INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination

First semester

B. Stat - Second year 2013-2014

Analysis III

Date: August 28, 2013 | Maximum Marks: 60

Duration: 2 hours 30 minutes

Answer all questions.

You must state clearly any result you use.

(1) Let $O_n(\mathbb{R})$ denote the set of all $n \times n$ orthogonal matrices with real entries. Prove that $O_n(\mathbb{R})$ is compact. Is it path-connected? Justify your answer. 5+5

(2) Consider the ℓ_1 and ℓ_2 norms on $\mathbb{R}^n$ defined as follows: For any $x = (x_1, x_2, \ldots, x_n)$,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}$$

Establish the following inequalities for $x \in \mathbb{R}^n$.

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2.$$

Draw open unit balls centred at $(0,0)$ with respect to the above norms on $\mathbb{R}^2$.
Is $\mathbb{R}^n$ with ℓ_1 norm homeomorphic with $\mathbb{R}^n$ with ℓ_2 norm? Justify your answer.

5+2+5

(3) Let $f : GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be defined by $f(X,Y) = XY$, where XY denote the matrix multiplication of X and Y. Prove that f is differentiable and compute the derivative of f at (A, B). 8

(4) Let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h(x,y) = \begin{cases} \frac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that h is not differentiable at $(0,0)$. 7

(5) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ be two differentiable functions. Determine the partial derivatives of $f \circ \phi$ in terms of the partial derivatives of f and ϕ_i $(i = 1, 2, 3)$, where $\phi = (\phi_1, \phi_2, \phi_3)$. 7

(6) Let $U = \{(r, u, v) \in \mathbb{R}^3 | r > 0, -\pi/2 < u < \pi/2 \text{ and } 0 < v < \pi\}$. Consider the function $\phi : U \to \mathbb{R}^3$ defined by

$$\phi(r, u, v) = (r \cos u \cos v, r \cos u \sin v, r \sin u).$$

Identify the image of ϕ. Prove that $\phi : U \to \phi(U)$ is a diffeomorphism. 3+7

(7) Suppose that U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is differentiable. Prove that if f has a local maxima at $p \in U$ then $Df_p = 0$. 6

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination : (2013-2014)
B. Stat II
Elements of Algebraic Structures
August 30, 2013 Maximum marks : 40 Duration : $2\frac{1}{2}$ hours

---

Answer Question 1 and any four from the rest $(2 \times 9 + 6 \times 4)$

1. *Indicate True/False for the statements below. Give a proof if you think a statement to be true. If it is false, give a counter-example. No marks will be given if you fail to justify your answer.*

   (a) Let H, K be two distinct, non-zero, proper subgroups of $(\mathbb{Q}, +)$. Then $|H \cap K| > 1$.

   (b) If every element of a group is of finite order, then the group is finite.

   (c) Let G be a group and $a, b \in G$ be both of finite order. Then ab is also of finite order.

   (d) Let G be a group and H be a normal subgroup of G of order 2. Then $H \subseteq Z(G)$.

   (e) Let H be a normal subgroup of a group G. If H and G/H are abelian then G is abelian.

   (f) Let R be an integral domain and $I \subsetneq R$ be an ideal. Then R/I is an integral domain.

   (g) Any subring of a ring R is also an ideal of R.

   (h) In a ring R, the equation $x^3 = 1$ has at most three solutions.

   (i) Let $(R, +, .)$ be a commutative ring with 1. Let $\phi : R \to R$ be a map which is a group morphism on the additive group $(R, +)$. Then ϕ is also a ring morphism on $(R, +, .)$.

2. Let G be a group and H, K subgroups of G. Let $HK := \{hk \,|\, h \in H,\ k \in K\}$. Give an example to show that HK may not be a subgroup of G. Prove that HK is a subgroup if and only if $HK = KH$.

3. Prove that the group A_4 does not have a subgroup of order 6.

4. Let G be a group. For $a \in G$ define $\sigma_a : G \longrightarrow G$ by $\sigma_a(x) = ax$. Show that σ_a is an element of the group $S(G)$ (the set of all bijections on G). Now prove that G is isomorphic to a subgroup of $S(G)$.

5. Prove that $\mathbb{Z}[X]$ is not a principal ideal domain.

6. Prove that the ring $\frac{\mathbb{Z}[i]}{<2+i>}$ is isomorphic to the ring $\mathbb{Z}/5\mathbb{Z}$.

Probability Theory III
B. Stat. IInd Year Semester 1
Indian Statistical Institute

Midsemestral Examination
Time: 2 Hours 30 minutes Full Marks: 35
Date: September 2, 2013

You may use any result proved in class without proving them again. However state clearly all the results you are using.

1. Let X be a nonnegative random variable defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\mathrm{E}[X] = 1$. For every $A \in \mathcal{F}$, define
$$\mathrm{Q}(A) = \mathrm{E}\left[X \mathbb{1}_A\right].$$
Show that Q is a probability on $(\Omega, \mathcal{F})$. Further show that, if $\mathrm{P}(A) = 0$, then $\mathrm{Q}(A) = 0$ as well. [3+2=5]

2. Which of the following is/are characteristic functions: $(1 + t^4)^{-1}$, $\exp(-2\sin^2 t)$? Justify your answers. [4+7=11]

3. Show that, for odd k,
$$\int_{-\infty}^{\infty} \frac{\cos^k x}{1+x^2} dx = \frac{\pi}{2^{k-1}} \sum_{\substack{0 \le l \le k \\ k+l \text{ is even}}} \binom{k}{\frac{k+l}{2}} \mathrm{e}^{-l}.$$
[7]

4. Let $\{X_n\}$ be a sequence of i.i.d. Cauchy random variables and $\{a_n\}$ be a sequence of real numbers. Show that $\sum_{k=1}^{n} a_k X_k$ converges weakly iff $\sum_{k=1}^{\infty} |a_k| < \infty$. [5]

5. Let ϕ_n be a sequence of characteristic functions, which converge at every point in a nonempty open interval around 0 to a function ϕ, which is also continuous at 0. Show that the sequence of the distribution functions corresponding to ϕ_n's are tight. [7]

Indian Statistical Institute
Mid-semestral Examination : 2013-14
Statistical Methods III

Date: 04.07.2013 Maximum marks: 40 Duration: 2hrs

This is a closed book, closed notes examination. The paper carries 45 marks. Attempt all questions. The maximum you can score is 40. If any unfair practice is suspected, 5 marks will be deducted from the aggregate of each suspect.

1. State if the following statements are true or false. Answers without proper justfication get no credit.
   (a) If a 1-sample t-test rejects H_0 at 5% level of significance, then it must reject H_0 at 1% level as well. [3]
   (b) An MP test (at any level) must be unbiased. [3]
   (c) Let ϕ be an MP test of level α and ξ be an MP test of level $\beta > \alpha$. Then ϕ must be more powerful than ξ. [3]

2. We have data $X_1, ..., X_n$ iid $N(\mu, \sigma^2)$ with μ unknown, but $\sigma^2 > 0$ known. Let ξ_1 be the standard test of $H_0 : \mu = 0$ against the two-sided alternative at 5% level of significance. Let ξ_2 be the standard test of the same hypotheses at the same level, but without assuming the knowledge of σ^2. Compare the two tests in terms of P(type I error) and P(type II error). Also sketch roughly the power functions of the two tests on the same axes. [1+2+2]

3. Obtain an MP test of level α for $H_0 : \theta = 1$ against $H_1 : \theta = 2$ based on data $X_1, ..., X_n$ iid $Unif(-\theta, \theta)$. [10]

4. The life times of 5 bulbs are observed to be

$$3.5, 4.1, 3.7, 5.14.0.$$

It is believed that the life time distribution is *Exponential* with unknown mean λ. Derive the MP test of

$$H_0 : \lambda = 4 \text{ against } H_1 : \lambda = \lambda_1,$$

where $\lambda_1 > 4$ is some given number. Use 5% level of significance. You are given that

x	6.493946	7.880598	36.614076	40.966355
$F(x)$	0.025	0.05	0.95	0.975

where

$$F(x) = \frac{4^{-5}}{4!} \int_0^x t^4 e^{t/4} dt.$$

Does your test depend on λ_1? Interpret this. Write down your conclusion for the given data set. [5+1+2+2]

5. You are performing a 2-tailed paired t-test based on a sample of size 17. Your test statistic turns out to be 2.76. You want to perform the test at 5% level of significance. Unfortunately, you dont have any t-table ready at hand. The only table you have is an F-table. Suggest how you may test the hypothesis using this table. [5]

6. Two physical quantities x and y are related as

$$y = \alpha + \beta x.$$

In order to ascertain α, β two known values $x_1 \neq x_2$ are taken for x, and the corresponding values y_1, y_2 of y are measured independently and found to be Y_1, Y_2, where

$$Y_i = y_i + \epsilon_i,$$

ϵ_i's being iid $N(0,1)$ measurement errors. Based on the data Y_1, Y_2 (plus the known values x_1, x_2) find a 95% CI for β. [6]

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination : 2013-14

Course Name : B. STAT. II Year (First Semester)

Subject Name : Physics I

Date : 06/09/13 **Maximum Marks : 40** **Duration : $2\frac{1}{2}$ hours**

---

1. (a) A 3D vector $\vec{a}$ of constant magnitude is varying over time. What can you say about the direction of $\vec{a}$?
(b) Show that the four different points P:(2,4,3), Q:(-2,1,4), R:(1,2,5), and S:(-5,0,3) are coplanar.
(c) Prove the identity $|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2|\vec{B}|^2 - \left(\vec{A}.\vec{B}\right)^2$.
(d) For a surface of constant potential U prove that grad$U$ is everywhere normal to the surface. 2+3+3+2

2. (a) Show that under static conditions the net charge of a charged conductor resides on its outer surface.
(b) Prove that the electric field just outside a charged conductor is perpendicular to its surface. Evaluate the magnitude of this field.
(c) Show that the function $\phi = 3x^2 + 8y - 3z^2$ can represent the electrostatic potential in a charge free region. 3+(2+3)+2

3. (a) State Gauss's theorem in electrostatics.
(b) A sphere of radius R is uniformly charged with charge density ρ. Using Guass's theorem find the electric field E at a distance r from the centre of this sphere for (i) $r < R$, (ii) $r = R$ and (ii) $r > R$. Sketch the variation of E with r. 2+(3+1+2+2)

4. (a) What do you mean by an electric dipole?
(b) Find the potential energy of an electric dipole placed in the field of another electric dipole.
(c) An electric dipole of moment $\vec{p}_1$ is fixed at the origin of co-ordinates. Another coplanar electric dipole of moment $\vec{p}_2$ is placed at the position $\vec{r}$ and is free to rotate. Show that for equilibrium $\tan\theta_1 = -2\tan\theta_2$, where θ_1 and θ_2 are the angles that $\vec{r}$ makes with $\vec{p}_1$ and $\vec{p}_2$, respectively. 2+4+4

5. (a) State Poisson's and Laplace's equations in electrostatics.

(b) What is meant by polarization of dielectrics?

(c) Define magnetic vector potential.

