INDIAN STATISTICAL INSTITUTE

End-Semestral Examination: 2013-14 (Backpaper)

M. Math. II Yr.

Representation Theory

Date: 08/08/2014 Maximum Marks: 100 Duration: 3 Hours

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(1) Let G be a topological group. Let F be a closed subset and K be a compact subset of G such that $F \cap K = \emptyset$. Show that there exists an open neighbourhood V of the identity of G such that $F \cap VK = \emptyset$. [10]

(2) Let H be a closed subgroup of a locally compact group G. Show that G/H is Hausdorff and is locally compact. [10]

(3) Let G be a locally compact group with left Haar measure λ. Show that $\lambda(G)$ is finite if and only if G is compact. [10]

(4) Let G be a unimodular group and $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(G)$ and $g \in L^q(G)$, show that $f * g \in C_0(G)$ and $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. [15]

(5) Let $\mathcal{P}$ be the set of all continuous functions of positive type on a locally compact group G and $\mathcal{P}_1 = \{\varphi \in \mathcal{P} : \|\varphi\|_\infty = 1\}$. Show that on $\mathcal{P}_1$, the weak* topology coincides with the topology of compact convergence on G. [15]

(6) Let G be a locally compact group and H be a subgroup of G. If H is locally compact in the relative topology, then show that H is a closed subgroup of G. [10]

(7) Let G be a compact group. If π is a finite dimensional representation of G, then its character χ_π is the function $\chi_\pi(x) = \text{tr}\,(\pi(x))$, $x \in G$.

Show that $\{\chi_\pi : [\pi] \in \widehat{G}\}$ forms an orthonormal basis for the space of central functions of $L^2(G)$. [10]

(8) For a nonnegative integer n, let H_n denote the space of all homogeneous polynomials of degree n in two complex variables. Define a representation π_n of $SU(2)$ on H_n by

$$[\pi_n(g)f](z) = f(zg), \quad f \in H_n, g \in SU(2), z \in \mathbb{C}^2.$$

Show that π_n is a unitary and irreducible representation. [20]

INDIAN STATISTICAL INSTITUTE
Back-paper Examination : 2013-14
M. Math. - Second Year
Descriptive Set Theory

Date : 08.08.14 Maximum Score : 100 Time :3 Hours

1. Answer all the questions. All questions carry equal marks. Maximum score—100.

2. You are free to use any theorem proved in the class. However, you must state any theorem that you use at least once in the answer script.

(1) (a) Let X be a second countable space and $\{F_\alpha : \alpha < \kappa\}$ a transfinite, non-increasing sequence of closed sets. Show that there is a countable ordinal α such that $F_\beta = F_\alpha$ for all $\beta > \alpha$.

(b) Let X be a Polish space. For any closed $A \subset X$ define its Cantor-Bendixson derivatives $A^{(\alpha)}$ by transfinite induction as follows:

$$\begin{aligned} A^{(0)} &= A \\ A^{(\alpha+1)} &= (A^{(\alpha)})' \\ A^{(\lambda)} &= \cap_{\alpha<\lambda} A^{(\alpha)} \text{ if } \lambda \text{ limit} \end{aligned}$$

Show that A is countable iff $A^{(\alpha)} = \emptyset$ for some countable ordinal α.

(2) Show that every Borel subset of a Polish space X is a continuous image of $\mathbb{N}^{\mathbb{N}}$.

(3) Let B be a Borel subset of a Polish space X and Y a Polish space. Supose $f : B \to Y$ be a one-to-one, Borel measurable function. Show that $f(B)$ is a Borel subset of Y.

(4) Let $(G, \cdot)$ be a group with a Polish topology such that the multiplication $g \cdot h$ is separately continuous in each vaiables and H a second countable group. Show that every Borel homomorphism $\varphi : G \to H$ is continuous.

(5) Let $(G, \cdot)$ be a group with a Polish topology such that for every $g \in G$, $h \to g \cdot h$ is continuous and for every $h \in G$, $g \to g \cdot h$ is Borel. Show that for every meager $I \subset G$, I^{-1} is meager.

Indian Statistical Institute, Kolkata

Mid-semester examination: First semester 2014 -'15

M.Math I

Subject	:	**Measure theory**
Time	:	3 hours
Maximum score	:	30

Instructions:

- *Justify every step in order to get full credit of your answers, stating clearly the result(s) that you use. Points will be deducted for missing arguments. Partial credit will be given for your approach to the problem.*
- *Switch off and deposit your mobile phones to the invigilator during the entire examination.*
- *Total marks carried by the questions turns out to be 46 which is more than 30. The total marks obtained will be multiplied with $\frac{15}{23}$.*

(1) Is $h(x) = \sum_{n \in \mathbb{N} \text{ s.t. } q_n < x} \frac{1}{2^n}$ a distribution function where $\{q_n\}_{n \in \mathbb{N}}$ is an enumeration of $\mathbb{Q}$? If yes, find its associated Lebesgue-Stieltjes measure. If not, prove it.

[5 marks]

(2) Will every open dense subset of $\mathbb{R}$ ALWAYS have infinite Lebesgue measure? If yes, prove it. If not, provide a counter-example with justification.

[4 marks]

(3) Are all monotone functions from $\mathbb{R}$ to $\mathbb{R}$ Borel measurable? If yes, prove it. If not, provide a counter-example with justification.

[6 marks]

(4) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\{f_n : \Omega \to \overline{\mathbb{R}}\}_{n \in \mathbb{N}}$ be a sequence of Borel measurable functions. Prove that $\liminf_{n \to \infty} f_n$ is Borel measurable ONLY USING definition of Borel measurability of functions and its necessary and sufficient condition.

[6 marks]

(5) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\{f_n : \Omega \to \overline{\mathbb{R}}\}_{n \in \mathbb{N}}$ be a sequence of Borel measurable functions. Is the inequality $\liminf_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu \geq \int_\Omega \liminf_{n \to \infty} f_n \, \mathrm{d}\mu$ ALWAYS true? If yes, prove it. If not, then provide (i) a counter example and (ii) a sufficient condition if any.

[5 marks]

(6) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : \Omega \to \overline{\mathbb{R}}$ be Borel measurable functions such that f^2 and g^2 are integrable. Show that

$$\left| \int_\Omega fg \, \mathrm{d}\mu \right| \leq \left(\int_\Omega f^2 \, \mathrm{d}\mu \right) \left(\int_\Omega g^2 \, \mathrm{d}\mu \right).$$

Hints: First prove for nonnegative simple functions.

[15 marks]

(7) Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $\{f_n : \Omega \to \overline{\mathbb{R}}\}_{n \in \mathbb{N}}$ be a sequence of Borel measurable functions. Show that $f_n \xrightarrow[n \to \infty]{} f$ a.e. $[\mu]$ implies $\mu\{|f_n - f| \geq \varepsilon\} \xrightarrow[n \to \infty]{} 0$ for all $\varepsilon > 0$. *Hints: Consider the set* $\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \{|f_k - f| \geq \varepsilon\}$.

[5 marks]

INDIAN STATISTICAL INSTITUTE

MIDSEMESTRAL EXAMINATION : (2014-2015)
M. MATH I
ALGEBRA I
AUGUST 26, 2014 MAXIMUM MARKS : 40 (WEIGHTAGE 20 %) DURATION : 2 HOURS

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(1) In $\mathbb{Z}[\sqrt{-5}]$ let $\alpha = 3(1+\sqrt{-5})$ and $\beta = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$. Determine whether $\gcd(\alpha, \beta)$ exists in $\mathbb{Z}[\sqrt{-5}]$. [6]

(2) (a) Let R be a commutative ring with 1 and $J \subset R$ be an ideal. Prove that [6]

$$\frac{R[X]}{< J, X >} \simeq \frac{R}{J}.$$

(b) let p be a prime in $\mathbb{Z}$. Is the ideal $< p, X_1, \cdots, X_n >$ maximal in $\mathbb{Z}[X_1, \cdots, X_n]$? Justify your answer. [4]

(c) Let m, n be positive integers and consider the ideal $I =< X^m, Y^n > \subset \mathbb{Q}[X, Y]$. Find the radical of I in $\mathbb{Q}[X, Y]$. [4]

(3) Let R be a commutative ring with 1 and M be a simple R-module (i.e., M has no submodule other than M and $\{0\}$). Prove that $End_R(M)$ is a division ring (There is no need to prove that $End_R(M)$ is a ring. **But mention how addition and multiplication are defined in** $End_R(M)$). [6]

(4) Let $R = \mathbb{Z}[X]$. Consider $\phi : R \to R$ defined by $X \mapsto X^2$. Is ϕ a ring morphism? Is ϕ a morphism of R-modules? Give complete justification. [4]

(5) Let R be a commutative ring with 1 and M be an R-module. Prove that the R-modules $Hom_R(R, M)$ and M are isomorphic. [6]

(6) Answer <u>any one</u> of the following questions. [4]

(a) Let R be a commutative ring with 1 and $a, b \in R$ be such that $< a, b >= R$. Prove that $< a^{15}, b^{11} >= R$.

