_1 \downarrow 0$ and $u_2 \downarrow 0$ separately.

REMARK 4.4. Since $\psi(x, u) \to 0$ as $u \to \infty$ for every fixed $x \in (a, b)$, the RHS in Cor 4.3 defines a proper probability distribution and consequently $t - \sigma_t$ and $d_t - t$, conditional on $B_t \in (a, b)$ converge to proper probability distributions.

Note that for $a < x < b$,

$$\psi(x, u) = P\{d_0 > u \mid B_0 = x\}.$$

For $a < x < b$, let η_x be a random variable such that η_x is distributed as $d_0 \mid B_0 = x$, i.e. $P(\eta_x \le u) = 1 - \psi(x, u)$. Let U be uniformly distributed on the interval (a, b). Then we can rephrase the above Corollary as follows.

COROLLARY 4.5.

$$\lim_{t\to\infty} P\{d_t - t \le u \mid B_t \in (a,b)\} = Ef(U)$$

where $f(x) = P(\eta_x \le u)$.

PROOF. The distribution of B_t conditioned on the event $\{B_t \in (a, b)\}$ converges as $t \to \infty$ to the uniform distribution on (a, b).

In fact, more is true. We have

THEOREM 4.6. *For a Borel set A in $\mathbb{R}^2$ we have*

$$\lim_{t\to\infty} P\{(d_t - t,\ B_{d_t} - B_t) \in A \mid B_t \in (a,b)\} = Eh(U)$$

where

$$h(x) := P\{(d_0, B_{d_0} - x) \in A \mid B_0 = x\}$$

and U is uniformly distributed over (a, b).

PROOF. We note that if $d_t(\omega) < \infty,\ d_t(\omega) = t + d_0(\theta_t\omega)$ and $B_{d_t}(\omega) = B_{d_0}(\theta_t\omega)$ where $\theta_t\omega$ is the path ω shifted by time t. Hence, using the Markov property at time t,

$$P\{(d_t - t,\ B_{d_t} - B_t) \in A, |B_t \in (a,b)\}$$
$$= \frac{\int_a^b P\{(d_0,\ B_{d_0} - B_0) \in A \mid B_0 = x\}\ \frac{1}{\sqrt{2\pi t}}\ \varphi\Big(\frac{x}{\sqrt{t}}\Big)\ dx}{\int_a^b \frac{1}{\sqrt{2\pi t}}\ \varphi\Big(\frac{x}{\sqrt{t}}\Big)\ dx}$$

and the proof can be completed as in the previous theorem.

REMARK 4.7. Theorem 4.2 and Corollary 4.3 suggest that the following analogue of Theorem 4.6 should be true viz.

$$\lim_{t\to\infty} P\{(t - \sigma_t,\ B_t - B_{\sigma_t}) \in A \mid B_t \in (a,b)\} = Eh(U)$$

where

$$h(x) := P\{(d_0, x - B_{d_0}) \in A \mid B_0 = x\}$$

and U is uniformly distributed random variable over (a,b). This has not been proved yet.

5 Some renewal equations

In this section we take $\Omega = C[0,\infty)$, $B_t(\omega) \equiv \omega(t)$ for $\omega \in \Omega$, $t \geq 0$. Let $\mathcal{F} = \sigma\{B_t, t \geq 0\}$ and $\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}$. Let P_0 denote the standard Wiener measure on $(\Omega, \mathcal{F})$ with $B_0 \equiv 0$. The translates of P_0 are denoted as $P_x(A) := P_0(A - x)$, $A \in \mathcal{F}$. Fix $a < b$. Let $\tau_k, k = 0,1,2...$ and $\sigma_t, t \geq 0$ be as in Section 3. Let $u > 0$ and $0 < y < b - a$ be given. Define for $t \geq 0, \phi(t), z(t), \gamma(t), y(t)$ as follows:

$$\begin{aligned}
\phi(t) &:= P_a\{0 < t - \sigma_t < u,\ 0 < B_t - B_{\sigma_t} < y\} \\
z(t) &:= P_a\{0 < t - \sigma_t < u,\ 0 < B_t - B_{\sigma_t} < y,\ 0 < t < \tau_2\} \qquad (5.1)\\
\gamma(t) &:= P_a\{0 < d_t - t < u,\ 0 < B_{d_t} - B_t < y\} \\
y(t) &:= P_a\{0 < d_t - t < u,\ 0 < B_{d_t} - B_t < y,\ 0 < t < \tau_2\}.
\end{aligned}$$

We now show that ϕ and γ satisfy renewal equations.