(d) Consider a particle of mass m and charge q initially moving with a uniform velocity v in the x-direction is subjected to a uniform magnetic field B applied in the z-direction. Describe the motion of the particle. Show that the kinetic energy of the particle does not change under the influence of B. 2+2+1+(3+2)

Mid-Semester Examination 2013

Course – B Stat II Year (2013-2014)

Subject – Microeconomics

06·09·13

Maximum Marks 40 Duration 2.5 Hrs

Answer all the following questions

1. Consider a commodity bundle $x' \in$ consumption set X. Define the upper contour set Cx' and lower contour set Lx'. State the continuity axiom.
   What is lexicographic preference? Explain with an example that it does not satisfy the continuity axiom.
   Show that under continuity axiom, Lx' and Cx' are closed and share a common boundary.
   (6+6+10=22)

2. a) Suppose a consumer's utility function is given by $U = (x_1^2 + x_2^2)^{\frac{1}{2}} - 30$; $p_1 = 3, p_2 = 2$ and budget M = 50. (All the symbols have usual meanings). What is the equilibrium consumption basket?
   b) Suppose the utility function of a consumer is given by $\min\{2x_1 + x_2, 2x_2 + x_1\}$. The two commodities are bought and sold in the market at prices Rs.2 and Rs.3 respectively. To start with, the consumer has 15 units of commodity 1 only. What will be the equilibrium consumption basket of the consumer? Can you specify the change in x_2 following an increase in p_2 in terms of the Slutsky equation in this example?
   c) Suppose the utility function of a consumer given by $U = U(x_1, x_2); U_1 \succ 0, U_2 \succ 0$ is continuous, differentiable, shows diminishing marginal rate of substitution and has all the other desired properties. The prices p_1, p_2 and budget M are given. Initially, the consumer is in equilibrium. Show the change in equilibrium in a diagram following an increase in p_1. Consider two cases, Case 1 and Case2. In Case 1, after the increase in p_1, the government adjusts the consumer's budget so that the consumer can just enjoy the previous utility level at minimum cost at the new set of prices. In Case 2, the government adjusts consumer's budget so that the consumer can just purchase the initial consumption basket at new prices. Compare the costs incurred by the government in the two situations. (9+9+4=22)

INDIAN STATISTICAL INSTITUTE

Mid- Semestral Examination: 2013-14

Course Name: B. STAT. II YEAR
Subject Name: Molecular Biology

Date: 06.09.13 **Maximum Marks: 40** **Duration: 1hr 30 mins**

Note: Answer any five out of first seven questions, and any five from Question no. 8 to 14. For Multiple choice questions & fill in the blanks only first attempted five will be considered. For descriptive questions best five will be considered.

Multiple choice questions and fill in the blanks: Answer any five out of seven questions (Each question carries 2 marks)

1. An investigator would be able to distinguish a solution containing RNA from one containing DNA by

a. heating the solution to 82.5 °C and measuring the absorption of light at 260nm.
b. comparing the Tm of each solution.
c. monitoring the change in absorption of light at 260nm whole elevating the temperature
d. measuring the absorption of light at 260nm

2. The percentage of G-C base pairs in a DNA molecule is related to the Tm because

a. the stability of G-C and A-T base pairs is intrinsically different
b. A-T base pairs require a higher temperature for denaturation
c. the triple bond of G-C base pairs are less stable than the double bond of A-T base pairs
d. the G-C content equals the A-T content

3. Special structures called telomeres are needed in eukaryotic cells but not bacteria because

a. eukaryotic cells contain linear chromosomes
b. eukaryotic cells have more than one chromosome
c. eukaryotic cells contain a nucleus
d. eukaryotic cells contain more forms of DNA polymerases

4. What feature of a protein can be used to predict it's function

a. number of amino acids
b. overall charge of the protein
c. molecular weight of the protein
d. structure of the protein

5. Non-covalent bond include all the following except

a. disulfide bond
b. an ionic bond
c. a hydrogen bond
d. a Van der Waals interaction

6. The amount of adenine is always equal to the amount of _______ in DNA.

7. A short length of dsDNA molecule has 80 thymine and 80 guanine bases. The total number of nucleotide in the DNA fragment is --------------------

Descriptive questions: Answer any five out of seven questions
(Each question carries 2X3 =6 marks)

8. **a)** Define bacterial growth curve.
b) If an investigator started a bacterial culture with 100 cells of *E. coli*, whose doubling time is 22 minutes, calculate the time required to reach to 10^8 cells?

9. **a)** Establish the Henderson-Hasselbalch equation.
b) Aspirin has a pK_a of 3.4. What is the ratio of A^- to HA in: (i) the blood (pH = 7.4) and (2) the stomach (pH = 1.4)?

10. **a)** What are the main differences in leading and lagging strand synthesis during DNA replication?
b) Explain the roles of two types of Topoisomerases in DNA replication.

11. **a)** Briefly describe Ramachandran Plot.
b) What are the most common types of protein folds present in protein secondary structures? Explain with diagram of each type.

12. **a)** Which bases are (i) purines and (ii) pyrimidines?
b) Why DNA is negatively charged?

13. **a)** Describe some basic differences between plant and animal cells.
b) What is the major energy containing molecule for chloroplasts, mitochondria and therefore animals and plants?

14. **a)** Calculate the length of DNA (in nm) having 23 base pairs.
b) Explain why DNA replication is necessary. Explain Okazaki fragments.

INDIAN STATISTICAL INSTITUTE

Semestral Examination

First semester

B. Stat - Second year 2013-2014

Analysis III

Date: November 11, 2013 | Maximum Marks: 60

Duration: 3 hours

Answer all questions.

You must state clearly any result you use.

(1) If f is a real valued continuous function on $\mathbb{R}$ then show that

$$\int_0^x \left\{ \int_0^v \left\{ \int_0^u f(t)\,dt \right\} du \right\} dv = \frac{1}{2} \int_0^x (x-t)^2 f(t)\,dt$$

10

(2) Let I be the interval $[0,1]$. Consider the function $f : I \times I \to \mathbb{R}$ defined as follows:

$$\begin{aligned} f(x,y) &= 1 && \text{if either } x \text{ or } y \text{ is irrational} \\ &= 1 - 1/q && \text{if } x = p/q \text{ in its lowest term and } y \text{ is any rational} \end{aligned}$$

Show that for a fixed rational $r \in I$, the function $g(y) = f(r,y)$ is not integrable. Is f integrable over $I \times I$? Justify your answer. 3+7

(3) Give a detailed proof of the equality

$$\int_R f(y)\,dy = \int_{T(R)} f(T(x))|J_T(x)|dx$$

for continuous functions f defined on a rectangle $R \subset \mathbb{R}^2$, in the following two cases:

(a) $T(u,v) = (v,u)$ for all $(u,v) \in \mathbb{R}^2$.

(b) $T(u,v) = (u+a, v+b)$ for all $(u,v) \in \mathbb{R}^2$ ($(a,b) \in \mathbb{R}^2$ is a fixed vector).

5+5

(4) Assume that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-map and $\phi(u) = (\phi_1(u), \phi_2(u), \ldots, \phi_n(u))$, where $u = (u_1, \ldots, u_n)$ be any point in $\mathbb{R}^n$. Prove that

$$d\phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\phi_n = J_\phi\, du_1 \wedge du_2 \wedge \cdots \wedge du_n$$

where J_ϕ denotes the Jacobian of the map ϕ. 6

(5) Let A be a positive definite symmetric matrix of order n. Prove that

$$\int_{\mathbb{R}^n} e^{-\langle Ax, x\rangle}\, dx = \frac{\pi^{\frac{n}{2}}}{\sqrt{\det A}}$$

10

(6) Consider the oriented affine 2-chain $c = [e_0, e_1, e_1 + e_2] + [e_1 + e_2, e_2, e_0]$ in $\mathbb{R}^2$ and a map $T : [0,1] \times [0,1] \to \mathbb{R}^3$ defined by $T(u, v) = (\cos u, \sin u, v)$. Compute the integral

$$\int_{T \circ c} dy \wedge dz + dz \wedge dx + dx \wedge dy$$

10

(7) Let $\gamma_1(t) = (\cos 2\pi t, \sin 2\pi t)$ and $\gamma_2(t) = (2\cos 2\pi t, 2\sin 2\pi t)$ for $t \in [0, 1]$. Show that

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$$

for any closed 1-form on $\mathbb{R}^2$. 10

PROBABILITY THEORY III
B. STAT. IIND YEAR SEMESTER 1
INDIAN STATISTICAL INSTITUTE

Semestral Examination

Time: 3 Hours 30 minutes   Full Marks: 50

Date: November 18, 2013

You may use any result proved in class without proving them again. However clearly state all the results you are using.

1. Let X and Y be two random variables such that $\mathrm{E}[|Y|] < \infty$ and there exists $x_0 < \infty$, such that for all $x > x_0$, we have $\mathrm{P}[|X| > x] \leq \mathrm{P}[|Y| > x]$. Show that $\mathrm{E}[|X|] < \infty$. [5]
2. Prove or disprove: If X and Y are independent random variables, such that $X + Y$ has same distribution function as X, then $\mathrm{P}[Y = 0] = 1$. [6]
3. Let $\{X_n\}$ be an i.i.d. sequence of random variables with common distribution function F such that $\sup\{x : F(x) < 1\} = \infty$. Define $\overline{F}(t) = \mathrm{P}[X > t]$ and $\tau(t) = \min\{n : X_n > t\}$. Show that $\tau(t)$ is a valid random variable for each t. As $t \to \infty$, show that $\overline{F}(t)\tau(t)$ converges weakly to a unit exponential random variable. [7]
4. Let $\{X_n\}$ be an i.i.d. sequence with common distribution function F. Let $\lambda_n \uparrow \infty$ and $A_n = \{\max_{1 \leq m \leq n} X_m > \lambda_n\}$. Show that $\mathrm{P}[\limsup A_n] = 0$ or 1 according as $\sum(1 - F(\lambda_n)) < \infty$ or $\sum(1 - F(\lambda_n)) = \infty$. [7]
5. Let $\{X_{n,k} : 1 \leq k \leq n, n \geq 1\}$ be a triangular array of nonnegative random variables. Define $S_n = \sum_{i=1}^{n} X_{n,i}$ and $M_n = \max_{i=1}^{n} X_{n,i}$. Show that $M_n \xrightarrow{\mathrm{P}} 0$ implies $S_n/n \xrightarrow{\mathrm{P}} 0$. [5]
6. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables satisfying $\sum_1^\infty \mathrm{P}[X_n \neq Y_n] < \infty$. If there exists a random variable X and a sequence $a_n \to \infty$, such that $a_n^{-1} \sum_1^n X_r \to X$ a.e., find the almost everywhere limit of $a_n^{-1} \sum_1^n Y_r$. [6]
7. Let $X_1, \ldots, X_n$ are i.i.d. standard normal random variables. For $k = 1, \ldots, n-1$, define $Y_k = (\sum_1^k X_i - kX_{k+1})/\sqrt{k(k+1)}$. Show that $Y_1, \ldots, Y_{n-1}$ are i.i.d. standard normal random variables. [6]
8. Let X and Y be independently and uniformly distributed on the interval $(0, 1)$. Find the conditional distribution function of X given $X \wedge Y$. [8]

Indian Statistical Institute
Statistical Methods III
B.Stat. 2nd year
First Semestral Examination

Date: Nov 20, 2013 Duration: 3 hrs.