(b) Let R be a commutative ring with 1 and M be an R-module. Recall that $ann_R(M) := \{r \in R \mid rx = 0 \, \forall x \in M\}$, which turns out to be an ideal of R (no need to prove this). Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. Find $ann_{\mathbb{Z}}(M)$.

MID SEMESTRAL EXAMINATION

Course Name: M. Math, 1st year
Subject Name : Calculus on several variables
Date: 28.08.2014, Maximum Marks: 40, Duration: 2 hours

Answer any four questions

(1) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. Let x and v be elements of $\mathbb{R}^n$.

a. Prove that f is continuous.

b. Prove that $Df(x)(v)$ equals the directional derivative of f at x in the direction v.

c. Let $n = 2$, $m = 1$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) = 0$ and $f(x,y) =$ [illegible] for $(x,y) \neq (0,0)$. Prove that f all directional derivatives of f exist at $(0,0)$ but f is not continuous at $(0,0)$. **2 + 4 + 4 = 10**

(2) $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is such that all its partial derivatives are bounded on the open set U.

a. Prove that f is continuous on U.

b. If we assume U to be convex and $\frac{\partial f}{\partial x_1} = 0$ on U, then prove that f depends only on $x_2, x_3, \ldots, x_n$. **5 + 5 = 10**

(3) Let $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be defined by $f(A) = A^2$. Let I_n denote the $n \times n$ identity matrix.

a. Let X, T be elements of $M_n(\mathbb{R})$. Prove that $Df(X)(T) = XT + TX$.

b. Is f one one on the complement of the set $B(I_n, R)$ for sufficiently big R?

c. Is there a neighbourhood of I_n on which f is one one ? **3 + 2 + 5 = 10**

(4) Let U be an open ball in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^1 function.

a. Prove that $\sup_{z \in \overline{U}} ||Df(z)|| < \infty$.

b. Suppose that $x_0 \in \mathbb{R}^n$ is such that $Df(x_0) = I_n$. Prove that there is a neighbourhood of x_0 on which f is one one. **3 + 7 = 10**

(5) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a submersion at a point $x_0 \in \mathbb{R}^3$. Define $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi(x_1, x_2, x_3) = (f(x_1, x_2, x_3), x_2, x_3)$.

a. Prove that there exists a neighbourhood U of x_0 such that $\phi : U \to \phi(U)$ is a diffeomorphism.

b. Prove that the map $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is the canonical submersion.

c. Is it possible to have an immersion from $\mathbb{R}^3 \to \mathbb{R}$? Explain giving reasons. **4 + 4 + 2 = 10**

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination
Programme: Master of Mathematics
Course: Linear Algebra

Duration: Three Hours Date : 29/08/2014

Answer all questions. Write brief but complete answers. The maximum you can score is 60.

1. Suppose V is a vector space over some field. Let $\{v_1, v_2, \cdots, v_n\}$ be a spanning set for V and let $\{w_1, w_2, \cdots, w_m\}$ be a linearly independent subset of vectors in V. Starting from the definitions, establish that we must have that $m \leq n$. For this question, you are not supposed to apply any theorem. (8 marks)

2. Suppose V is a vector space over some field and V_1, V_2, and W are subspaces of V. Suppose, moreover, $V = V_1 \oplus V_2$; i.e., V is the internal direct sum of V_1 and V_2. Is it necessarily true that $W = (W \cap V_1) \oplus (W \cap V_2)$? Prove it if it is true, give a counter example otherwise. (5 marks)

3. Suppose we have an $n \times n$ real matrix of rank $r < n$. Show that it is possible to remove $(n - r)$ rows and $(n - r)$ columns so that the resulting $r \times r$ matrix is nonsingular. (6 marks)

4. Suppose V is a finite dimensional vector space over some field. Suppose $T : V \longrightarrow V$ is a linear map having the property that for every $v \in V$, there is a positive integer k, which may depend on v, such that $T^k(v) = 0$. Show that there is a positive integer n such that $T^n(v) = 0$ for every $v \in V$.
(7 marks)

5. Let V be the vector space of all of $n \times n$ matrices over some field. Suppose W is the subspace of all matrices with zero trace. Give an example of a subspace W' of V that is complimentary to W (i.e., $W \oplus W' = V$).
(7 marks)

SEE THE OTHER SIDE

6. Let $A = (a_{ij})$ be an $n \times n$ real matrix of rank $(n-1)$. Show that there are indices i and j, $1 \leq i, j \leq n$, such that if we replace the entry a_{ij} by any other real number, the resulting matrix is of rank n. (8 marks)

7. (a) Define *determinant* as a multilinear map.
(b) Let $A = (a_{ij})$ be an $n \times n$ matrix over some field. Write an expression for the determinant of A as a sum involving prducts of the entries a_{ij}.
(c) Suppose ρ and τ are two permutation of the set $\{1, 2, \cdots, n\}$. Suppose B is the matrix (b_{ij}), where

$$b_{ij} = a_{\rho(i)\tau(j)} \text{ for } 1 \leq i, j \leq n.$$

Find with explanations the relation between $det(A)$ and $det(B)$.
(3+2+6=11 marks)

8. Suppose V is a vector space over an infinite field. Let $\{v_1, v_2, \cdots, v_n\}$ and $\{w_1, w_2, \cdots, w_n\}$ be two bases of V. Prove that there are infinite number of scalers α such that $\{v_1 + \alpha w_1, v_2 + \alpha w_2, \cdots, v_n + \alpha w_n\}$ is a basis of V.
(13 marks)

Indian Statistical Institute
Semestral Examination: 2014-15

Course Name: M. Math, 1st year
Subject Name : Analysis of several variables
Date: ~~28.08.2014~~, Maximum Marks: 60, Duration: 3 hours
31.10.14
Answer any four questions

1. a. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^1 function such that $0 \in \text{Range}(f)$. Let $S = f^{-1}(0)$. Assume that $\frac{\partial f(p)}{\partial x_{n+1}} \neq 0 \ \forall p \in S$.

   i. Define $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $\Phi(x_1, x_2, ..., x_{n+1}) = (x_1, x_2, ..., f(x_1, x_2, ..., x_{n+1}))$. If $p \in S$, prove that $D\Phi(p)$ is invertible.

   ii. Fix a point p in S. Prove that there exists a suitable neighbourhood V of p and co-ordinates $(y_1, y_2, ..., y_{n+1})$ on V such that $S \cap V = \{(y_1, ..., y_{n+1}) : y_{n+1} = 0\}$.

   b. Show that the system of equations:

   $3x + y - z + u^2 = 0, \ x - y + 2z + u = 0, \ 2x + 2y - 3z + 2u = 0$

   can be solved for x, y, u in terms of z. **1 + 5 + 4 = 10**

2. a. If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable such that for all $1 \leq i, j \leq n$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are continuous, prove that there exist suitable bases such that the matrix of $D^2 f$ is given by $((\frac{\partial^2 f}{\partial x_i \partial x_j}))_{1 \leq i,j \leq n}$.

   b. Let $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be defined by $f(A) = A^{-1}$. Using the fact $Df(A)(H) = -A^{-1}HA^{-1}$, deduce a formula for $D^2(f)(A)$. **4 + 6 = 10**

3. a. Let C be a set in $\mathbb{R}^n$ such that the boundary of C has measure zero and $\int_C 1 = 0$. Prove that the set C has content 0.

   b. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function such that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous.

   i. Prove that there can be no rectangle R such that $\int_R (\frac{\partial^2 f}{\partial x \partial y}(x, y) - \frac{\partial^2 f}{\partial y \partial x}(x, y)) dx dy > 0$.

P.T.O

ii. Use i. to show that $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$. **3 + 5 + 2 = 10**

4. a. Let U be an open set in $\mathbb{R}^n$ and f a smooth real valued function f on U such that $df = 0$. What can you say about f?

   b. Let U be an open set in $\mathbb{R}^n$ such that $0 \in U$. Let C_0^∞ be the vector space of smooth real valued functions defined on some neighbourhood W of 0 in U. For a vector $v = (v_1, ..., v_n) \in \mathbb{R}^n$ let $D_{0,v}$ denote the linear functional on C_0^∞ defined by $D_{0,v}(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(x)$.

   i. Prove that the definition of $D_{0,v}$ does not depend on the choice of W.

   ii. For $i = 1, 2, ..., n$, x_i will denote the projection onto the i-th co-ordinate. Let $f \in C_0^\infty$ such that $f(0) = 0$, prove that $f(x) = \int_0^1 \frac{\partial f}{\partial t}(tx)dt$ and deduce that there are smooth real valued functions $g_1, ..., g_n$ on U such that $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ and $f = \sum_{i=1}^n x_i g_i$.

   iii. Let δ be a linear functional on C_0^∞ satisfying Leibnitz condition. Prove that $\delta(1) = 0$.

   iv. Prove that there is a vector v in $\mathbb{R}^n$ such that δ is of the form $D_{0,v}$.

