Fourier Analysis: M. Math II: Mid Semester Examination

August 26, 2013.

Maximum Marks 40 | Maximum Time 2:30 hrs.

Answer all questions.

1. Give <u>short</u> answers (at most 3-4 lines) to these questions:

(a) Consider the function on $\mathbb{R}$: $\psi(x) = e^{-\frac{2}{1-|x|^2}}$ for $|x| < 1$ and $f(x) = 0$ otherwise. Let $\psi_\delta(x) = \psi(\delta x)$. For a C^∞-function f define $g(x) = f(x)\,\psi_\delta(x)$. Suppose $\widehat{g}$ is supported on a set of finite measure. What can you say about f?

(b) Let F be a radial tempered distribution on $\mathbb{R}^d$, i.e. for any $A \in \mathrm{SO}(n)$, $AF = F$. Show that $\widehat{F}$ is also radial.

(c) Assume that for any $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $\|f * g\|_{r,\infty} \leq C\|f\|_p\|g\|_{q,\infty}$ for all Schwartz class functions f, g. Using this prove that $\|f * g\|_r \leq C\|f\|_p\|g\|_{q,\infty}$ for p, q, r and f, g as above.

(d) Let f be a homogeneous function of degree k on $\mathbb{R}^d$ for some $k \in \mathbb{R}$. Can such an f be an L^p-function? If it is an L^p function, then find p in terms of k and d.

4×5

2. (a) Suppose that for some $0 < \alpha, \beta, \gamma < 1$,

$$|A + B|^\alpha \geq C|A|^\beta|B|^\gamma$$

for any two measurable subsets A, B of finite measures of $\mathbb{R}^d$. Then show that $\alpha = \beta + \gamma$.

(b) Show that for α, β, A, B as above it is actually true that

$$|A + B|^{\beta+\gamma} \geq C|A|^\beta|B|^\gamma.$$

(Hint for (b): First prove and then use that $\chi_A \leq |B|^{-1}\chi_{A+B} * \chi_{-B}$.) $6 + 6$

<u>P. T. O.</u>

3. Assume the inequality $\|f * g\|_r \le C\|f\|_p\|g\|_q$ for Schwartz class functions f, g. Use this to show that $\|f * g\|_{q'} \le C\|f\|_p\|g\|_{r'}$ for Schwartz class functions f, g. Do not use Young's inequality. 6

4. Consider the following PDE on $\mathbb{R}^d \times \mathbb{R}^+$ $(x \in \mathbb{R}^d, t > 0)$:

$$\frac{\partial^2}{\partial t^2} u(x,t) = \Delta_x u(x,t) \text{ where } \Delta_x = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

with initial condition

$$u(x,0) = 0, \frac{\partial}{\partial t} u(x,0) = f(x), \text{ for some } f \in L^2(\mathbb{R}^d).$$

Find its solution $u(x,t)$ and show that for any fixed $t_0 > 0$,

$$\|u(x,t_0)\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}.$$

6

INDIAN STATISTICAL INSTITUTE

First Class Test Examination: 2013

28·08·13

Subject Name : **Basic Probability Theory**

Course Name : M.Math II yr. Maximum Score: 50 Duration: 180 minutes

Note: Attempt all questions. Marks are given in brackets. Total marks is 55, but you can score maximum 50. State the results clearly you use. Use separate page for each question.

Problem 1 (6+4 = 10). Suppose $P(X = x, Y = y) = \frac{c}{xy}$ for integers $1 \leq x, y \leq n$ and $c \in \mathbb{R}$. Express the probability mass function (p.m.f.) of X without using c. Are X and Y independent? Justify.

Problem 2 (8). Let $X_1, \ldots, X_k \stackrel{iid}{\sim} unif(\{0, 1, \ldots, n\})$ and let $Y = X_1 + \ldots + X_k$. Prove that for all $0 \leq i \leq nk$, $P[Y = i] = P[Y = nk - i]$.

Problem 3 (5+5 = 10). Consider the following random experiment for a set $S = \{x_1, \ldots, x_n\}$: For each $1 \leq i \leq n$, Alice and Bob independently picks x_i "at random" with probability p and q respectively. That is, Alice would choose x_1 with probability p and then choose x_2 with probability p and so on. Similarly Bob will choose x_1, x_2 and so on each with probability q and all these choices for both are independent.

(a) Write down the sample space and corresponding p.m.f. of the combined experiment.

(b) Compute the probability that there is no common element picked by both.

Problem 4 ((4+5)+8 =17).

(a) Let n be a fixed positive integer and we choose X_1 uniformly from $[1..n]$. For $i \geq 2$ and for any $x \leq x_{i-2} \leq \cdots x_1 \leq n$, we choose integers X_i's uniformly from $\{1, 2, \ldots, x\}$ given that $X_1 = x_1, \ldots, X_{i-1} = x_{i-1} = x$. Express the given condition by conditional probabilities and hence find joint p.m.f. of $(X_1, \ldots, X_i)$ for any fixed i.

(b) Suppose we continue the process until we observe 1. Let N denote the number of X-values until we observe 1 (e.g., if $X_1 = 4, X_2 = 2, X_3 = 1$ then $N = 3$). If we denote $P[N = k] = p_{n,k}$ then show that

$$p_{n,k} = \frac{1}{n} \sum_{i=1}^{n} p_{i,k-1}.$$

Problem 5 (10). Let X and Y be independent random variables, each having the geometric distribution with parameter p. Find $E(Y|X + Y = n)$, $n \geq 2$.

Mid-Semestral Examination
August 2013
M.Math. Second Year, First Semester
Differential Topology

30.08.'13

Full Marks : 40 Time : 2 hours

Answer all the four questions.
Answers should be complete as far as practicable.
Figures at the end of each question indicates marks allotted to the question.

1 (a) Show that the Grassmann manifold $G_k(\mathbb{R}^n)$ of all k-dimensional subspaces of $\mathbb{R}^n$ is a smooth manifold of dimension $k(n-k)$.

(b) Show that there is a natural diffeomorphism between $G_k(\mathbb{R}^n)$ and $G_{n-k}(\mathbb{R}^n)$. [6 + 4 = 10]

2. (a) If M and N are smooth manifolds of dimensions n and m respectively, and if $f : M \to N$ is a smooth map which is an immersion at a point $p \in M$, then there is a local representation of f at p which is the canonical embedding

(b) Show that there is no submersion of a compact manifold into an Euclidean space. [6 + 4 = 10]

3. (a) Show that if K is a compact set in $\mathbb{R}^n$ such that $K \cap (t \times \mathbb{R}^{n-1})$ has measure zero in the hyperplane $\mathbb{R}^{n-1}$, then K has measure zero in $\mathbb{R}^n$.

(b) Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}^m$ be a smooth map. Let C be the set of critical points of f in U, and D be the set of points in C where the Jacobian matrix $J(f)$ vanishes. Show that the set $f(C - D)$ has measure zero in $\mathbb{R}^m$. [5 + 5 = 10]

4. (a) Let M and N be smooth manifolds, Z a submanifold of N, and $f : M \to N$ a smooth map, where none of M, N and Z has boundary. Then, if

$$f \pitchfork Z \quad \text{(that is, } f \text{ if transversal to } Z\text{)},$$

then show that $f^{-1}(Z)$ is a submanifold of M, and the codimension of $f^{-1}(Z)$ in M equals the codimension of Z in N.

(b) Use the map $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x_1, x_2, \dots, x_n) = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$$

to show that the set of all unit vectors in $\mathbb{R}^n$ is a smooth manifold of dimension $n-1$. [6 + 4 = 10]

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2013-14
M. Math. - Second Year
Mathematical Logic

Date : O4. 09. 2013 | Maximum Score : 100 | Time :3 Hours

You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) Let M be a structure of a first order language L and N a substructure of M. Assume that for every formula $\varphi[x, x_0, \cdots, x_{n-1}]$ and for every $a_0, \cdots, a_{n-1} \in N$, whenever there is a $b \in M$ such that $M \models \varphi[b, a_0, \cdots, a_{n-1}]$, there is a $c \in N$ such that $M \models \varphi[c, a_0, \cdots, a_{n-1}]$. Show that N is an elementary substructure on M.

[20]

(2) Show that the class of all order-dense, linearly ordered sets satisfying the least upper bound axiom is not elementary.

[15]

(3) Show that $\mathbb{R}$ is not a definable subset of the ring $\mathbb{C}$.

[15]

(4) Let M and N be two countable, homogeneous structures of a first order language L such that for every $k \geq 1$,

$$\{tp^M(\overline{a}) : \overline{a} \in M^k\} = \{tp^N(\overline{b}) : \overline{b} \in N^k\}.$$

Show that M and N are isomorphic.