THEOREM 5.1. *Let $\mu_a(A) := P_a\{\tau_2 \in A\}$ be the distribution of τ_2 under P_a. Then, for all $t \geq 0$ we have*

1.

$$\phi(t) = z(t) + \int_0^t \phi(t-s)\ \mu_a(ds) \tag{5.2}$$

2.

$$\gamma(t) = y(t) + \int_0^t \gamma(t-s)\ \mu_a(ds)$$

PROOF. 1. Let $\theta_t : \Omega \to \Omega$, $\theta_t\omega(s) = \omega(s+t)$, $s \geq 0$, $t \geq 0$. For $\omega \in \Omega$ we observe the following: Let $t \geq \tau_2(\omega)$. Let $\omega' = \theta_{\tau_2}\omega$, the path ω shifted by the stopping time $\tau_2(\omega)$. Let $s := t - \tau_2(\omega)$. Then, $\sigma_s(\omega') = \sigma_t(\omega) - \tau_2(\omega)$. In particular, it follows that if $t \geq \tau_2(\omega)$, then $t - \tau_2(\omega) - \sigma_{t-\tau_2(\omega)}(\theta_{\tau_2}\omega) = t - \sigma_t(\omega)$. Further, $B_{\sigma_s(\omega')}(\omega') = B_{\sigma_t(\omega)-\tau_2(\omega)}(\omega') = B_{\sigma_t(\omega)}(\omega)$. In particular, it follows that if $t \geq \tau_2(\omega)$,

$$\begin{aligned} B_{t-\tau_2(\omega)}(\omega') - B_{\sigma_{t-\tau_2(\omega)}(\omega')}(\omega') &\equiv B_{t-\tau_2(\omega)} \circ \theta_{\tau_2}(\omega) - B_{\sigma_{t-\tau_2(\omega)}} \circ \theta_{\tau_2}(\omega) \\ &= B_t(\omega) - B_{\sigma_s(\omega')}(\omega') \\ &= B_t(\omega) - B_{\sigma_t(\omega)}(\omega). \end{aligned}$$

We then have,

$$\begin{aligned} \phi_a(t) - z(t) &= P_a\{0 < t - \sigma_t < u, 0 < B_t - B_{\sigma_t} < y, \tau_2 \leq t\} \\ &= P_a\{0 < t - \tau_2(\omega) - \sigma_{t-\tau_2(\omega)} \circ \theta_{\tau_2}(\omega) < u, \\ &\qquad 0 < B_{t-\tau_2(\omega)}(\theta_{\tau_2}\omega) - B_{\sigma_{t-\tau_2(\omega)}}(\theta_{\tau_2}\omega) < y, \tau_2 \leq t\} \\ &= E_a[1_{(\tau_2 \leq t)} P_a\{0 < t - \tau_2 - \sigma_{t-\tau_2} \circ \theta_{\tau_2} < u, \\ &\qquad 0 < B_{t-\tau_2} \circ \theta_{\tau_2} - B_{\sigma_{t-\tau_2}} \circ \theta_{\tau_2} < y \mid \mathcal{F}_{\tau_2}\}] \\ &= E_a[1_{(\tau_2 \leq t)} \phi_{B_{\tau_2}}(t - \tau_2)] \\ &= \int_0^t \phi_a(t-s)\ P_a(\tau_2 \in ds) \quad [\text{ since } B_{\tau_2} \equiv a] \end{aligned}$$

where the last but one equality follows from an application of the strong Markov property (see Revuz and Yor (1991)).

2. The proof is similar using the strong Markov property and the relations $d_t(\omega) = d_{t-\tau_2(\omega)}(\omega')$ and $B_{d_t}(\omega) = B_{d_{t-\tau_2(\omega)}(\omega')}(\omega')$ where $\tau_2(\omega) \leq t$ and $\omega' = \theta_{\tau_2}\omega$.

REMARK 5.2. A similar result holds when $-(b-a) < y < 0$ and also when P_a is replaced by P_b with appropriate changes in the definition of the function $z(t)$ and μ_a.

Let $U_a(t) := \sum_{n=0}^{\infty} F^{*n}(t)$, where $F(t) := P_a\{\tau_2 \in [0,t]\}$ and for $n \geq 1$, $F^{*n}(t)$ is the n^{th} convolution power of F and $F^{*0}(t) = I_{[0,\infty)}(t)$.