   2 + 1 + (1 + 2) + 1 + 3 = 10

5. a. Let g_1, g_2 be C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}$ such that $\frac{\partial g_2}{\partial x} = \frac{\partial g_1}{\partial y}$.

   i. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt$. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   ii. Without using Poincare' Lemmà, prove that any closed one form on $\mathbb{R}^2$ is exact.

   b. Let $U = \mathbb{R}^2 - \{0\}$ and ω be the 2-form on U defined by $\omega(x,y) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$.

   i. Prove that ω is a closed form.

   ii. Let ϵ be a positive number and $\gamma : (-\epsilon, 2\pi + \epsilon) \to U$ is defined by $\gamma(t) = (r\cos t, r\sin t)$. Evaluate $\int_\gamma \omega$.

   iii. Deduce that ω is not exact. **2 + 3 + 2 + 3 = 10**

P.T.O

6. a. Let V be an n-dimensional vector space. Show that every non-zero alternating n-form on V is the volume element determined by some inner product and orientation for V.

   b. Let ω be a differential 1-form and C a 2 chain in $\mathbb{R}^3$. Prove that $\int_C d\omega = \int_{\partial C} \omega$.

   c. Let η be a differential 2-form and ξ a differential 1-form on $\mathbb{R}^4$ such that $d\xi = \eta$. Is ξ unique? **3 + 6 + 1 = 10**

7. Class Test **20**

INDIAN STATISTICAL INSTITUTE

SEMESTRAL EXAMINATION : (2014-2015)
M. MATH I
ALGEBRA I
NOVEMBER 3, 2014 MAXIMUM MARKS : 50 DURATION : $3\frac{1}{2}$ HOURS

By a ring we mean a commutative ring with 1.

(1) (a) Let R be a ring and M, N, P be R-modules. Establish the isomorphism: [5]

$$(M \oplus N) \otimes_R P \simeq (M \otimes_R P) \oplus (N \otimes_R P)$$

(b) Let m, n be positive integers and $d(> 0)$ be the gcd of m, n. Prove that [5]

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$$

(2) Let R be a ring and $0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$ be an exact sequence of R-modules. Assume that there is an R-module morphism $v : M \longrightarrow M'$ such that $v\phi = id_{M'}$. Prove that [5+5]

(a) there is an R-module morphism $u : M'' \longrightarrow M$ such that $\psi u = id_{M''}$

(b) $M = \phi(M') + u(M'')$ and $\phi(M') \cap u(M'') = \{0\}$.

(3) (a) Let R be a Noetherian ring. Prove that the ring $R[X]$ is also Noetherian. [5]

(b) Determine whether $\dfrac{\mathbb{Q}[X, Y]}{< X^2 + Y^2 - 1 >}$ is: (i) a Noetherian ring; (ii) an integral domain. Clearly state the results that you used. [2+3]

(4) (a) Let R be an integral domain with quotient field F and let $p(X) \in R[X]$ be a monic polynomial. Assume that $p(X) = a(X)b(X)$, where $a(X), b(X)$ are monic in $F[X]$ of smaller degree than $p(X)$. Prove that if $a(X) \notin R[X]$, then R is not a UFD. Deduce that $\mathbb{Z}[2\sqrt{2}]$ is not a UFD. [5]

(b) Prove that the ideal $< 2, 1 + \sqrt{-5} >$ in $\mathbb{Z}[\sqrt{-5}]$ is a prime ideal. [5]

(5) (a) Let R be a ring such that for every element $a \in R$, there is an integer $n(a) > 1$ such that $a^{n(a)} = a$. Prove that every prime ideal in R is maximal. [5]

(b) Let k be a field and consider the polynomial ring $k[X]$. Show that there are infinitely many mutually non-associate prime elements in $k[X]$. [3] [PTO]

(c) Show that every non-zero *minimal prime ideal* $\mathfrak{p}$ (i.e., $\mathfrak{p}$ does not contain any non-zero prime ideal other than $\mathfrak{p}$) in a UFD is principal. [3]

(d) Let M, N, P be R-modules and $f : M \longrightarrow P$ be an injective map of R-modules. Is the induced map $f \otimes 1 : M \otimes N \longrightarrow P \otimes N$, sending $m \otimes n$ to $f(m) \otimes n$, always injective? Justify your answer. [3]

(e) Let R be the direct product ring $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (under componentwise addition and multiplication). Let P be the principal ideal in R genereted by $(1, 0)$. Show that P is a projective R-module which is not free. [3]

Semestral Examination
M. Math First year, First Semester
2014-2015

Topology I

Date: 7 November, 2014

Maximum Marks: 60

Duration: 3 hours

Write in legible handwriting. Justify each step - do not skip any argument.
If you use any standard theorem then state it explicitly.

Group A
(Answer any three)

Notation: $C(X, Y)$ = the set of continuous functions from X to Y.

(1) Prove that every compact Hausdorff space X can be embedded in $[0,1]^J$ for some index set J. 8

(2) Let X be a completely regular space and βX its Stone-Čech compactification. Prove that if X is disconnected then so is βX. 8

(3) Show that the n-dimensional real projective space $\mathbb{R}P^n$ contains an open subset which is homeomorphic to an open ball in $\mathbb{R}^n$. Hence conclude that $\mathbb{R}P^n$ is a compactification of $\mathbb{R}^n$. 8

(4) Prove that if X and Y are locally compact Hausdorff, then the composition map $C(X, Y) \times C(Y, Z) \to C(X, Z)$ defined by

$$(f, g) \mapsto g \circ f \text{ for } f \in C(X, Y),\ g \in C(Y, Z)$$

is continuous with respect to the compact open topologies on the spaces. 6

Group B
(Answer all questions)

Notation: $\mathbb{S}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$.

(1) Suppose that A is a strong deformation retract of a topological space X. Show that the inclusion map $i : A \hookrightarrow X$ is a homotopy equivalence. Hence show that $i_\# : \pi_1(A, a) \to \pi_1(X, a)$ is an isomorphism, where a is any point of A. 5+3

P.T.O

(2) Let X be a topological space and CX denotes the quotient space $X \times [0,1]/X \times \{1\}$. Prove that CX is contractible. 6

(3) Determine the fundamental group of $\mathbb{S}^2 \vee \mathbb{S}^3$. Recall that if X and Y are two topological spaces and $p \in X$, $q \in Y$, then $X \vee Y = X \sqcup Y / p \sim q$. 6

(4) (a) Let $p : (E, e_0) \to (B, b_0)$ is a covering projection and $p(e_0) = b_0$. Suppose that α and β are two paths in B originating from b_0 which are path homotopic. If $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts of α and β respectively with initial point e_0 then prove that $\tilde{\alpha}(1) = \tilde{\beta}(1)$. (You may assume the homotopy lifting theorem) 8

(b) Let α be a loop in B with base point b_0 and $x \in p^{-1}(b_0)$. Suppose that $\tilde{\alpha}_x$ is the unique lift of α in E starting at x. Prove that $x.[\alpha] = \tilde{\alpha}_x(1)$ defines an action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$. 8

(c) Consider the covering projection $f : \mathbb{S}^1 \to \mathbb{S}^1$ given by $f(z) = z^n$. Identify the fibre $f^{-1}(1)$ as a cyclic group G. Then interpret the action of $\pi_1(\mathbb{S}^1, 1)$ on $f^{-1}(1)$ as an action of $\mathbb{Z}$ on G. 6

Indian Statistical Institute

Semestral Examination

First Semester, M.Math. First Year, 2014-15

Course: Linear Algebra

Maximum marks: 60

Duration: Three Hours Date: 10 November, 2014

Answer all questions. State precisely the results that you are using in each answer. Give brief, coherent and complete answers. The maximum you can score is 60.

1. Suppose $(V, \langle , \rangle)$ is a real inner product space. Give a complete proof of the Cauchy-Schwarz inequality:

$$|\langle u, v\rangle| \leq ||u||||v|| \text{ for all } u, v \in V.$$

Here $||x||$ denotes the non-negative square root of $\langle x, x\rangle$. For this question you are not allowed to use any other result proved in the class. (10 points)

2. Suppose V is a finite-dimensional vector space over a field. Suppose $T : V \longrightarrow V$ is a projection; i.e., $T^2 = T$. Is it true that T is diagonalizable? Prove your assertion. (7 points)

3. Give an example of a pair of real matrices whose characteristic polynomials are the same but the matrices are not similar (over $\mathbb{C}$). Explain why the matrices are not similar. (7 points)

4. Suppose A is a complex square matrix such that $A^4 = I$. Show that A is diagonalizable. (7 points)

5. Suppose U and V are two finite-dimensional vector spaces over $\mathbb{C}$. Show that $U^* \otimes V^*$ and $(U \otimes V)^*$ are isomorphic as vector spaces over $\mathbb{C}$. Here, U^* and V^* denote the dual spaces of U and V respectively. (10 points)

6. Suppose two complex normal square matrices A and B are similar. Show that there is a unitary matrix U such that $U^{-1}AU = B$. (10 points)

7. Show that if two 2×2 real matrices have the same characteristic and minimal polynomials then they are similar over $\mathbb{R}$. (12 points)

INDIAN STATISTICAL INSTITUTE, KOLKATA

FINAL EXAMINATION: FIRST SEMESTER 2014 -'15

M.MATH I

Subject : **Measure Theoretic Probability**

Time : 4 hours

Maximum score : 65 14·11·2014

Instructions:

- *Justify every step in order to get full credit of your answers, stating clearly the result(s) that you use. Points will be deducted for missing arguments. Partial credit will be given for your approach to the problem.*
- *Total marks carried by the questions turns out to be 95 which is more than 65. The total marks obtained will be multiplied with* $\frac{13}{19}$.