[20]

(5) Let $\mathcal{F}$ be a set of formulas of the language of a propositional logic which is finitely satisfiable. Show that $\mathcal{F}$ is satisfiable.

[20]

(6) Show that $ACF(p)$, p a prime, is not finitely axiomatizable.

[15]

(7) Show that the class of all well-ordered sets if not elementary.

[10]

Indian Statistical Institute
Mid-Semestral Examination: 2013-2014
Programme: Master of Mathematics
Course: **Number Theory**

Maximum Marks: 60 DATE – 04.09.13
Duration: Three Hours

Instruction: Answer all questions. You may use any standard result or results proved in class after stating it clearly. Write short, precise answers with adequate explanations.

---

1. Is the converse of the Wilson's theorem true? Explain. (4 points)

2. Suppose a, b, c, d are positive integers such that $(a, b) = (c, d) = 1$. Show that if $\frac{a}{b} + \frac{c}{d}$ is an integer then $b = d$. (6 points)

3. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Determine with proof the value of $(3a + 5b, 3c + 5d)$. (6 points)

3. Let $n \geq 5$ be an integer. What is the smallest integer N such that for any odd positive integer a, $a^N \equiv 1 \pmod{2^n}$? Prove your assertion. (4 points)

4. (a) Among the integers $1, 2, 3, \cdots, 16$, how many are primitive roots for 17?
(b) Suppose $n > 1$ is an integer and $p = 4n + 3$ and $q = 2n + 1$ are both primes. Show that $p - 3$ is a primitive root for p. (1+11=12 points)

5. Show that there is no solutions to the equation $x^2 + xy + y^2 = x^2y^2$ over the set of positive integers. (10 points)

6. Suppose $A > 0$ is a positive integer. Show that

$$\sum_{\substack{1 \leq n \leq X \\ (n,A)=1}} \frac{1}{n} \sim \frac{\phi(A)}{A} \log X$$

as $X \longrightarrow \infty$. (12 points)

7. Suppose $\omega(n)$ denotes the number of distinct prime factors of n for positive integers n. Show that

$$\sum_{1 \leq n \leq X} \omega^2(n) = O\left(X(\log\log X)^2\right).$$

(11 points)

Fourier Analysis: M. Math II: Semester Examination

November 11, 2013.

Maximum Marks 60 Maximum Time 2:30 hrs.

Answer all questions.

(1) (a) Find the Hilbert transform of $g = \chi_{(0,1)} - \chi_{(-1,0)}$.

(b) Let $f \in L^p(\mathbb{R}^d)$ for $1 \leq p \leq 2$ and $\Delta f = 0$. Prove that $f = 0$.

(c) Let $f \in L^1(\mathbb{R})$ and $f(x) = 0$ for $x > 0$. Show that $\|\widehat{f}(z)\|_\infty \leq \|f\|_1$ for $\Im z > 0$.

7+5+3=15

(2) (a) Define $\phi_r = \frac{1}{2r}\chi_{I(0,r)}$ where $I(0,r) = (-r,r)$. Show that ϕ_r as $r \to 0$ defines an approximate identity.

(b) Suppose that a function $f \in L^\infty(\mathbb{R})$ which is compactly supported has a jump discontinuity at x_0 such that both the limits $f(x_0+) = \lim_{x\to x_0+} f(x)$ and $f(x_0-) = \lim_{x\to x_0-} f(x)$ are finite. Prove that

$$\lim_{r\to 0}\frac{1}{2r}\int_{I(x_0,r)} f(y)dy = \frac{1}{2}(f(x_0+) + f(x_0-)),$$

where $I(x_0,r) = (x_0 - r, x_0 + r)$. 5+5

(3) Recall that radial part of a locally integrable function f on $\mathbb{R}^d$ is a radial function f_0 defined as

$$f_0(|x|) = \int_{\mathbb{S}^{d-1}} f(|x|\omega)d\omega.$$

Show that if for a function $f \in L^1(\mathbb{R}^d)$, its translate $\ell_x f$ has no radial part for all $x \in \mathbb{R}^d$, then $f \equiv 0$. 10

P.T.O.

(4) Let $\alpha > 0$ be fixed. Take a point $x_0 \in \mathbb{R}^d$. Let $\{E_n\}$ be a sequence of Borel sets of $\mathbb{R}^d$ such that for a sequence of balls $B(x_0, r_n)$ with $r_n \to 0$ as $n \to \infty$, $E_n \subset B(x_0, r_n)$ and $|E_n| \geq \alpha |B(x_0, r_n)|$. Show that

$$\sup_{E_n} \frac{1}{|E_n|} \int_{E_n} |f(y)| dy \leq Mf(x_0)$$

where Mf is the Hardy-Littlewood maximal function.

Using this and the Hardy-Littlewood maximal inequality show that in $\mathbb{R}^2$, maximal function defined by centered ellipses (instead of balls) with their axes proportional to $1 : 10^9$ satisfies the weak $1-1$ property. 5+5

(5) Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Show that there exists a measurable set $E \subset \mathbb{R}^d$ such that $|E| < \frac{1}{\lambda}$ and

$$\left(\int_{\mathbb{R}^d \setminus E} |f(x)|^2 dx \right)^{1/2} \leq \lambda \|f\|_1.$$

10

(6) For $f \in L^1(\mathbb{R})$ and $R > 0$ let

$$S_R f(x) = \int_{-R}^{R} \widehat{f}(\xi) e^{-i\xi x} d\xi.$$

Write S_R in terms of Hilbert transform and prove that for any fixed $\alpha > 0$,

$$\lim_{R \to \infty} |\{x \mid |S_R f(x) - f(x)| > \alpha\}| = 0.$$

10

INDIAN STATISTICAL INSTITUTE

End Semestral Examination: 2013

'2·11·2

Subject Name : **Basic Probability Theory**
Course Name : M.Math II yr.   Maximum Score: 50   Duration: 180 minutes

Note: Attempt Q6 and Q7 and any four from Q1 to Q5. Marks are given in brackets. Total score is 56 and maximum you can score 50. State the results clearly you use. Use separate page for each question. In Q3 and Q7, let $X_1, X_2, \ldots$ be an infinite sequence of independent and identically distributed random variables with mean μ and variance σ^2.

Problem 1 (5+5 =10). Suppose we choose a random graph on n vertices by choosing every edge independently with probability $0 < p < 1$. For every pair $(i, j) \in S := \{(a, b) : 1 \leq a < b \leq n\}$, we define $Y_{i,j} = 1$ if there is an edge between i and j, $i < j$. Let D_v denote the degree of the vertex. (1) Show that for any two vertices $v \neq v'$, D_v and $D_{v'}$ are independent. (2) Given two vertices $v_1 \neq v_2$, what is the expected number of vertices v which are neighbors to both v_1 and v_2?

Problem 2 (10). Prove that $\frac{1}{2}\sum_{s\in S} |\mu_1(s) - \mu_2(s)| = \max_{E\subseteq S}(\mu_1(E) - \mu_2(E))$ where μ_1, μ_2 are two p.m.f. on S.

Problem 3 (5 + 5 =10).

(i) Let $Y \geq 0$ be a continuous r.v. Show that $E(Y^2) = \int_0^\infty 2yPr[Y \geq y]dy$.

(ii) Let $Y_k = X_k 1_{X_k \leq k}$ for all $k \geq 1$. Using (i) prove that $\sum_{k=1}^\infty Var(Y_k)/k^2 \leq 4\mu$. You may use the fact that $\sum_{k>y} k^{-2} \leq 4$.

Problem 4 (10). Let $Pr[X \neq Y] = 1$. We define $r(X, Y) = 0$ if $X < Y$ otherwise 1. Prove that if $r(X, Y)$ is individually independent with X and Y then $r(X, Y)$ is constant with probability one.

Problem 5 (10). Prove that $N(0, 1)$ has maximum differential entropy among all continuous probability density function with variance 1.

Problem 6 (8). We choose a number repeatedly at random with replacement from $\{1, 2, \ldots, 1000\}$ until the number is divisible by 4. It is known that the number of trials T follows $geo(1/4)$. Let T' be the number of trials required to stop until the sum of the numbers is divisible by 4. Does T' still follow geometric distribution? Justify.

Problem 7 (8). Let $N \in_R \{1, 2, \ldots, \}$ be independent with $X_1, X_2, \ldots$. We define $S = X_1 + \ldots + X_N$. Prove that $Var(S) = E(N)\sigma^2 + Var(N)\mu^2$.

Semestral Examination
November 19, 2013
M.Math. Second Year, Third Semester
Differential Topology

Full Marks : 60 Time : 3 hours

19·11·2013

Answer any three of the five questions.
Answers should be complete as far as practicable.
Marks allotted for each of the questions is twenty.

