INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination : 2015-16

Course Name: M.S. (Q.E.) I YEAR / M.STAT. II YEAR / M.MATH. II YEAR

Subject Name: Game Theory I

Date: 25-08-2015 Maximum Marks: 40 Duration: 90 minutes

Problem 1. (a) Let S be the set of pure strategy Nash equilibria of a game G. Under what minimal (most general) condition a correlated strategy q of the form $q(s) > 0$ if and only if $s \in S$ is a correlated Nash equilibrium?
(b) Justify your answer by a proof or a counterexample.
Let G be a zero-sum game with $|S_i| \geq 3$ for all $i \in N$ which does not have any pure strategy Nash equilibrium. Then G does not have a correlated equilibrium that is *not* a mixed Nash equilibrium. (5+5)

Problem 2. Justify your answer by a proof or a counterexample.
Consider a finite extensive form game with imperfect information where the number of edges in the longest path in the game tree is three. Then
(i) Sequential rationality implies consistency,
(ii) Consistency implies sequential rationality,
(iii) Sequential equilibrium is subgame perfect,
(iv) Sequential equilibrium always exists.

(2+3+5+5)

Problem 3. Consider a Bayesian game $G^B = \langle N, (T_i)_{i\in N}, (P_i)_{i\in N}, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ where $T_i = A_i = [0,1]$ and $P_i = (U[0,1])^{|N|}$ for all $i \in N$.
(i) Justify your answer by a proof or a counterexample. There exist u_i: $i \in N$ such that a strategy-tuple s^* of the form $s_i^*(x) = x$ for all $i \in N$ is

(a) a Bayesian Nash equilibrium,

(b) unique Bayesian Nash equilibrium.

(ii) Suppose $N = \{1,2\}$ and $u_i(t,a) = t_i - t_j + a_i - a_j$ for all $i \in N$. What can you say about the Bayesian Nash equilibrium?
(iii) Justify your answer by a proof or a counterexample. Suppose $u_i(t_i, t_{-i}, a) \geq u_i(t_i', t_{-i}, a)$ for all $t_i \geq t_i'$ and a. Then for any equilibrium s^* it must be that $s_i^*(t_i) \geq s_i^*(t_i')$ for all $t_i \geq t_i'$. (5+5+5)

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2015-2016
M. Math. - II Year
Topology-III

Date : 07. 09. 2015 Maximum Score : 40 Time : 2:30 Hours

Any result that you use should be stated clearly.

(1) (a) What is an excisive triad?
(b) Prove that $(S^n; D^n_+, D^n_-)$ is an excisive triad.
(c) Prove that for $n > 1$,

$$H^q(S^{n-1}; \mathbb{Z}) \cong H^{q+1}(S^n; \mathbb{Z})$$

for all $q \geq 0$.

[2+4+ 4= 10]

(2) (a) State Eilenberg-Steenrod axioms for a cohomology theory.
(b) Prove that for a contractible space X, $H^k(X; \mathbb{Z}) = 0$ for $k > 0$.

[6 + 4 = 10]

(3) (a) Define the notion of cup product in cohomology.
(b) Prove that for any space X, the cohomology groups

$$H^q(X; \mathbb{Z}),\ q \geq 0,$$

is a graded ring.
(c) Let $X = S^3 \bigvee S^6$. Compute the cohomology ring $H^*(X; \mathbb{Z})$.

[6 + 6 + 6= 18]

(4) (a) State universal coefficient theorem for cohomolgy.
(b) Suppose cellular chain complex of a finite CW complex X of dimension n is given by

$$0 \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots C_1 \xrightarrow{\partial_1} C_0,$$

where $C_k = \mathbb{Z}$, $0 \leq k \leq n$ and $\partial_k = 1 + (-1)^k$ for all k, $1 \leq k \leq n$. Compute $H^k(X; \mathbb{Z}_p)$, where p is a odd prime.

[4+6=10]

INDIAN STATISTICAL INSTITUTE

M.Math./M.Stat. II Year
Mid-Semester Examination : Semester I : 2015-2016
ADVANCED PROBABILITY I

Date : 07.09.2015 | **Maximum Score : 40** | Time : 2 Hours

Note : This paper carries questions worth a **total** of **48 marks**. Answer as much as you can. The **maximum** you can score is **40**.

1. Let Let $(E, \mathcal{E})$ be a measurable space. Denoting φ_i, for each $i \geq 1$, to be the i-th coordinate projection map on E^∞ onto E, show that $\mathcal{E}^\infty$ is the smallest σ-field on E^∞ making all the φ_i, $i \geq 1$, measurable. Deduce (or show otherwise) that if $\mathcal{E}$ is countably generated, then so is $\mathcal{E}^\infty$. (5+5)=**[10]**

2. (a) Show that the set $\{x \in \mathbb{R}^\infty : \lim x_n \text{ exists and is finite}\}$ belongs to $\mathcal{B}^\infty$.
(b) For $x \in \mathbb{R}^\infty$, let $f(x)$ denote the smallest $n \geq 1$ such that $x_n + \cdots + x_{2n} < 0$; set $f(x) = 0$, if no such n exists. Show that f is a $\mathcal{B}^\infty$-measurable function on $\mathbb{R}^\infty$. (5+5)=**[10]**

3. Construct a probability space $(\Omega, \mathcal{A}, P)$ and a sequence $\{X_n\}$ of positive real-valued random variables on it, so that the random variables $Y_n = \prod_{i=1}^n X_i$, $n \geq 1$, are independent with each having an exponential distribution with parameter 1. **[9]**

4. Consider the function $\phi(x) = 1/\sqrt{x}$ on $(0, \infty)$ onto $(0, \infty)$. Denote λ to be the Lebesgue measure on the Borel σ-field $\mathcal{B}_+$ on $(0, \infty)$ and let $\mu = \lambda\phi^{-1}$ on $\mathcal{B}_+$. Show that $\mu \ll \lambda$ and find $\frac{d\mu}{d\lambda}$. [Consider $\mu[(a, b]]$ for $(a, b] \subset (0, \infty)$.] **[9]**

5. (a) Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\mathcal{G}$ and $\mathcal{E}$ sub-σ-fields of $\mathcal{A}$. Show that if X an integrable randonm variable such that $\sigma(X) \vee \mathcal{E}$ is independent of $\mathcal{G}$, then $E(X \mid \mathcal{G} \vee \mathcal{E}) = E(X \mid \mathcal{E})$, P-almost surely.
(b) Show that if X and Y are square integrable random variables on some probability space satisfying $E[X \mid Y] = Y$ and $E[Y \mid X] = X$, then $X = Y$ almost surely. (5+5)=**[10]**

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2015-16 (First Semester)

M. MATH. II YEAR
Commutative Algebra I

Date: 09.09.2015 | Maximum Marks: 40 | Duration: 3 Hours

Throughout the paper, R will denote a commutative ring with unity.
Clearly state any results that you use.

Group A

Attempt ANY 3 questions.
Each question carries 9 marks.

1. Let I be a finitely generated ideal of R. Prove that $\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1$, i.e., if I/I^2 is generated by r elements as an R/I-module, then I is generated by $r+1$ elements.

2. Let $A = \mathbb{R}[X,Y]/(X^2+Y^2-1)$, $P = (\overline{X}, \overline{Y}-1)A$ and $B = \mathbb{C}[X,Y]/(X^2+Y^2-1)$.
   (i) Show that PB is principal and determine all possible $f \in B$ such that $PB = fB$.
   (ii) Deduce that P is not a principal ideal of A.

3. Let M be a finitely generated R-module. Prove that $S^{-1}M = 0$ if and only if $S \cap Ann_R M \neq \emptyset$. Give an example to show the necessity of the hypothesis that M is finitely generated.

4. Prove that any finitely generated torsion-free module over an integral domain R is isomorphic to a submodule of R^m for some m.

Group B

Attempt ANY 3 questions.
Each question carries 5 marks.

1. Let $R \subseteq A$ be integral domains and let $f, g \in R$ be such that $(f,g)R = R$. Suppose that $R[1/f] = A[1/f]$ and $R[1/g] = A[1/g]$. Prove that $R = A$.

2. Let $M = R \oplus R$ and $a = (2,1)$. Let N be a R-module. Prove that given any $b \in N$, there exists an R-linear map $f : M \to N$ such that $f(a) = b$.

3. Let M and N be finitely generated R-modules such that $M \otimes_R N = 0$. Prove that $Ann_R M + Ann_R N = R$.

4. Prove that any projective R-module is R-flat.

INDIAN STATISTICAL INSTITUTE

Mid Semester Examination: 2015

Subject Name : **Basic Probability Theory** Course Name : M.Math II yr.
Maximum Score: 40 Date: 11th Sept 2015 Duration: 180 minutes

Note: Attempt all questions. Marks are given in brackets. Total marks is 50, but you can score maximum 40. State the results clearly you use. Use separate page for each question.

Problem 1 (8). Let X, Y be independent and uniformly distributed over a finite set G. Suppose $(G, +)$ forms a group. Show that $X + Y$ is independent to X.

Problem 2 (8). Let M, N be independently distributed as $Poi(\lambda)$. By using conditional expectation, compute $E(M^N)$.

Problem 3 (6+4 = 10). Describe probability distribution of $(X_1, \ldots, X_k)$, k draws of Polya's sampling scheme for three parameters r, b, c where r and b denote the number of red and blue balls respectively and c denotes the number of balls of same going to be added at each draw. Here X_i denotes the color of the ball at i^{th} draw.

If $r = b = c = 1$ then find the distribution of the number of red balls drawn in the k draws.

Problem 4 (6). Let X be uniformly distributed over a finite set S and let $T \subsetneq S$. Compute $\min_Y \Delta(X, Y)$ where minimum is taken over all random outcomes Y over T.

Problem 5 (8). Let $\langle S_n \rangle$ be a possibly biased non-trivial random walk (i.e., $0 < p < 1$). Show that $|S_n|$ is a markov chain.

Problem 6 (2+8 = 10). Define recurrent element of a Markov chain. Show that if i is recurrent and $i \leftrightarrow j$ then j is recurrent.

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination 2015-2016

M.Math (Second year)

Differential Topology

Maximum Marks: 40 Date: 14 September, 2015

Duration: 2 hours 30 minutes

Answer all questions.

State clearly any result you use in your answer.

(1) Prove that the real projective space $\mathbb{R}P^n$ of dimension n is a manifold. 8

(2) Let $f : M \to N$ be a smooth map between manifolds. Prove that $g : M \to M \times N$ defined by $g(x) = (x, f(x))$ for all $x \in M$ is transversal to $\{x_0\} \times N$, where x_0 is an arbitrary point of M. 6

(3) Consider the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy(x+y-1)$ for $(x, y) \in \mathbb{R}^2$. Find all real numbers a for which $f^{-1}(a)$ is a submanifold of $\mathbb{R}^2$. Justify your answer. 8

(4) Prove that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not diffeomorphic if $m \neq n$. 4

(5) If Q is a positive definite quadratic form on $\mathbb{R}^n$, then show that $Q^{-1}(y)$ is diffeomorphic to the $(n-1)$-sphere $\mathbb{S}^{n-1}$ for each $y > 0$. 6

(6) Let $f : M \to \mathbb{R}$ be a smooth function on a manifold M of dimension n. Prove the following: f is Morse if and only if

$$\det(H_f)^2 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 > 0$$

on any chart $(U; x_1, x_2, \ldots, x_n)$, where H_f denotes the Hessian matrix of f with respect to the given coordinate system. Hence show that the property of being Morse is a stable property if M is compact. 3+4

(7) Show that every n-manifold M admits an immersion in $\mathbb{R}^{2n}$. 6

Indian Statistical Institute

Mid-semsetral examination : (2015-16)

M. Math. II year

Differential Geometry

Date : 16.09.15 Maximum marks : 40 Duration : 2 hours.