(1) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables on it such that $X_n \overset{d}{\sim}$ Binomial $(n, \frac{\lambda}{n})$ for all $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} P\{X_n = k\} = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for all } k \in \{0\} \cup \mathbb{N}.$$

Hints: $e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.

[10 marks]

(2) Let μ and ν be two finite measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Show that there exists a unique measure λ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ such that $\lambda(F) = \int_{\mathbb{R}} \nu\{x + y : (x, y) \in F\}\ \mu(dx)$ for all $F \in \mathcal{B}(\mathbb{R}^2)$. Before using any result for proving this, you should make sure that the hypothesis for the result holds.

[15 marks]

(3) (a) Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of random variables on the probability space $(\Omega, \mathcal{F}, P)$. Show that $\bigcup_{\text{finite } F\subset\Lambda} \sigma\{X_\lambda : \lambda \in F\}$ is a field which generates $\sigma\{X_\lambda : \lambda \in \Lambda\}$.
(b) If $\{\Lambda_i\}_{i\in I}$ is a partition of Λ and $\{X_\lambda\}_{\lambda\in\Lambda}$ is independent, then the family $\{\ \mathcal{F}_i := \sigma\{X_\lambda : \lambda \in \Lambda_i\}\ \}_{i\in I}$ (of sub-σ-fields of $\mathcal{F}$) is independent.

[15 marks]

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(4) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\{A_1, \ldots, A_n\} \subset \mathcal{F}$ and $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$. If $\mathcal{G} = \sigma\{B_1, B_2, B_3\} \subset \mathcal{F}$ and $f = \sum_{1 \leq i \leq n} \lambda_i \, 1_{A_i}$, then evaluate $E(f|\mathcal{G})$ from its definition.

[10 marks]

(5) Let $\mathcal{X}$ be the set of all random variables on the probability space $(\Omega, \mathcal{F}, P)$. Define $\rho : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by $\rho(X, Y) = E\left(\min\{|X - Y|, 1\}\right)$. Show that

(a) $(\mathcal{X}, \rho)$ is a metric space and

(b) a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ converges in probability to $X \in \mathcal{X}$ if and only if X_n converges to X in ρ.

[15 marks]

(6) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Set $\mathcal{S} = \{f : \Omega \to \mathbb{C} : f \text{ is Borel measurable}\}$. Define $p : \mathcal{S} \to [0, \infty]$ by $p(f) = \sup\{M \in [0, \infty) : \mu\{|f| \geq M\} > 0\}$ for all $f \in \mathcal{S}$. Show that

(a) $p(f) < \infty$ implies $p(f) = \inf\{M \in [0, \infty) : \mu\{|f| > M\} = 0\}$,

(b) $p(f) < \infty$ and $p(g) < \infty$ implies $p(f + g) \leq p(f) + p(g)$,

(c) $p(f) = 0$ implies $f = 0$ a.e. $[\mu]$, and

(d) $p(\alpha f) = |\alpha| \, p(f)$.

[15 marks]

(7) Let $\mathcal{F}$ be a σ-field on Ω, and λ and μ be measures on $(\Omega, \mathcal{F})$ such that μ is sigma finite and $\lambda \ll \mu$. Show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $A \in \mathcal{F}$ and $\mu(A) < \delta$, we have $\lambda(A) < \varepsilon$.

Hint: Assume the conclusion does not hold. Then, apply various results such as Borel-Cantelli's lemma, Fatou's lemma, etc.

[15 marks]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination

M. Math First year, First Semester

2014-2015

Topology I

Date: 31·12·2014 Maximum Marks: 45

Duration: 3 hours 15 minutes

Write in legible handwriting. Justify each step - do not skip any argument.
If you use any standard theorem then state it explicitly.

Notation: $C(X,Y) =$ the set of continuous functions from X to Y.

(1) Show that every subspace of a normal space is completely regular. 7

(2) Let $p : E \to B$ be a covering projection, where the base space B is path-connected. Prove that any two fibres of p are equinumerous. 10

(3) Suppose that G is a topological group. Prove the following statements:
(a) Every open subgroup of G is closed.
(b) If H is a subgroup of G then so is the closure of H. 5+5

(4) Let $J_1 = (0,1)$ and $J_2 = (2,3)$ be two open intervals in $\mathbb{R}$. What is the 1-point compactification of $J_1 \cup J_2$? Justify your answer. 8

(5) Let X, Y be two topological spaces and Z a subspace of X. Prove that the restriction map $C(X,Y) \to C(Z,Y)$, defined by $f \mapsto f|_Z$ for all $f \in C(X,Y)$ is continuous with respect to the compact open topologies on the function spaces. 7

(6) Prove that the quotient space $S^1 \times [0,1]/S^1 \times \{1\}$ is homeomorphic to a 2-disc. 10

(7) Show that a contractible space is path-connected. 6

(8) Prove that the Möbius band is homotopically equivalent to a circle. 10

(9) Determine the fundamental group of the space $\mathbb{S}^n \vee \mathbb{S}^1$, $(n > 1)$. (Recall that, if X and Y are two topological spaces and $p \in X$, $q \in Y$, then $X \vee Y = X \sqcup Y/p \sim q$.) 7

P.T.O.

(10) If G is a topological group then prove that $\pi_1(G, e)$ is abelian. 7

(11) Suppose that $p : (E, e_0) \to (B, b_0)$ is a covering projection and γ is a path which lies in an evenly covered open set U in B. If $\gamma(0) = b_0$ then prove that γ has a unique lift in E starting at e_0. 8

(12) Find a generator τ of $\pi_1(S^1, 1))$. Let $p : S^1 \to S^1$ be defined by $p(z) = z^2$. Prove that the homomorphism $p_{\#} : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ maps τ onto 2τ. 4+6

INDIAN STATISTICAL INSTITUTE

SEMESTRAL EXAMINATION (BACK PAPER) : (2014-2015)

M. MATH I

ALGEBRA I

02/01/15
DATE

MARKS : 100 DURATION : 3 HOURS

By a ring we mean a commutative ring with 1.

(1) Let $R = C([0,1])$. For each $c \in [0,1]$, let $M_c = \{f \in C([0,1]) \mid f(c) = 0\}$. [3+5+2]

(a) Prove that M_c is a maximal ideal of R.

(b) Prove that any maximal ideal of R is of the form M_c for some $c \in [0,1]$.

(c) If $b, c \in [0,1]$ and $b \neq c$, then show that $M_b \neq M_c$.

(2) Let R be an integral domain and $\mathfrak{P}$ be a prime ideal of R. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in R[X]$ be such that $a_{n-1}, \cdots, a_1, a_0 \in \mathfrak{P}$ and $a_0 \notin \mathfrak{P}^2$. Then prove that f is irreducible in $R[X]$. [10]

(3) Let R be a ring and I be an ideal of R. Prove that $\text{rad}(I)$ is the intersection of all the prime ideals of R containing I. [10]

(4) Let R be a PID. Prove that every non-zero prime ideal is maximal in R. Give an example, with full justification, to show that $R[X]$ may not be a PID. [4+6]

(5) Let $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$. Find all the units of the ring $\mathbb{Z}[\sqrt{-5}]$. Is this ring isomorphic to the ring $\mathbb{Z}[\sqrt{-2}]$? [6+6]

(6) Let R be a ring and L, M, N be R-modules. Prove the following [4+4+4]

(a) $\text{Hom}_R(R, M) \simeq M$ (as R-modules).

(b) $\text{Hom}_R(R, R) \simeq R$ (as rings).