This paper carries 70 marks. Attempt all questions. The maximum you can score is 60. Justify all your steps. This is a closed book, closed notes examination. You may use your own calculator. No need to perform more than two steps of any iterative method. Standard statistical tables will be provided.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 5 will be subtracted from the overall aggregate of each of these students.

1. Let $X_1, ..., X_n$ be iid from the mixture distribution $pN(\mu, 0.1^2) + (1-p)N(0,1)$. We want to find the mle of $\mu \in \mathbb{R}$ and $p \in (0,1)$. Derive the EM algorithm for this problem. Clearly mention the complete data, the E-step, the M-step and the iteration to be used. [10]

2. Use Fisher's scoring method to obtain the mle of θ, based on the random sample
$$-20.72, -0.74, 0.78, 0.77, 2.28$$
from Cauchy$(\theta, 1)$ distribution. You may use the fact that
$$\int_{-\infty}^{\infty} \frac{1-x^2}{(1+x^2)^3} dx = \frac{\pi}{4}.$$
[10]

3. Let $\phi_n(X_1, ..., X_n)$ be the following test of $H_0 : p = \frac{1}{2}$ against $H_1 : p \neq \frac{1}{2}$ based on $X_1, X_2, ..., X_n$ iid Bernoulli(p) where $p \in (0,1)$:
$$\phi_n(X_1, ..., X_n) = \begin{cases} 0 & \text{if } |\overline{X}_n - \frac{1}{2}| < 0.1 \\ 1 & \text{else} \end{cases}$$
Find the limiting probability of type I error and sketch the limiting power curve as $n \to \infty$. Justify your answer. [5]

4. A book with 100 pages is checked for typos, and the following frequency distribution has resulted.

k	0	1	2	3	4	5	6
n_k	17	27	22	19	4	8	3

Here n_k is the number of pages with exactly k typos in each. Test the hypothesis that the number of typos in a randomly selected page follows a Poisson distribution. Use 5% level of significance. Clearly state your assumptions, procedure and conclusion. [10]

5. We have 3 independent random samples each of size n as follows:

$$\begin{aligned} X_{11}, ..., X_{1n} &\sim N(\mu_1, \sigma^2), \\ X_{21}, ..., X_{2n} &\sim N(\mu_2, \sigma^2), \\ X_{31}, ..., X_{3n} &\sim N(\mu_3, \sigma^2), \end{aligned}$$

where $\mu_1 \in \mathbb{R}, \mu_2 \in \mathbb{R}, \mu_3 \in \mathbb{R}, \sigma^2 > 0$ are all unknown. Derive the likelihood ratio test for testing

$$H_0 : \mu_1 = \mu_2 = \mu_3 \text{ vs. } H_1 : \text{ not } H_0.$$

Cast it in a form so that you can use the F-distribution to find the critical value of the test for any given level α. Clearly mention the degrees of freedom for the distribution. [15]

6. Let X_1, X_2 be a random sample from $N(\mu, 1)$ distribution. Find the sampling distributions of

   (a) $\frac{X_1}{|X_2|}$, when $\mu = 0$.

   (b) $\frac{X_1}{|X_1|}$ for general μ.

[5+5]

7. (a) Distinguish between the seasonal component and the cyclical component of a time series.

   (b) Find the autocorrelation function of the time series as given below:

$$X_t = Z_t + 0.7Z_{t-1} - 0.2Z_{t-2},$$

   where $E(Z_t) = 0$ and $V(Z_t) = \sigma^2$ for all t, and $cov(Z_t, Z_{t'}) = 0$ for all $t \neq t'$.

[5+5]

INDIAN STATISTICAL INSTITUTE

First Semester Examination 2012-13

Course Name: B Stat 2nd Year

Subject Name: Economics I

Date: 22 November, 2013 **Maximum Marks: 60** **Duration: 3 Hours**

Answer any three from the following questions

1.i) Consider the following optimisation problem:

$$\max U(x_1, x_2, \ldots\ldots, x_n) \quad U_i > 0 \;\; x_i \geq 0 \;\; \text{s.t.} \sum r_i x_i = w$$

Suppose the utility function is strictly quasiconcave. Derive the optimality condition for the following two cases: (i) When there is interior solution and (ii) When there is corner solution. Explain your answers with the help of diagrams.

ii) Suppose x and y are two commodities in the consumption basket of an individual consumer. It is given that the marginal utility of x denoted $mu_x = 40 - 5x$ and marginal utility of y denoted $mu_y = 30 - y$. Prices of x and y are Rs.5 and Re.1 respectively. The budget of the consumer is Rs.40. (i) Derive the optimum consumption bundle the consumer will choose. (ii) How will your answer to (i) change if the consumer's budget falls to Rs.10?

2. Long-run cost of a representative firm in a perfectly competitive market is given by $C = \frac{1}{10^3} q^2 + 10^3$, while the market demand function is $Q = 26{,}00000 - 600000P$. Compute the price (P) and the quantity (Q) that will prevail when the market attains long run industry equilibrium. Also compute the number of firms that will operate in the industry in this equilibrium situation. [12+10 = 22]

3.Consider the cost function of a perfectly competitive firm in two different situations:

(a)
$$C = q^2 + 1 \quad \text{for } q > 0$$
$$= \frac{1}{2} \quad \text{for } q = 0$$

and

(b) $\begin{aligned} C &= q^2 + 1 \quad \text{for } q > 0 \\ &= 0 \quad\quad\; \text{for } q = 0 \end{aligned}$

(i) Draw the graphs of the AVC, AC and MC schedules corresponding to the two cost functions in a diagram.

(ii) Derive the supply functions of two firms, when one has the cost function (a) and the other has the cost function (b).

(iii) What quantity of output the firm having the cost function (a) will supply and how much profit it will earn, when the price of its product is 1.5. How do you explain the situation?

[3+13+6=22]

4.A profit maximising monopolist has the option of producing its output in two separate plants having the cost functions $C_1 = 30 + 20Q_1$ and $C_2 = Q_2^2 + 1$ respectively. The market demand function is given by $P = 120 - 2Q$. (i) How much output will the monopolist sell in the market, what price will he charge and how much output will he produce in each of the two plants in equilibrium? (ii) Suppose the government imposes a sales tax at the rate of Rs.66 per unit of output sold in the market. How will it affect the variables mentioned in (i)?

[11+11 = 22]

5. A monopolist faces the market demand curve $P = 120 - Q$, while his cost function is $C = \frac{Q^2}{2} + 1$.

(i) Compute the price (P) and quantity (Q) that will prevail in equilibrium. Draw the AR, MR and MC schedules of the monopolist and indicate the equilibrium (P, Q) in a diagram.

(ii) Suppose the government imposes a ceiling on the price the monopolist can charge so that $P \leq 50$and succeeds in enforcing it fully. Compute the monopolist's equilibrium (P, Q) in this situation. Draw the AR and MR schedules that will obtain in this situation and indicate the equilibrium (P, Q) in the diagram. [10+12=22]

4. During translation, the type of amino acid that would be added to the growing polypeptide depends on the

(a) codon on the tRNA only.
(b) anticodon on the mRNA only.
(c) anticodon on the tRNA to which the amino acid is attached only.
(d) codon on the mRNA and the anticodon on the tRNA to which the amino acid are attached.

5. If you had two guinea pigs of opposite sex, both homozygous, one black and one brown, but you didn't know which was the dominant characteristic, how could you be certain that the guinea pigs are truly homozygous?

(a) The guinea pigs would be homozygous for black (or brown) coat color if each strain could be bred for many generations and only black (or brown) colored offspring were produced.
(b) If the immediate parents of the black (or brown) guinea pigs were both of that color, it proves they are homozygous.
(c) If a cross between the black and brown guinea pig produced four all black offspring, the black guinea pig would have to be homozygous for black coat color.
(d) Any of the aforesaid results would prove the black guinea pig was homozygous.
(e) Only microscopic examination of the guinea pig's genes could absolutely confirm homozygosity.

6. Sigma factor is the component of

(a) RNA polymerase
(b) DNA polymerase
(c) DNA ligase
(d) Endonuclease

7. Suppose that you are given a polypeptide sequence containing the following sequence of amino acids: **tyrosine - proline - aspartic acid - isoleucine - cysteine.** Use the portion of the genetic code given in the table below to determine the DNA sequence that codes for this polypeptide sequence.

mRNA	**Amino acid**
UAU, UAC	tyrosine
CCU, CCC, CCA, CCG	proline
GAU, GAC	aspartic acid
AUU, AUC, AUA	isoleucine
UGU, UGC	cysteine

(a) 5'-AUGGGUCUAUAUACG-3'
(b) 3'-ATGGGTCTATATACG-5'
(c) 3'-GCAAACTCGCGCGTA-5'
(d) 5'-ATTGGGCTTTAAACA-3'

Group B

Answer any five out of the following seven questions

(Each question carries 8 marks)

8. In rabbits, the dominant allele *B* causes black fur and the recessive allele *b* causes brown fur. Two alleles, dominant ***R*** and recessive ***r***, of another independently assorting gene can cause long and short fur respectively. A homozygous rabbit with long and black fur is crossed with a rabbit with short and brown fur. And the offspring is intercrossed. In the F_2 generation, what proportion of rabbits, with long and black fur, will be homozygous for both genes? **[8]**

9. A man, who is color blind and possesses "O" blood group, has children with a woman who has normal color vision and "AB" blood group. The woman's father had color blindness. X-linked gene determines color blindness. **(a)** What are the genotypes of the man and the woman? **(b)** What proportion of the children will have color blindness and type "B" blood group? **(c)** What proportion of their children will be color blind and have type "AB" blood group? **[2+3+3]**

10.(a) Why multiple "origin of replication" is needed for replication of entire human genome? **(b)**Why transcription and translation occur simultaneously in bacterial cell but not in human cell? **(c)** What are the differences in leading and lagging strand synthesis during DNA replication? **[2+3+3]**

11.(a) Why the frequency of recombinant gametes is always half the frequency of crossing over?

(b) In corn, Colored kernels (C) is dominant over colorless (c); Plump kernels (S) is dominant over shrunken (s); Starchy kernels (W) is dominant over waxy (w).

A trihybrid (Cc Ss Ww) plant is testcrossed and the following progeny are obtained:

2708 Colorless, plump, waxy
2538 Colored, shrunken, starchy
626 Colorless, plump, starchy
601 Colored, shrunken, waxy
116 Colorless, shrunken, starchy
113 Colored, plump, waxy
4 Colored, plump, starchy
2 Colorless, shrunken, waxy

Determine linkage (including map distance) for the genes in this cross. **[2+6]**

12. **(a)** Distinguish between transcription and translation. **(b)** What is tRNA charging? **[5+3]**

13. Elizabeth is married to John, and they have four children. Elizabeth has a straight nose (recessive) and is able to roll her tongue (dominant). John is also able to roll his tongue, but he has a convex (Roman) nose (dominant). Of their four children, Ellen is just like her father, and Dan is just like his mother. The other children—Anne, who has a convex nose, and Peter, who has a straight nose—are unable to roll their tongues. Please answer the following questions about this family.

INDIAN STATISTICAL INSTITUTE

Final- Semestral Examination: 2013-14

Course Name: B. STAT. II YEAR
Subject Name: Molecular Biology

Date: 22·11·13 **Maximum Marks: 50** **Duration: $2^1/_2$ hrs**

Note: Answer any five from Group A, and any five from Group B. In Group A, only first attempted five answers will be considered. In Group B, best five answers will be considered.