Answer ANY THREE questions. Each of them carries 15 marks. The maximum you can score is 40

(1) Let P_1, P_2, P_3 be three smooth functions on $\mathbb{R}^3$ such that at any point, at least one of tem is nonzero. Let $X = \sum_{i=1}^{3} P_i \frac{\partial}{\partial x_i}$ and let θ be the smooth 2-dimensional distributon defined by $\theta(m) = X(m)^{\perp}$, where orthogonal completemt is taken w.r.t. the Euclidean Riemannian metric of $\mathbb{R}^3$. Give an explicit example of (P_1, P_3, P_3) such that θ is not integrable (with proof).

(2) Prove that any Lie group is orientable.

(3) Let G be a Lie group with a bi-invariant Riemannian metric and ∇ denotes the covariant derivative operator corresponding to the Levi-civita connection. Prove that for any two left invariant vector field X, Y,

$$\nabla_X(Y) = \frac{1}{2}[X, Y].$$

(4) Prove that the sphere S^n with the Riemannian metric inherited from $\mathbb{R}^{n+1}$ is locally conformally equivalent to $\mathbb{R}^n$ with the Euclidean metric.

Fourier Analysis: M. Math II: Mid Semester Examination

September 18, 2015.

Maximum Marks 40 Maximum Time 2:30 hrs.

Answer all questions. State any result you use.

1. Give short answers (at most 3-4 lines) to these questions:
   (a) For $\xi \in \mathbb{R}^n$, define the function $h_\xi : x \mapsto e^{i\xi \cdot x}$ for $x \in \mathbb{R}^n$. Find $\widehat{h_\xi}$.

   (b) Define the measurable function h on $\mathbb{R}$ by $h(x) = 1$ for $x > 0$ and $h(x) = 0$ for $x < 0$. Find the (distributional) derivative of h.

   (c) Suppose that for a function $f \in L^2(\mathbb{R})$, $g(x) = |x|\widehat{f}(x) \in L^2(\mathbb{R})$. Show that f is continuous. If $\mathbb{R}$ is replaced by $\mathbb{R}^n$, i.e. if $f \in L^2(\mathbb{R}^n)$, $g(x) = \|x\|\widehat{f}(x) \in L^2(\mathbb{R}^n)$, then is f continuous? If not then suggest correction in definition of g.

   (d) Take a function $f \in L^1(\mathbb{R}^n)$. Show that for any given $\epsilon > 0$, there is a set E with $|E| < \epsilon$, such that f restricted to E^c is in $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$.
   (Hint: $\|f\|_{1,\infty} \leq \|f\|_1$. Use distribution function.) $3+3+5+5$

2. Let f be a continuous function on $\mathbb{R}$. Suppose that for any interval I, $\int_I f = 0$. Show that $f = 0$. What is your conclusion if f is an arbitrary locally integrable function satisfying the property that for any interval I, $\int_I f = 0$? 7

3. Let $g(x) = e^{-ix^2}$, $x \in \mathbb{R}$. Show that the operator $T : f \mapsto f * g$ is $2-2$ and cannot be $p-p$ when $p \neq 2$. (Hint: $\widehat{g}$ is not required. Write $f * g$ explicitly.) 10

4. Let $p \in [1, \infty]$ be fixed. For a function $f \in L^p(\mathbb{R})$, define $h(y) = f(y)e^{-y^2}$.
   (a) Show that $h \in L^1(\mathbb{R})$.
   (b) Show that Fourier transform of h extends holomorphically to $\mathbb{C}$.
   (Only sketch mentioning all the points is enough.)
   (c) If $\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}dy = 0$ for all $x \in \mathbb{R}$, then show that $f \equiv 0$. 2+3+3

INDIAN STATISTICAL INSTITUTE

M.Math./M.Stat. II Year
Semestral Examination : Semester I : 2015-2016
ADVANCED PROBABILITY I

Date : 16.11.2015 | **Maximum Score : 60** | Time : 3 Hours

Note : This paper carries questions worth a **total** of **72 marks**. Answer as much as you can. The **maximum** you can score is **60**.

1. State the following clearly: (i) Kolmogorov consistency theorem; (ii) Doob's maximal inequality; (iii) Doob's upcrossing inequality; (iv) Optional sampling theorem; (v) Hewitt-Savage zero-one law. (5×2)=**[10]**

2. Let μ and ν be two σ-finiite measures on a measurable space $(\Omega, \mathcal{A})$. Observe that $\mu \ll \mu+\nu$ and find the Radon-Nikodym derivative of $\dfrac{d\mu}{d(\mu+\nu)}$. (Hint: You may use the Lebesgue decomposition $\nu(\cdot) = \int g\, d\mu + \nu(\cdot \cap N)$ where $\mu(N) = 0$.) **[6]**

3. X and Y be two integrable random variables satisfying $E[X \mid Y] = Y$ and $E[Y \mid X] = X$.
(a) Assuming that X and Y are square integrable, show that $X = Y$ almost surely.
(b) Without assuming square integrability of X and Y, prove that $X = Y$ almost surely.
(Hint: For any real c, $E\left[(X-Y)\mathbf{I}_{\{X \le c\}}\right] = 0 = E\left[(X-Y)\mathbf{I}_{\{Y \le c\}}\right]$) (6+6)=**[12]**

4. Consider a real-valued function F of bounded variation on $[0,1]$. For each $n \ge 0$, let M_n be the function on $[0,1]$ defined by $M_n = \sum_{k=1}^{2^n} 2^n \left[F(\frac{k}{2^n}) - F(\frac{k-1}{2^n})\right] \mathbf{I}_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}$.
(a) Denoting $\mathcal{A}_n$, for each $n \ge 0$, to be the σ-field on $[0,1]$ generated by the intervals $\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$, $1 \le k \le 2^n$, show that $\{M_n\ n \ge 0\}$ defines an L_1-bounded martingale with respect to the filtration $\{\mathcal{A}_n, n \ge 0\}$ on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$.
(b) Show that if F is absolutely continuous, then $\{M_n, n \ge 0\}$ is uniformly integrable and hence deduce that there exists a borel measurable function f on $[0,1]$ such that $F(t) = \int_0^t f(s)ds,\ \forall\, t \in [0,1]$. (6+6)=**[12]**

5. Let $\{M_n, n \ge 0\}$ be an L_2-martingale and $\{\langle M \rangle_n\}$ be the associated "bracket" process.
(a) Denoting $T_k = \inf\{n \ge 0 : \langle M \rangle_{n+1} > k\}$, show that $\{M_{n \wedge T_k}, n \ge 0\}$ is an L_2-bounded martingale and hence deduce that $\{M_n\}$ converges almost surely on the set $\{\langle M \rangle_\infty < \infty\}$.
(b) Let $f : [0,\infty) \to [0,\infty)$ be any increasing function such that $\int_0^\infty [1 + f(x)]^{-2} dx < \infty$. Using part (a) and an appropriate Kronecker Lemma (which should be clearly stated), prove that $M_n = o(f(\langle M \rangle_n))$ almost surely on the set $\{\langle M \rangle_\infty = \infty\}$. (4+6)=**[10]**

6. Let $X_1, \ldots, X_n$ be integrable random variables taking values in $\{0, 1, 2, \ldots\}$ and S_k, $1 \le k \le n$ be the partial sums. For $1 \le k \le n$, denote $M_k = S_{n-k+1}/(n-k+1)$ and $\mathcal{A}_k = \sigma(\{S_j : n-k+1 \le j \le n\})$. Suppose that joint distribution of $(X_1, \ldots, X_n)$ remains invariant under permutations.
(a) Show that $\{M_k, 1 \le k \le n\}$ is a (finite parameter) martingale w.r.t $\{\mathcal{A}_k, 1 \le k \le n\}$.
(b) If $\tau = \inf\{k : 1 \le k \le n, M_k \ge 1\}$ and $\tau = n$ if no such $1 \le k \le n$ exists, then show that M_τ equals one on $\bigcup_{k=1}^n \{S_k \ge k\}$ and equals zero elsewhere.
(c) Use (b) to deduce that on the set $\{S_n < n\}$, $P\left[\bigcup_{k=1}^n \{S_k \ge k\} \mid S_n\right] = S_n/n$ and hence conclude that $P\left[S_k < k\ \forall\, 1 \le k \le n \mid S_n\right] = (1 - S_n/n)^+$.
(d) In an election, candidate A gets a votes and his rival B gets b votes with $a > b$ positive integers (so that A is the winner). Assuming that the $a+b$ votes were counted in random order, use the above to show that the probability that A was always ahead during the entire counting process is $(a-b)/(a+b)$. (6+6+4+6)=**[22]**

INDIAN STATISTICAL INSTITUTE

End Semestral Examination: 2015

Subject Name : **Basic Probability Theory**. Date : 16th Nov. 2015
Course Name : M.Math II yr. Maximum Score: 50 Duration: 180 minutes

Note: Attempt all questions from part-1 and only one question from part-2. Marks are given in brackets. Total marks alloted is 55 and maximum you can score is 50. State the results clearly you use. Use separate page for each question.

Part-1.

Problem 1 (6+3 = 9). Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{unif}(\{0,1\}^m)$. For any subset $S \subseteq \{1, 2, \ldots, n\}$ define $Y_S := \oplus_{i \in S} X_i$, where $\oplus$ is coordinate-wise modulo 2 addition. Prove that Y_S's are pairwise independent but not k-wise independent for $k > 2$.

Problem 2 (6+2=7). Compute the moment generating function of the negative binomial distribution and hence find its expectation. Note that the negative binomial can be viewed as a sum of independent geometric distributions.

Problem 3 (5). For any events $A_1, \ldots A_n$, show that

$$P(\cup_i A_i) \geq \sum_i P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j).$$

Problem 4 (5). Let E_i's be pairwise disjoint events such that $\cup_i E_i = E$. Let F be an event such that for all i, $P(F \mid E_i) \geq c$. Then show that $P(F \mid E) \geq c$.

Problem 5 (8). A line of 100 airline passengers is waiting to board a plane. They each hold a ticket to one of the 100 seats on that flight. (For convenience, let's say that the i^{th} passenger in line has a ticket for the seat number i.) Unfortunately, the first person in line is crazy, and will ignore the seat number on their ticket, picking a random seat to occupy. All of the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random. What is the probability that the last (100^{th}) person to board the plane will sit in their proper seat (#100)?

Problem 6 (3+7=10). Define perfect secrecy. Show that if an encryption algorithm has perfect secrecy then the size of key space should be at least the size of the message space.

P.T.O.

Part-2 [10].

Problem 7. Define delta universal hash function. Show that the dot-product hash is an delta universal hash function. Define the conditional min-entropy $H_\infty(X|Y)$. Show that $H_\infty(X|Y) \leq H_\infty(X)$.

Problem 8. State and prove the weak law of large numbers. Show that the probability that the number of Head in 24 tosses of a fair coin is at least 18 is less than $2/e$.

Problem 9. Define sampler and an ϵ-estimator based on the sampler. What is the randomness complexity and ϵ for the classical sample mean and median of average (state the sampler and estimator) for estimating population mean?

INDIAN STATISTICAL INSTITUTE
Semestral Examination : 2015-2016
M. Math. - II Year
Topology-III

Date : 20. 11. 2015 Maximum Score : 60 Time : 2:30 Hours

Any result that you use should be stated clearly.

(1) (a) Define Cap Product operation in Singular cohomology.
(b) Explain the meaning of an orientation on a closed n-dimensional manifold.
(c) State Poincare duality theorem for an orientable closed manifold.
(d) Compute cohomology ring $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$.

[4 + 4 + 2 + 6 = 16]

(2) (a) Compute the singular cohomology groups

$$H^k(S^2 \times S^4; \mathbb{Z}),\ k \geq 0.$$

(b) If $u \in H^2(S^2 \times S^4; \mathbb{Z})$ is a generator, prove that $u \cup u = 0$.
(c) Justify whether the spaces $S^2 \times S^4$ and $\mathbb{C}P^3$ have the same homotopy type or not.

[8 + 8 + 6 = 22]

(3) (a) State Cellular approximation theorem.
(b) Prove that $\pi_k(S^n) = 0$ for $0 \leq k < n$.

[4 + 4 = 8]

(4) (a) Let X be a path connected space and $p : \tilde{X} \longrightarrow X$ be its universal cover. Assume that $\tilde{X}$ is contractible. Prove that $\pi_k(X) = 0$ for all $k > 1$.
(b) Compute the first two nontrivial homotopy groups of $\mathbb{C}P^6$. Justify your answer.