1 (a) Let M and N be smooth manifolds without boundary, and M be compact. Show that for any embedding $f : M \to N$ and any smooth homotopy $f_t : M \to N$ of f, there is an $\epsilon > 0$ such that f_t is an embedding for all $t < \epsilon$.

(b) Let X and Y be locally compact Hausdorff spaces, and X^+ and Y^+ denote their one-point compactifications. Show that a map $f : X \longrightarrow Y$ is proper if and only if its extension $f^+ : X^+ \longrightarrow Y^+$ is a continuous map.

(c) Show that a proper injective immersion from a manifold M into a manifold N is an embedding.

2. (a) Let $f : M \longrightarrow N$ be a smooth map between manifolds, and Z be a submanifold of N of codimension k. Let $p \in f^{-1}(Z)$. Show that

(i) there is an open neighbourhood U of $f(p)$ in N, and a submersion

$$g : U \longrightarrow \mathbb{R}^k$$

such that $g^{-1}(0) = Z \cap U$.

(ii) The map f is transverse to Z at p if and only if p is a regular point of $g \circ f$.

(b) Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}^m$ be a smooth map. Let C be the set of critical points of f in U, and D be the set of points in C where the Jacobian matrix $J(f)$ vanishes. Show that the set $f(C - D)$ has measure zero in $\mathbb{R}^m$.

3. (a) Let $E \longrightarrow M$ be a vector bundle over a manifold M, and $A \subset M$ be a submanifold and a closed subset of M. Show that any smooth section of the bundle $E|A$ can be extended to a smooth section of E.

(b) If $f_t : M \longrightarrow N$ is a smooth homotopy between smooth manifolds, and H is a vector bundle over N, then the pull-backs f_0^*H and f_1^*H are isomorphic.

4. (a) If K is a compact subset of a smooth manifold M, then any smooth function $f : K \longrightarrow \mathbb{R}$ can be extended to a smooth function $F : M \longrightarrow \mathbb{R}$.

(b) Show that any compact manifold can be embedded in an Euclidean space.

5. (a) Show that a manifold is orientable if and only if it admits a nowhere vanishing differentiable n-form.

(b) State and prove the Stokes' theorem for integration of differential forms on a compact oriented manifold with boundary.

Indian Statistical Institute
Semestral Examination: 2013-2014
Programme: Master of Mathematics
Course: **Number Theory**

Maximum Marks: 60
Duration: Three Hours

Instruction: Answer all questions. Write precise answers. Quote any standard result that you may need to use.

---

1. Suppose G is a group of order n. Suppose for all divisors d of n,

$$\#\{g \in G : g^d = 1\} \leq d.$$

Show that G is a cyclic group. (8 points)

2. Suppose q_1 and q_2 are two relatively prime positive integers. For a positive integer n, let $U(n)$ denote the group of residues modulo n that are prime to n.
(a) Show that $U(q_1 q_2)$ and $U(q_1) \times U(q_2)$ are isomorphic as groups by describing an explicit isomorphism. Prove that the map you have described is an isomorphism.
(b) Show that the arithmetic function defined by $q \mapsto c_q(n)$ is multiplicative for any fixed positive integer n. Here, $c_q(n)$ is defined by

$$c_q(n) = \sum_{\substack{1 \leq a \leq q-1 \\ (a,q)=1}} e^{2\pi i \frac{na}{q}}.$$

(5+5=10 points)

3. Suppose for a positive integer n, $d(n)$ denotes the number of divisors of n. Show that for any $\varepsilon > 0$, there is a positive real number $C(\varepsilon)$ such that

$$d(n) \leq C(\varepsilon) n^{\varepsilon}$$

for every positive integer n. (6 points)

4. Suppose for a positive integer n, $\sigma(n)$ is defined as

$$\sigma(n) = \sum_{d|n} d.$$

Show that

$$\sum_{1 \leq n \leq X} \sigma(n) = \frac{\pi^2}{12} X^2 + O(X \log X).$$

(12 points)

5. Suppose p is an odd prime and a is an integer such that $(a, p) = 1$. Show that the congruence $ax \equiv y (\bmod p)$ has a solution (x_0, y_0) such that

$$0 < |x_0| < \sqrt{p}, \quad 0 < |y_0| < \sqrt{p}.$$

(8 points)

6. (a) Define Jacobi symbols.
(b) State the Quadratic Reciprocity law for Jacobi symbols.
(c) Suppose $p = 4n + 1$ is a prime where n is an odd positive integer. Show that n is a quadratic residue modulo p. (2+3+3=8 points)

7. Suppose $\chi (\bmod q)$ is a primitive Dirichlet character that assumes nonreal values. Give a complete proof of the fact that the Dirichlet L-function $L(s, \chi)$ can not vanish at $s = 1$. (12 points)

INDIAN STATISTICAL INSTITUTE
Semesteral Examination : 2012-13
M. Math. - Second Year
Mathematical Logic

Date : 22.11. 2013 | Maximum Score : 100 | Time :3 Hours

1. Answer all the questions.

2. You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) Let $\mathcal{L}$ be the language of a propositional logic and T a set of $\mathcal{L}$-formulas that is finitely satisfiable. Without using Post tautology theorem, show that T is satisfiable. [20]

(2) Let $\mathcal{L}$ be a first-order language having only a binary predicate symbol $<$ as a non-logical symbol. Is the class of all well-ordered sets elementary? Justify your answer. [10]

(3) Describe all the definable subsets of the field $\mathbb{R}$ of real numbers. [20]

(4) Show that the theory of torsion-free, divisible abelean groups has algebraically prime models. [15]

(5) Show that every consistent theory has a simple complete extension. [15]

(6) Show that there exist algebraically closed fields of arbitrarily large cardinalities. [15]

(7) Show that the theory of real closed fields admits elimination of quantifiers. [20]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2013-14 (First Semester)

M. MATH. II YEAR
Commutative Algebra I

Date: 22.11.2013 | Maximum Marks: 70 | Duration: 4 Hours

Note: Attempt 5 questions from Group A and 4 from Group B.
R will denote a commutative ring with unity.

GROUP A

Attempt ANY FIVE questions.
Each question carries 12 marks.

1. (i) Let $I, P_1, \ldots, P_n$ be ideals of R such that
(a) P_i is a prime ideal for every $i \geq 3$.
(b) $I \subseteq P_1 \cup \ldots \cup P_n$.
Prove that $I \subseteq P_i$ for some i, $1 \leq i \leq n$.

   (ii) Give an example of a ring R and ideals I, I_1, I_2, I_3 of R such that $I \subseteq I_1 \cup I_2 \cup I_3$ but $I \not\subseteq I_j$ for any j, $1 \leq j \leq 3$. [8+4=12]

2. Let k be a field, A a finitely generated k-algebra, G a finite group of automorphisms of A and $R = \{x \in A \mid \sigma(x) = x \; \forall \; \sigma \in G\}$. Prove that:

   (i) A is integral over R.

   (ii) R is a Noetherian ring. [4+8=12]

3. Let $A = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 - Z^3)$.

   (i) Show that A is a Noetherian integral domain.

   (ii) Examine whether A is a UFD. (Clearly state the results that you use.) [4+8=12]

4. (i) Let I be an ideal of an integral domain R and $x \in R$. Prove that if both $I + (x)$ and $(I : x)$ are principal ideals of R, then I is a principal ideal of R.

   (ii) Let M be an R-module and P an ideal of R that is maximal among all annihilators of non-zero elements of M. Show that P is a prime ideal of R. [7+5=12]

5. (i) Let k be a field. Prove that any maximal ideal of $k[X_1, \ldots, X_n]$ is generated by n elements. (Clearly state any result that you use.)

   (ii) Let $R \subset A$ be integral domains such that A is integral over R. If $\phi : A \to B$ is a ring homomorphism into an integral domain B such that $\phi|_R$ is injective, then show that ϕ is injective. [7+5=12]

[P.T.O.]

6. (i) Prove that the tensor product of two Noetherian R-modules is a Noetherian R-module.

   (ii) Show that if there exists a faithful Noetherian module over a ring R, then R must be a Noetherian ring. [6+6=12]

7. (i) Let M be an R-module. Show that M is flat over R if and only if M_P is flat over R_P for every maximal ideal P of R.

   (ii) Let B be a subring of an integral domain A. Suppose that there exist $f, g \in B$ such that
   (a) $(f, g)B = B$,
   (b) $B[1/f] = A[1/f]$,
   (c) $B[1/g] = A[1/g]$.
   Prove that $B = A$. [6+6=12]

8. (i) Let I be a finitely generated ideal of R such that I/I^2 is generated by r elements as an R/I-module. Prove that I is generated by $r + 1$ elements.