(c) $\text{Hom}_R(M, L \oplus N) \simeq \text{Hom}_R(M, L) \oplus \text{Hom}_R(M, N)$. [PTO]

(7) (a) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$. [4]

(b) Let R be a ring and M be an R-module. Prove that the R-modules M and $R \otimes_R M$ are isomorphic. [4]

(c) Let $f : R \to S$ be a morphism of rings and M be a finitely generated R-module. Prove that $M \otimes_R S$ is a finitely generated S-module. [4]

(8) (a) Let R be a ring and P be a projective R-module. Prove that $Hom_R(P, R)$ is also a projective R-module. [6]

(b) Let $R = \mathbb{Z}/6\mathbb{Z}$. Give an example of a projective R-module which is not free. [6]

(9) Let R be a ring and $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. Prove that M is Noetherian if and only if both M' and M'' are Noetherian R-modules. [12]

Indian Statistical Institute
Mid Semestral Examination: 2014-15

Course Name: M. Math, 1st year
Subject Name : Differential Geometry I
Date: 23.02.2015, Maximum Marks: 40, Duration: Two and a half hours

Answer any four questions

1. a. Let M be a manifold of dimension n and f a function from M to $\mathbb{R}$. Let $p \in M$ and (U_α, ϕ_α) be a chart around p. f is called C^∞ at p if the function $f \circ \phi_\alpha^{-1}$ is C^∞. Prove that this definition is independent of the choice of (U_α, ϕ_α).

   b. Give an example of a parametrized n-surface S in $\mathbb{R}^{n+1}$ and a point p in S which need not admit a neighborhood in S which is an n-surface.

   c. Can the figure eight be an integral curve to a smooth vector field on $\mathbb{R}^2$?

   d. Is $\mathbb{R}^n$ itself an example of an n-surface?

   e. If a 1-surface in $\mathbb{R}^2$ can be globally parametrized, prove that it has to be connected.
   $\mathbf{3 + 2 + 2 + 2 + 1 = 10}$

2. a. Let $F : M_n(\mathbb{R}) \to \mathbb{R}^n$ be defined as $F = (F_1, ..., F_n)$ where $F_i(A) = \sum_j a_{ij}^2 - d_i^2$ where A_{ij} denote the entries of the matrix A and $d_1, ..., d_n$ are strictly positive real numbers. Consider the surface $F^{-1}(0)$. If f is a real valued smooth function on S, does it always attain its extremeum on S?

   b. Let S be an n-surface in $\mathbb{R}^{n+1}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function such that g attains its extremeum on S at a point p and the set $g^{-1}(g(p))$ is an n-surface. Prove that the tangent space to the level set of g through p equals T_pS.

   c. Let $S = \cap_i f_i^{-1}(c_i)$ be an n-surface in $\mathbb{R}^{n+k}$. Give an example (a clear figure should be sufficient) to show that $\{\nabla f_i(p) : i\}$ need not be a mutually orthogonal set of vectors. $\mathbf{3 + 4 + 3 = 10}$

3. a Prove that a connected manifold is always path connected.

   b. Let S be a connected n-surface in $\mathbb{R}^{n+1}$. Let

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$$d(x,y) = \inf_\gamma\{l(\gamma) : \ \gamma \text{ is a smooth curve joining x and y}\}.$$

Prove that if $d(x,y) = 0$, then $x = y$.

c. Deduce the formula for the length of a curve from the formula for the volume of a parametrized n-surface in $\mathbb{R}^{n+1}$. **2 + 4 + 4 = 10**

4. Let $\phi : U \to \mathbb{R}^{n+1}$ be a parametrized n-surface in $\mathbb{R}^{n+1}$. Let X be the vector field along ϕ whose i-th component is given by $(-1)^{n+1+i}$ times the determinant of the matrix obtained by deleting the i-th column from the matrix $(\frac{\partial\phi}{\partial x_1}, ..., \frac{\partial\phi}{\partial x_n})^T$, where $\frac{\partial\phi}{\partial x_i}$ are the co-ordinate vector fields along ϕ.

   a. Show that $X(p) \neq 0$ for all p in U.

   b. Show that X is a normal vector field along ϕ.

   c. Show that $N = \frac{X}{||X||}$ is the orientation vector field along ϕ.

   d. Conclude that N is smooth. **2 + 4 + 2 + 2 = 10**

5. Let U be a vector field $\mathbb{R}^n$. Using the Anubhav-Asfaq example, demonstrate that the uniqueness of the integral curve at a point p can fail if X is not smooth. **10**

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination (2014–2015)

M MATH I

Functional Analysis

Date : 24.02.2015 | Maximum Marks : 90 | Time : 3 hrs.

This paper carries 105 marks. Maximum you can score is 90. Precisely justify all your steps. Carefully state all the results you are using.

1. (a) Define a bounded set in a topological vector space X. [2]
   (b) If the topology of X is generated by a family $\{p_i : i \in I\}$ of seminorms, show that $A \subseteq X$ is bounded if and only if each p_i is bounded on A. [8]
2. Show that if f is a linear functional on a topological vector space X, then $\ker f = \{x \in X : f(x) = 0\}$ is either closed or dense in X. [8]
3. Let X be an infinite dimensional normed linear space.
   (a) Show that the weak topology on X is strictly weaker than the norm topology. [8]
   (b) However, if $A \subsetneq X$ is a convex set, then the closure of A is the norm and the weak topology are the same. [9]
4. Prove that in any infinite dimensional *separable* Banach space there exists a dense independent sequence. [10]
   [Hint : No proper subspace can have nonempty interior.]
5. Let Y be a linear subspace of a normed linear space X, and let $x_0 \in X$ be such that
$$d = d(x_0, Y) = \inf_{y \in Y} \|y - x_0\| > 0.$$
   Show that there exists $f \in X^*$ such that $f(x_0) = 1$, $\|f\| = \frac{1}{d}$, and $f|_Y \equiv 0$. [10]
6. Let X and Y be normed linear spaces and $T : X \to Y$ be a linear map. Define
$$\|x\|_1 = \|x\| + \|T(x)\|, \quad x \in X.$$
   (a) Show that $\|\cdot\|_1$ is a norm on X. [5]
   (b) Show that $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ if and only if T is continuous. [5]
7. Show that if any two norms on a vector space X are equivalent, then X must be finite dimensional. [10]

8. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a map which is linear in the first variable and conjugate-linear in the second variable.

Further assume that there exists $C > 0$ such that $|B(x, y)| \leq C\|x\|\|y\|$ for all $x, y \in \mathcal{H}$.

Show that there exists a unique bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ such that $B(x, y) = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$. [10]

9. Let $P[0, 1]$ denote the space of all polynomials on $[0, 1]$ with complex coefficients. Define $\| \cdot \| : P[0, 1] \to [0, \infty)$ by

$$\|f\| = \left(\int_0^1 |f(t)|^2 dt \right)^{\frac{1}{2}}$$

for $f \in P[0, 1]$.

(a) Show that $\| \cdot \|$ comes from an inner product.

(b) Find a basis for $P[0, 1]$ that is orthonormal with respect to this inner product. [2 + 8 = 10]

(c) Show that $L^2[0, 1]$ is the completion of $P[0, 1]$ in this norm. [10]

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2014-15
M. Math.-I Year
Topology-II

Date : 25. 02. 2015 | Maximum Score : 40 | Time : 2:30 Hours

Any result that you use should be stated clearly. All topological spaces under consideration are assumed to be Hausdorff.

(1) (a) Define the notion of a covering space of a path connected, locally path connected space. Prove that any covering projection $p : X \longrightarrow Y$ is an open map.

(b) Let $p : X \longrightarrow Y$ be a covering projection. Let $y \in Y$ and $\pi = \pi_1(Y, y)$. Let Δ be the Deck transformation group of p. Let $F_y = p^{-1}(y)$. Prove that F_y is a right π-space and also a left Δ-space by defining suitable actions.

(c) Compute the isotropy subgroups at any point $x \in F_y$ with respect to the right π-action and the left Δ-action on F_y.

(d) Prove that for any $x \in F_y$, $[\sigma] \in \pi$ and $h \in \Delta$, $h(x[\sigma]) = (hx)[\sigma]$.

[6 + 6 + 6 + 5 = 23]

(2) Justify the following statements.

(a) Let X be a topological space and $f : X \longrightarrow S^n$, S^n being the unit sphere in $\mathbb{R}^{n+1}$, be a continuous function which is not surjective. Then f is null homotopic.

(b) Let $q : \tilde{B} \longrightarrow B$ be the universal covering map of a path connected, locally path connected and semi-locally simply connected space B. Let $p : E \longrightarrow B$ be any other covering of B. Then there always exists a map $\phi : \tilde{B} \longrightarrow E$ satisfying $p \circ \phi = q$ such that ϕ is a covering map of E.

(c) Let $\mathbb{R}P^2$ be the real projective plane and $f : \mathbb{R}P^2 \longrightarrow S^1 \times S^1$ be any continuous function. Then f must be homotopic to a constant map.

[5 + 5 + 5= 15]

(3) (a) Define the notion of properly discontinuous action of a group on a space.

(b) Let G be a finite group and X be a path connected, locally path connected G-space. Assume that the action is free. Prove that the canonical map $p : X \longrightarrow X/G$ sending each point x to its orbit Gx is a covering projection.

(c) State Van Kampen theorem. Let $n > 0$ be any integer and $X(n)$ be the complement of n distinct points in $\mathbb{R}^2$. Compute fundamental group of $X(n)$.

[2 + 4 + 6 = 12]

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2014-15 (Second Semester)

M. MATH. I YEAR
Algebra II

Date: 26.02.2015 | Maximum Marks: 40 | Duration: 3 Hours

Throughout the paper, k will denote a field.