[4+6=10]

(5) (a) Justify the statement with proper reasons: 'There exists a CW complex X which is 4-connected and $H_2(X)$ is non-trivial'.
(b) Let $\mu : \mathbb{C} \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be the scalar multiplication. Consider the induced action

$$S^1 \times S^3 \longrightarrow S^3$$

obtained by restricting μ to vectors of unit lengths, respectively. Determine the orbit space X of this action and compute $\pi_3(X)$.

[6 + 6 = 12]

Indian Statistical Institute

Fourier Analysis : M. Math II, Year 2015-2016 : Semester Examination

November 23, 2015.

Maximum Marks 60 Maximum Time 3:00 hrs.

Answer all questions. State any result/theorem you use.

(1) Let $f \in L^1(\mathbb{R}^2)$ be supported on the closed first quadrant, i.e. if $x < 0$ or $y < 0$ then $f(x, y) = 0$. Find the domain in $\mathbb{C} \times \mathbb{C}$ where its Fourier transform can be extended. 6

(2) Consider $\mathbb{R}$ with the measure $e^{|x|}dx$. Construct a function on $\mathbb{R}$ which is weak L^1 but not L^1 with respect to the measure $e^{|x|}dx$. Justify your answer. 7

(3) Let f be a nonzero twice differentiable function in $L^p(\mathbb{R}^2)$ for some $p \in [1, \infty]$. Suppose f satisfies

$$\frac{\partial^2}{\partial x^2} f(x, y) = \frac{\partial^2}{\partial y^2} f(x, y).$$

Show that $p > 2$. 7

(4) For $f \in S(\mathbb{R})$ and $a, b \in \mathbb{R}$ with $a < b$, define

$$S_{a,b}f(x) = \int_a^b \widehat{f}(\xi) e^{i\xi x} d\xi.$$

Write all the steps (without proof) which show that for $f \in L^p(\mathbb{R})$, $S_{a,b}f \to f$ in L^p as $a \to -\infty$ and $b \to \infty$. 7

(5) Suppose $f \in L^1(\mathbb{R}^2)$ satisfies the property that $f(k_\theta x) = e^{im\theta} f(x)$ for all $x \in \mathbb{R}^2$ and for all $0 \leq \theta \leq 2\pi$ where $m \in \mathbb{Z}$ and $m \neq 0$ and

$$k_\theta = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

Show that $\int_{\mathbb{R}^2} f(x)dx = 0$. 6

P. T. O.

(6) For $s \in \mathbb{R}$ define two "norms" on $\mathbb{R}^n$ by

$$\|f\|_H = \left(\int_{\mathbb{R}^n} (1+\|\xi\|^2)^s |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \text{ and } \|f\|_L = \left(\int_{\mathbb{R}^n} \|\xi\|^{2s} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2}$$

(a) Show that if $s < -n/2$ then $\|f\|_H \leq C\|f\|_1$ for all f in Schwartz space of $\mathbb{R}^n$.

(b) If it is given that

$$\|f\|_p \leq C\|f\|_L$$

for all $f \in S(\mathbb{R}^n)$ then show that $1/2 - 1/p = s/n$.

8+7

(7) Find if the ideal under convolution generated by the function $x^2 e^{-x^2} \in L^1(\mathbb{R})$ is dense in $L^1(\mathbb{R})$. 5

(8) Fix a $p \in (1, \infty)$. Take a function $f \in L^p(\mathbb{T})$. Show that if $\widehat{f}(n) \neq 0$ for all $n \in \mathbb{Z}$, then the span of translates of f is dense in $L^p(\mathbb{T})$. 5

INDIAN STATISTICAL INSTITUTE

Semestral Examination 2015–2016

M.Math (Second year, First Semester)

Differential Topology

Maximum Marks: 60

Date: 26 November, 2015

Duration: 3 hours

Answer all questions.
State clearly any result that you use in your answer.

Notation: M will always denote a smooth manifold.

(1) Prove that for any manifold M, the product manifold $M \times M$ is orientable. 7

(2) Define de Rham cohomology groups of a manifold. Prove that a smooth map $f : M \to N$ between manifolds defines a homomorphism $f^{\#} : H^k(N) \to H^k(M)$ between the de Rham cohomology groups, for every $k \geq 0$. 7

(3) (a) Show that if $f : \mathbb{C} \to \mathbb{C}$ is an analytic function then f (viewed as a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$) is orientation preserving at all z where $f'(z) \neq 0$.

(b) Define degree of a smooth map $f : M \to M$. Suppose that $p : \mathbb{C} \to \mathbb{C}$ is a polynomial function of complex variable. Prove that the degree of this map is equal to its algebraic degree. 4+6

(4) (a) Let $\wedge^n(\mathbb{R}^n)^*$ denote the space of alternating n-multilinear forms on $\mathbb{R}^n$. Determine the dimension of this vector space. Give an explicit example of a non-trivial alternating n-multilinear form on $\mathbb{R}^n$. 5

(b) Let $\ell : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Determine the map $\ell^* : \wedge^n(\mathbb{R}^n)^* \to \wedge^n(\mathbb{R}^n)^*$. 5

(c) Let $\Omega^n(M)$ be the space of all smooth n-forms on M. If M is orientable then prove that $\Omega^n(M)$ is a $C^\infty(M)$-module of rank one. For any smooth map $f : M \to M$ determine the induced map $f^* : \Omega^n(M) \to \Omega^n(M)$. 5

(5) (a) Define orientation on a zero dimensional manifold.

(b) Let $[0, 1]$ has the standard orientation as a submanifold of $\mathbb{R}$. Explain the boundary orientation on $\partial([0, 1])$.

(c) State Stokes' Theorem clearly. Interpret the theorem for the manifold $M = [0, 1]$. 2+4+6

P. T. O.

(6) Consider the 2-form

$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

on $\mathbb{R}^3 \setminus \{0\}$.

(a) Show that ω is closed.

(b) Prove that $\int_{\mathbb{S}^2} \omega \neq 0$, where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ centred at origin.

(c) Is ω an exact form? Justify your answer. 3+6+4

INDIAN STATISTICAL INSTITUTE

First Semestral Examination : 2015-16

Course Name: M.S. (Q.E.) I YEAR / M.STAT. II YEAR / M.MATH. II YEAR

Subject Name: Game Theory I

Date: 28.11.2015 Maximum Marks: 50 Duration: 2 hours

Problem 1. The two-player game of Tic-Tac-Toe is played on a 3×3 board. Player 1 starts by putting a cross on one of the nine fields. Next, player 2 puts a circle on one of the eight remaining fields. Then player 1 puts a cross on one of the remaining seven fields, etc. If player 1 achieves three crosses or player 2 achieves three circles in a row (either vertically or horizontally or diagonally) then that player wins. If this does not happen and the board is full, then the game ends in a draw.

(a) Design a pure maximin strategy for player 1. Show that this maximin strategy guarantees at least a draw to him.

(b) Show that player 1 cannot guarantee a win.

(c) What is the value of this game? (10)

Problem 2. An $m \times n$ matrix game $A = (a_{ij})$ is called symmetric if $m = n$ and $a_{ij} = a_{ji}$ for all $i, j = 1, ..., m$.

Prove that the value of a symmetric game is zero and that the sets of optimal strategies of players 1 and 2 coincide. (15)

Problem 3. Consider the following game:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ J \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & J \end{array} \\ \begin{pmatrix} -1,1 & 1,-1 & 1,-1 & -1,1 \\ 1,-1 & -1,1 & 1,-1 & -1,1 \\ 1,-1 & 1,-1 & -1,1 & -1,1 \\ -1,1 & -1,1 & -1,1 & 1,-1 \end{pmatrix} \end{array}$$

a. Find a completely mixed Nash equilibrium in which each player assigns the same probability to the actions 1, 2, and 3.

b. Show that the equilibrium you found in part a is the only equilibrium of the game.

(15)

Problem 4. Consider the following bimatrix game:

$$\begin{array}{cc} & \begin{array}{cc} L & S \end{array} \\ \begin{array}{c} L \\ S \end{array} & \begin{pmatrix} 2,2 & -1,-1 \\ -1,-1 & 1,1 \end{pmatrix} \end{array}$$

(a) Which payoffs can be reached as limiting average payoffs in subgame perfect equilibria of the infinitely repeated game $G^{\infty}(\delta)$ for suitable choices of δ?
(b) Which payoffs can be reached as limiting average payoffs in Nash equilibria of the infinitely repeated game $G^{\infty}(\delta)$ for suitable choices of δ?
(c) Describe a subgame perfect Nash equilibrium of $G^{\infty}(\delta)$ resulting in the limiting average payoffs $(3/2, 3/2)$. Also give the corresponding restriction on δ.
Hint: Use folk theorem.

(10)

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2015-16 (First Semester)

M. MATH. II YEAR

Commutative Algebra I

Date: 30/11/2015 — Maximum Marks: 60 — Duration: $3\frac{1}{2}$ Hours

Throughout the paper, k will denote a field and R will denote a commutative ring with unity. Clearly state any result that you use.

Group A

Attempt ANY 5 questions.
Each question carries 8 marks.

1. Examine whether $\mathbb{C}[X,Y,Z]/(X^2+Y^3+Z^5)$ is a UFD or not.
2. Let A be a finitely generated k-algebra which is an integral domain with field of fractions L and $L \neq A$. Prove that $L \neq A[1/f]$ for any $f \in A$.
3. Suppose that R is a subring of A and α is a unit in A. Show that $R[\alpha] \cap R[\alpha^{-1}]$ is integral over R.
4. Let $R = k[X]$ and $f \in R$. Let $A = k[X,Y]/(Y^2 - f)$. Suppose that A is an integral domain. Show that A is a normal domain if and only if f does not have any square factor.
5. Let I, $P_1, \ldots, P_n$ be ideals of R such that $P_3, P_4, \ldots, P_n$ are prime ideals and $I \subseteq P_1 \cup \cdots \cup P_n$. Prove that $I \subseteq P_i$ for some i, $1 \leq i \leq n$.
6. Let M be an R-module and $f : M \to M$ an R-linear map. Show that
(i) If M is Artinian and f is injective, then f is an isomorphism.
(ii) If M is Noetherian and f is surjective, then f is an isomorphism.
7. Prove that any k-subalgebra of the polynomial ring $k[X]$ is a finitely generated k-algebra.

Group B

Attempt ANY 2 questions.
Each question carries 12 marks.

1. Let R be an integral domain in which every prime ideal is principal. Prove that R is a principal ideal domain.
2. Let I be a finitely generated nilpotent ideal of R. Prove that R is Noetherian if and only if R/I is Noetherian. Give an example to show that finite generation of I is necessary. [8+4]
3. Let $R = k[X,Y]/(X^2, XY)$. Let $x = \bar{X}, y = \bar{Y}$.
(i) Show that y^n is (x,y)-primary in R $\forall\, n \geq 1$.
(ii) Show that $(0) = (x) \cap (y^n)$ is an irredundant primary decomposition of (0) in R, for each $n \geq 1$.
(iii) Determine the associated prime ideals of R. [3+6+3]

Indian Statistical Institute

First semestral Examination : (2015-16)

M. Math II year

Differential Geometry II

Date : 01/12/15 Maximum marks : 60 Duration : 3 *hours*.

Answer all the questions. Marks are indicated in brackets

(1) Let M be a (not necessarily Riemannian) manifold and $\nabla^{(1)}, \nabla^{(2)}$ be the covariant derivative operators corresponding to two affine connections on M. Define a map $A : \chi(M) \times \chi(M) \to \chi(M)$ (where $\chi(M)$ denotes the vector space of all smooth vector fields on M) by

$$A(X,Y) = \nabla^{(1)}_X Y - \nabla^{(2)}_X Y.$$

Prove that $\nabla^{(1)}$ and $\nabla^{(2)}$ determine the same geodesics if and only if $A(X,Y) = -A(Y,X)$ for all $X, Y \in \chi(M)$. Hence deduce that an affine connection is completely determined by its torsion form and the set of geodesics. [10+5=15]

(2) Let S be the sphere of unit radius $\{(x,y,z) : x^2+y^2+z^2 = 1\} \subseteq R^3$, equipped with the usual Riemannian metric inherited from the Euclidean metric of R^3.