   (ii) Give an example of an ideal I of a Noetherian domain R such that I/I^2 is a cyclic R/I-module but I is not a principal ideal of R. (You need not give detailed proofs for this part.) [8+4=12]

GROUP B

State whether the following statements are TRUE or FALSE with brief justifications. Attempt ANY FOUR. Each question carries 4 marks.

1. $\mathbb{R}[U, V]/(UV + V^2 + 1)$ is a PID.

2. Any finitely generated projective module is finitely presented.

3. If R_P is an integral domain for every prime ideal P of R, then R is necessarily an integral domain.

4. If $\{I_n \mid n \geq 1\}$ is a family of ideals of R and S a multiplicatively closed subset of R then $S^{-1}(\bigcap_n I_n) = \bigcap_n S^{-1}(I_n)$.

5. If there exists a Noetherian ring A satisfying $R \subset A \subset R[X]$ then R must be a Noetherian ring.

6. If R is an integral domain which is not a field then $R[X]/(f(X))$ cannot be a field.

[$4 \times 4 = 16$]

Note for first year JRF Mathematics research scholars:
Submit the unanswered questions by 25.11.2013.

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2013-14 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date: 24.2.2014 | Maximum Marks: 30 | Duration: $2\frac{1}{2}$ Hours

Note: Clearly state the results that you use.

1. Let $A = \mathbb{R}[X,Y,Z]/(XY - Z^2)$; $x = \overline{X}$, $z = \overline{Z}$ and $P = (x,z)A$.

   (i) Show that A is a Noetherian integral domain.

   (ii) Show that A_P is a discrete valuation ring with $PA_P = zA_P$.

   (iii) Describe an irredundant primary decomposition of P^2, mention the associated prime ideals and identify the isolated and embedded components.

   (iv) Show that the symbolic power $P^{(2)}(:= P^2A_P \cap A) = xA$. [1+3+7+4=15]

2. Let R be a local Dedekind domain. Prove that the maximal ideal of R is principal. [8]

3. Prove ANY THREE of the following statements.

   (i) If M is a finitely generated R-module such that $M_P = 0$ for every $P \in \operatorname{Ass}_R M$ then $M = 0$.

   (ii) Any radical ideal of a valuation ring is a prime ideal.

   (iii) If R is an integral domain and P a nonzero prime ideal in $R[X]$ such that $P \cap R = (0)$, then $R[X]_P$ is a DVR.

   (iv) Any nonzero ideal of any Dedekind domain R is a finitely presented projective module of rank one. [$4 \times 3 = 12$]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2013-2014, Second Semester
M-Stat II (MSP) and M-Math II
Ergodic Theory

Date: 24.02.14 Max. Marks 30 Duration: $2\frac{1}{2}$ Hours

Note: Answer all questions, maximum you can score is 30.

1. Let $(X, \mathcal{B}, \mu, \mathcal{T})$ be a measure preserving dynamical system and suppose $\mu(X) < \infty$. Prove that T is ergodic if and only if for all measurable maps $f : X \to \mathbf{R}$, $f(Tx) \geq f(x)$ a.e. implies f is constant a.e. [10]

2. Let $(X, \mathcal{B}, \mu, \mathcal{T})$ be an ergodic dynamical system with μ a probability measure. Let ν be another finite measure (but not probability) on $\mathcal{B}$ such that $\nu << \mu$ and $\nu T^{-1} \leq \nu$. Show that ν is T-invariant and ν is a constant multiple of μ. [8]

3. Let $(X, \mathcal{B}, \mu, \mathcal{T})$ be a measure preserving dynamical system where μ is a probability. In this set-up, prove that Poincaré's Recurrence Theorem is a consequence of Birkoff's Ergodic Theorem. [10]

4. State and prove Rokhlin's Lemma for n=3. [8]

INDIAN STATISTICAL INSTITUTE, KOLKATA
MIDTERM EXAMINATION: SECOND SEMESTER 2013 -'14
M. MATH / M.STAT II YEAR

Date - 25.02.1

Subject	:	**Advanced Functional Analysis**
Time	:	2 hours
Maximum score	:	40

Attempt all the problems. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answerscript. Points will be deducted for missing or incomplete arguments.

(1) Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space. Let $\mathcal{A} = L^\infty(\Omega, \mathscr{F}, \mu)$ and $\mathcal{H} = L^2(\Omega, \mathscr{F}, \mu)$. For $f \in \mathcal{A}$, define the essential spectrum of f, denoted by $\sigma_{ess}(f)$ to be

$$\{\lambda \in \mathbb{C} : \mu\{x : |f(x) - \lambda| < \epsilon\} > 0 \text{ for all } \epsilon > 0\}$$

Define $M : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by $f \mapsto M_f$ where $M_f(g) = fg$ for $f \in \mathcal{A}, g \in \mathcal{H}$.

a) Show that M is an isometric isomorphism

b) Show that $\sigma(M_f) = \sigma_{ess}(f)$

[5+5=10 marks]

(2) Let $\mathcal{A} = C^1[0,1]$ be equipped with the norm $\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$. Prove that $\mathcal{A}$ is a commutative Banach algebra. Show that the maximal ideal space of $\mathcal{A}$ is homeomorphic to $[0,1]$. The Gelfand transform for $\mathcal{A}$ is neither isometric nor onto.

[5+3+2=10 marks]

(3) Suppose $\mathcal{A}$ is a Banach algebra and $\{x_n\}$ is a sequence of invertible elements in $\mathcal{A}$ which converges to a non-invertible element x. Then, $\|{x_n}^{-1}\| \to +\infty$

[7 marks]

(4) Suppose $\mathcal{A}$ is a C^*-algebra and $x, y \in \mathcal{A}$ are self-adjoint elements. Say $x \leq y$ if $(y-x)$ is a positive element. Prove that

i) $x \leq y \implies zxz^* \leq zyz^* \ \forall\, z \in \mathcal{A}$

ii) $0 \leq x \leq y \implies \|x\| \leq \|y\|$

iii) $x \geq 0$ and invertible $\Leftrightarrow \exists\, \lambda > 0$ such that $x \geq \lambda 1$.

iv) $0 \leq x \leq y$ with x invertible $\implies y$ is invertible and $0 \leq y^{-1} \leq x^{-1}$.

[3+3+2+5=13 marks]

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2013-14 (Fourth Semester)

M. MATH. II YEAR
Algebraic Geometry

Date: 21.2.2014 Maximum Marks: 40 Duration: $2\frac{1}{2}$ Hours

Each question carries 6 marks. Answer any SEVEN questions. Throughout the paper, k will denote a field.

1. Prove that if $R \subset A$ are integral domains such that A is finitely generated as an R-algebra, then there exists $f(\neq 0) \in R$ and elements $y_1, \ldots, y_d \in A$ such that $\{y_1, \ldots, y_d\}$ is algebraically independent over R and A_f is integral over $R_f[y_1, \ldots, y_d]$.

2. Let k be a field and m be a maximal ideal of $k[X_1, \ldots, X_n]$. Show that there exists a finite field extension $L|k$ and a point $P = (a_1, \ldots, a_n) \in L^n$ such that $m = (X_1 - a_1, \ldots, X_n - a_n)L[X_1, \ldots, X_n] \cap K[X_1, \ldots, X_n]$.

3. (i) Let V be an affine algebraic set in $\mathbb{A}^n_{\mathbb{C}}$ such that $\mathbb{C}[V]^* = \mathbb{C}^*$. Show that for any $f \in \mathbb{C}[V] \setminus \mathbb{C}$ the induced polynomial function $f : V \to \mathbb{C}$ is surjective.
(ii) Give an example of an affine algebraic set V in $\mathbb{A}^2_{\mathbb{C}}$ for which $\mathbb{C}^* \subsetneqq (\mathbb{C}[V])^*$ and a non-constant polynomial function $f : V \to \mathbb{C}$ which is not surjective.

4. Prove that every injective morphism from $\mathbb{C} \to \mathbb{C}$ is an isomorphism. Give an example to show that this need not be true if $\mathbb{C}$ is replaced by $\mathbb{R}$.

5. Let $V = \mathcal{Z}(f)$ be an irreducible hypersurface in $\mathbb{A}^n$. Show that there does not exists any irreducible affine algebraic set W satisfying $V \subsetneqq W \subsetneqq \mathbb{A}^n$.

6. Let k be an infinite field. Show that the intersection of any two non-empty open subsets of $\mathbb{A}^n_k$ in Zariski topology is non-empty.