Group A
Attempt ANY FIVE questions.
Each question carries 6 marks.

1. Prove that any splitting field E of a polynomial $f(X) \in k[X]$ is a finite normal extension of k.
2. Find the number of distinct irreducible monic polynomials of degree 2 and 3 over $\mathbb{F}_7$.
3. Compute $[\mathbb{Q}(\sqrt{2} + \sqrt[3]{5}) : \mathbb{Q}]$.
4. Let α be algebraic over k. Prove that the number of distinct embeddings of $k(\alpha)$ in a normal closure of $k(\alpha)$ is at most $[k(\alpha) : k]$. Prove that it is a factor of $[k(\alpha) : k]$.
5. Let G be a finite group of order p^2q, where p and q are distinct primes. Prove that G has a proper nontrivial normal subgroup.
6. Let H be a finite subgroup of a group G. Consider the action of the group $H \times H$ on G
$$\phi : (H \times H) \times G \to G,$$
defined by $\phi((h_1, h_2), g) = h_1 g h_2^{-1}$ for $h_1, h_2 \in H$ and $g \in G$. Show that every orbit under this action has $|H|$ elements if and only if H is a normal subgroup of G.

Group B
State with justifications whether the following statements are true or false.
Each question carries 3 marks.

1. If E is a splitting field of a polynomial $f \in \mathbb{Q}[X]$ such that $[E : \mathbb{Q}] = 3$, then all roots of f are in $\mathbb{R}$.
2. If $K \subseteq F \subseteq L$ are fields with $L|_F$ separable, then $L|_K$ is necessarily separable.
3. The polynomial $X^3 + 15X^2 + 6$ is reducible in $\mathbb{Q}(\sqrt{3}, \sqrt{2})[X]$.
4. Any finite extension $L|_k$ of degree p over a field k of characteristic $p(> 0)$ is necessarily a purely inseparable extension.
5. Any finite group G acting transitively on a set X with $|X| \geq 2$ necessarily has an element g which has no fixed point.

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2014-15

M. Math. I Year **Complex Analysis**

Date: 27.02.2015 Maximum Marks: 40 Duration: 3 Hours

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(1) (a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, where

$$a_n = \frac{n^2}{4^n + 3n}.$$

[4]

(b) Suppose that f is holomorphic in an open set Ω. If $|f|$ is constant, then show that f is constant. [6]

(2) Let $\xi \geq 0$. Let γ denote the contour consisting of the semicircle of radius R in the lower half-plane and the line segment from R to $-R$ considered counter-clockwise. Integrate a suitable function along the contour γ to prove that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i \xi x}}{(1+x^2)^2}\, dx = \frac{\pi}{2} e^{-2\pi\xi}(1 + 2\pi\xi).$$

[10]

(3) Let f be an entire function which is also injective (that is, one-to-one). Show that $f(z) = az + b$, with $a, b \in \mathbb{C}$ and $a \neq 0$. [10]

[Hint. Assume that f is not a polynomial. Then show that $g(z) = f(1/z)$ has an essential singularity at 0. Apply the Casorati-Weierstrass theorem. The sets $U = \{z : 0 < |z| < 1\}$ and $V = \{z : |z| > 1\}$ may be useful.]

(4) Let f be a non-constant holomorphic function in Ω, where Ω is an open set in $\mathbb{C}$ containing the closed unit disc. Suppose that $|f(z)| = 1$ whenever $|z| = 1$.

(a) Show that $f(z) = 0$ has a root. [5]

[Hint. If not, then $g(z) = 1/f(z)$ is holomorphic in Ω. Apply maximum modulus principle to conclude that $|f|$ is constant.]

(b) Show that $f(\Omega)$ contains the unit disc $\mathbb{D}$. [5]

[Hint. Let $w_0 \in \mathbb{D}$ and consider the constant function $h(z) = -w_0$. Apply Rouche's theorem to conclude that $f(z) - w_0 = 0$ has a root.]

(5) (a) Let S be a finite subset of $\mathbb{Z}$ and $f(z) = \sum_{k \in S} c_k z^k$. What is the residue of f at 0? [2]

(b) For $m \in \mathbb{N}$, let $I_m = \int_0^{2\pi} (\cos\theta)^m \, d\theta$. Show that

$$I_m = \frac{1}{i 2^m} \int_C (z + z^{-1})^m z^{-1}\, dz,$$

where C is the unit circle. Find the value of I_m. [8]

INDIAN STATISTICAL INSTITUTE

Semestral Examination, Semester II (2014–2015)

M MATH, 1ST YEAR

Functional Analysis

Date : 24.04.2015 | Maximum Marks : 100 | Time : $3\frac{1}{2}$ hrs.

This paper carries 120 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. Let X be a normed linear space. Show that the norm on X is lower semicontinuous with respect to the weak topology. [5]

2. (a) Let X be a Banach space. Show that any Hamel (algebraic) basis of X is either finite or uncountable. [10]

   (b) Give an example of a normed linear space with a countable Hamel basis. [5]

3. Let X, Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$. Show that there is a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$ if and only if T is 1-1 and $T(X)$ is closed in Y. [20]

4. Let $\{\mathcal{H}_k\}_{k\in\mathbb{N}}$ be a sequence of Hilbert spaces and $T_k \in \mathcal{L}(\mathcal{H}_k)$ for all $k \in \mathbb{N}$ such that $\sup\limits_{k\in\mathbb{N}} \|T_k\| < \infty$.

   (a) Find $\sigma_p\left(\bigoplus\limits_{k\in\mathbb{N}} T_k\right)$ in terms of $\sigma_p(T_k)$, $k \in \mathbb{N}$ and justify.

   (b) Show that $\overline{\bigcup\limits_{n\in\mathbb{N}} \sigma(T_k)} \subseteq \sigma\left(\bigoplus\limits_{k\in\mathbb{N}} T_k\right)$.

   (c) Show that equality occurs when T_k's are scalar multiple of identities of the respective Hilbert spaces. [20]

5. Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be a nilpotent operator, that is, $T^k = 0$ for some $k \in \mathbb{N}$. Find $\sigma(T)$ and justify. When does such a T become normal? [10]

[P T O]

6. Find examples of a Hilbert space $\mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$ (with proper justifications) such that

   (a) both $\sigma_p(T)$ and $\sigma(T) \setminus \sigma_p(T)$ are dense in $\sigma(T)$.

   (Hint : Use 4(c).)

   (b) $\sigma_p(T) = \emptyset$. [15]

   (Hint : $T(f)(x) = xf(x)$ for $f \in L^2([0,1], m)$ where m is the Lebesgue measure.)

7. Let the sequence $\{T_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be convergent in Weak Operator Topology to $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ (where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces). Show that $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. [10]

8. Show that any normal operator $T \in \mathcal{L}(\mathcal{H})$ which is bounded below, has a bounded inverse (where $\mathcal{H}$ is a Hilbert space). [10]

9. Let Γ be a (possibly uncountable) set and $\{c_\gamma\}_{\gamma\in\Gamma}$ be a bounded family of complex scalars. If $\{e_\gamma\}_{\gamma\in\Gamma}$ is the canonical orthonormal basis of the Hilbert space $\mathcal{H} = \ell^2(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} \,\middle|\, \sum_{\gamma\in\Gamma} |f(\gamma)|^2 < \infty \right\}$, and $T \in \mathcal{L}(\mathcal{H})$ is given by

$$T(e_\gamma) = c_\gamma e_\gamma \text{ for } \gamma \in \Gamma$$

then show that T is compact if and only if $|\{\gamma \in \Gamma : |c_\gamma| \geq \varepsilon\}| < \infty$ for all $\varepsilon > 0$. [15]

Indian Statistical Institute
End Semester Examination: 2014-15

Course Name: M. Math, 1st year
Subject Name : Differential Geometry I
Date: 27.04.2015, Maximum Marks: 40, Duration: Three hours

Answer all questions

1. a. Let S be an n-surface in $\mathbb{R}^{n+1}$. Let X be a smooth unit tangent vector field defined on an open set U in S. Prove that the index of X at the point p is 0 for all p in U.

   b. Using part a. (and NOT otherwise), prove that there can be no smooth nowhere vanishing vector field on S^2.

   c. Prove that the sum of angles of a geodesic triangle on a S^2 is always greater than π.

   d. In part c., does your solution depend on the choice of orientation on S^2? **3 + 2 + 3 + 2 = 10**

2. a. For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0, \pi] \to S^2$ be the geodesic defined by

$$\alpha_\theta(t) = (\cos\theta \sin t, \sin\theta \sin t, \cos t).$$

   Note that α_θ joins the north pole $p = (0, 0, 1)$ to the south pole $(0, 0, -1)$. Let $v = (1, 0, 0) \in T_p(S^2)$ and N the outward normal on S^2.

   a. Show that $\dot{\alpha}_\theta(t)$ and $N(\alpha_\theta(t)) \times \dot{\alpha}_\theta(t)$ forms an orthonormal basis for $T_{\alpha_\theta(t)}S^2$.

   b. Compute the parallel translate P_α of v along α_θ at the point $\alpha_\theta(t)$.

   c. Let $v, w \in T_pS^2$ such that $||v|| = ||w||$. Show that there exists a piecewise smooth curve $\alpha : [a, b] \to S^2$ with $\alpha(a) = \alpha(b) = p$ such that $P_\alpha(v) = w$. **2 + 4 + 4 = 10**

3. Answer any four questions:

   a. Let S be an n-surface in $\mathbb{R}^{n+1}$. Let $p, q \in S$ and α a smooth curve from p to q.