(a) Write down explicitly the Christoffel symbols for the Levi-Civita connection in the spherical polar coordinates.

(b) Compute the Ricci and scalar curvatures at an arbitrary point. [7+8=15]

(3) Let M be a complete Riemannian manifold and $N \subset M$ be a Riemannian submanifold. Prove that N is complete too. [10]

(4) Let M be a Riemannian manifold of dimension n such that the sectional curvature (w.r.t. the Levi-Civita connection) at any point of M is a constant C. Prove that the scalar curvature at any point is $n(n-1)C$. [10]

(5) Suppose that F and G are two isometries of a connected Riemanian manifold M. Moreover, assume that there is a point p in M such that $F(p) = G(p)$ and $dF(p) = dG(p)$. Prove that $F(m) = G(m)$ for all $m \in M$. [15]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination (Back Paper): 2015-16

Course Name: M.S. (Q.E.) I YEAR / M.STAT. II YEAR / M.MATH. II YEAR

Subject Name: Game Theory I

Date: 29·01·16 Maximum Marks: 50 Duration: 2 hours

Problem 1. Consider the following version of prisoner dilemma game.

1/2	C	D
C	4,4	0,6
D	6,0	2,2

(a) Is it true that the outcome (C,C) forever can be a subgame perfect equilibrium of the infinitely repeated prisoners dilemma game? If yes, provide the equilibrium strategies. Find the restriction on the discount factor δ.
(b) What do you think about the other Subgame Perfect Equilibria that are possible in this repeated prisoners dilemma game (use folk theorem)? Plot graphically: (i) The set of subgame perfect Nash Equilibria payoffs, (ii) The set of feasible payoffs.
(c) Consider the following tit-for-tat (TFT) strategy (row player version)
First round: Play C.
Second and later rounds: If the history from the last round is (C,C) or (D,C) play C. If the history from the last round is (C,D) or (D,D) play D.
(i) Is it true that TFT supports (C,C) forever as an NE in the Infinitely Repeated PD? - Justify your answer by a proof.
(ii) Is it true that TFT as an equilibrium strategy is Subgame Perfect?
- Justify your answer by a proof.

(20)

Problem 2. A buyer wants to buy a car but does not know whether the particular car he is interested in has good or bad quality (a lemon is a car of bad quality). About half of the market consists of good quality cars. The buyer offers a price p to the seller, who is informed about the quality of the car: the seller may then either accept of reject this price. If he rejects, there is no sale and the payoff will be 0 to both. If he accepts, the payoff to the seller will be the price minus the value of the car, and to the buyer it will be the value of the car minus the price. A good quality car has a value of 15,000, a lemon has a value of 5,000. (a) Set up the extensive as well as strategic form of this game. (b) Compute the subgame perfect equilibrium or equilibria of this game.

(15)

Problem 3. Let (b, β) be a consistent assessment in an extensive form game Γ. Show that (b, β) is Bayesian consistent.

(15)

INDIAN STATISTICAL INSTITUTE

Backpaper Examination 2015–2016

M.Math (Second year, First Semester)

Differential Topology

Maximum Marks: 100 Date: 11/02/2016

Duration: 3 hours 30 minutes

Answer all questions.

State clearly any result that you use in your answer.

Notation: M will always denote a smooth manifold.

$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$.

$\mathbb{R}P^2$ = real projective plane.

(1) Let $SL_2(\mathbb{R})$ denote the set of all 2×2 matrices with real entries and determinant 1. Prove that it is a manifold; find its dimension. Determine the tangent space of $SL_2(\mathbb{R})$ at a point A. 8

(2) Let $f : M \to \mathbb{R}$ be smooth function and $p \in M$ a non-degenerate critical point of f. Show that p is an isolated critical point. Hence show that a Morse function on a compact manifold can have only finitely many critical points. 5+3

(3) Assume that $H = \{(x, y, z) \in \mathbb{R}^3 | x^2 - y^2 + z^2 = 1\}$. Draw a diagram for H showing clearly the coordinate axes. Prove that $(0, 0, 1)$ is a non-degenerate critical point of the function $f : M \to \mathbb{R}$ defined by $f(x, y, z) = z$. Hence, determine the index of the critical point. 2+7+1

(4) Let $g = (g_1, g_2) : M \to \mathbb{R}^2$ be a smooth function. Prove that $(a, b) \in \mathbb{R}^2$ is a regular value of g if and only if $g_1^{-1}(a)$ intersects $g_2^{-1}(b)$ transversally. 8

(5) (a) Let $f : \mathbb{S}^2 \to \mathbb{R}^4$ be defined as follows:

$$f(x, y, z) = (x^2 - y^2, xy, xz, yz), \quad (x, y, z) \in \mathbb{S}^2.$$

Prove that f is an immersion.

(b) Use f to show that $\mathbb{R}P^2$ embeds in $\mathbb{R}^4$. 7+5

(6) Prove that if M is connected then its zero-th de Rham cohomology group is isomorphic to $\mathbb{R}$. 6

P. T. O.

(7) (a) Let $\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ denote the canonical volume form on $\mathbb{R}^3$. Prove the following relation for any smooth function $f : \mathbb{R}^3 \to \mathbb{R}$:

$$\mathrm{d}(i_{\mathrm{grad}\, f}\Omega) = (\Delta f)\Omega,$$

where d denotes the exterior derivation, $i_{\mathrm{grad}\, f}$ denotes the interior product with respect to the gradient vector field $\mathrm{grad}\, f$ and and Δ is the Laplacian on $\mathbb{R}^3$.

(b) Note that $\mathrm{div}(\mathrm{grad}\, f) = \Delta f$. Can you define the notion of divergence of a vector field on an oriented manifold M? Justify your answer. 4+4

(8) Let ω be a closed 1-form on $\mathbb{R}^n$. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \int_{\gamma_x} \omega, \quad x \in \mathbb{R}^n$$

where $\gamma_x : [0,1] \to \mathbb{R}^n$ is a curve defined by $\gamma_x(t) = tx$, $t \in \mathbb{R}$. Prove that, $\mathrm{d}f = \omega$. 10

(9) Consider the 1 form

$$\omega = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \text{ on } \mathbb{R}^2 \setminus (0,0).$$

Prove that ω is closed but not exact. 8

(10) (a) Assume that $\mathbb{S}^2$ is orientable. Consider the antipodal map $a : \mathbb{S}^2 \to \mathbb{S}^2$. Determine whether a is orientation preserving or orientation reversing.

(b) Prove that $\mathbb{R}P^2$ is not orientable. 6+4

(11) Let M be an oriented manifold of dimension n without boundary.

(a) Consider the product manifold $M \times [0,1]$ with the product orientation. Describe the boundary orientation on $M \times \{0,1\}$.

(b) Suppose that M is compact and without boundary. Let $f_0, f_1 : M \to N$ be homotopic maps. Prove that

$$\int_M f_0^*\omega = \int_M f_1^*\omega$$

for any closed n-form ω on N. 6+6

Indian Statistical Institute

Fourier Analysis : M. Math II, Year 2015-2016 ; Back Paper Examination

Date...13/02/16

Full Marks 100. Maximum Marks 45. Maximum Time 3:00 hrs.

Answer as many questions as possible. Maximum you can score is 45.
State any result/theorem you use.

(1) For $f \in S(\mathbb{R}^2)$ we say f is of type $n \in \mathbb{Z}$, if $f(k_\theta x) = e^{in\theta} f(x)$ for all $x \in \mathbb{R}^2$ and $k_\theta \in \mathrm{SO}(2)$, where

$$k_\theta = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}.$$

Suppose that $f_1, f_2 \in S(\mathbb{R}^2)$ are of types $m, n \in \mathbb{Z}$ respectively.

(a) Show that $\int_{\mathbb{R}^2} f(x)g(x)dx = 0$.

(b) Find the type of $f * g$.

7+8=15

(2) Suppose that for functions $f \in L^p(\mathbb{R})$ and $g \in C_c(\mathbb{R})$ $f * g \equiv 0$. Show that either $f = 0$ or $g = 0$. 6

(3) Let F be a tempered distribution and $\delta > 0$. Define dilation of F by δ. Define Fourier transform $\widehat{F}$ of F. Find the relation between $\widehat{F_\delta}$ and $\widehat{F}$. 4+4+7=15

(4) For a function $m(\xi)$ and a $\rho > 0$ define operators T and T_ρ on $S(\mathbb{R})$ by

$$Tf(x) = \int_{\mathbb{R}} \widehat{f}(\xi) m(\xi) e^{i\xi x} d\xi, \quad \text{and } T_\rho f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) m(\rho\xi) e^{i\xi x} d\xi.$$

Suppose that T is $p - p$ for some $1 < p < \infty$. Show that (a) T is $2 - 2$ and (b) T_ρ is $p' - p'$, and (c) $\|T\|_{L^p - L^p} = \|T_\rho\|_{L^{p'} - L^{p'}}$. 6+7+7=20

(5) Let V_f be the closure of the span of translates of $f \in L^\infty(\mathbb{T})$. Show that V_f contains the function $\theta \mapsto e^{in\theta}$ for some $n \in \mathbb{Z}$. 10

P. T. O.

(6) Define central and noncentral Hardy-Littlewood maximal functions $M_c f$ and Mf respectively on $\mathbb{R}^n$.

(a) Show that there are constants $C_1 > 0, C_2 > 0$ such that

$$C_1 Mf \leq M_c f \leq C_2 Mf.$$

(b) Show that if $\|M_c f\|_p \leq C\|f\|_q$ for all functions $f \in S(\mathbb{R}^n)$ then $p = q$.

7+7

(7) Let $D_n(\theta) = \sum_{n=-N}^{N} e^{in\theta}$ for $\theta \in \mathbb{T}$.

(a) Show that if $\|f * D_N\|_p \leq C\|f\|_p$ for all $f \in C(\mathbb{T})$ then $f * D_N \to f$ in $L^p(\mathbb{T})$ as $N \to \infty$.

(b) Assume $\|f * D_N\|_{1,\infty} \leq C\|f\|_1$. State and establish the result regarding the convergence of $S_N f$ to f as $N \to \infty$ for $f \in L^1$.

(c) Consider the operator $S : f \mapsto f * D_N$. Write S as a multiplier operator and expand it in terms of T, M_N, M_{-N} where $M_N f(\theta) = e^{iN\theta} f(\theta)$ and T is given by $Tf(\theta) = \sum_{n=-\infty}^{\infty} \text{Sgn } n \ \widehat{f}(n) e^{in\theta}$ for all $f \in C(\mathbb{T})$.

6+7+7=20

INDIAN STATISTICAL INSTITUTE

M.Math./M.Stat. II Year
Backpaper Examination : Semester I : 2015-2016
ADVANCED PROBABILITY I

Date: 15.2.16 | **Maximum Score : 45** | Time : 3 Hours

Note : This paper carries questions worth a **total** of **100 marks**. Answer as much as you can. The **maximum** you can score is **45**.