Group A

Multiple choice questions: Answer any five out of seven questions (Each question carries 2 marks)

1. In a Mendelian monohybrid cross, which generation is always completely heterozygous?

   (a) F_1 generation
   (b) F_2 generation
   (c) F_3 generation
   (d) P generation
   (e) All of these are possible

2. Human blood type is determined by codominant alleles. There are three different alleles, known as I^A, I^B and i. The I^A and I^B alleles are codominant, and the i allele is recessive.
   The possible phenotype for human blood group are type A, type B, type AB and type O. Type A and B individuals can be either homozygous or heterozygous.
   A woman with type A blood and a man with type B blood could potentially have offspring with which of the following blood types?

   (a) type A
   (b) type B
   (c) type AB
   (d) type O
   (e) All of the above.

3. Transcription is the transfer of genetic information from

   (a) DNA to mRNA
   (b) DNA to RNA
   (c) tRNA to mRNA
   (d) mRNA to protein

(a) What are the genotypes of Elizabeth and John? **(b)** Elizabeth's father was a straight-nosed roller, while her mother was a convex-nosed non-roller. Can you figure out their genotypes? **(c)** John's father was a straight-nosed roller, while his mother was a convex-nosed roller. Can you determine their genotypes? **[2+3+3]**

14.**(a)** Do you expect the following mRNA would be translated? **(b)** Where could translation begin in a mRNA? **(c)** In general, can a ribosome bind at more than one site on mRNAs? Note: AUG is the first codon.

5'- GGCCAGGAGGCUUCCAUGCGAUUGUUCAAGUGACA-3'

(d) What would be the result (in terms of the produced protein) if RNA polymerase initiated transcription one base upstream of its normal starting point and if so, what would be the scenario? What would be the result (in terms of the produced protein) if translation began one base downstream of its normal starting point and if so, what would happen? **[1+1+1+5]**

INDIAN STATISTICAL INSTITUTE

First-Semester Examination : 2013-14

Course Name : B. STAT. II Year (First Semester)

Subject Name : Physics I

Date : 22/11/-13 **Maximum Marks : 60** **Duration : 3 hours**

---

1. (a) A particle of unit mass moves in a force field given by $\vec{F} = (3t^2 - 4t)\hat{i} + (12t - 6)\hat{j} + (6t - 12t^2)\hat{k}$, where t is the time. Find the change in momentum of the particle from time $t = 1$ to $t = 2$.
(b) A particle of mass m moves along the x-axis under the influence of a conservative force field having potential $V(x)$. If the particle is located at positions x_1 and x_2 at respective times t_1 and t_2, prove that if E is the total energy,
$$t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}.$$
(c) Show that the impulse of a force $\vec{F}$ acting on a particle is identical to the change in momentum of the particle. 4+4+2

2. (a) A system consists of two masses m_1 and m_2 suspended over a frictionless pulley of radius a and connected by a flexible string of constant length l. Using D'Alembert's principle find the equation of motion of the system. Calculate the tension in the string.
(b) A dumb bell, consisting of two particles of masses m_1 and m_2 that are rigidly fastened to a rod of length l and of negligible mass, is free to move in the vertical plane XY under the action of gravity. Find the Lagrangian L and set up the equations of motion of the dumb bell in the vertical plane. (3+1)+(3+3)

3. (a) Calculate the work done W in moving a particle along a curve C from point A to point B in a conservative force field, when V_A and V_B are the potential energies of the particle at these two points, respectively.
(b) Prove that in absence of external forces the centre of mass of a system of particles is either at rest or in motion with constant velocity.
(c) Show that total angular momentum of a system of particles about

a fixed point is equal to the angular momentum of the centre of mass about that point, plus the angular momentum of the system about the centre of mass. 2+4+4

4. (a) Derive Lagrange's equations of motion for a conservative system of particles by variational procedure.
(b) Using the calculus of variations find the shortest distance between two points in a plane.
(c) Obtain the equation of motion of one dimensional harmonic oscillator using Hamilton's principle. 4+3+3

5. (a) Consider an infinite cylinder of radius a having a uniform charge density λ per unit length. Using Gauss's theorem find the electric field E at a distance r from the axis of the cylinder for (i) $r > a$ and (ii) $r < a$. Sketch the variation of E with r.
(b) Find the electric potential and electric field in free space due to an electric dipole. (2+2+2)+(3+1)

6. (a) Four point charges of +120, +60, +60 and +120 esu, respectively, are placed at the corners of a square $ABCD$ of side 12 cm. If the surrounding medium is air, find the direction and magnitude of the electric field at O, the point of intersection of the diagonals. Calculate the electrostatic potential at the point O.
(b) The electrostatic potential in a medium is given by,

$$\phi(r) = \frac{q}{4\pi\epsilon_0} \cdot \frac{e^{-r/\lambda}}{r}.$$

Calculate the corresponding electric field. (4+2)+4

7. (a) Two concentric conducting spherical shells of radii a_1 and a_2 ($a_2 > a_1$), are charged to potentials ϕ_1 and ϕ_2, respectively. Determine the electric potential and field in the region between the shells.
(b) If ϕ_1, ϕ_2,. . .,ϕ_n are solutions to Laplace's equation in electrostatics, then show that $\phi = \sum_{i=1}^{n} c_i\phi_i$ is also a solution, where c's are the arbitrary constants.
(c) Check whether the function $\phi = 6x^2 + 16y - 6z^2$ can represent the electrostatic potential in a charge free region or not. (4+2)+2+2

INDIAN STATISTICAL INSTITUTE

Semestral Examination 2013-2014

B. Stat II Year

Elements of Algebraic Structures

Date: November 25, 2013 Maximum marks: 60 Duration: 4 hours

Group A (*Answer any three questions:* 5×3)

1. (a) Let G be a finite group and H be a cyclic subgroup generated by h. Prove that, if $ghg^{-1} = h^n$ for some $n \in \mathbb{Z}$, then $g \in N(H)$, where $N(H)$ is the normalizer of H in G. [3]
   (b) Let G be a group such that there is exactly one element a of order 2. Prove that $a \in Z(G)$.[2]
2. Let G be a nonabelian group of order 8. Show that either $G \simeq D_4$ or $G \simeq Q_8$.
3. Let G be a finite group and H be a subgroup of G such that $|H| = p^n$ for some prime p and $n \geq 1$. Prove that $[N(H) : H] \equiv [G : H] \pmod{p}$.
4. Prove that a group of order 30 must contain a nontrivial proper normal subgroup.

Group B (*Answer any three questions:* 6×3)

5. Let $R = \mathbb{Z}[\sqrt{-5}]$ and let $\alpha = 2(1+\sqrt{-5})$, $\beta = (1+\sqrt{-5})(1-\sqrt{-5})$. Prove that *gcd* of α and β does not exist in R.
6. Let $R = \{f(X) \in \mathbb{Z}[X] \mid \text{coefficient of } X \text{ in } f \text{ is zero}\}$, which is subring of $\mathbb{Z}[X]$. Show that R is not a UFD.
7. (a) Let R be a commutative ring and I, J, K be ideals of R. Prove that $I(J+K) = IJ + IK$.
   (b) Let $R = \mathbb{Z}[i]$. Let $P = <1+i>$. Prove that $P^2 = <2>$.
8. Let $f, g \in \mathbb{Q}[X]$ be such that the product $fg \in \mathbb{Z}[X]$. Prove that the product of any coefficient of f and any coefficient of g is an integer.

Group C (*Answer any three questions:* 6×3)

9. (a) Prove that if $F \subseteq K$ is an extension of fields such that $[K : F]$ is finite, then K is algebraic over F.
   (b) Give an example of a field extension $F \subseteq K$ such that K is algebraic over F but $[K : F]$ is not finite.
10. Compute the degree of the splitting field of $X^3 - 2$ over $\mathbb{Q}$ with complete justification.
11. Express $\mathbb{Q}(\sqrt{5}, \sqrt{7})$ as $\mathbb{Q}(\alpha)$ for some $\alpha \in \mathbb{Q}(\sqrt{5}, \sqrt{7})$. Find the minimal polynomial of α over $\mathbb{Q}$.
12. Let p be a prime. Given any $n \geq 1$, show that there is a field of degree n over $\mathbb{F}_p$.

[PTO]

Group D (*Answer any one question:* 9×1)

13. Let R be the ring of all real valued continuous functions on the closed interval $[0, 1]$.

(a) For each $\alpha \in [0, 1]$, show that the ideal $M_\alpha = \{f \in R \mid f(\alpha) = 0\}$ is maximal. [4]

(b) Let M be a maximal ideal of R. Show that there is some $\alpha \in [0, 1]$ such that $M = M_\alpha$. [5]

14. (a) Prove that $X^4 + 1$ is an irreducible polynomial in $\mathbb{Z}[X]$. [4]

(b) For each prime p, show that $X^4 + 1 \in \mathbb{F}_p[X]$ is reducible. [5]

Indian Statistical Institute
Statistical Methods III
B.Stat. 2nd year
First Backpaper Examination

Date: 27.01.14 Duration: 4 hrs.

This paper carries 100 marks. Attempt all questions. The maximum you can score is 45. Justify all your steps. This is a closed book, closed notes examination. You may use your own calculator. No need to perform more than two steps of any iterative method. Standard statistical tables will be provided.

1. We continue to draw uniformly distributed random numbers $X_1, X_2, ...$ from the set $\{1, 2, ..., N\}$ with replacement until we get the first repetition. If the observed data set is

$$2, 5, 3, 6, 3,$$

then find the MLE of N. [10]

2. Show that based on a random sample $X_1, ..., X_n$ from $N(0, \sigma^2)$ we cannot have a UMP test of

$$H_0 : \sigma^2 = 1 \text{ vs. } H_1 : \sigma^2 \neq 1.$$

[10]

3. Let our data be $(x_1, Y_1), ..., (x_n, Y_n)$ where for each i,

$$Y_i = \alpha + \beta x_i + \epsilon_i.$$

Here x_i's are known real numbers, $\alpha, \beta$are unknown real numbers, and $\epsilon_1, ..., \epsilon_n$ are iid $N(0, \sigma^2)$. Obtain mle of σ^2 and a 95% confidence interval for α. [5+5]

4. Let $X_1, ..., X_n$ be the life times of n bulbs. Assume that these are iid with density

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

Each of the n bulbs is observed for at most 10 hours. Thus the recorded life times are $Y_1, ..., Y_n$, where

$$Y_i = \begin{cases} X_i & \text{if } X_i < 10 \\ 10 & \text{else.} \end{cases}$$

Based on $Y_1, ..., Y_n$ we have to find mle of λ. Formulate this as an incomplete data problem. Clearly mention the complete data. Write down the E-step for EM algorithm for this problem. [10]

[Please turn over]

5. Does there exist a UMP test (at 5% level) for testing

$$H_0 : \theta = 1 \text{ vs. } H_1 : \theta > 1$$

based on a random sample $X_1, ..., X_n$ from Unif$(0, \theta)$, where $\theta \in [1, \infty)$? If so, construct such a test. Otherwise, prove the nonexistence. [10]

6. A box contains 5 white and 3 black balls. A coin with $P(head) = p \in \{0.1, 0.8\}$ is tossed, and independently a ball is drawn at random from the box. If the outcome of the toss is a head, then the ball is returned to the box, otherwise it is not returned. Then a fresh ball is drawn from the box. Based on only the colour of this second ball, obtain an MP test (at 5% level) to test $H_0 : p = 0.1$ vs. $H_1 : p = 0.8$. [10]

7. In an inkjet printer a head carrying the ink cartridge moves over the paper and deposits a drop of ink at the print location. Two different printer models are being compared for accuracy of positioning the head. 5 lines are printed with each of them. Each line consists of a single point at distance 10 cm. from the left margin. The actual locations of the ink drops as deposited by the two printers are recorded as follows (distance in cm. from the left margin):

Printer 1	9.98	9.89	10.03	10.07	10.00
Printer 2	10.01	10.00	9.99	10.00	10.02

Making suitable distributional assumptions test the hypothesis that the second printer is more accurate than the first. Use 5% level of significance. [10]

8. A random sample is drawn from a locality, and the educational level and gender of each person in the sample are recorded. These are presented in the following contingency table.