COROLLARY 5.3. *Fix $t > 0, 0 < y < b-a$. Then we have*

$$\phi(t) \quad = \quad \int_{[0,t]} z(t-u) U_a(du). \tag{5.3}$$

PROOF. It is well known (see Resnick (1992)) that the unique solution of equation (5.2) is given by equation (5.3).

REMARK 5.4. Using the fact that the process $(T_b)_{b\geq 0}$ has stationary and independent increments and the explicit form of the distribution of T_b (see Revuz and Yor (1991), p. 107), it follows from the definition of the renewal function that

$$U(t) \quad = \quad \int_0^t f(s, b-a)\ ds$$

where for $s > 0, x > 0$ we define

$$f(s,x) := \sum_{n=0}^{\infty} \frac{2nx}{\sqrt{2\pi s^3}} \left[e^{-\frac{(2nx)^2}{2s}} \right].$$

Further, let $m(t,x,z)$ denote the joint density of (B_t, M_t) under P_0, where $M_t := \sup_{s\leq t} B_s$ (see Revuz and Yor (1991), Ex. 3.14). Using the spatial homogeneity of Brownian motion, z_t defined in eqn (5.1) can be written as

$$z(t) = P_0\{0 < t - \hat{\sigma}_t < u, 0 < B_t < y, t < \tau_1\}$$

where $\hat{\sigma}_t := \sup\{s \leq t : B_s \notin (0, b-a)\}$. By conditioning the RHS with respect to (B_{t-u}, M_{t-u}) we can write $z(t)$ as follows:

$$z(t) \quad = \quad \int_0^{b-a} \int_{-\infty}^{z} h(u,y,x,z) m(t-u,x,z)\ dx\ dz$$

where the function $h(u,y,x,z)$ is given as follows: Define $N_u := \inf_{r\leq u} B_r$. Then,

$$h(u,y,x,z) \quad = \quad I_{(-\infty,0]}(x) P_x\{M_u < b-a, 0 < B_u < y\}$$

$$
\begin{aligned}
& + I_{(0,z]}(x)P_x\{N_u \leq 0 < M_u < b-a, 0 < B_u < y\} \\
= \; & I_{(-\infty,0]}(x)P_x\{M_u < b-a, 0 < B_u < y\} \\
& + I_{(0,z]}(x)P_x\{M_u < b-a, 0 < B_u < y\} \\
& - I_{(0,z]}(x)P_x\{0 < N_u < M_u < b-a, 0 < B_u < y\} \\
= \; & I_{(-\infty,0]}(x)P_x\{M_u < b-a, 0 < B_u < y\} \\
& + I_{(0,z]}(x)P_x\{M_u < b-a, 0 < B_u < y\} \\
& - I_{(0,z]}(x)P_x\{0 < u < T_0 \wedge T_{b-a}, 0 < B_u < y\}
\end{aligned}
$$

Note that

$$P_x\{0 < u < T_0 \wedge T_{b-a}, 0 < B_u < y\} = \int_0^y K(u,x,0,b-a,r)\,dr$$

where the density $K(u,x,0,b-a,y)$ is given explicitly by equation (4.1).

References

athreya, k.b. and ney, p.e. (2000). *Branching processes.* Dover, Minolta, N.Y.

athreya, k.b. (1986). Darling and Kac revisited. *Sankhya A*, **48**, 255–266.

athreya, k.b. and lahiri, s.n. (2006). *Probability theory.* Hindustan Book Agency, India.

darling, d.a. and kac, m. (1957). On occupation times for Markoff processes. *Trans. Amer. Math. Soc.*, **84**, 444–458.

feller, w. (1986). *An introduction to Probability theory and its applications*, vol. II. Wiley Eastern Limited, New Delhi.

kallianpur, g. and robbins, h. (1953). Ergodic properties of Brownian motion process. *Natl. Acad. Sci.*, **39**, 525–533.

papanicolau, g.c., stroock, d.w. and varadhan, s.r.s. (1977). Martingale Approach to some Limit Theorems. In *Proc. 1976 Duke Conf. on Turbulence, Duke Univ. Math. Series III.*

rajeev, b. (1989a). Sojourn times of Martingales. *Sankhya A*, **51**, 1–12.

rajeev, b. (1989b). Crossings of Semi-Martingales. In *Seminaire de Probabilites XXIV, Lecture Notes in Mathematics 1426.* Springer Verlag, pp. 107–116.

resnick, sidney i. (1992). *Adventures in Stochastic Processes.* Birkhauser, Boston.

revuz, d. and yor, m. (1991). *Continuous Martingales and Brownian motion*, 3rd edition. Springer Verlag, Berlin.

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