1. (a) Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space. Show that if ν is a finite measure on $\mathcal{A}$, then $\nu \ll \mu$ if and only if, $\lim_{\mu(A)\to 0} \nu(A) = 0$ [in the sense that, $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that, $A \in \mathcal{A}, \mu(A) < \delta \Longrightarrow \nu(A) < \epsilon$].
(b) Does the above result hold good, if ν is only σ-finite, but not finite? If yes, then prove it and if not, then establish that by giving a counter-example. (10+8)=**[18]**

2. (a) State and prove conditional Minkowski's inequality.
(b) Let X, Y be independent random vectors of dimensions m and n respectively on a probability space $(\Omega, \mathcal{AS}, P)$ and let $Z = h(X,Y)$ where h is a real-valued borel measurable function on $\mathbb{R}^{m+n}$. Denoting $Q(x,\cdot)$, for each $x \in \mathbb{R}^m$, to be the distribution of the real random variable $h(x,Y)$, show that $P(w, B) = Q(X(\omega), B)$, $\omega \in \Omega, B \in \mathcal{B}(\mathbb{R})$ defines a regular conditional distribution of Z, given $\sigma(X)$. (8+10)=**[18]**

3. (a) Let $\{M_n, n \geq 0\}$ be a submartingale with $\sup_{n\geq 0}(M_{n+1} - M_n)^+ \in L_1$. If τ_a, for any $a > 0$ is defined as $\tau_a = \inf\{n \geq 0 : M_n \geq a\}$, then show that $\{M_{\tau_a \wedge n}\}$ is L_1-bounded.
(b) Let $\{M_n, n \geq 0\}$ be a martingale satisfying $\sup_{n\geq 0}|M_{n+1} - M_n| \in L_1$, Denote $A_1 = \{\omega : \lim_n M_n(\omega) \text{ exists}\}$ and $A_2 = \{\omega : \limsup_n M_n(\omega) = +\infty, \liminf_n M_n(\omega) = -\infty\}$. Show that $P(A_1 \cup A_2) = 1$.
(c) Let $(\Omega, \mathcal{A}, P)$ be a probability space, let $\{\mathcal{A}_n, n \geq 1\}$ be a filtration in $\mathcal{A}$. Show that, for any sequence of events $A_n \in \mathcal{A}_n$, the two sets $\{\omega : \omega \in A_n \text{ for infinitely many } n\}$ and $\{\omega : \sum_n P(A_{n+1} \mid \mathcal{A}_n)(\omega) = \infty\}$ are almost surely equal. (8+8+8)=**[24]**

4. (a) Show that for any random variable Y taking values in $[-c, c]$ and with $E[Y] = 0$, one has $E[e^{\theta Y}] \leq \exp(\theta^2 c^2/2)$, for any $\theta \in \mathbb{R}$.
(b) Let $\{M_n, n \geq 0\}$ be a martingale with $M_0 \equiv 0$ and $|M_n - M_{n-1}| \leq c_n \;\forall\, n \geq 1$, for some constants $c_n > 0$. Show that, for any $\theta \in \mathbb{R}$, one has $P\left(\sup_{1\leq k\leq n} M_k > x\right) \leq \exp\left\{-\theta x + (\theta^2/2)\cdot \sum_{k=1}^n c_k^2\right\}$, for any $x > 0$ and for any $n \geq 1$.
(c) Let $\{M_n, n \geq 0\}$ be a martingale satisfying the asmptions in (b) above. From (b) (or otherwise), deduce that $P\left(\sup_{1\leq k\leq n} M_k > x\right) \leq \exp\left\{-\frac{1}{2}x^2 \Big/ \sum_{k=1}^n c_k^2\right\}$. (8+10+6)=**[24]**

5. Let $\{M_n, n \geq 0\}$ be an L_2-martingale and $\{\langle M\rangle_n\}$ be the associated "bracket" process.
(a) Denoting $T_k = \inf\{n \geq 0 : \langle M\rangle_{n+1} > k\}$, show that $\{M_{n\wedge T_k}, n \geq 0\}$ is an L_2-bounded martingale and hence deduce that $\{M_n\}$ converges almost surely on the set $\{\langle M\rangle_\infty < \infty\}$.
(b) Let $f : [0,\infty) \to [0,\infty)$ be any increasing function such that $\int_0^\infty [1 + f(x)]^{-2}\, dx < \infty$. Using part (a) and an appropriate Kronecker Lemma (which should be clearly stated), prove that $M_n = o(f(\langle M\rangle_n))$ almost surely on the set $\{\langle M\rangle_\infty = \infty\}$. (8+8)=**[16]**

INDIAN STATISTICAL INSTITUTE
Semestral Backpaper Examination: 2015-16 (First Semester)

M. MATH. II YEAR
Commutative Algebra I

Date: ~~13/11/2015~~ 15/02/16 Maximum Marks: 100 Duration: 3 Hours

Throughout the paper, R will denote a commutative ring with unity.
Clearly state any result that you use.

1. Prove that any UFD is a normal domain. [8]

2. Let M be an R-module and N, K are R-submodules of M. Prove that $M/(N \cap K)$ is an Artinian R-module if M/N and M/K are Artinian R-modules. [8]

3. Let $B = \mathbb{C}[X, Y, Z]/(XY - Z^2)$.
(i) Examine whether the following elements are prime in B: $x - z$, $x - z - 1$.
(ii) Show that $B \cong \mathbb{C}[U^2, UV, V^2]$. [6+8=14]

4. Let $R \subseteq A$ be commutative rings with A integral over R. Let P be a prime ideal of A and $Q = P \cap R$.
(i) Show that P is a maximal ideal of A if and only if Q is maximal ideal of R.
(ii) Deduce that there does not exist prime ideals $P_1 \subsetneq P_2$ of A such that $P_1 \cap R = P_2 \cap R$. [8+4=12]

5. Prove that if R is Noetherian then $R[[X]]$ is Noetherian. [14]

6. (i) Let I be an ideal of a Noetherian ring such that $I = \sqrt{I}$. Prove that I does not have any embedded prime ideal.
(ii) Let $R = \mathbb{C}[X, Y]/(XY)$ and $x = \bar{X}$.
Compute $Ass_R(R/xR)$ and $Ass_R(R/x^2R)$. [6+8=14]

7. Let $A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 - Z^3)$.
(i) Show that A is a Noetherian domain.
(i) Examine whether A is a UFD or not. [4+12=16]

8. Let V be an affine algebraic set in $\mathbb{C}^n$.
(i) Let g be an element of $\mathbb{C}[V]$ such that $g(x) \neq 0 \; \forall \; x \in V$, considering g as a polynomial function on V. Show that g is a unit in $\mathbb{C}[V]$.
(ii) Suppose that $(\mathbb{C}[V])^* = \mathbb{C}^*$. Show that any $f \in \mathbb{C}[V] \setminus \mathbb{C}$ induces a *surjective* polynomial function $f : V \to \mathbb{C}$.
(iii) Give an example of an affine algebraic set V in $\mathbb{C}^2$ for which $\mathbb{C}^* \subsetneq (\mathbb{C}[V])^*$ and a non-constant polynomial function $f : V \to \mathbb{C}$ which is not surjective. [5+5+4=14]

INDIAN STATISTICAL INSTITUTE
MID-SEMESTRAL EXAMINATION: 2015-2016
M.S. (Q.E.) I, II Years and M. Math. II Year
Game Theory II

Date: 22.02.2016 Maximum Marks: 100 Time: 3 hours

1 (a). Define a market game by giving necessary preliminaries.

(b). How is the underlying market functioning here? Provide precise arguments.

©. Show that the core of a market game is non-empty. (Clearly state any result that you use here.) (2+4+9 15)

2 (a). When do you say that two coalition form games with the same player set are strategically equivalent? Interpret the relation in terms of exchange rate and tax/subsidy criteria.

(b). Consider the coalition form games $u, v \in G^N$, where $v(S) = |S|$ and $u(S) = -|S|$ for all $S \in 2^N$. Are these two games monotonic and strategically equivalent? If they are strategically equivalent, are the exchange rate and tax/subsidy vector unique?

©. How are the cores of two arbitrary strategically equivalent games $u, v \in G^N$ related? (3+5+8=16)

3. Interpret reasonableness and ideal perfectness of a payoff vector associated with a coalition form game by giving necessary definitions. Demonstrate rigorously that a core allocation of the game possesses these characteristics. (2+2+8 12)

4. An industrialist I owns a factory and each member of a set L of laborers owns only his own labor powers. Laborers can produce nothing on their own and members of any non-empty subset S of $s = |S|$ laborers can produce exactly s units of output if they work in the factory. Formulate a coalition form game that models this situation. Is it a constant-sum game? Further, which of the following properties hold for your game: monotonicity, inessentiality, essentiality, superadditivity, additivity, symmetry and duality? Justify your claim in each case. (17)

5. Define the bargaining set of a coalition form game by giving all necessary preliminaries. Interpret it in terms of incentives that a player can receive. Make a systematic comparison between the bargaining set and the stable set of a coalition form game. (8+2=10)

P.T.O

6. Answer the following questions:

a. Of two coalition form games, one of which is essential while the other is inessential, which is preferred from the viewpoint of grand coalition formation, given that the games possess the same player set?

b. Determine the cardinality of the set of imputations of an additive coalition form game. All necessary concepts are required to be defined explicitly.

c. Mention explicitly the merits and demerits of the core as a solution concept of a cooperative game theoretic problem. (3+4+5=12)

7. Provide a sufficient condition for the dominance core of a coalition form game to be its only stable set. Justify your claim rigorously. (You must prove all results that you require in your demonstration.) (10)

8. Identify the necessary and sufficient condition for an imputation to be a core element. (8)

Indian Statistical Institute

Mid-semsetral examination : (2015-16)
M. Math II year

Special Topics (Introduction to Noncommutative Geometry)

Date : 22/2/16 Maximum marks : 40 Duration : 2 hours 30 minutes.

Answer ANY THREE questions. Each question carries 15 marks. The maximum you can score is 40. All the algebras considered are over the field of complex numbers. For an algebra $\mathcal{A}$, we denote its n-fold tensor product by $\mathcal{A}^n$

(1) Let Γ be a finite abelian group and $\mathcal{A} = \mathbb{C}\Gamma$. The dual group $\hat{\Gamma}$ has a canonical action on $\mathcal{A}$ by automorphism and let $\mathcal{B}$ denote the corresponding semi-direct product (or crossed-product) algebra. Compute the Hochschild homology $HH_*(\mathcal{B})$.

(2) With the notation used in class, prove that

$$b'N = Nb.$$

(3) In the notation used in class, prove that $\alpha := (x = x_0, x_1, x_2, ..., x_n)$ is a cycle in $\mathrm{Tot}(CC)_n$, where $x \in \mathcal{A}^{n+1}$ is such that $\overline{x} := x + \mathrm{Im}(1-t)$ is a cycle in C_n^λ and $x_k \in \mathcal{A}^{n+1-k}$ is given by:

$$x_{2k+1} = -hb(x_{2k}), \quad x_{2k+2} = -\frac{1}{n-1} b'(x_{2k+1}).$$

(4) Let G be a group, $\mathcal{A} := \mathbb{C}G$ and let $\chi_g \in \mathcal{A}$ be the function which is 1 at g and 0 elsewhere. Given a function $c : G^{(3)} \equiv G \times G \times G \longrightarrow \mathbb{C}$ satisfying
(i) $c(gg_0, gg_1, gg_2) = c(g_0, g_1, g_2)$ for all $g, g_i \in G$,
(ii) $c(g_1, g_2) - c(g_0, g_2) + c(g_0, g_1) = 0$ for all $g_0, g_1, g_2 \in G$,

define $\phi_c : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathbb{C}$ by $\phi_c(\chi_{g_0} \otimes \chi_{g_1} \otimes \chi_{g_2}) = c(1_G, g_1, g_1 g_2)$ if $g_0 g_1 g_2 = 1_G$, and 0 otherwise. Here 1_G denotes the identity of G. Prove that ϕ_c is a cyclic 2-cocycle.

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination 2015–2016

M.Math (Second year, Second semester)

Topology IV

Maximum Marks: 40 | Date: 23 February, 2016

Duration: 2 hours

Answer all questions.
State clearly any result that you use in your answer.

(1) Let X be orientable. Show that the product orientation on $X \times X$ is independent of the orientation of X. 6

(2) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function and W is a compact 2-dimensional submanifold of $\mathbb{R}^2$ with boundary. If f does not vanish on W, then show that the mod 2 winding number of $f|_{\partial W}$ is 0. 6

(3) $\mathbb{R}P^2$ can not be embedded in $\mathbb{R}^3$ - Justify the statement. 6

(4) Suppose that X is the central circle in the open Mobius band.
   (a) Determine the mod 2 intersection number of X with itself.
   (b) Hence prove that the Mobius band is not orientable. Give a complete answer.