	No education	School education	College education
Male	23	146	36
Female	45	147	32

Test (at 5% level) the hypothesis that educational level has no association with gender in this locality. [10]

9. Describe Fisher's exact test. Why is it called a conditional test? [10]

10. (a) Distinguish between the seasonal component and the cyclical component of a time series.

    (b) Find the autocovariance function of the time series X_t, where

$$X_t = Z_t + 0.5Z_{t-1} + 0.1Z_{t-2},$$

where $E(Z_t) = 0$ and $V(Z_t) = \sigma^2$ for all t, and $cov(Z_t, Z_{t'}) = 0$ for all $t \neq t'$.

[5+5]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination

First semester

B. Stat - Second year 2013-2014

Analysis III

Date: 31 01 14 Maximum Marks: 100

Duration: 3 hours

Answer all questions.

You must state clearly any result you use.

(1) Consider the Hilbert-Schmidt norm of an $n \times n$ matrix given by $\|A\| = \sqrt{\text{trace}A^*A}$ and the operator norm $\|A\|_\infty = \sup_{\|x\|=1} \|Ax\|$ for $A \in M_n(\mathbb{R})$. Prove the following inequalities:

$$\|A\|_\infty \leq \|A\| \leq \sqrt{n}\|A\|_\infty$$

Hence show that the topologies defined by the two norms are equivalent. 12

(2) Let $\gamma : [0, 1] \to O(n)$ be a differentiable map such that $\gamma(0) = I_n$. Prove that $\gamma'(0)$ is a skew-symmetric matrix. 8

(3) Let $\mathrm{Gl}_n(\mathbb{R})$ be the set of all $n \times n$ invertible matrices over real numbers. Prove that $f : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ defined by $f(X) = X^{-1}$ is differentiable and compute the derivative of f at some $A \in \mathrm{GL}_n(\mathbb{R})$.

Is the function $g(X) = \|X^{-1}\|$ differentiable everywhere? Justify your answer ($\|\ \|$ denotes the Hilbert Schmidt norm on $M_n(\mathbb{R})$). 8+4

(4) Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2y + e^x + z$. Note that $f(0, 1, -1) = 0$. Justify the following statement: *There is a C^1-function $g : U \to \mathbb{R}$ defined on some open set $U \subset \mathbb{R}^2$ containing $(1, -1)$ such that*

$$g(1, -1) = 0 \text{ and } f(g(y, z), y, z) = 0.$$

Find D_1g and D_2g at $(1, -1)$. 6+6

(5) Find stationary points of the function $f(x, y, z) = x + y + z$ on the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 | \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}.$$

Determine maximum and minimum values of f on the surface. 10

(6) Let $T : D \to D$ be a C^1 map defined on a closed ball D in $\mathbb{R}^k$. Suppose that T is bijective and the Jacobian $J_T > 0$. Then prove that

$$\int_{\Phi \circ T} \omega = \int_{\Phi} \omega$$

for any k-form ω on $\mathbb{R}^k$ and any k-surface $\Phi : D \to \mathbb{R}^k$. 12

(7) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map and S a rectangle in $\mathbb{R}^2$. Prove that

$$\text{area } (T(S)) = |\det T| \text{ area } (S)$$

12

(8) Find a real valued function Φ on $\mathbb{R}^2$ such that

$$d\Phi = (1 + y) \cos x \, dx + (\sin x + y) \, dy$$

10

(9) Make a sketch of the region $S \subset \mathbb{R}^2$ in the first quadrant lying between the two hyperbolas $xy = 1$ and $xy = 2$ and the two straight lines $y = x$ and $y = 4x$. Evaluate the integral $\int_S x^2 y^2 dx \, dy$. 12

PROBABILITY THEORY III
B. STAT. IIND YEAR SEMESTER 1
INDIAN STATISTICAL INSTITUTE

Backpaper Examination

Time: $3\frac{1}{2}$ Hours Full Marks: 100

Date: ~~January~~ Feb 23, 2014

You may use any result proved in class without proving them again. However clearly state all the results you are using.

1. For a nonnegative random variable X, show that $\mathrm{E}[\log^+ X] < \infty$ iff $\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{P}[X > n] < \infty$. [10]

2. Calculate $\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^2}$. [10]

3. Show that $X_n \Rightarrow X$ iff $\mathrm{E}[g(X_n)] \to \mathrm{E}[g(X)]$ for every compactly supported continuous function g. [12]

4. Let X_n be random variables taking integer values and $X_n \Rightarrow X$. Show that X also takes only integer values and $\mathrm{P}[X_n = j] \to \mathrm{P}[X = j]$ for all integers j. Show that
$$\sum_j |\mathrm{P}[X_n = j] - \mathrm{P}[X = j]| \to 0.$$
[5+5=10]

5. Show that $\mathrm{P}(A_n \Delta A) \to 0$ iff $\mathrm{E}[(\mathbb{1}_{A_n} - \mathbb{1}_A)^2] \to 0$. [6]

6. Let $\{X_n\}$ be a sequence of independent random variables. Show that $\mathrm{P}[\sup X_n < \infty] = 1$ iff $\sum_n \mathrm{P}[X_n > M] < \infty$ for some M. [10]

7. Let $\{X_n\}$ be the partial minima sequence of an i.i.d. uniform sequence on $(0, 1)$. Compute mean and variance of $\sum_1^n X_k$. Show that $\sum_1^n X_k / \log n \xrightarrow{\mathrm{P}} 1$. [3+3+4=10]

8. Let $\{X_n\}$ be an increasing sequence of nonnegative random variables with $\mathrm{E}[X_n] \sim an^{\alpha}$ with $a, \alpha > 0$ and $\mathrm{Var}[X_n] \sim bn^{\beta}$ with $b > 0$ and $\beta < 2\alpha$. Show that $X_n/n^{\alpha} \to a$ a.e. [14]

9. Let $X_1, \ldots, X_n$ are i.i.d. centered normal random variables with variance σ^2. Define $T = X_1 / \sqrt{\sum_1^n X_i^2 / n}$. Identify, with justification, the distribution of
$$\sqrt{\frac{n-1}{n}} \cdot \frac{T}{\sqrt{1 - \frac{T^2}{n}}}.$$
[6]

10. Let X and Y be independently and uniformly distributed on the interval $(0, 1)$. Find the conditional distribution function of $X \vee Y$ given X. [12]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: 2013-2014

B. Stat. (Hons.) 2nd Year. 2nd Semester

Statistical Methods IV

Date: February 24, 2014 **Maximum Marks: 60** **Duration: 2 and 1/2 hours**

• This question paper carries 66 points. Answer question no. 1 and as much as you can from the rest. However, the maximum you can score is 60.

• You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

---

1. (i) **Answer either (a) or (b).** [8]

(a) Consider k samples consisting of n_i, $i = 1, \ldots, k$, p-variate observations from k populations. Let $n := n_1 + \cdots + n_k$. We have stated and proved analysis of variance formula, when $p = 1$ and $k = 2$, which enables us to decompose the overall variability of all n observations into between- and within-group variabilities. State and prove an analogous result for *arbitrary* p and k.

(b) Consider a sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ consisting of n p-variate observations. A real-valued measure of variability of the observations about a fixed vector $\boldsymbol{\mu}$ is given by $\det[\mathrm{S}(\boldsymbol{\mu})]$, where

$$\mathrm{S}(\boldsymbol{\mu}) := \sum_{i=1}^{n} (\boldsymbol{X}_i - \boldsymbol{\mu})(\boldsymbol{X}_i - \boldsymbol{\mu})^{\mathrm{T}}/n.$$

Show that $\det[\mathrm{S}(\boldsymbol{\mu})] \geq \det[\mathrm{S}(\bar{\boldsymbol{X}})]$, where $\bar{\boldsymbol{X}}$ is the sample mean vector.

(ii) The data in the following table represent respiratory rates of eight cows over 15 minutes after the administration of the drug Ketamin.

	Cow							
Time (minutes)	1	2	3	4	5	6	7	8
0	20	28	22	20	28	36	30	28
5	36	20	24	36	32	44	32	38
10	28	52	16	18	30	52	28	28
15	20	44	26	12	30	48	30	20

Explain how you will formulate the problem of testing uniformity of respiratory rates across time. [4]

[Please turn over.]

2. Suppose (X_i, Y_i), $i = 1, \ldots, n$, are points in the plane. Let $\boldsymbol{X} := (X_1 X_2 \ldots X_n)^{\mathrm{T}}$, $\boldsymbol{Y} := (Y_1 Y_2 \ldots Y_n)^{\mathrm{T}}$. Denote by b_n, the ordinary least squares coefficient — based on $\boldsymbol{X}$ and $\boldsymbol{Y}$, for a line passing through the origin. Suppose $\boldsymbol{X} \sim \mathrm{N}_n(\mathbf{0}, \boldsymbol{I}_n)$, $\boldsymbol{Y} \sim \mathrm{N}_n(\mathbf{0}, \boldsymbol{I}_n)$ and $\boldsymbol{X}$ is independent of $\boldsymbol{Y}$. Find the distribution of b_n, suitably normalized. Describe an application of this result. [8+3=11]

3. Suppose $\boldsymbol{X} \sim \mathrm{N}_p(\mathbf{0}, \boldsymbol{\Sigma})$. Show that $\mathrm{Var}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}) = 2\,\mathrm{trace}(\boldsymbol{\Sigma}^2)$. [8]

4. Suppose $\mathbf{M} \sim \mathrm{W}_p(\boldsymbol{\Sigma}, n)$, $n \geq p+2$, where $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix. Show that $\mathrm{E}(\mathbf{M}^{-1}) = \boldsymbol{\Sigma}^{-1}/(n-p-1)$. [10]

5. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n, \boldsymbol{X}_{n+1}$ be i.i.d. $\mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ $(n \geq p+1)$ variables, where $\boldsymbol{\Sigma}$ is positive definite. For $j = n, n+1$, let $\bar{\boldsymbol{X}}_j$ and $\mathbf{S}_j$ be defined by $j\bar{\boldsymbol{X}}_j := \sum_{i=1}^{j} \boldsymbol{X}_i$, $\mathbf{S}_j := \sum_{i=1}^{j}(\boldsymbol{X}_i - \bar{\boldsymbol{X}}_j)(\boldsymbol{X}_i - \bar{\boldsymbol{X}}_j)^{\mathrm{T}}$. Let $T := (\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_{n+1})^{\mathrm{T}}\mathbf{S}_{n+1}^{-1}(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_{n+1})$.