7. Let $V(\subseteq \mathbb{A}^n_k)$ and $W(\subseteq \mathbb{A}^m_k)$ be two affine algebraic sets. Show that $V \times W$ is an affine algebraic set. Give an example to show that the Zariski Topology on $V \times W$ need not be the product topology.

8. Let C be the plane algebraic curve in $\mathbb{A}^2_{\mathbb{C}}$ defined by the equation $y^2 = x^3 - x^2$. Determine the singular point(s) of C. Explicitly determine the tangent spaces to C at the points $(0, 0)$ and $(1, 0)$. What are the dimensions of the tangent spaces?

9. Let $V \subsetneqq \mathbb{A}^n_{\mathbb{C}}$ be an affine algebraic variety. Show that $\mathbb{A}^n_{\mathbb{C}} \setminus V$ is path-connected in the Euclidean topology and hence in the Zarisky topology.

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2013-2014
M. Math. - II Year
Topology-III

Date : 27. 02. 2014 Maximum Score : 40 Time :2:30 Hours

Any result that you use should be stated clearly.

(1) Answer the following with proper reasons.

(a) Suppose a topological space X has two different CW complex structures. Denote the space X by X_1 and X_2 as CW complexes. Is it possible that the integral cellular homology groups of X_1 and X_2 are different?

(b) Let $X = (S^1 \vee S^1) \times \mathbb{R}P^2$. What is $H_1(X;\mathbb{Z})$?

(c) Let G be any abelian group. Is it true that for any space X, there always exists a surjective homomorphism

$$\kappa : H^n(X;G) \longrightarrow \text{Hom}(H_n(X), G)?$$

[3+ 3+ 4=10]

(2) (a) State excision axiom for cohomology.

(b) Suppose $V \subset U \subset A \subset X$. Assume that V can be excised from (X, A) and

$$(X - U, A - U) \hookrightarrow (X - V, A - V)$$

is a deformation retract. Then prove that U can be excised.

(c) Let

$$E^n_+ = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n | \; x_{n+1} \geq 0\},$$
$$E^n_- = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n | \; x_{n+1} \leq 0\}.$$

Use part (b) to prove that

$$H^q(E^n_+, S^{n-1}) \cong H^q(S^n, E^n_-)$$

for all q.

[2+5+5=12]

(3) (a) Define the notion of a CW complex.

(b) Define complex projective space $\mathbb{C}P^n$. Prove that $\mathbb{C}P^n$ can be obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell to it and hence describe a Cw complex structure of $\mathbb{C}P^n$.

(c) Compute integral homology groups of $\mathbb{C}P^n$.

[4+6+4=14]

(4) (a) Explain what is Mayer-Vietories Exact sequence.

(b) Let m, $n \geq 2$ and $X = S^m \vee S^n$. Apply Mayer-Vietories exact sequence to compute the integral homology groups of X.

[4+5=9]

INDIAN STATISTICAL INSTITUTE

MID-SEMESTER EXAMINATION : (2013-2014)

M. MATH II

ALGEBRAIC NUMBER THEORY

FEBRUARY 28, 2014 MAXIMUM MARKS : 40 DURATION : 3 HOURS

28·02·14

---

(1) (a) Prove that the ring $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain. [6]

(b) Find all **integer** solutions of the equation $y^2 + 2 = x^3$. [6]

(2) Consider the number field $K = \mathbb{Q}(2^{\frac{1}{3}})$. Let $x = a + b2^{\frac{1}{3}} + c2^{\frac{2}{3}} \in K$, where $a, b, c \in \mathbb{Q}$. Find $TR_{K|\mathbb{Q}}(x)$, $N_{K|\mathbb{Q}}(x)$ and the characteristic polynomial $f_x(t)$, in terms of a, b, c. [6]

(3) (a) Let $d \equiv 3 \pmod 4$. Show that $\frac{1}{2}(-1 + \sqrt{-d})$ is an algebraic integer. [2]

(b) Let $d \in \mathbb{Z}$ be square-free and $d \equiv 1 \pmod 4$. Find an integral basis for the number field $K = \mathbb{Q}(\sqrt{d})$. Find the discriminant d_K. [4+1]

(4) Let K be an algebraic number field of degree n and $\mathcal{O}_K$ be its ring of algebraic integers. For a nonzero ideal I of $\mathcal{O}_K$, let $\mathcal{N}(I)$ denote the norm of I. Now answer the following :

(a) Give a brief argument to show that $\mathcal{N}(I)$ is finite. [2]

(b) Let $I = \mathcal{P}_1^{m_1} \cdots \mathcal{P}_r^{m_r}$. Then, prove that

$$\mathcal{N}(I) = \mathcal{N}(\mathcal{P}_1)^{m_1} \cdots \mathcal{N}(\mathcal{P}_r)^{m_r}.$$

Deduce that if $\mathcal{N}(I)$ is a prime number, then I is a prime ideal of $\mathcal{O}_K$. Is the converse true? Justify your answer. [3+1+1]

(c) Prove that $\mathcal{N}(I) \in I$. Deduce that if I is a prime ideal then it contains exactly one rational prime p and then, $\mathcal{N}(I) = p^m$ where $m \leq n$. [2]

(d) Prove that only finitely many ideals of $\mathcal{O}_K$ has given norm. [2]

(5) Let R be Dedekind domain with field of fractions K.

(a) Let I be a proper non-zero ideal of R. Prove that there is $\alpha \in K \setminus R$ such that $\alpha I \subset R$. [5]

(b) Let I be a proper non-zero ideal of R. Prove that there is an ideal J of R such that IJ is principal. [5]

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination: 2013-14

M. Math. II Yr.

Representation Theory

Date: 03/03/2014 Maximum Marks: 40 Duration: $2\frac{1}{2}$ Hours

---

(1) (a) Let H be a subgroup of a topological group G. Show that the quotient map $q : G \to G/H$ is an open mapping.

(b) If H is connected subgroup of a topological group G such that G/H is also connected, then show that G is connected.

(2) (a) Let H be a subgroup of a topological group. If H has an isolated point, then show that it is discrete.

(b) Let G be a topological group and H be a subgroup of G such that $\overline{U} \cap H$ is closed in G for some neighbourhood U of 1 in G. Then show that H is closed.

(3) Let

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x, y \in \mathbb{R}, x \neq 0 \right\}.$$

with matrix multiplication as the group operation. Identify G with an appropriate subset of $\mathbb{R}^2$ and equip G with the subspace topology derived from $\mathbb{R}^2$. Compute the left and right Haar integral on G. Also, find the modular function.

(4) Let π be a unitary representation of a locally compact group G. If π has a nontrivial invariant subspace M, then show that π is the direct sum of π^M and $\pi^{M^\perp}$, where π^M is the restriction of π to M.

Give an example to show that the above result may fail to hold if the representation π is not unitary.

(5) Let G be a locally compact group and $\mathcal{P}_1$ denote the set of all continuous functions on G of positive type with sup norm 1. Show that $\mathcal{P}_1$ is a bounded convex set in $L^\infty(G)$. Let $\mathcal{E}(\mathcal{P}_1)$ be the set of all extreme points of $\mathcal{P}_1$. Show that the convex hull of $\mathcal{E}(\mathcal{P}_1)$ is weak-* dense in $\mathcal{P}_1$.

INDIAN STATISTICAL INSTITUTE
Mid-Semesteral Examination : 2012-13
M. Math. - Second Year
Descriptive Set Theory

Date : 03. 03. 2014 Maximum Score : 100 Time :3 Hours

1. You are free to answer all the questions. Maximum score—100.

2. You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) Given two well-ordered sets $(W_1, \leq_1)$ and $(W_2, \leq_2)$, show that exactly one of the following three holds:
   (a) $(W_1, \leq_1)$ is order isomorphic to an initial segment of $(W_2, \leq_2)$.
   (b) $(W_1, \leq_1)$ is order isomorphic to $(W_2, \leq_2)$.
   (c) $(W_2, \leq_2)$ is order isomorphic to an initial segment of $(W_1, \leq_1)$.

[20]

(2) Let X be a Polish space and Y a metrizable space. Suppose there is a continuous function $f : X \to Y$ with range uncountable. Show that Y contains a homeomorph of the Cantor set. [20]

(3) Show that every uncountable Borel subset of a Polish space is of cardinality $2^{\aleph_0}$. [20]

(4) Let X be a metrizable space and $\{A_n\}$ a sequence of subsets of X, each a continuous image of $\mathbb{N}^{\mathbb{N}}$, such that $\cap_n A_n = \emptyset$. Show that there exist Borel sets $B_n \supset A_n$ with $\cap_n B_n = \emptyset$. [20]

(5) Let X be a metrizable space. Show that the collection of all subsets of X of the form $U \Delta N$, U open and N meager in X, is a σ-field of subsets of X. [15]

(6) Show that for every Lebesgue measurable subset A of $\mathbb{R}$, there is a G_δ subset $G \subset \mathbb{R}$ such that $A \Delta G$ is of measure 0. [15]

(7) Show that $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$. [10]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2013-14 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date: 2.5.2014 Maximum Marks: 70 Duration: $3\frac{1}{2}$ Hours

Note: Answer THREE questions from GROUP A and THREE from GROUP B.
Clearly state the results that you use.
R denotes a commutative ring with 1.