6+8

(5) Suppose that M and N are compact oriented manifolds. Prove that

$$\chi(M \times N) = \chi(M).\chi(N),$$

where $\chi(.)$ denotes the Euler characteristic. 8

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2015-2016, Second Semester
M-Math II and M-Stat II (MSP)
Ergodic Theory

Date: 25/02/16 **Max. Marks 30** **Duration: 2 Hours**

Note: Answer all questions, maximum you can score is 30.
All measures considered here are probabilities.

1. (a) Let T be a periodic transformation on $(X, \mathcal{B}, \mu)$, i.e., for each $x \in X$ $\exists$ an integer $p \geq 1$ such that $T^p x = x$. For a fixed x, let p_x be the smallest such p. Then for any function $f : X \to \mathbf{R}$, show that $\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m x)$ exists $\forall x \in X$. For any fixed x find the limit in terms of p_x. [3]

   (b) Give an example of a periodic transformation on a probability space which is ergodic. [3]

   (c) Let T be invertible measure preserving and ergodic on $(X, \mathcal{B}, \mu)$ and $\mu(\{x\}) > 0$ for some $\{x\} \in \mathcal{B}$. Then show that T is periodic on $\{x\}$ and that μ is uniformly distributed on the orbit of x. [4]

2. (a) Show that a measure preserving transformation T on $(X, \mathcal{B}, \mu)$ is ergodic if and only if whenever $f : X \to \mathbf{R}$ is measurable and $f(Tx) \geq f$ $a.e.$ then f is constant $a.e.$ [5]

   (b) If T is ergodic, is it necessarily true that T^2 is ergodic? Justify your answer by a proof or a counterexample, whichever is appropriate.

   [6]

3. (a) Show that a measure preserving transformation is weak mixing if and only if $T \times S$ is ergodic for any ergodic transformation S. [8]

   (b) Let T be weak mixing on $(X, \mathcal{B}, \mu)$, $f \in L_1(\mu)$ and $\alpha \in K := \{z \in \mathbf{C} : |\mathbf{z}| = 1\}$, such that α is not a root of unity. Then show that $\frac{1}{n}\sum_{m=0}^{n-1} \alpha^m f(T^m x) \to 0$ for almost all $x \in X$. [5]

INDIAN STATISTICAL INSTITUTE

Mid Semester Examination: 2015

Subject Name : **Graph Theory and Combinatorics** (M.Math II yr).
Maximum Score: 40 Date 26 Feb. 2016 Duration: 120 minutes

Note: Attempt all questions. Marks are given in brackets. Total marks is 45. but you can score maximum 40. State the results clearly you use. Use separate page for each question. All graphs are simple and undirected graphs.

Problem 1 (2+4+4 = 10). Prove that a 3-regular graph over odd number of vertices does not exist. On the other hand, construct a 3-regular graph for any even n. What is the maximum number of connected component, a 3-regular graph with n vertices can have? Justify.

Problem 2 (6). Let G be a graph, not necessarily connected, such that every vertex has even degrees. Let W be a maximal trail (a walk whose edges are not repeated) from u to v then show that $v = u$ (without using the Euler's theorem).

Problem 3 (3 + 6 = 9). Define vertex cover of a graph. Show that every minimal vertex cover of a bipartite graph $G[X, Y]$ is a disjoint union $N(S) \cup (X \setminus S)$ for some $S \subseteq X$.

Problem 4 (10). An intersection graph is defined as follows: Let X be a set and V be a set of some subsets of X. We define an intersection graph over V for which (A, B) is an edge if $A \cap B$ is non-empty. $A, B \in V$. Show that any graph is isomorphic to an intersection graph.

Problem 5 (10). Suppose G is a edge-disjoint union three cycles C_1, C_2 and C_3 so that exactly one vertex is common to all three cycles. Let c be a cost function over the set of edges. Suppose we know that there are six edges $e_1, e_2 \in C_1, e_3, e_4 \in C_2$ and $e_5, e_6 \in C_3$ whose costs are equal to 10 and all other edges have cost 1. Let $\mathcal{T}$ be the set of all **minimum spanning trees**. We define the MST graph over $\mathcal{T}$ whose set of edges is defined as follows: (T, T') is an edge if there is an elementary tree transformation δ such that $\delta(T) = T'$. Draw the MST graph with an explanation.

INDIAN STATISTICAL INSTITUTE

M.Stat./M.Math. II Year
Mid-Semester Examination : Semester II : 2015-2016
STOCHASTIC PROCESSES I

Date : 29.02.2016 | **Maximum Score : 40** | Time : 2 Hours

Note : This paper carries questions worth a **total** of **46 marks**. Answer as much as you can. The **maximum** you can score is **40**.

Standard Notation: S= a metric space, $\mathcal{S}$= Borel σ-field on S.

1. (a) Show that, for $k > 1$, a probability distribution function F on $\mathbb{R}^k$ is either continuous everywhere or has an uncountable set of points of discontinuity.

   (b) Let $\mathbf{P}$ be any probability on $(\mathbb{R}, \mathcal{B})$. Show that there is a sequence $\{\mathbf{P}_n\}$ of probabilities, each supported on a finite set of rationals, such that $\mathbf{P}_n \xrightarrow{w} \mathbf{P}$. (6+6)=**[12]**

2. (a) Show that for any probability $\mathbf{P}$ on $(S, \mathcal{S})$, the class consisting of all $\mathbf{P}$-continuity sets in $\mathcal{S}$ form a field.

   (b) For a family $\mathcal{P}$ of probabiltiy measures on $(S, \mathcal{S})$, let $\bar{\mathcal{P}}$ consist of all probabilities obtained as weak limits of sequences in $\mathcal{P}$. Show that if $\mathcal{P}$ is tight, then so is $\bar{\mathcal{P}}$. (6+6)=**[12]**

3. Let S be a non-compact separable metric space, which is locally compact (that is, for every $x \in S$, there is $\epsilon > 0$ such that $B(x, \epsilon)$ has compact closure). Show that if $\mathbf{P}, \mathbf{Q}$ are probability measures on the Borel σ-field $\mathcal{S}$ satisfying $\int f \, d\mathbf{P} = \int f \, d\mathbf{Q}$ for all continuous functions f with compact support, then $\mathbf{P} \equiv \mathbf{Q}$.

   [You may use the fact that any separable locally compact metric space S can be written as $S = \bigcup_n K_n$ where K_n, $n \geq 1$ are compact and satisfy $K_n \subset (K_{n+1})^o$.] **[10]**

4. Let $\{x_n\}$ be a sequence in S. Suppose $\{\mathbf{P}_n\}$ is a sequence of probability measures on $(S, \mathcal{S})$ with $\mathbf{P}_n(\, B(x_n, \epsilon_n)\,) = 1$ for each $n \geq 1$, where $\{\epsilon_n\}$ is a sequence of positive numbers with $\epsilon_n \downarrow 0$. Show that if $\mathbf{P}_n \xrightarrow{w} \mathbf{P}$ for some probability $\mathbf{P}$, then $\{x_n\}$ must converge to some $x \in S$ and $\mathbf{P}$ equals δ_x. (8+4)=**[12]**

Indian Statistical Institute

Mid-semester Examination 2016

M.Math 2nd Year

Course name: **Advanced Number Theory**

Date: **1 March 2016**

Maximum marks: **80**

Duration: **2 hours 30 minutes**

Note: There are 8 questions with total marks 90. Answer as many questions as you like. The maximum you can score is 80. This is an open notes (class notes only) but closed books exam.

1.

(i) Use Euler summation formula to show that

$$\sum_{2\le n\le x} \frac{1}{n\log n} = \log\log x + A + O\left(\frac{1}{x\log x}\right)$$

for some constant A. (5 marks)

(ii) Let $d(n)$ be the divisor function. Prove that for $x \ge 2$ one has

$$\sum_{n\le x} \frac{d(n)}{n} = \frac{(\log x)^2}{2} + 2\gamma\log x + O(1)$$

where γ is the Euler constant. (You may assume Dirichlet's asymptotic formula.) (5 marks)

2. Let p and q be two distinct prime numbers.

(i) Suppose χ (resp. ψ) is a primitive Dirichlet character modulo p (resp. q). Show that the arithmetic function

$$f(n) = \chi(n)\psi(n)$$

is a primitive Dirichlet character modulo pq. (6 marks)

(ii) How many primitive Dirichlet characters are there modulo p^2q? (6 marks)

(iii) Show that

$$\sum_{a=1}^{p} \chi(a) = 0.$$

(4 marks)

3. Let χ be an even primitive character modulo p (which is a prime). Write down (without proof) the functional equation of the Dirichlet L-function $L(s,\chi)$. Determine all the trivial zeros of $L(s,\chi)$. In particular argue why $L(0,\chi) = 0$. (2+5+2=9 marks)

p.T.O

4. Let
$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$
be the completed zeta function. Recall that we have a product expression
$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}.$$

Show that $A = \log\frac{1}{2}$ and $B = -\gamma/2 - 1 + (\log 4\pi)/2$. Using the functional equation compute $\zeta(0)$ and $\zeta(-1)$. (2+5+3+3=13 marks)

5. Using the partial fraction expansion of ζ and the functional equation, show that
$$B = -\sum_{\rho}\frac{1}{\rho}.$$
Using the numerical value $B = -0.023...$ show that $|\gamma| > 6$ for all ρ. (6+5=11 marks)

6. For $1 < \sigma \leq 2$ show that there exists a constant A (independent of σ) such that
$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1} + A.$$
Prove that for $t \geq 2$ one has
$$\zeta(1+it) \ll t.$$
(5+5=10 marks)

7. State the prime number theorem for $\pi(x)$. Derive it from the following asymptotic for von Mangoldt function
$$\sum_{n<x}\Lambda(n) = x + O\left(xe^{-c\sqrt{\log x}}\right)$$
where $c > 0$ is a constant. Find asymptotic for the sum
$$\sum_{\substack{p<x \\ p \text{ prime}}} d(p)$$
where d is the divisor function. (2+4+2=8 marks)

8. Let p be a prime and χ be a primitive Dirichlet character modulo p. Let V be a smooth function supported on $[1,2]$, whose derivatives satisfy $|V^{(j)}(x)| \leq 1$ for all $j > 0$. Using the Poisson summation formula show that
$$\sum_{n\in\mathbb{Z}}\chi(n)V\left(\frac{n}{x}\right) = \frac{x}{p}\sum_{n\in\mathbb{Z}}\left(\sum_{a=1}^{p}\chi(a)e\left(\frac{an}{p}\right)\right)\int V(y)e\left(-\frac{xny}{p}\right)dy.$$
Using integration by parts show that
$$\int V(y)e\left(-\frac{xny}{p}\right)dy \ll \min\left\{\frac{p}{x|n|}, \frac{p^2}{x^2|n|^2}\right\}.$$
and then derive
$$\sum_{n\in\mathbb{Z}}\chi(n)V\left(\frac{n}{x}\right) = O(p^{1/2}\log p).$$
(6+3+4=13 marks)

Indian Statistical Institute
Mid-Semestral Examination: 2015-2016
Programme: Master of Mathematics
Course: **Algebraic Number Theory**

Maximum Marks: 60 02.03.16
Duration: 2 Hours and 30 minutes

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1. (a) Suppose V is a vector space of dimension n over a field F. Suppose $f : V \times V \mapsto F$ is a non-degenerate symmetric bilinear form. Show that given any basis $\{v_1, v_2, \cdots, v_n\}$ of V over F, there is a basis $\{w_1, w_2, \cdots, w_n\}$ of V over F such that

$$f(v_i, w_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases}$$

(b) Let K be a number field of degree n. Let $\mathcal{O}_K$ be the ring of integers in K. Using (a) above or otherwise, show that $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n. You may assume the fact that $Tr_{K/\mathbb{Q}}$ defines a non-degenerate symmetric bilinear form. (5+11=16)

2. Let $K = \mathbb{Q}(\sqrt{-13})$. Prove that $\mathcal{O}_K$ is not a UFD. (10)

3. Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a Dedekind domain. (10)

4. Let K be a number field of degree n. Let $\mathcal{O}_K$ be the ring of integers in K.
(a) Show that a nonzero fractional ideal in $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n.
(b) Suppose $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\{\beta_1, \beta_2, \cdots, \beta_n\}$ are two bases for $\mathcal{O}_K$ over $\mathbb{Z}$. Show that $Disc_{K/\mathbb{Q}}(\alpha_1, \alpha_2, \cdots, \alpha_n) = Disc_{K/\mathbb{Q}}(\beta_1, \beta_2, \cdots, \beta_n)$.
(c) Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathcal{O}_K$. If $Disc_{K/\mathbb{Q}}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a nonzero square-free integer, show that $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is an integral basis for K. Recall that an integer n is called *square-free* if $p^2 \nmid n$ for any prime p. (3+4+9=16)

5. Let K be a number field of degree n. Let $\mathcal{O}_K$ be the ring of integers in K. Suppose $0 \neq \alpha \in \mathcal{O}_K$.
(a) Define norm of an integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$.
(b) Show that the norm of the principal ideal (α) in $\mathcal{O}_K$, where α is a nonzero element in $\mathcal{O}_K$, is same as $|N_{K/\mathbb{Q}}(\alpha)|$.
(c) Suppose $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_K$. Show that the norm of $\mathfrak{p}$ is a power of a prime number. (2+4+4=10)

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2015-16 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date: 03.03.2016 Maximum Marks: 40 Duration: 3 Hours

Attempt ANY 6 questions.
Each question carries 7 marks.

1. Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$ be a graded ring. Show that if R is Noetherian, then R is a finitely generated algebra over R_0.

2. Let R be a ring and I be an ideal in R. Suppose that R is complete with respect to the I-adic topology. Let M be a finite R-module such that $IM = M$. Prove that $M = 0$.

3. Let R be a semilocal ring with maximal ideals $m_1, \ldots, m_n$. Prove that the completion of R with respect to the $I := m_1 \cap \cdots \cap m_n$-adic topology is isomorphic to the direct product $\widehat{R_1} \times \widehat{R_2} \times \cdots \times \widehat{R_n}$, where $\widehat{R_i}$ is the completion of the local ring R_{m_i}, for $1 \leq i \leq n$.

4. Let K be a field. Show that any valuation ring V of K is a normal local domain in which every finitely generated ideal is principal.

5. Prove that any semilocal Dedekind domain is a PID.

6. Let $R \subset A$ be integral domains such that A is a finitely generated R-algebra. Show that there exists $c \in R$ with $c \neq 0$ and elements $y_1, \cdots, y_d \in A$ such that $\{y_1, \cdots, y_d\}$ is algebraically independent over R and A_c is integral over $R[y_1, \cdots, y_d]_c$.

7. Let R be a normal domain. Prove that the associated prime ideals of any nonzero principal ideal of R are of height one.

Indian Statistical Institute

Final Examination 2016

M.Math 2nd Year

Course name: **Advanced Number Theory**

Date: **26 April 2016**

Maximum marks: **60**

Duration: **2 hours 30 minutes**

Note: There are 6 questions with total marks 70. Answer as many questions as you like. The maximum you can score is 60. This is an open notes (class notes only) but closed books exam.

1.

(i) Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Show that the modular group $G = SL(2, \mathbb{Z})/\{\pm I\}$ is generated by S and T. (8 marks)

(ii) For any positive integer ℓ, show that

$$(STS)^{\ell} = ST^{\ell}S.$$

Compute $(STS)^{\ell} i$ (where $i = \sqrt{-1} \in \mathbb{H}$). (2+2=4 marks)

2. For $i = 1, 2$ let f_i be a modular function of weight k_i.

(i) Is

$$g(z) = f_1(z)^{3k_2} f_2(z)^{2k_1} + 2016 f_1(z)^{k_2} f_2(z)^{4k_1}$$

a modular function? Justify your answer. (1+4=7 marks)

(ii) What is the dimension of the space S_{2016} (of cusp forms of weight 2016)? Write an explicit basis for this vector space. (2+3=5 marks)

3. For $k \geq 2$ an integer, let

$$G_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^{2k}}$$

be the Eisenstein series of weight $2k$.

(i) Derive the Fourier expansion of $G_k(z)$ at ∞. (6 marks)

(ii) Let Δ be the Ramanujan delta function. Show that $\Delta(z) \neq 0$ for any $z \in \mathbb{H}$. Show that the map

$$\begin{aligned} M_{2k-12} &\to S_{2k} \\ f &\mapsto \Delta . f \end{aligned}$$

is an isomorphism of vector spaces. (2+4=6 marks)

4. Let $f \in S_{2k}$ be a cusp form of weight $2k$ with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}.$$

(i) Show that $|a(n)| = O(n^k)$. (6 marks)

(ii) For a prime p, let $T(p)$ be the associated Hecke operator. Compute the p^3-th Fourier coefficient of $T(p)f$. (3 marks)

5.

(i) Show that for any two coprime positive integers m and n, one has

$$T(mn) = T(m)T(n).$$

(5 marks)

(ii) Suppose

$$f(z) = 1 + \sum_{n=2}^{\infty} a(n) e^{2\pi i n z}$$

is a cusp form of weight $2k$, which is an eigenfunction of all the Hecke operators $T(n)$. Show that

$$a(mn) = a(m)a(n)$$

for any pair of positive coprime integers m, n. (5 marks)

6. Let $D < 0$ be a discriminant (i.e. D is a negative integer with $D \equiv 0, 1 \bmod 4$).

(i) Write down a positive definite binary quadratic form of discriminant D. Show that there are only finitely many binary quadratic forms of discriminant D modulo the action of $SL(2, \mathbb{Z})$. (2+6=8 marks)

(ii) Define class number $h(D)$. Show that $h(D) = O(D)$. (2+5=7 marks)

INDIAN STATISTICAL INSTITUTE
Semesteral Examination : 2015-16
M. Math. - Second Year
Mathematical Logic

Date : 26. 04. 2016 | Maximum Score : 100 | Time :3 Hours

1. The paper carries 120 marks. You are free to answer all the questions.

2. Notation as used in the class.

3. You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) (a) Prove compactness theorem for first order logic.

(b) Show that there is an elementary extension of the ordered field $\mathbb{R}$ which is non-Archimedean.

[12 + 6]

(2) (a) Show that ACF has quantifier elimination.

(b) Show that RCF has quantifier elimination.

(c) Show that the theories ACF and RCF are model complete.

[14 + 12 + (6 + 6)]

(3) (a) Show that the theories RCF and $ACF(p)$, $p = 0$ or a prime, are complete.

(b) Let φ be a first order sentence in the language of fields. Assume that there is an infinite set I of primes such that for every $p \in I$ there is an algebraically closed field of characteristic p in which φ is true. Show that φ is true in all algebraically closed fields of characteristic 0.

[(6 + 6) + 10]

(4) (a) Show that $\mathbb{R}$ is not a definable subfield of $\mathbb{C}$.

(b) Show that $\mathbb{R}_{alg}$ is not a definable subfield of $\mathbb{R}$.

[6 + 6]

(5) Use model theoretic methods to prove the following.

(a) Let $\mathbb{F}$ be an algebraically closed field and I an ideal in $\mathbb{F}[X_1, \cdots, X_n]$. Then

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

(b) Let $f \in \mathbb{R}(X_1, \cdots, X_n)$ be such that $\neg\exists\bar{a} \in \mathbb{R}^n(f(\bar{a}) < 0)$. Then there exist $f_1, \cdots, f_k \in \mathbb{R}(X_1, \cdots, X_n)$ such that $f = f_1^2 + \cdots + f_k^2$.

[15 + 15]

INDIAN STATISTICAL INSTITUTE
M.Math 2nd year Semester Examination: 2016

Subject Name : **Graph Theory and Combinatorics.**
Maximum Score: 60 Date 27th April, 2016 Duration: 180 minutes

Note: Attempt all questions. Marks are given in brackets. Total marks is 70, but you can score maximum 60. State clearly the results you use. Use separate page for each question. Let $[x] = \{1, 2, \dots, x\}$ for a positive integer x.

Problem 1 (3 + 9 = 12). Define Ramsey number $R(m, n)$ and show (by induction on $m + n$ or otherwise) that

$$R(m, n) \leq R(m, n-1) + R(m-1, n).$$

Problem 2 (3+9 = 12). Define a cut set (for edges) of a graph. Show that the symmetric difference of two distinct cut sets is again a cut-set.

Problem 3 (12). Let $M_{m\times n} = ((m_{i,j}))$ be the incident (binary) matrix of a connected directed graph G with n vertices and m edges (i.e., if k^{th} edge is (v_i, v_j) then $m_{k,i} = 1$, $m_{k,j} = -1$ and $m_{k,r} = 0$ for all other r). Prove that the rank of M is $n - 1$.

Problem 4 (6+10 = 16). State Max-flow Min-cut theorem. You need to define all necessary terms (e.g., flow, capacity of a cut etc.) for stating the theorem. By using it, prove that the number of maximum matching of a bipartite graph is the same as the size of the minimum vertex cover (König theorem).

Problem 5 (5 + (5+8) = 18). Given $f : [N] \to [N]$, define a directed graph $G_f = ([N], E)$ where $E = \{(x, f(x)) : x \in [N]\}$.

1. Show that each connected component of G_f contains exactly one cycle.
2. Let $s_1 < s_2 < \cdots < s_t$ be the vertices from all cycles of G_f (note, t may be 1). We denote $s_i' = f(s_i)$. Let H_f be the graph obtained after removing all edges from all cycles of G_f. Let T_f be the graph after adding the edges $(s_1', s_2'), (s_2', s_3'), \dots, (s_{t-1}', s_t')$ to H_f (if $t = 1$ then we do not add any edge).
   (a) Show that the undirected version of T_f (viewing a directed edge as an undirected edge) is a tree.
   (b) Given any tree T over $[N]$ and two numbers (may be equal) $x_1, x_2 \in [N]$, show that there is a function f such that $(T_f, s_1', s_t') = (T, x_1, x_2)$.

Indian Statistical Institute

Semsetral examination : (2015-16)
M. Math II year

Special Topics (Introduction to Noncommutative Geometry)

Date : 29.4.16 Maximum marks : 60 Duration : 3 hours.

Answer ALL questions. Each question carries 20 marks. The maximum you can score is 60. All the algebras considered are over the field of complex numbers.

(1) Let $(C^\infty(S^1), L^2(S^1), i\frac{d}{dt})$ be the standard (odd) spectral triple on the circle. Prove that this is finitely summable and compute the index pairing between the odd Chern character and the canonical generator $[u]$ of the K_1 group, where $u(z) = z, z \in S^1$. [20]

(2) Let $\mathcal{A}$ be a unital complex algebra and e be an idempotent in $\mathcal{A}$. Let $\mathcal{A}^m$ denote tensor product of m copies of $\mathcal{A}$. For $n \geq 1$, define

$$C_n(e) := (y_n, z_n, y_{n-1}, ..., z_1, y_1) \in \mathrm{Tot}_{2n}(CC(\mathcal{A})) \equiv \mathcal{A}^{2n+1} \oplus \mathcal{A}^{2n} \oplus ... \oplus \mathcal{A},$$

$y_k = \alpha_k e^{\otimes^{2k+1}}$ and $z_k = \beta_k e^{\otimes^{2k}}$, where α_k, β_k are real constants depending only on k.

(a) Prove that we can choose α_k, β_k such that $C_n(e)$ is a cycle in the CC-bicomplex for cyclic homology.

(b) Let $\mathcal{A}$ be a unital C^* algebra. Using (a), construct a well-defined group homomorphism Θ from the cyclic homology $HC_{2n}(\mathcal{A})$ to $K_0(\mathcal{A})$ such that for $\tau \in HC^{2n}(\mathcal{A})$ (cyclic cocycle) and $[P] \in K_0(\mathcal{A})$, $< \Theta([P]), \tau >$ coincides, possibly upto a constant, with the pairing between $[P]$ and τ defined in class.

[10+10=20]

(3) Prove that there is a nontrivial projection (i.e. neither 0 nor 1) in the noncommutative torus $\mathcal{A}_\theta$ for irrational θ.