(a) Show that $\mathbf{S}_{n+1} = \mathbf{S}_n + n(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_n)(\boldsymbol{X}_{n+1} - \bar{\boldsymbol{X}}_n)^{\mathrm{T}}/(n+1)$.

(b) Show that $(n+1)T/n$ follows a Beta distribution, with parameters to be obtained by you. [4+9=13]

6. Suppose $\mathbf{X} = (\boldsymbol{X}_1 \boldsymbol{X}_2 \cdots \boldsymbol{X}_n)^{\mathrm{T}}$ is an $n \times p$ $(n \geq p+1)$ data matrix from $\mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. It is known that $\boldsymbol{\mu} = k\boldsymbol{\mu}_0$ for some unknown constant k, where $\boldsymbol{\mu}_0$ is a known $p \times 1$ non-null vector, and $\boldsymbol{\Sigma}$ is an unknown positive definite matrix.

(a) Find the maximum likelihood estimator (MLE) of $(k, \boldsymbol{\Sigma})$.

(b) Denote the MLE of $(k, \boldsymbol{\Sigma})$ by $(\hat{k}, \hat{\boldsymbol{\Sigma}})$. Decide, with adequate reasons, if $\hat{k}$ is unbiased for k. [8+4=12]

$* * * * **$ *Best of Luck!* $* * * * **$

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: 2013-2014

Course Name: B.Stat. (Hons.) 2nd Year

Subject Name : Economic and Official Statistics and Demography

(Answer Group A and Group B on two separate answer booklets)

Group A

Maximum Marks: 30 Time: 1 hour

(Answer any two from Question no. 1,2 and 3. Question no 4 is compulsory.)

1. State the properties of Laspeyres and Paasche index number and show in this context that the reciprocal of Paasche (current weighted) index of price or quantity is the weighted Arithmatic mean of the backward price or quantity realtives, weights being the current values item by item. 10

2. State the axioms on Index numbers and mention the index numbers which satisfy the axioms. 10

3. Construct a Quantity index number from the table and show that it satisfies time reversal test. 10

Commodity	1973=base		1974	
	Price	Quantity	Price	Quantity
A	6	70	8	120
B	8	90	10	100
C	12	140	16	280

4.Write short note (any two) : 5*2=10

a) Better life Index of OECD

b) Micro data of World Bank

c) Survey rounds of NSSO

D) Function of Central Statistics Office

Group B

Maximum Marks: 30 Time : 1 Hour

(Answer as much as you can. Maximum score remains however 30)

1. What do you understand by neo-Malthusian thinking. Explain briefly the statement "population pressure contributed to (induced) technological development and thus led to a rise in per capita income and production". Describe the population growth pattern in the post-independence period and explain how the neo-Malthusian thinking persisted in the period. [1 + 4 + 5 = 10]

2. Explain how demographic dividend arises. How high is high enough to be called dividend? When had India crossed the 60 p)]er cent marks in the share of population in the age group of 15 to 64 years. When is it expected to reach the peak share of 69 per cent? For how long the share would remain above 65 per cent? [3 + 4 + 1 + 1 +1 = 10]

3. Explain the logic behind the construction of the Whipple index (W). State the assumption behind the use of Sex Ratio Score (SRS) for measuring accuracy of data on population age distribution. Write the mathematical expressions for SRS along with explanations for the symbols. [4 + 2 + 4 = 10]

4. Write short notes on the following.

   (a) Age heaping and age shifting.

   (b) U.N. Joint Score

[3 + 4 = 7]

== END ==

B.Stat. II / Introduction to Markov Chain

Midsem. Exam. / Semester II 2013-14

Date - February 26, 2014 / Time - 2 hours

Maximum Score - 30

NOTE : SHOW ALL YOUR WORK TO GET THE FULL CREDIT. RESULTS USED MUST BE CLEARLY STATED.

1. Let P be the transition probability matrix of a Markov chain on the states space $S = \{1, 2, 3, 4, 5, 6, 7\}$ and it is given by

$$P = \begin{bmatrix} 0 & 0.4 & 0.2 & 0 & 0 & 0.4 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 \end{bmatrix}$$

   (a) (4 marks) Describe, with a diagram, if this chain is irreducible and find the communicating class(es). Find the periodicity of the states. Classify the states according to transience, null recurrence and positive recurrence, with very brief justification.

   (b) (3 marks) For each communicating class(es) find the stationary distribution (separately), whenever they exist.

   (c) (3 marks) Find the prob that starting from the state 2, the chain will eventually go to $\{3, 4\}$.

2. Suppose there are two kinds of balls in an urn, red and black. In total, there are K of them. At each time n, a ball is chosen at random and replaced by a ball of the opposite colour. Let X_n be number of red balls in the urn at time n. Note that $\{X_n\}$ is a Markov chain on the state space $S = \{0, 1, \ldots, K\}$.

   (a) (4 marks) Write down its transition probabilities. Is this chain irreducible? Justify. Is this chain aperiodic? If yes, justify; if not, find its period.

   (b) (5 marks) If the invariant distribution exists for this chain, find it. Calculate the expected time it takes to reach the state K from the state 0.

3. Let X_n be the Markov chain on the state space $S = \{0, 1, \ldots, \}$ with transition probabilities, $p_{i,0} = p_i$ and $p_{i,i+1} = 1 - p_i$ with $0 < p_i < 1$ and all $\imath \geq 0$; $p_{i,j} = 0$, otherwise. Note that the chain is irreducible.

   (a) (4 marks) Find the condition for the transience of the chain in terms of p_i. Show that this is equivalent to having a bounded (say, by 1) nontrivial solution for the system, for $i \neq 0$, $y_i = \sum_{j \neq 0} p_{i,j} y_j$.

   (b) (4 marks) Find the condition for the null recurrence of the chain. Show that this is equivalent to having a nontrivial solution to the system $x_i = \sum_{j \in S} x_i p_{i,j}$, that provides $\sum_{i \in S} x_i = \infty$.

4. Suppose a machine has two components that run independently. Each fails with probability $p \in (0, 1)$ on a given day and keeps operating with probability $q = 1 - p$. The machine fails when both components fail. When a component is down, the repair can take k no. of days with probability $a(1-a)^{k-1}$ for $0 < a < 1$, $k \geq 1$. Let X_n be the no. of component(s) working on the day n.

   (a) (3 marks) Is $\{X_n\}$ a Markov chain? Argue briefly. If so, write down its transition probability matrix.

   (b) (3 marks) Find the long run probability that the machine is operating (i.e., the proportion of time it is operating) in terms of a and p.

   (c) (3 marks) Determine the expected time between two consequtive breakdowns of the machine.

All the best.

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination

B. Stat. - II Year (Semester - II)

Discrete Mathematics

Date : 27-02-14 Maximum Marks : 60 Duration : 2:30 Hours

Note : You may answer any part of any question, but maximum you can score is 60.

1. The Ramsey number $R(s,t)$ is the minimum number n such that any graph on n vertices contains either an independent set of size s or a clique of size t. Prove that for any $s, t \geq 1$, there is $R(s,t) < \infty$. [12]

2. For any graph G on n vertices, not containing a 4-cycle. Prove that $E(G) \leq \frac{n}{4}(1+\sqrt{4n-3})$. [12]

3. Suppose that $d_1, d_2, \ldots, d_n$ are integers with $d_1, \geq d_2 \geq \ldots \geq d_n$. Prove that, for $n > 1$, $d(d_1, d_2, \ldots, d_n)$ is graphic if and only if $d(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, , d_{d_1+2} \ldots, d_n)$ is graphic. [12]

4. Let $G(V,E)$ be an undirected graph having degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n$. Suppose $|V| = n$ and $|E| = m$. Let $t_i(G)$ be the number of triples of $V(G)$ having i edges in between each triples where $0 \leq i \leq 3$. Similarly, $t_i(\bar{G})$ be the number of triples of $V(G)$ having $0 \leq i \leq 3$ edges in $\bar{G}$. Prove that $t_0(G) + t_3(G) = \binom{n}{3} - m(n-2) + \sum_{i=1}^{n} \binom{d_i}{2}$. Hence or otherwise prove that $t_3(\bar{G}) + t_3(G) = \binom{n}{3} - m(n-2) + \sum_{i=1}^{n} \binom{d_i}{2}$ [10+5=15]

5. For two vertices $u, v \in V$, let $d(u,v)$ denotes the length of the shortest path between u and v. The diameter of a connected graph G is the maximum $d(u,v)$ among all pair $u, v \in V$. Prove that every nontrivial self-complementary graph has diameter 2 or 3. [15]

INDIAN STATISTICAL INSTITUTE

Mid-Sem Examination : (2013-2014)

B. Stat. II Year

Physics II: Thermodynamics and Statistical Mechanics

Date: 28.02.14 Duration: 2 hrs Maximum Marks: 40

Attempt as many questions. Total marks can not exceed 40.

1.(a) Two identical bodies of constant heat capacity C_P have initial temperatures T_1 and T_2 respectively and final temperature T_f. If the bodies are at constant pressure without any change of phase, show that the work obtainable is maximum when $T_f = \sqrt{T_1 T_2}$.

(b) A Carnot engine with the sink at 10°C has an efficiency of 30%. By how much must the temperature of the source be changed to increase its efficiency to 50%? [7+3]

2.(a) Calculate the increase of entropy of $1g$ of hydrogen when its temperature is raised from $-173°C$ to $27°C$ and its volume becomes four times its original volume, if its molar heat capacity $C_v = 4.86$ cal.K^{-1} and $R = 2.01$ cal.mol^{-1}K^{-1}.

(b) How do the following quantities change in first and second order phase transitions?

(i) specific entropy,

(ii) specific heat at constant pressure,

(iii) specific Gibb's function,

(iv) coefficient of volume expansion,

(v) specific volume. [5+5]

3. The pressure on 1 Kg of copper is increased reversibly and isothermally from very near 0 atm. to 1000 atm. at $0°C$. Taking the density $\rho = 9 \times 10^3 Kg/m^3$, volume expansivity $\beta = 5 \times 10^{-5} K^{-1}$, and specific heat $C_P = 385 JK^{-1}/Kg$ to be constant, calculate

(a) heat transfer during the process.

(b) temperature change if the compression had been reversible and adiabatic. [5+5]

4.(a) State and prove Liouville's Theorem in the context of ensemble theory. Use it for the classification of stationary ensembles.