GROUP A
Attempt ANY THREE

1. (i) Suppose that x is an element in R which is neither a unit nor a zero-divisor. Prove that $R/x^{n-1}R \cong xR/x^nR$ as R-modules for each $n \geq 1$; and hence construct a short exact sequence
$$0 \to R/x^{n-1}R \to R/x^nR \to R/xR \to 0.$$
Deduce that $Ass_R\ (R/x^nR) = Ass_R\ (R/xR)\ \forall n \geq 1$.

   (ii) Compute $Ass_R\ (R/xR)$ and $Ass_R\ (R/x^2R)$ when $R = \mathbb{C}[X,Y]/(XY)$ and x is the image of X in R. [10+6=16]

2. Prove that any semilocal Dedekind domain is a PID and deduce that any ideal of a Dedekind domain is generated by at most two elements. [16]

3. (i) State and prove Krull's Hauptidealsatz (Principal Ideal Theorem).

   (ii) Deduce that a Noetherian domain R is a unique factorisation domain if and only if every prime ideal in R of height one is principal. [12+4=16]

4. Let I be an ideal of R such that R is I-adically complete.

   (i) Prove that $1 - a$ is a unit in R for every $a \in I$.

   (ii) Let M be an R-module which is separated in the I-adic topology. Show that if M/IM is finitely generated then so is M. [7+9=16]

5. Examine whether the following statements are TRUE or FALSE with brief justification. Answer ANY FOUR.

   (i) If M and N are finitely generated R-modules, then $Supp\ (M \otimes_R N) = Supp\ M \cap Supp\ N$.

   (ii) If a Noetherian domain R with field of fractions K contains a nonzero prime element p for which $R[1/p] = K$, then R is a DVR.

   (iii) Any two minimal prime ideals of an ideal of $\mathbb{C}[X,Y]$ have the same height.

   (iv) If m is a maximal ideal of an integral domain R, then the m-adic completion of R is necessarily an integral domain.

   (v) 3 is a cube in the ring of 5-adic integers. [4 × 4 =16]

GROUP B

Attempt ANY THREE

1. Let R be a normal domain, K the field of fractions of R, L a finite Galois extension of K with Galois group G and A the integral closure of R in L. Let P and Q be prime ideals of A such that $P \cap R = Q \cap R$. Prove that there exists $\sigma \in G$ such that $\sigma(P) = Q$. [10]

2. (i) Let $R = \mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$ and K denote the field of fractions of R. Show that there exist exactly two valuation rings of K containing $\mathbb{C}$ but not containing R.

   (ii) If $\mathcal{P}$ is the place and v the valuation defined by the valuation ring $\mathbb{C}[\frac{1}{X}]_{(\frac{1}{X})}$, compute

$$\mathcal{P}(\frac{X}{2X^2+1}) \text{ and } v(\frac{X}{2X^2+1}).$$

[6+4=10]

3. Let $P \subsetneqq Q$ be prime ideals in a Noetherian ring R. Show that if there exists one prime ideal P_1 in R with $P \subsetneqq P_1 \subsetneqq Q$, then there exist infinitely many prime ideals P_i in R such that $P \subsetneqq P_i \subsetneqq Q$. [10]

4. Let k be a field, $A = k[X_1, \cdots, X_n]$ and $m = (X_1 - a_1, \cdots, X_n - a_n)$. Let $I = (f_1, \cdots, f_m)$ be an ideal of A contained in m and $R = A_m/IA_m$. Let r be the rank of the corresponding Jacobian matrix :— the $m \times n$-matrix whose (i, j)th entry is $(\frac{\partial f_i}{\partial X_j}|_{(a_1,\cdots,a_n)})$.

   (i) Show that R is a regular local ring if and only if dim $R = n - r$.

   (ii) Let $B = \mathbb{C}[X, Y, Z]/(XY - Z^2)$. Describe all maximal ideals m of B for which B_m is a regular local ring. [7+3=10]

5. Give an explicit example each (with brief justification):

   (i) A maximal ideal of height one in $\mathbb{C}[[X]][Y]$.

   (ii) A valuation ring containing $\mathbb{C}[X, Y, Z]/(XY - Z^2)$ and having the same field of fractions. [5+5=10]

INDIAN STATISTICAL INSTITUTE
End-Semestral Examination: 2013-14
M. Math. II Yr.
Representation Theory
Date: 05/05/2014 Maximum Marks: 60 Duration: 3 Hours

Each question carries 10 marks.

---

(1) Let G be a locally compact abelian group and $\widehat{G}$ be its dual. Show that
  (a) if G is discrete, then $\widehat{G}$ is compact,
  (b) if G is compact, then $\widehat{G}$ is discrete.

(2) Let G be a locally compact abelian group and K be a compact subset of $\widehat{G}$. Show that there exists $f \in L^1(G) \cap B(G)$ such that $\hat{f} \geq 0$ on $\widehat{G}$ and $\hat{f} > 0$ on K. ($B(G)$ is the set of all finite linear combinations of continuous positive definite functions.)

(3) Let G be a locally compact abelian group and E be a nonempty open subset of $\widehat{G}$. Show that there exists $\hat{f} \in A(\widehat{G})$ with $\hat{f} \neq 0$ such that $\hat{f} = 0$ outside E. Here $A(\widehat{G}) = \{\hat{f} : f \in L^1(G)\}$.

(4) Show that every unitary representation of a compact group has a finite dimensional subrepresentation.

(5) Let π be a unitary representation of a compact group G on a Hilbert space H_π and $\mathcal{E}_\pi$ be the span of all matrix elements of π, that is, $\mathcal{E}_\pi = \text{span}\{\varphi_{u,v} : u, v \in H_\pi\}$, where $\varphi_{u,v}(x) = \langle \pi(x)u, v \rangle$. Show that $\mathcal{E}_\pi$ is a two sided ideal in $L^1(G)$.

(6) (a) Let π_1 and π_2 be unitary representations of a locally compact group G on the Hilbert spaces H_1 and H_2 respectively. Let $T \in \mathcal{C}(\pi_1, \pi_2)$, that is, $T : H_1 \to H_2$ is a bounded linear operator satisfying $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$. Show that $T^*T \in \mathcal{C}(\pi_1)$.
  (b) In addition, if π_1 is irreducible, then show that TH_1 is a closed invariant subspace for π_2.

  [**Hint** for (b). By (a), $T^*T = \lambda I$ for some λ. Show that $S = \lambda^{-1/2}T$ is an isometry, and hence a unitary from H_1 onto TH_1. Now, show that SS^* is in $\mathcal{C}(\pi_2)$ and is the orthogonal projection onto SH_1.]

(7) Let G be a compact group and V be a finite dimensional subspace of $L^2(G)$ which is invariant under the left regular representation of G. Let π denote the restriction of the left regular representation to V. Prove that V consists of continuous functions and each $f \in V$ can be written as

$$f(x) = \text{tr}\left(A\overline{\pi}(x)\right)$$

for some linear operator A (depending on f) on V. Here $\overline{\pi}$ is the contragredient of π.

  [**Hint.** Each $f \in V$ can be written as $f = \sum a_i e_i$, where $\{e_i\}$ is an orthonorma basis of V. Write $\pi(x)f$ in terms of a_is and the matrix coefficients $\pi_{ij}(x)$ and observe that $f(x) = [\pi(x^{-1})f](e)$.]