Hint: Try with $P = f(U)V^{-1} + g(U) + h(U)V$, where f, g, h are suitable continuous functions on S^1. [20]

INDIAN STATISTICAL INSTITUTE

Semestral Examination 2015–2016

M.Math (Second year, Second semester)

Topology IV

Maximum Marks: 60 Date: 2 May, 2016

Duration: 2 hours 30 minutes

Answer all questions.
State clearly any result that you use in your answer.

(1) Let G be a Lie group. Prove or disprove the following statements:
  (a) G is orientable.
  (b) Euler characteristic of G is zero. 5+5

(2) Prove that the tangent space of a Lie group G at the identity e has a non-trivial Lie algebra structure. 10

(3) Show that any map $f : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $f(-x) = f(x)$ for all $x \in \mathbb{S}^n$ has an even degree. 6

(4) (a) Suppose that $f : M \to \mathbb{R}$ is a Morse function on a compact oriented manifold M (without boundary). Prove the following relation:

$$\chi(M) = \sum_{k \geq 0} (-1)^k n_k,$$

where $\chi(M)$ denotes the Euler characteristic of the manifold M and n_k is the number of critical points of f of index k. (Hint: Use the gradient vector field of f on M). 10

(b) Consider the complex projective space $\mathbb{C}P^2$ as the quotient of unit sphere in $\mathbb{C}^3$. Let $f : \mathbb{C}P^2 \to \mathbb{R}$ be a function defined by

$$f([z_0 : z_1 : z_2]) = \sum_{i=0}^{2} c_i |z_i|^2, \quad [z_0 : z_1 : z_2] \in \mathbb{C}P^2.$$

where $c_0 < c_1 < c_2$. Find the critical points of f and show that f is Morse. Use part (a) to find the Euler characteristic of $\mathbb{C}P^2$. 10

P. T. O.

(5) (a) Prove that the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $f(z) = z^q$ has degree q. (Here the notion of degree is as defined in the course) 6

(b) Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an arbitrary smooth map of the unit circle. Prove that there exists a smooth map $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(\cos t, \sin t) = (\cos g(t), \sin g(t)), \quad \text{for all } t \in \mathbb{R}.$$

Show that there is an integer q satisfying

$$g(t + 2\pi) = g(t) + 2q\pi \quad \text{for all } t \in \mathbb{R}.$$

Hence prove that the degree of f is q. 4+5+5

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2015-2016, Second Semester
M-Stat II and M-Math II
Ergodic Theory

Date: 03·04·16 Max. Marks 50 Duration: 3 Hours

Note: 1. Answer all questions.
2. All the measures considered are probability measures
3. Total Marks: 55. Maximum you can score: 50.

1. a) Let X be a Polish space and $\mathcal{B}$ be its Borel σ-field with a probability μ. If $T : X \to X$ be a measure-preserving ergodic transformation with discrete spectrum then show that T is invertible mod 0. [8]

   (b) T is as in (a). Show that T and T^{-1} are conjugates of each other. [5]

2. a) Construct an example of a Markov shift T which is ergodic but not weak- mixing and T^2 is not ergodic. [7]

   b) If T, defined on a probability space $(X, \mathcal{B}, \mu)$ is measure preserving and weak-mixing, then show that $T^{n_1} \times T^{n_2}$ is also weak-mixing for any two positive integers n_1 and n_2. [7]

3. Let $E \subset Z$, Z being the set of integers, such that $d^*(E) > 0$ where $d^*(E) := \limsup \frac{|E \cap \{-n,\ldots,n\}|}{2n+1}$. Construct a probability space $(X, \mathcal{B}, \mu)$ and an invertible measure preserving transformation $T : X \to X$ and a set $A \in \mathcal{B}$ such that $\mu(A) = d^*(E)$ and for any $k (\geq 1)$ integers $m_1, \ldots, m_k$, $\mu(T^{-m_1}A \cap \cdots \cap T^{-m_k}A) \leq d^*(E - m_1 \cap \cdots E - m_k)$, where $E - m := \{n \in Z : n + m \in E\}$. [13]

4. (a) Let $([0,1], \mathcal{B}, \lambda)$ be the probability space where $\mathcal{B}$ is the Borel σ-field and λ Lebesgue measure. Find countable (not finite) measurable partitions ξ, η of $[0,1]$ such that $H(\xi) < \infty$ and $H(\eta) = \infty$. [3]

[P.T.O.]

(b) For any two finite measurable partitions ξ and η of X, let $d(\xi, \eta) = H(\xi|\eta) + H(\eta|\xi)$. Then show that d is a metric on the space of finite measurable partitions. [3]

(c) Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving ergodic dynamical system with finite entropy. Let $\mathcal{A}$ be a finite sub-σ-field of $\mathcal{B}$. Then show that as $n \uparrow \infty$, the probability of an atom of $\vee_{i=0}^{n-1} T^{-i}(\mathcal{A})$ containing a given $x \in X$ decays exponentially to 0 a.s. [9]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2015-16 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date: 05-05-2016 | Maximum Marks: 60 | Duration: $3\frac{1}{2}$ Hours

GROUP A
Attempt ANY FOUR questions.
Each question carries 7 marks.

1. Let k be a field and A be a k-subalgebra of a finitely generated k-algebra B. Prove that $m \cap A$ is a maximal ideal of A for every maximal ideal m of B. Give an example to show that the hypothesis that B is a finitely generated k-algebra is necessary.

2. Let $R = \mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$ and K denote the field of fractions of R. Show that there exist exactly two valuation rings of K containing $\mathbb{C}$ but not containing R.

3. Let $P \subsetneqq Q$ be prime ideals in a Noetherian ring R. Show that if there exists one prime ideal P_1 in R such that $P \subsetneqq P_1 \subsetneqq Q$, then there exist infinitely many prime ideals P_i in R such that $P \subsetneqq P_i \subsetneqq Q$.

4. Let L be a finite extension of $\mathbb{C}(X)$ and let S be the set of all discrete valuations v of L over $\mathbb{C}(X)$ for which $v(X) > 0$. Prove that S is a finite set and $\sum_{v \in S} v(X) = [L : \mathbb{C}(X)]$.

5. Let R be a Noetherian normal domain and P be a prime ideal of R. Prove that $P^{-1} = R$ if and only if height $P > 1$.

6. Let R be a Noetherian local ring such that there exists a principal prime ideal P of R of height one. Prove that R is an integral domain.

7. Let (R, m) be a regular local ring of dim n. Let $x \in m$. Prove that R/xR is a regular local ring of dim $n-1$ if and only if $x \notin m^2$.

GROUP B
Attempt ANY THREE questions.
Each question carries 12 marks.

1. Let k be a field and A be a finitely generated k-algebra which is an integral domain. Let K be the field of fractions of A and L be a finite extension of K. Prove that the integral closure of A in L is a finite A-module.

2. Answer any ONE of the following:
(i) Let (R, m) be a Noetherian ring and M be a finitely generated R-module. Let I be an ideal of definition generated by r elements. Prove that the function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $f(n) = \ell(M/I^n M)$ is a polynomial function of degree at most r.
(ii) State and prove Krull-Akizuki Theorem.

3. Let $R = \mathbb{C}[X, Y]/(Y^2 - X^2 - X^3)$.
(i) Show that R is a Noetherian integral domain of dimension one.
(ii) Describe all maximal ideals m of A for which A_m is not a regular local ring.
(iii) Prove that the normalisation of R is a polynomial ring in one variable (say t) over $\mathbb{C}$ and display an explicit integral equation over R satisfied by t.
(iv) Show that $t \notin R$. [3+2+5+2]

4. Give an example of each of the following with brief justification:
(i) A non-Noetherian valuation ring.
(ii) An integral extension $A \hookrightarrow B$ with an ideal I in B such that $\text{ht}(I \cap A) > \text{ht}(I)$.
[6+6]

5. State whether the following statements are TRUE or FALSE with brief justification. Attempt ANY FOUR.
(i) If $x_1, \cdots, x_n$ is a regular sequence in a regular local ring of dimension n, then $x_1, \cdots, x_n$ is a regular system of parameters.
(ii) The intersection of a totally ordered collection of valuation rings of a field K is a valuation ring of K.
(iii) The ring $k \times k$ is a regular ring when k is a field.
(iv) All invertible ideals of a UFD are principal.
(v) Every radical ideal of a valuation ring is a prime ideal. [3+3+3+3]

Indian Statistical Institute

Semestral Examination: 2015-2016
Programme: Master of Mathematics
Course: **Algebraic Number Theory**

Maximum Marks: 60
Duration: 3 Hours

Date: 06.05.2016

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1. Let A and B be Dedekind domains and suppose $A \subset B$. Suppose $\mathfrak{p}$ is a prime ideal in A and $\mathfrak{P}$ is a prime ideal in B. Show that $\mathfrak{P}|\mathfrak{p}B$ if and only if $\mathfrak{P} \cap A = \mathfrak{p}$. (6)

2. (a) Define the terms ramification index and inertia degree.
(b) Determine with justification whether the prime 37 splits, ramifies, or remains prime in the extension $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$. (5+10=15)

3. Determine with justification the class numbers of the following number fields:
(a) $\mathbb{Q}(\sqrt{-19})$
(b)$\mathbb{Q}(\sqrt{-20})$. (8+9=17)

4. Suppose L/K is a Galois extension of number fields and let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$. (a) Describe the natural action of the Galois group $Gal(L/K)$ on the set of primes $\mathfrak{P}$ lying above $\mathfrak{p}$ and
(b) show that this action is transitive (6+10=16).

5. (a) State Dirichlet's Unit Theorem.
(b) Describe the structure (both the free and the torsion part) of the unit group of the ring of integers of $\mathbb{Q}(\alpha)$ where α is a root of the polynomial $x^3 + 3x^2 - 3$. (2+6=8)

INDIAN STATISTICAL INSTITUTE
Semestral Backpaper Examination: 2015-16 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date: 25/2016/07 Maximum Marks: 100 Duration: 3 Hours

1. (i) Let R be an integral domain and P a prime ideal in $R[X]$ such that $P \cap R = 0$. Show that $R[X]_P$ is a discrete valuation ring.
(ii) Let R be a Noetherian ring with field of fractions K such that $R[1/p] = K$ for some prime element p of R. Show that R is a discrete valuation ring.
(iii) Examine whether $\mathbb{C}[[X]][Y]$ is a discrete valuation ring. [8+4+3]

2. Prove that if the ring B is integral over a ring A, then $\dim B = \dim A$. [7]

3. Let P be a prime ideal of a Noetherian ring of height r. Show that there exist $a_1, \ldots, a_r \in P$ such that $\operatorname{ht}(a_1, \ldots, a_r) = r$. [8]

4. Let $I \subseteq J$ be ideals of a ring R such that J is not contained in any minimal prime ideal of I. Show that $\dim(R/I) \geq 1 + \dim(R/J)$. [8]

5. (i) Prove that height of any non-zero non-unit element of a Noetherian domain is one.
(ii) Prove that prime ideals satisfy the descending chain condition in a Noetherian ring. [9+3]

6. Let k be a field and A be finitely generated k-algebra A which is an integral domain. Prove that all the maximal ideals of A have the same height. Give an example to show that the hypothesis that A is an integral domain is necessary. [8+4]

7. (i) Prove that any non-zero ideal in a Dedekind domain R is a finitely generated projective R-module of rank one.
(ii) Examine whether the ring $A = \mathbb{C}[X,Y]/(X^3 - Y^2)$ is a Dedekind domain. [7+3]

8. Let $A = \mathbb{C}[X, Y, Z]/(XY - Z^2)$ and $P = (x, z)$, the ideal of A generated by x and z, the images of X and Z respectively. Show that
(i) A is a Noetherian integral domain.
(ii) A_P is a discrete valuation ring with $PA_P = zA_P$.
(iii) A is a two-dimensional normal domain.
(iv) A_m is a regular local ring for all but one maximal ideals of A. [4+4+6+4]

9. Let R be a normal domain, K the field of fractions of R, L a finite Galois extension of K with Galois group G and A be the integral closure of R in L. Let P and Q be prime ideals of A such that $P \cap R = Q \cap R$. Prove that there exists $\sigma \in G$ such that $\sigma(P) = Q$. [10]