P.T.O

If $v, w \in T_p(S)$, prove that the parallel translate P_α along α satisfies $P_\alpha(v + w) = P_\alpha(v) + P_\alpha(w)$. **5**

b. i. Show that the mean curvature at a point p of an oriented n-surface S can be computed from the values of normal curvature k on any orthonormal basis $\{v_1, v_2, ..., v_n\}$ for T_pS by the formula $H(p) = \frac{1}{n}\sum_{i=1}^{n} k_p(v_i)$.

ii. Rigorously compute $\int_{\text{Image }(\alpha_\theta)} d\zeta$ where α_θ is as in Problem 2 and $d\zeta$ is the measure coming from the volume form on S^2. **2 + 3 = 5**

c. Let S be a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ whose Gauss Kronecker curvature is nowhere zero. Prove that the Gauss map $N : S \to S^n$ is a diffeomorphism. **5**

d. Consider the cylinder $x_1^2 + x_2^2 = r^2$ in $\mathbb{R}^3$ with the normal given by $N(x_1, x_2, x_3) = \frac{1}{r}(x_1, x_2, 0)$. Compute all its geodesics. **5**

e. Let $S = \{p \in S_1 : g_1(p) \leq c_1, ..., g_k(p) \leq c_k\}$ where S_1 is an n-surface in $\mathbb{R}^{n+1}$ and $g_1, ..., g_k : S \to \mathbb{R}$ are real valued smooth functions on S_1 such that $g_i^{-1}(c_i) \cap g_j^{-1}(c_j)$ is empty whenever $i \neq j$ and such that $\nabla g_i(p) \neq 0$ whenever $p \in g_i^{-1}(c_i)$. Prove that S is an n-surface with boundary in $\mathbb{R}^{n+1}$. **5**

f. Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, oriented by $\frac{\nabla f}{||\nabla f||}$. Suppose that $p \in S$ is such that $\frac{\nabla f}{||\nabla f||} = e_{n+1}$, where for $1 \leq i \leq n+1$, e_i denotes the i-th canonical basis element of $\mathbb{R}^{n+1}$. Show that the matrix for the Weingarten map L_p w.r.t the basis $\{e_1, e_2, ..., e_n\}$ of T_pS is $\left(-\frac{1}{||\nabla f(p)||}\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$ **5**

INDIAN STATISTICAL INSTITUTE
Semestral Examination : 2014-2015
M. Math. - I Year
Topology-II

Date : 30. 04. 2015 | Maximum Score : 60 | Time :2:30 Hours

Any result that you use should be stated clearly.

(1) Answer the following with proper reasons.

(a) Let X_n be the circle in $\mathbb{R}^2$ with center at $(1/n, 0)$ and radius $1/n$. Let $X = \cup_{n\geq 1} X_n$ considered as a subspace of $\mathbb{R}^2$. Is X a CW complex?

(b) Let $X = (S^1 \vee S^1) \times \mathbb{R}P^2$. What is $H_1(X; \mathbb{Z})$?

(c) Let $p : E \longrightarrow B$ be a covering map, $b_0 \in B$, and $e_0 \in p^{-1}(b_0)$. Suppose the homomorphism $p_* : \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$ is surjective. Is p a homeomorphism ?

(d) Does there exists a covering $p : E \longrightarrow S^1 \vee S^1$ such that the Deck transformation group $\Delta(p)$ of p is isomorphic to S_3, the symmetric group on three symbols?

[5+5+5+5 =20]

(2) (a) State excision axiom for homology.

(b) Suppose $V \subset U \subset A \subset X$. Assume that V can be excised from (X, A) and

$$(X - U, A - U) \hookrightarrow (X - V, A - V)$$

is a deformation retract. Then prove that U can be excised.

(c) For $n \geq 1$, let

$$E^n_+ = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n |\ x_{n+1} \geq 0\},$$
$$E^n_- = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n |\ x_{n+1} \leq 0\}.$$

Use part (b) to prove that

$$H_q(E^n_+, S^{n-1}) \cong H_q(S^n, E^n_-)$$

for all q.

[4+6+6=16]

(3) (a) Define the notion of a CW complex.

(b) Describe what is known as cellular chain complex of a CW complex.

(c) Let X be the space obtained from S^1 by attaching a 2-cell by a map $f : S^1 \longrightarrow S^1$ of degree 2. Describe a CW complex structure of X and compute its homology by using cellular chain complex.

(d) Prove that $\mathbb{C}P^n$ can be obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell to it and hence describe a CW complex structure of $\mathbb{C}P^n$.

(e) Compute integral homology groups of $\mathbb{C}P^n$.

[4+6+6+4=20]

(4) (a) Deduce Mayer-Vietoris Exact sequence using Excision axiom.

(b) Let m, $n \geq 2$ and $X = S^m \vee S^n$. Apply Mayer-Vietoris exact sequence to compute the integral homology groups of X.

[8+6=14]

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination: 2014-15

M. MATH. I YEAR
Algebra II

Date: ~~06.05.2015~~ 05/05/15 Maximum Marks: 60 Duration: 4 Hours

Clearly state the results that you use.
Throughout the paper, k will denote a field.
$\mathbb{C}$: field of complex numbers
$\mathbb{Q}$: field of rational numbers

Group A
Answer ANY THREE questions.

1. Let ζ_7 be a primitive 7^{th} root of unity and let $\alpha = 2\cos(2\pi/7)$.
   (i) Determine the minimal polynomial of ζ_7 over $\mathbb{Q}(\alpha)$.
   (ii) Determine the minimal polynomial of α over $\mathbb{Q}$.
   (iii) Prove that $\mathbb{Q}(\alpha)|_{\mathbb{Q}}$ is a cyclic extension.
   (iv) Deduce that the regular 7-gon is not constructible by straightedge and compass. [12]

2. Let $L|_k$ be a finite normal extension. Show that $L = L_s L_i$, where L_s denotes the separable closure of k in L and L_i denotes the purely inseparable closure of k in L. [12]

3. Let $k = \mathbb{C}(t)$; t transcendental over $\mathbb{C}$. Let $f(X) = X^{12} - t \in k[X]$.
   (i) Determine a splitting field E of f over k and compute $[E : k]$.
   (ii) Find all intermediary subfields of E containing k. [12]

4. Let G be a simple group of order 60. Prove that G is isomorphic to A_5. [12]

5. Answer ANY TWO questions with justifications.
   (i) Give an example of an algebraic field extension $L|_k$ such that $|\text{Aut}_k L| < \infty$ but $[L : k]$ is not finite. [6]
   (ii) Give an example of a finite field extension $L|_k$ which is inseparable but not purely inseparable. [6]
   (iii) Give an example of a finite field extension $L|_k$ which is not simple. [6]

Group B
Answer ANY TWO questions.

1. Suppose that the Galois group of an irreducible and separable polynomial $f(X) \in k[X]$ is Abelian. Let E be a splitting field of $f(X)$ over k and let $\alpha_1, \ldots, \alpha_n$ be the roots of $f(X)$ in E. Show that $E = k(\alpha_i)$ for each i, $1 \le i \le n$, and $[E : k] = \deg f$. [8]

2. Show that the polynomial $X^7 - 10X^5 + 15X + 5$ is not solvable by radicals over $\mathbb{Q}$. [8]

3. Prove that any group of order 2015 is solvable.
   (Note: 2015 has three prime factors.) [8]

P.T.O.

Group C

State with justifications whether the following statements are true or false.
Answer ANY FOUR questions. Each question carries 3 marks.

1. The polynomial $X^2 + 1 \in \mathbb{F}_{625}[X]$ has a root in $\mathbb{F}_{625}$.

2. All prime fields have countable algebraic closure.

3. $\mathbb{Q}(\cos 20^\circ)$ is a Galois extension of $\mathbb{Q}$.