INDIAN STATISTICAL INSTITUTE
Semesteral Examination : 2013-14
M. Math. - Second Year
Descriptive Set Theory

Date 05. 05. 2014 Maximum Score : 100 Time :3 1/2 Hours

1. You are free to answer all the questions. Maximum score—100.
2. You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) (a) Let X be a second countable space. Show that there is a closed set U in $\mathbb{N}^{\mathbb{N}} \times X$ such that for every closed $C \subset X$ there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ satisfying $C = U_\alpha$.
(b) Show that for every uncountable Polish space X, there is an analytic $U \subset X \times X$ such that for every analytic $A \subset X$, there is an $x \in X$ with $A = U_x$.
(c) Show that every uncountable Polish space contains an analytic non-Borel subset.
[10+12+5]

(2) Let X be a Polish space and B a Borel subset. Show that there is a finer Polish topology τ such that B is clopen with respect to the new topology and the Borel σ-algebra with respect to both the topologies are the same. [15]

(3) Let X be a Polish space and $A \subset X$. Show that there is a set $B \supset A$ with Baire property such that for every set $C \supset A$ with Baire property $B \setminus C$ is meager. [12]

(4) Let X be a Polish space and $\mathcal{A}$, $\mathcal{B}$ two countably generated sub σ-algebras of the Borel σ-algebra $\mathcal{B}_X$ such that the sets of atoms of $\mathcal{A}$ and $\mathcal{B}$ are the same. Show that $\mathcal{A} = \mathcal{B}$. [12]

(5) Let X and Y be Polish spaces and $B \subset X \times Y$ Borel such that for every $x \in X$, the section B_x is open in Y. Show that

$$B = \cup_n (B_n \times U_n),$$

where B_n's are Borel in X and U_n's are open in Y. [15]

(6) Let X and Y be Polish spaces and $B \subset X \times Y$ Borel such that the sections, B_x, $x \in X$, are compact. Show that the projection $\pi_X(B)$ is Borel in X. [12]

(7) Let $(G, \cdot)$ be a Polish group acting on a Polish space X such that for every $x \in X$, $g \to g \cdot x$ is Borel and for every $g \in G$, $x \to g \cdot x$ is continuous. Show that the action is continuous. [20]

INDIAN STATISTICAL INSTITUTE

SECOND SEMESTRAL EXAMINATION : (2013-2014)
M. MATH II
ALGEBRAIC NUMBER THEORY
DATE : MAY 7, 2014. MAXIMUM MARKS : 65 DURATION : 4 HOURS

---

(1) (a) Let K be a number field and $\mathcal{O}_K$ be its ring of integers. For $\alpha \in \mathcal{O}_K$, define $\phi(\alpha) := |N_{K|\mathbb{Q}}(\alpha)|$. For $\alpha, \beta \in \mathcal{O}_K \setminus \{0\}$, show that the following statements are equivalent. [4]

(i) $\exists\, \gamma, \delta \in \mathcal{O}_K$ so that $\alpha = \beta\gamma + \delta$ where $\delta = 0$ or $\phi(\delta) < \phi(\beta)$.

(ii) For any $\epsilon \in K$, there exists $\eta \in \mathcal{O}_K$ such that $\phi(\epsilon - \eta) < 1$.

(b) Using (a) or otherwise prove that $\mathcal{O}_K$ is a Euclidean domain with Euclidean function ϕ defined as above when $K = \mathbb{Q}(\sqrt{d})$ and $d = -1, -2, -3, -7, -11$. [7]

(c) Let $d < -11$ be a square-free integer. Prove that the ring of integers of $\mathbb{Q}(\sqrt{d})$ is not a Euclidean domain. [7]

(2) Let $p \in \mathbb{Z}$ be a prime and $n = p^l$, $l \geq 1$. Let ζ be a primitive nth root of unity and write $K = \mathbb{Q}(\zeta)$. Let $\mathcal{O}_K$ be the ring of integers of K. Prove that

(a) $p\mathcal{O}_K = (1-\zeta)^d \mathcal{O}_K$ where $d = \varphi(p^l) = [K : \mathbb{Q}]$; [5]

(b) $(1-\zeta)\mathcal{O}_K \in \mathrm{Spec}(\mathcal{O}_K)$ and inertia degree of $(1-\zeta)\mathcal{O}_K$ is 1; [3]

(c) the basis $1, \zeta, \cdots, \zeta^{d-1}$ of the $\mathbb{Q}$−vector space K has discriminant $\pm p^s$, where $s = p^{l-1}(lp - l - 1)$. [5]

(d) $1, \zeta, \cdots, \zeta^{d-1}$ is an integral basis of $\mathcal{O}_K$. [5]

(3) Let R be a Dedekind domain and K its quotient field. Let L be a *Galois* extension of K of degree n and S be the integral closure of R in L.

(a) Prove that S is a Dedekind domain. [6]

(b) Let $\mathfrak{p} \in \mathrm{Spec}(R) \setminus (0)$. Prove that $P \mapsto \sigma(P)$ defines an action of $\mathrm{Gal}(L|K)$ on the set $T = \{P \in \mathrm{Spec}(S) | P \cap R = \mathfrak{p}\}$. Show that this action is transitive. [4]

(c) Let $P \in \mathrm{Spec}(S) \setminus (0)$. Define the *decomposition group* G_P of P over K. For $\sigma \in \mathrm{Gal}(L|K)$, show that $G_{\sigma(P)} = \sigma G_P \sigma^{-1}$. [4]

(d) Let $\mathfrak{p} \in \mathrm{Spec}(R) \setminus (0)$. Prove that

$$\mathfrak{p}S = \left(\prod_{\sigma} \sigma(P)\right)^e,$$

where $P \in \mathrm{Spec}(S)$ such that $P \cap R = \mathfrak{p}$ and σ varies over a system of representatives of $\mathrm{Gal}(L|K)/G_P$. [6]

[PTO]

(4) (a) Let $d > 0$ be a square-free integer and $K = \mathbb{Q}(\sqrt{d})$. If $d \equiv 1 \pmod 4$ then show that the equation $x^2 - dy^2 = 4$ has infinitely many integer solutions. Clearly state the results that you used. [5]

(b) Find the fundamental unit for each of the following number fields: (i) $\mathbb{Q}(\sqrt{3})$, (ii) $\mathbb{Q}(\sqrt{5})$. [4]

(c) Consider the number field $K = \mathbb{Q}(2^{\frac{1}{3}})$. Assuming the fact that $\mathcal{O}_K = \mathbb{Z}[2^{\frac{1}{3}}]$, determine the prime ideal factorization of $31\mathcal{O}_K$. [4]

(d) Let K be a quadratic number field with discriminant d and let p be an odd prime. Prove that $p\mathcal{O}_K$ is a product of two distinct prime ideals of $\mathcal{O}_K$ if and only if d is a square modulo p. [5]

INDIAN STATISTICAL INSTITUTE
Semestral Examination : 2013-2014
M. Math. - II Year
Topology-III

Date : 09. 05. 2014 Maximum Score : 60 Time :3:00 Hours

Any result that you use should be stated clearly.

(1) (a) Define the notions of local orientation and orientation on a manifold.
(b) Define Cap product operation.
(c) Describe the duality homomorphism.
(d) State Poincare duality isomorphism theorem.
[4+3+2+3 = 12]

(2) (a) Define Cup product operation on graded cohomology groups.
(b) Compute graded cohomology algebra of $\mathbb{R}P^{\infty}$ with coefficients $\mathbb{Z}_2$.
[4+8 = 12]

(3) (a) Describe the Hurewicz homomorphism and state the Hurewicz isomorphism theorem.
(b) Use Hurewicz theorem to prove for $n \geq 2$, $\pi_k(S^n) = 0$, for $k < n$ and $\pi_n(S^n) = \mathbb{Z}$.
[5+5= 10]

(4) (a) Define Hurewicz fibration.
(b) Prove that there exists a Hurewicz fibration over $\mathbb{C}P^n$ with fibre S^1.
(c) Compute $\pi_3(S^2)$.
(d) Prove that for $n > 1$,
$$\pi_{2n-1}(\mathbb{C}P^{n-1}) = \mathbb{Z}.$$
[4+4+4 =2= 14]

(5) Either prove or disprove the following statements.
(a) A closed manifold of odd dimension has Euler characteristic zero.
(b) Homotopy groups of a finite CW complex are finitely generated.
[6+ 6=12]

(6) (a) State Whitehead theorem for CW complexes.
(b) Suppose X is an n-dimensional CW complex, $n \geq 0$. Assume that X is n-connected. Prove that X is contractible.
[4 + 6 =10]

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2013-14 (Second Semester)

M. MATH. II YEAR

Algebraic Geometry

Date: 13.5.2014 Maximum Marks: 60 Duration: 4 Hours

Throughout the paper, k will denote a field.

GROUP A

Answer ANY FIVE questions

Each question carries 10 marks.