(b) A single particle with energy $\epsilon \leq E$, where $E = p^2/2m$ is enclosed in a volume V. Determine (asymptotically) the number of microstates available in energy range ϵ to $\epsilon + d\epsilon$. [7+3]

5. If two samples of the same ideal gas at same temperature T and same particle density (N/V) are mixed together, find the change in the entropy S of the system before and after the mixing, given that $S = Nkln(V) + \frac{3}{2}Nk[1 + ln(\frac{2\pi mkT}{h^2})]$. Explain the fallacy on physical grounds. Modify the expression of S to remove the fallacy. [10]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: (2013 – 14)
B. Stat II Year
Agriculture

Date 28·02·14 Maximum Marks 30 Duration Three hours .
(Attempt any **Three** questions)
(Number of copies of the question paper required Seven)

1. Define Moisture Availability Index? Draw a suitable rice calendar with the following data. 2+8

Week No.	*22*	*23*	*24*	*25*	*26*	*27*	*28*	*29*	*30*	*31*	*32*	*33*	*34*	*35*	*36*	*37*	*38*	*39*	*40*
Rainfall (mm) at 0.5 Prob.	*0*	*0*	*10*	*19*	*29*	*37*	*98*	*135*	*18*	*90*	*98*	*142*	*93*	*87*	*15*	*32*	*2*	*0*	*0*
PET (mm)	*51*	*42*	*30*	*27*	*23*	*23*	*22*	*20*	*42*	*20*	*19*	*17*	*21*	*23*	*29*	*26*	*31*	*33*	*34*

2. Name the different weather parameters that are related to crop production? Write down the names of the apparatus used for measuring/estimating different weather components with their units. 3+7

3. Write in brief about "System of Rice Intensification". 10

4. Write short notes on any five of the following: 2 x 5

   a. Decade rainfall
   b. Phytoclimate
   c. Hair Hygrograph
   d. Monsoon withdrawal
   e. Season

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2013-14
Course: B-Stat Second Year
Subject: Economics II

Date: 28.02.2014 Maximum Marks: 40 Duration: 2 Hours

Answer all questions

1. a) Define the concepts of personal income, personal disposable income, personal saving and national saving (NS). Using these definitions, derive the identity involving NS and aggregate investment in an economy. Is this investment a gross or net concept?

b) Suppose aggregate investment in an economy is found to exceed NS of the economy. What does it imply in terms of the economy's total purchasing power and total absorption? How is this situation made possible? Also comment on how this situation is financed. [10+10]

2. The following are data for a hypothetical economy in crores of rupees

Net rental income of persons	24
Depreciation	669
Wages and salaries	3780
Personal consumption expenditure	4378
Sales and excise taxes	525
Business transfer payments	28
Gross investment	882
Exports of goods and services	659
Subsidies of government to business	9
Government purchases of goods and services	1148
Imports of goods and services	724
Net interest	399
Proprietor's income	441
Corporate profits	485
Net income from abroad	5

a. Compute GDP using the spending approach.
b. Compute NDP.
c. Compute national income in two ways
d. Compute the statistical discrepancy

[6+1+12+1]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2013–2014

B.Stat. (Hons.) 2nd Year. 2nd Semester

Statistical Methods IV

Date: April 30, 2014 **Maximum Marks: 75** **Duration: 3 and 1/2 hours**

- This question paper carries 85 points. Answer as much as you can. However, the maximum you can score is 75.
- You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

---

1. Suppose we have a random sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ from $\mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix. Let $\boldsymbol{\mu}$ be unknown. We wish to test the hypothesis $\mathrm{H}_0 : \boldsymbol{\Sigma}$ *is diagonal* against $\mathrm{H}_1 : \mathrm{H}_0$ *is false.* Denote by $\mathbf{R}$, the sample correlation matrix. Let λ denote the likelihood ratio test (LRT) statistic for testing H_0 against H_1.

(a) Show that $-2\log\lambda = -n\log(\det \mathbf{R})$.

(b) Argue that under H_0, $-2\log\lambda \xrightarrow{d} \chi^2_{p(p-1)/2}$ as $n \to \infty$. [7+5 = 12]

2. Suppose $\mathbf{X}_i$ is an $n_i \times p$ ($n_i \geq p+1$) data matrix from $\mathrm{N}_p(\boldsymbol{\mu_i}, \boldsymbol{\Sigma})$, $i = 1, 2$. The parameters $\boldsymbol{\mu_1}, \boldsymbol{\mu_2} \in \mathbb{R}^p$, and $\boldsymbol{\Sigma}$ is an *unknown* $p \times p$ positive definite matrix. Let $n \stackrel{def}{=} n_1 + n_2$. Assume that $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent. Consider the problem of testing

$$\mathrm{H}_0 : \boldsymbol{\mu_1} = \boldsymbol{\mu_2} \text{ against } \mathrm{H}_1 : \boldsymbol{\mu_1} \neq \boldsymbol{\mu_2}.$$

Obtain the LRT statistic, denoted by λ, of this testing problem. Obtain the null distribution of λ or of a suitable equivalent of λ. [8+8 = 16]

3. (a) Discuss how from the decomposition of *total sum of squares* into *regression sum of squares*, and *residual sum of squares* in a multiple linear regression model one can develop a suitable test for significance of regression.

(b) Obtain the null distribution of the test statistic developed in (a) or of a suitable equivalent of it. [12+9 = 21]

[P.T.O.]

4. Suppose $X_1, X_2, \ldots$ are i.i.d. gamma variables with unknown scale parameter $\theta > 0$ and known shape parameter $\beta > 0$. Find the MLE of θ. Discuss how you can obtain an approximate $100(1-\alpha)\%$ confidence interval for θ based on its MLE. [3+9 = 12]

5. Consider n i.i.d observations, denoted by $X_1, \ldots, X_n$, from an exponential distribution with unknown location parameter θ and unknown scale parameter σ, $\theta \in \mathbb{R}, \sigma > 0$. Obtain the LRT statistic for testing $\mathrm{H}_0 : \theta = \theta_0$ against $\mathrm{H}_1 : \theta \neq \theta_0$ and the null distribution of it or of a suitable equivalent of it. [7+7 = 14]

[**Note.** You may assume the following result to hold, and use it, if required.

Let $E_{(1)} \leq E_{(2)} \leq \cdots \leq E_{(n)}$ *denote the order statistics corresponding to a sequence of i.i.d.* $\mathrm{Exp}(0,1)$ *random variables* $E_1, \ldots, E_n$*. If we define* $E_{(0)} = 0$*, the normalized exponential spacings* $(n-i+1)(E_{(i+1)} - E_{(i)})$*,* $i = 0, \ldots, n-1$*, are i.i.d.* $\mathrm{Exp}(0,1)$ *random variables.*]

6. Suppose that we have two independent random samples: $X_1, \ldots, X_m$ from a $\mathrm{Beta}(\mu, 1)$ population and $Y_1, \ldots, Y_n$ from a $\mathrm{Beta}(\theta, 1)$ population. Show that the LRT of $\mathrm{H}_0 : \theta = \mu$ versus $\mathrm{H}_1 : \theta \neq \mu$ is based on the statistic

$$T(\boldsymbol{X}, \boldsymbol{Y}) \stackrel{def}{=} \frac{\sum_{i=1}^{m} \log X_i}{\sum_{i=1}^{m} \log X_i + \sum_{j=1}^{n} \log Y_j}. \qquad [10]$$

****** *Best of Luck!* ******

B.Stat. II / Introduction to Markov Chain

Final Exam. / Semester II 2013-14

Time - 3 hours

Maximum Score - 50

05·05·14

NOTE : SHOW ALL YOUR WORK TO GET THE FULL CREDIT. RESULTS USED MUST BE CLEARLY STATED.

1. Let X_n be a Markov chain with stationary transition probability matrix p.

   (a) (4 marks) For finite state space $S = \{0, 1, 2, \ldots, k\}$, if the chain is irreducible aperiodic then show that $y_i = \sum_j p_{ij} y_j$ has unique solution $y_i = 1$ for all i.

   (b) (4 marks) If aperiodic condition is dropped would the result in part (1a) still hold?

   (c) (4 marks) If S has infinite state space, for example, $S = \{0, 1, 2, \ldots, \}$, would the result still hold whenever the chain is null recurrent? or positive recurrent?

2. During each unit of time either 0 or 1 customer arrives for service and joins a sing line. The probability. of one customer arriving is λ, and no customer arrives with probability $1 - \lambda$. Also during each unit of time independently of new arrivals, a single service is completed with probability p or continues on to the next period with probability $1 - p$. Let X_n be the total number of customer (waiting in line or being serviced) at the nth unit of time.

   (a) (3 marks) Show that X_n is a birth-death chain on $S = \{0, 1, 2, \ldots\}$ by writing its transition probability.

   (b) (3 marks) Determine the condition for which it is transient, null recurrent or positive recurrent.

   (c) (3 marks) Calculate the invariant distribution when it is positive recurrent.

   (d) (3 marks) Find expected length of queue in equilibrium (i.e., under stationary distribution).

3. Arrival of customers in a big supermarket (open for 24 hour) is assume to follow a Poisson process with intensity parameter λ. Assume after arrival each customer decides to buy a product with probability $0 < p < 1$ independently of others.

   (a) (5 marks) Show that the customers who buy a product follow a Poisson process. Find its intensity parameter.

(b) (3 marks) Following above, now briefly argue that the customers who do not a buy product also follow a Poisson process and it is independent of the part (3a). Write down its intensity parameter.

(c) (4 marks) Calculate the probability that at time t the number of customers who buy a product is bigger than the number of customers who do not buy any.

4. Let X_n be the number of offspring in the nth generation (for Branching process) with $X_0 = 1$, where offspring distribution given by ξ with probabilities $P(\xi = k) = (1/3, 1/3, 1/3)$, for $k = (0, 1, 2)$, respectively.

(a) (5 marks) Find $q_n = P(X_n = 0)$ and then its limit. [Hint: You may use p.g.f. of ξ]

(b) (4 marks) Let $\tau = \inf\{n \geq 0 : X_n = 0\}$. Find $P(\tau = k)$ and $E(\tau)$

(c) (3 marks) For the offspring distribution given by ξ with probabilities $P(\xi = k) = (1/4, 1/2, 1/4)$, for $k = (0, 1, 2)$, respectively, compare $E(\tau)$ with the earlier one (i.e., whether bigger or smaller with brief reason).

5. A scientist who owns r umbrellas distributes them between home and office according to the following routine. If it is raining upon departure from either place, an event that has probability p (independent of any other day), then an umbrella is carried to the other location (if available at the location of departure). If it is not raining then no umbrella is carried. Let X_n be the number of available umbrellas at whatever place the scientist happens to be departing at the nth trip.

(a) (4 marks) Observing that the X_n is a Markov chain, write down the trasition probability of the chain, clearly stating the state space. If required give the condtion for the chain to be irreducible.

(b) (4 marks) Under the irreducibility, determine the stationary invariant distribution for r odd integer.

(c) (4 marks) Let $0 < \alpha < 1$. How many umbrellas should the scientist own so that the probability of getting wet under equilibrium distribution (i.e., stationary distribution) is at most α. What number works for all weather condition (p).

All the best.

Indian Statistical Institute

Second Semester Examination 2013-14

Course Name : BSTAT Second Year

Subject Name : Economic and Official Statistics and Demography

Group A: Economic and Official Statistics

Date 8th May,2014 | Maximum Marks 50 | Duration 2 hours

Answer as many as you can. Each Question is assigned 8 marks (part questions carry equal marks).Use separate Sheets for Group A and Group B .Examination Sheets for Group A and B would be collected together at the end of the Examination.

1. State the Statistical Relation between Laspeyres and Paasches Index Numbers and comment on the relation.

2. Derive Divisia Integral Index for price and state the relevance of such an Index .

3. a) State and Explain the various types of errors in Index number theory.

 b) Elucidate the concept of constant utility Index.