4. S_5 has no subgroup of order 30.

5. If N is a normal subgroup of a group G, then the commutator subgroup $[N, N]$ is necessarily normal in G.

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2014-15

M. Math. I Year **Complex Analysis**

Date: 08.05.2015 Maximum Marks: 60 Duration: 3 Hours

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(1) Let $z_1, z_2, \ldots, z_n$ be distinct complex numbers contained in the open disc $D(0,r)$ of radius r centred at the origin. Let f be holomorphic in the closed disc $\overline{D(0,r)}$. Let $Q(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$ and

$$P(z) = \frac{1}{2\pi i}\int_{C_r} f(w)\frac{1-\frac{Q(z)}{Q(w)}}{w-z}\,dw,$$

where C_r is the circle of radius r centred at the origin. Show that P is a polynomial and $P(z_k) = f(z_k)$ for $k = 1, 2, \ldots, n$. [6]

(2) Let $\mathbb{D}$ denote the unit disc (the open disc of radius 1 centred at the origin). A complex number $w \in \mathbb{D}$ is called a *fixed point* for the map $f : \mathbb{D} \to \mathbb{D}$ if $f(w) = w$.

Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and has two fixed points α and β with $\alpha \neq \beta$. Show that $f(z) = z$ for all $z \in \mathbb{D}$.

[Hint. Use the automorphism $\psi_\alpha(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$ of $\mathbb{D}$ to construct a holomorphic map h from $\mathbb{D}$ to $\mathbb{D}$ with $h(0) = 0$ and apply Schwarz lemma. Note that $(f \circ \psi_\alpha)(0) = \alpha$.] [10]

(3) Show that

$$f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$$

is a conformal map from the half-disc $\{z = x+iy \in \mathbb{C} : |z| < 1, y > 0\}$ to the upper half-plane $\mathbb{H} = \{z = x+iy \in \mathbb{C} : y > 0\}$. [10]

(4) Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}, \quad \theta \in \mathbb{R} \text{ and } \beta \in \mathbb{H}.$$

[Hint. Note that $G(z) = i\frac{1-z}{1+z}$ is a conformal map from $\mathbb{D}$ to $\mathbb{H}$. Also, a conformal map from $\mathbb{D}$ to $\mathbb{D}$ is of the form $e^{i\theta}(a-z)/(1-\overline{a}z)$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.] [10]

(5) Prove that

$$e^z - 1 = ze^{z/2}\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4\pi^2 n^2}\right), \quad z \in \mathbb{C}.$$

[10]

(6) Prove that the second derivative of $\log\Gamma(s)$ at a positive integer m is equal to $\sum_{n=0}^{\infty}(m+n)^{-2}$.

That is, show that if m is a positive integer, then

$$\frac{d^2}{ds^2}(\log\Gamma(s))\Big|_{s=m} = \sum_{n=0}^{\infty}\frac{1}{(m+n)^2}.$$

[Hint. Consider the Hadamard factorization of $1/\Gamma$.] [10]

(7) Prove that for Re $s > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx.$$

[10]

INDIAN STATISTICAL INSTITUTE
Second Semester Back-Paper Examination: 2014-15

M. MATH. I YEAR
Algebra II

Date: 10.7.2015 | Maximum Marks: 100 | Duration: 3 Hours

Attempt all questions.
Clearly state the results that you use.
$\mathbb{C}$: field of complex numbers
$\mathbb{Q}$: field of rational numbers

1. Prove that A_6 has no subgroup of prime index. [8]

2. Let H and K be normal solvable subgroups of a group G. Prove that HK is a solvable group. [8]

3. Show that if $R \subsetneq \mathbb{C}$ is a UFD, then $\mathbb{C}$ cannot be the field of fractions of R. [6]

4. Let $L|_k$ be a finite field extension. Prove that L cannot be isomorphic to any of its proper subfields containing k. [7]

5. Let α be a real number such that $\alpha^4 = 5$. Show that
(i) $\mathbb{Q}(i\alpha^2)$ is normal over $\mathbb{Q}$.
(ii) $\mathbb{Q}(i\alpha + \alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$.
(iii) $\mathbb{Q}(i\alpha + \alpha)$ is not normal over $\mathbb{Q}$. [4+4+7=15]

6. Let E be a splitting field of the polynomial $(X^2 - 2)(X^2 - 3)(X^2 - 5)$ over $\mathbb{Q}$.
(i) Compute $[E : Q]$ with justification.
(ii) Determine the Galois group G of $E|_{\mathbb{Q}}$ and explicitly describe all the subfields of E as fixed fields of subgroups of G.
(iii) Deduce that any subfield of E is a normal extension of $\mathbb{Q}$. [6+14+4=24]

7. (i) Prove that any finite separable extension E over a field k is a simple extension.
(ii) Let $L = \mathbb{Q}(\omega, i)$. Find $\alpha \in L$ such that $L = \mathbb{Q}(\alpha)$. [6+4=10]

8. Determine a polynomial in $\mathbb{Q}[X]$ of degree 7 which is not solvable by radicals over $\mathbb{Q}$. [10]

9. State with justifications whether the following statements are true or false.
(i) There exists a field with exactly 225 elements.
(ii) The finite field $\mathbb{F}_{49}$ has a quadratic extension.
(iii) If $L|_F$ and $F|_k$ are finite field extensions with $L|_F$ normal, then $L|_k$ is necessarily normal. [4+4+4=12]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2014-15 (Backpaper)

M. Math. I Year **Complex Analysis**

Date: 13/7/ .2015 Maximum Marks: 100 Duration: 3 Hours

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(1) Show that each of the series $\sum_{n=1}^{\infty} \frac{nz^n}{1-z^n}$ and $\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2}$ converges uniformly on each closed disc $\{z \in \mathbb{C} : |z| \leq c\}$ with $0 < c < 1$. Prove that these two series are equal. [10]

(2) Let $a > 0$. Evaluate the integral

$$\int_0^\infty e^{-ax} \cos bx \, dx$$

by integrating e^{-Az}, $A = \sqrt{a^2 + b^2}$, over an appropriate sector with angle w such that $\cos w = a/A$. [10]

(3) Let $a_1, a_2, \ldots, a_n$ be points on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Prove that there exists a point z_0 on the unit circle such that the product of the distances from z_0 to the points a_j's is at least 1. [8]

(4) Let f be holomorphic on an open set Ω, let $z_0 \in \Omega$ and $f'(z_0) \neq 0$. Show that

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)} \, dz,$$

where C is a small circle centred at z_0 and is contained in Ω. [10]

(5) Compute the number of zeros of the polynomial

$$2z^5 - 6z^2 + z + 1$$

in the annulus $1 \leq |z| \leq 2$. [15]

(6) Let $a > 0$. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} \, dx.$$

[15]

(7) Let $D(0, R)$ denote the open disc of radius R centred at the origin. Suppose that $f : D(0, R) \to \mathbb{C}$ is holomorphic and $|f(z)| \leq M$ for all $z \in D(0, R)$. Show that

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{MR}.$$

[Hint. Use the Schwarz lemma.] [12]

(8) Prove that

$$\Gamma(s) = \lim_{n \to \infty} \frac{n^s n!}{s(s+1) \ldots (s+n)}$$

whenever $s \neq 0, -1, -2, -3, \ldots$. [10]

(9) Show that the Hadamard factorization of $\cos \pi z$ is

$$\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right).$$

[10]

INDIAN STATISTICAL INSTITUTE
Semestral Examination (Back Paper) : 2014-2015
M. Math. - I Year
Topology-II

Date : 24 · 07 · 15 Time : 3 Hours

Any result that you use should be stated clearly.

(1) Answer the following with proper reasons.

(a) What is the degree of the antipodal map $a : S^n \longrightarrow S^n$?

(b) For $m \neq n$, do $\mathbb{R}^m$ and $\mathbb{R}^n$ have the same homotopy type?

(c) Consider $\mathbb{R}^n$ as a topological group with addition and let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the subgroup consisting of points with integral coordinates. Let

$$q : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$$

be the quotient map. Is q a covering map?

(d) Let X be a Hausdorff space such that for every $x \in X$, there exists an open neighbourhood U of x and a homeomorphism $\phi : U \longrightarrow \mathbb{R}^n$, where n is a fixed positive integer. Does there exist a universal cover for X ?

(e) Describe a universal cover of $S^1 \vee S^2$ and justify your answer.

[5+5+5+5+5=25]

(2) (a) State Eilenberg-Steenrod axioms for homology.

(b) Compute singular homology groups of a one-point space.

(c) Let $X = X_1 \bigsqcup X_2$ be a disjoint union. Compute singular homology groups of X in terms of singular homology groups of X_1 and X_2.

[5+5+5=15]

(3) Prove that every short exact sequence of chain complexes induces a long exact sequence of homology groups.

[20]

(4) (a) Define the notion of a CW complex.

(b) Let $n \geq 1$. Describe a CW complex structure for S^n which is invariant under antipodal action and hence deduce a CW complex structure for $\mathbb{R}P^n$.

(c) Let $C_k(S^n)$ be the k-th cellular chain group of the cellular chain complex of S^n obtaind from the CW complex structure of S^n of part (b). Prove that for each $k \geq 1$, the subgroup of $C_k(S^n)$ consisting of k-cycles is infinite cyclic.

[4+6+10=20]

(5) Deduce Mayer-Vietoris Exact sequence from Excision axiom. Use Mayer-Vietoris Exact sequence to compute homology groups of S^n for $n \geq 1$.

[20]