1. Prove that any maximal ideal of $k[X_1, X_2, \ldots, X_n]$ is generated by n elements.
2. Let $\alpha : k[X, Y, Z] \to k[T]$ be a k-algebra homomorphism defined by $\alpha(X) = T^9$, $\alpha(Y) = T^6$, $\alpha(Z) = T^4$. Show that Ker $(\alpha) = (X^2 - Y^3, Y^2 - Z^3)$.
3. Show that $\mathbb{V} = \mathcal{Z}(Y^2 - X^3) \subseteq \mathbb{A}_k^2$ and $\mathbb{W} = \mathcal{Z}(Y^2 - X^2(X+1)) \subseteq \mathbb{A}_k^2$ are rational but not isomorphic.
4. Let k be an algebraically closed field. Find all singular points of the affine curves $y^2 = x^3$ and $y^3 = x^4$ in A_k^2. Show that the two curves are not isomorphic.
5. Let $\mathbb{V} = \mathcal{Z}_+(XY - Z^2) \subseteq \mathbb{P}_k^2$ be a projective variety. Prove that $k(\mathbb{V}) \cong k(\mathbb{P}_k^1)$.
6. Prove that the pole set of a rational function on a projective variety is an algebraic subset of V.
7. Compute $\mathrm{Cl}(A)$ for $A = \mathbb{C}[X, Y, Z]/(X^2 + Y^2 + Z^2)$.
8. Let $\mathbb{V}$ be an affine variety. Prove that $\dim \mathbb{V} = \dim \mathcal{O}_P(\mathbb{V})$ for all $P \in \mathbb{V}$. Deduce that $\dim \mathbb{V} \leq \dim T_{P,\mathbb{V}}$.

GROUP B

Answer ANY FOUR questions

Each question carries 5 marks.

Suppose that k is an algebraically closed field and $\mathbb{V}$ an affine variety in $\mathbb{A}_k^n$. Prove that

1. If $g \in k[\mathbb{V}]$ satisfies $g(P) \neq 0$ for every $P \in \mathbb{V}$, then $g \in k^*$.
2. $\dim \mathbb{V} = n - 1$ if and only if $V = \mathcal{Z}(f)$ for some irreducible polynomial $f \in k[X_1, X_2, \ldots, X_n]$.
3. The curve $\mathcal{Z}(X^n + Y^n - 1) \subseteq \mathbb{A}_k^2$ is not a rational curve for $n \geq 3$.
4. Any non-singular projective algebraic plane curve is irreducible.
5. Any cubic projective curve with multiplicity 3 at a point is reducible.
6. If the intersections of the opposite sides of a hexagon lie on a straight line, then the vertices of the hexagon lie on a conic.

Fourier Analysis: M. Math II: Back Paper Examination

Date ....

Total Marks 100 Maximum Time 3 hrs.

Answer all questions.

(1) (a) Find examples of two functions $f_1, f_2 \in L^1(\mathbb{R}^d)$ such that $g(x) = f_1(x)f_2(x)$ is not in $L^1(\mathbb{R}^d)$.

(b) For $a \in \mathbb{R}^d$ and a function f on $\mathbb{R}^d$ let $\ell_a(f)(x) = f(x-a)$. Define $h_a(x) = f_1(x)\ell_a(f_2)(x)$ where $f_1, f_2 \in L^1(\mathbb{R}^d)$. Show that for almost every $a \in \mathbb{R}^d$, $h_a \in L^1(\mathbb{R}^d)$. 4+6=10

(2) Give brief answers (maximum 3 lines) to these.

(a) Suppose $f : \mathbb{R} \to \mathbb{C}$ is measurable and

$$\int_{\mathbb{R}} e^{|x|}|f(x)|dx < \infty.$$

Can $\widehat{f}$ have uncountably many zeroes?

(b) If $f \in L^p(\mathbb{R})$, $p > 1$ and $g \in L^{p'}(\mathbb{R})$ then we know that $f * g \in C_0(\mathbb{R})$, i.e. $f * g$ is a continuous function vanishing at ∞. Is the result true for $p = 1$? Prove or give counter example to disprove.

(c) Let $f \in L^p(\mathbb{R})$ for some $p \in (1,2)$ and $g \in L^{p'/2}(\mathbb{R})$. Prove that $f * g \in L^{p'}(\mathbb{R})$.

(d) Let g be a measurable function on $\mathbb{R}^d$ and let $p \in (1,2)$ be fixed. Suppose that for every $f \in L^p(\mathbb{R})$, $f * g \in L^{p'}(\mathbb{R})$. Is it necessarily true that $g \in L^{p'/2}(\mathbb{R})$?

(e) Show that given an L^1-function f on $\mathbb{R}^d$ and an arbitrary $\epsilon > 0$, there is a measurable set $E \subset \mathbb{R}^d$ with $|E| < \epsilon$ such that outside this set $f \in L^p$ for all $p \in [1, \infty]$.

(f) Suppose $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$ is such that integral of f over every finite interval is zero. Does it mean $f \equiv 0$? $5 \times 6 = 30$

(3) Suppose for $f \in L^p(\mathbb{R})$ with $p \in [1,2]$, $\int_{\mathbb{R}} f(y)e^{-y^2}e^{2xy}dy = 0$ for all $x \in \mathbb{R}$. Show that $f \equiv 0$. 6

P.T.O.

(4) For $s \in \mathbb{R}$ and functions f on $\mathbb{R}^d$ define:

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

Suppose for some function f on $\mathbb{R}^d$, $\|f\|_{H^s} < \infty$. Show that if $s > d/2$ then f is continuous and

$$\|f\|_p \leq C\|f\|_{H^s} \text{ for } 2 \leq p \leq \infty.$$

10

(5) (a) For $\phi \in S(\mathbb{R}^d)$ and $\delta > 0$ define $\phi_\delta(x) = \phi(\delta x)$. For a tempered distribution F define F_δ by

$$F_\delta(\phi) = F(\phi_{1/\delta})$$

for all $\phi \in S(\mathbb{R}^d)$. Suppose a tempered distribution F is homogeneous of degree k that is $F_\delta = \delta^k F$. Show that $\widehat{F}$ is also homogeneous and find its degree of homogeneity.

(b) Let $f(x) = \|x\|^{-k}$ for some $k > 0$. Find the range of k for which it is a tempered distribution. Using (a), find the Fourier transform of f up to a constant. (You may assume that the Fourier transform of a radial tempered distribution is radial.)

7+7

(6) For any $z \in \mathbb{C}$, let $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$. Find the volume of the ellipsoid in $\mathbb{R}^d$ whose half-axes are of length $r_1, \ldots, r_d$ in terms of d, r_i and Gamma-function. 10

(7) For $k \in \mathbb{Z}$, let $\mathcal{Q}_k$ be the set of dyadic cubes of side length 2^k. Fix $f \in L^1(\mathbb{R}^d)$. For each $k \in \mathbb{Z}$ define

$$f_k(x) = \sum_{Q \in \mathcal{Q}_k} \left(\frac{1}{|Q|} \int_Q f(y) dy \right) \chi_Q(x).$$

(a) Give argument to show that $f_k(x) \to 0$ as $k \to \infty$ for a.e. $x \in \mathbb{R}^d$ and state appropriate results to establish that $f_k(x) \to f(x)$ as $k \to -\infty$ for a.e. $x \in \mathbb{R}^d$.

(b) Using (a) prove that $f = \sum_{k \in \mathbb{Z}} (f_k - f_{k+1})$, where the convergence is point wise a.e.

(c) Show that $f_k - f_{k+1}$ is constant on each dyadic cube $Q \in \mathcal{Q}_k$ and integral of $f_k - f_{k+1}$ is 0 on any bigger dyadic cube.

(d) Conclude that $f_k - f_{k+1}$ is orthogonal to $f_l - f_{l+1}$ whenever $k \neq l$. $4 \times 5 = 20$

Indian Statistical Institute
Backpaper Examination: 2013-2014
Programme: Master of Mathematics
Course: **Number Theory**

Maximum Marks: 100
Duration: Three Hours

Instruction: Answer all questions. Write precise answers. If you use any standard result you must quote it carefully.

---

1. Let g be a primitive root for p^n for some $n > 1$. Show that g is also a primitive root for p. (10 points)

2. Suppose a and n are positive integers and $n > 1$.
Suppose $a^{n-1} \equiv 1 \pmod{n}$ but $a^m \not\equiv 1 \pmod{n}$ for all other divisors m of $n-1$. Show that n must be a prime. (15 points)

3. State and prove Gauss Lemma. Using the Gauss Lemma, show that for an odd prime p, the Legendre symbol $\left(\frac{2}{p}\right)$ satisfies

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

(12+10=22 points)

4. If p is a prime of the form $p = 2^{2^n} + 1$, then show that 3 is a primitive root for p. (10 points)

5. Suppose G is a finite abelian group and let $g \in G$ be an element other than the identity. Show that there is a character χ of G such that $\chi(g) \neq 1$. (15 points)

6. Let $q > 1$ be an integer and let $\chi \pmod{q}$ be a nonprincipal real primitive Dirichlet character. Give a complete proof of the statement that $L(1, \chi) \neq 0$. (30 points)