4. Using a simple form state and explain the Economic and Statistical Criteria for the formulation of an Engel Curve .

5. a)State the Tornquist form of Engel curve. What is its relevance?

 b) State in addition three more popular forms of Engel Curves and comment on the Engel Elasticities.

6. State how future demand is projected using Engel curve (assume that the Engel Curve remains invariant over time).

7. a) State the characteristics of a Cobb Douglas Production Function.

 b) State the problem of estimation of Cobb Douglas Production function. State a remedy for the same.

8. Write Short notes on any two of the following:

a) Economic Census of Central Statistical Organisation, Government of India.

b) Living Standard Measurement Survey of World Bank

c) Sample Registration Scheme of Census of India .

d) Computation of Whole sale Price Index in India.

INDIAN STATISTICAL INSTITTUE, KOLKATA

Second Semestral Examination 2013-14

B.Stat (Hons.) II Year

Subject: Economic & Official Statistics and Demography

Group B: Demography & Economic Statistics

Date: 08/05/14 | Maximum Marks: 50 | Duration: 2 hours

(Answer all questions)

1. Answer the following.

i) The equation

$q_x = 2m_x / (2 + m_x)$

shows how the m-type and q-type mortality rates are related to one another. Derive a similar equation for the more general case of an age group of width n years.

ii) The standardized death rate for the town A was 1.23 when the population of town B was used as the standard. What does this tell you about mortality in A to that in B.

iii) Show that the crude birth rate in a stationary population corresponding to a life table is equal to $(1/e_0)$ where e_0 is the life expectation at birth. (4 + 2 + 4 = 10)

2(a) Express ${}_nL_x$ in terms of l_x, ${}_na_x$ and ${}_nd_x$.

(b) Why q_x cannot be calculated directly? Show how it gets estimated.

(c) If the crude birth rate in a country remains constant over a number of years, but the general fertility rate increases steadily, what does this tell you about the country's population? (3 + 1 + 3 + 3 = 10)

3(a) Write how Newton's halving formula is used for tackling errors due to inaccurate age reports or faulty enumeration.

(b) Explain in detail a method of rectifying an erroneously noted census count for a particular age group. (4 + 6 = 10)

4. The data below relate to fertility in a country in 1976 and 1993.

(a) Calculate age specific fertility rates for the two years.

(b) Using the 1976 population as the standard, calculate the standardized fertility rate for 1993.

(c) Comment on your results.

Age Group (in years)	1976		1993	
	No. of births ('000)	Mid-year female population ('000)	No. of births ('000)	Mid-year female population ('000)
15 - 19	57.9	1809	45.1	1455
20 - 24	182.2	1672	152.0	1831
25 - 29	220.7	1855	236.0	2070
30 - 34	90.8	1593	171.1	1967
35 - 39	26.1	1374	58.8	1729
40 - 44	6.5	1300	10.5	1750

(3 + 6 + 1 = 10)

5(a) State the weak and the strong Pareto Law with full explanation of all symbols.

(b) Prove that the Lorenz curve is monotonically increasing. Also, prove that it is a convex function.

(c) Derive the Lorenz curve for a Pareto distribution.

(d) How are Lorenz ratio and Lorenz curve related? (2 + 3 + 4 + 1 = 10)

END

INDIAN STATISTICAL INSTITUTE

Semestral Examination

B. Stat. - II Year (Semester - II)

Discrete Mathematics

Date : 13·05·14 Maximum Marks : 100 Duration : 3:00 Hours

Note : You may answer any part of any question, but maximum you can score is 100.

1. Prove that a plane graph G is bipartite if and only if every face has even length. [10]

2. Prove that if all capacities in a network are integers, then there is a maximum flow assigning integral flow to each edge. Prove all results (regarding flows in network) that you may use. [16]

3. Prove that every planar graph is 5-colorable. [14]

4. Let there be n bus drivers, n morning routes with duration $x_1, x_2, \dots x_n$ and n afternoon routes with duration $y_1, y_2, \dots, y_n$. A driver is paid overtime when the morning route and afternoon route exceed total time t. The objective is to assign one morning run and one afternoon run to each driver to minimize the total amount of overtime. Express this as weighted matching problem. Prove that giving the ith longest morning route and ith shortest afternoon route to the same driver, for each i, yields an optimal solution. [15]

5. Let $\tau(G)$ denote the number of spanning trees of a graph G. Prove that if edge $e \in E(G)$ is not a loop, then $\tau(G) = \tau(G - e) + \tau(G.e)$. [12]

6. Suppose there is a town where residents love forming different clubs. To limit the number of possible clubs, the town council establishes the following rules: (i) Every club must have an odd number of members and (ii) Every two clubs must share an even number of members. Prove that more than n clubs are impossible to form. [16]

7. Prove that for any graph G on n vertices, not containing a 4-cycle,
$$E(G) < \frac{1}{4}(1 + \sqrt{4n - 3})n.$$
[16]

8. Prove that any r-uniform hypergraph with less than 2^{r-1} hyperedges is 2-colorable. [16]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: (2013 – 2014)

B. Stat II Year

AGRICULTURE

Date 16.05.2014 Maximum Marks 50 Duration 3:00 hours.

(Attempt any five questions)

(Number of copies of the question paper required 10)

1. Write in brief about organic farming. 10

2. What are the criteria for essentiality of plant nutrients? Calculate the quantity of VC, Urea, SSP and KCL required for 1 hectare Potato crop to supply the nutrients requirement of 200 kg N, 120 kg P_2O_5 and 100 kg K_2O per hectare. 50% of required N should be given through VC.

   3+7

3. What are the different ancillary and yield attributing characters of rice plant. Estimate the expected yield of rice grain in t/ha from the following data:
   (i) Spacing – 25 x25cm, (ii) Average no. of tillers/hill –85, (iii) Average no. of effective tillers/hill –70, (iv) Average no. of grain/panicle –35, (v) Average panicle length -15 cm, (vi) Average no. of unfilled grain/panicle –7, (vi) Test weight -22 g.

   4+6

4. Write the differences between: 2.5 x 4
   a) Manures and Fertilizers
   b) Intercropping and mixed cropping
   c) Soil texture and soil structure
   d) Macro and micro nutrients

5. Cereal-legume intercropping experiment was done in 2:1 and 2:2 row replacement series system. Please calculate the advantages/disadvantages of the intercropping system from the following data

Cropping System	Cereal yield in kg/ha	Legume yield in kg/ha
Cereal Sole	5670	-
Legume Sole	-	4355
Cereal+Legume (2:2)	3890	3550
Cereal+Legume (2:1)	3765	2985

   10

6. What do you mean by Irrigation? What are the sources of water for irrigation? Write in brief about the different types of irrigations.

   2+4+4

INDIAN STATISTICAL INSTITUTE

Semester II Examination : (2013-2014)

B. Stat. II Year

Physics II: Statistical Mechanics and Electrodynamics

Date: 16.05.2014 Duration: 3 hrs Maximum Marks: 60

Question no. 1 and 2 are compulsory. Total marks can not exceed 60.

1.(a) Find the root mean square fluctuation in the energy in canonical ensemble. Why does the energy remain practically equal to that of the microcanonical ensemble?

(b) Starting from $S = -k \sum_r P_r \ln P_r$, (where P_r is the probability of the system to be in the energy state E_r), deduce $S = k \ln \Omega$ for a microcanonical ensemble (Ω being the number of microstates). Now let the system be in the ground state. Calculate S when the ground state is (i) degenerate, (ii) non-degenerate. $[(5+1)+(2+1+1)]$

2.(a) Derive the expression for *fugacity* of a system when the particles are (i) distinguishable, (e.g., harmonic oscillators) (ii) indistinguishable.

(b) A system exchanges energy and particles with a heat reservoir such that thermal equilibrium is reached at a common temperature T and chemical potential μ. Assume that the energy and number of particles of the reservoir are much greater than that of the system. Find the probability $P_{r,s}$ that the system, at any instant of time, would be in a state specified by energy E_r and number of particles N_s . How would $P_{r,s}$ change if the physical nature of the reservoir is altered? $[(2\frac{1}{2} + 2\frac{1}{2})+(4 + 1)]$.

3. Two concentric metal spherical shells of radius a and b are separated by a weakly conducting material of conductivity σ.

(a) If they are maintained at a potential difference V, what current flows from one to the other?

(b) What is the resistance between the shells?

(c) If $b \gg a$, the outer radius b is irrelevant. How do you account for that? Exploit this observation to determine the current flowing between two metal spheres, each of radius a and potential difference V between them, immersed deep in the sea and held quite far apart. [6+1+(1+2)]

4.(a) Express Maxwell's equations in vacuum by using the potential representation of electric and magnetic fields. Hence obtain the inhomogeneous wave equations in the Lorentz Gauge framework.

(b) Define skin depth. Show that the skin depth in a poor conductor is independent of its frequency. [(4+3)+(1+2)]

5.(a) A long co-axial cable, of length ℓ, consists of an inner conductor of radius a and outer conductor of radius b. It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields? Why is it not zero?

(b) Now let the current decrease from I to zero. What is the total momentum imparted to the cable? [(5+1)+4]

6. A square loop of wire of side a lies on a table, a distance s from a very long straight wire which carries a current I.

(a) What is the magnetic flux through the loop?

(b) If the loop is pulled away from the wire, at speed v, what emf is generated? In what direction (clockwise, or counterclockwise) does the current flow?

(c) How will the flux and emf change if the loop is pulled along the wire? [4+(3+1)+2]

7.(a) Prove that transverse electromagnetic waves can not occur in a hollow wave guide.

(b) Consider a monochromatic plane wave in vacuum travelling in the z-direction. Show that

$$\tilde{B}_0 = \frac{k}{\omega}(\hat{z} \times \tilde{E}_0),$$

where the symbols have their usual meaning. [4+6]

Indian Statistical Institute
Second Semester Examination 2014
Course: B-Stat II Year
Subject: Economics II (Macroeconomics)

Maximum Marks 60 Duration 2.5 Hours

Answer all questions

1. In a simple Keynesian model for a closed economy without government, there are two groups of income earners. Group 1 earns 800, while the income of group 2 is Y – 800, where Y denotes NDP. Average consumption propensities of Group 1 and Group 2 are 0.6 and 0.5 respectively. Investment function is given by $400 + .1Y$. Derive the aggregate saving function and the equilibrium amount of saving. Now suppose there takes place a transfer of income from Group 1 to Group 2 of 100 units. How will it affect the aggregate saving function? Do you observe paradox of thrift here? Explain. [22]

2. Suppose that the government takes an additional loan of Rs.100 from the central bank. Explain how this will lead to an increase in the stock of high-powered money by the same amount. Assuming CRR to be unity, show how people will come to hold an additional amount of money of the same amount at the end of the operation of the money multiplier process when the stock of high-powered money goes up by Rs.100. [22]

3.(i) Explain how a shift in the demand for real balance function is likely to affect the equilibrium in the IS-LM model. (ii) Following an increase in the autonomous demand for produced goods and services in the above model, the government decides to implement an accommodating monetary policy. Subsequently, Y is found to increase by 2400 units. Compute the increase in the supply of real balance brought about by the central bank, if an exogenous increase in the demand for real balance by one unit is found to shift the LM curve horizontally by -4 units. [20]