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WEAKLY STABLE FAMILIES OF TRANSFORMATIONS

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SUMMARY. The notion of weak stability of a semigroup $\mathcal{V}$ of contractions in a Hilbert space H is introduced in this paper. The weak stability of $\mathcal{V}$ is studied in relation to that of the tensor product of $\mathcal{V}$ with $\mathcal{V}$. Generalizations of the 'mixing theorem' of ergodic theory to an arbitrary family of measure-preserving transformations on a probability space follow as corollaries. An analysis of the situation when such a family of transformations has a probability distribution defined on it is also given.

1. INTRODUCTION

Let H be a Hilbert space and $\mathcal{V}$ a semigroup of contractions on H. The splitting theorem of Jacobs (1962-63) gives H as the direct sum of two subspaces R and F, which consist, respectively, of 'reversible' and 'flight' vectors for the semigroup $\mathcal{V}$. Vectors invariant under $\mathcal{V}$ are reversible. This suggests the consideration of those semigroups for which every reversible vector is invariant. Such semigroups we call weakly stable. We study the weak stability of $\mathcal{V}$ in relation to that of $\tilde{\mathcal{V}} = \{\tilde{V} = V \times V : V \epsilon \mathcal{V}\}$, the tensor product of $\mathcal{V}$ with itself, defined on the tensor product Hilbert space $H \times H$. Under certain conditions we prove, in Section 2, that $\mathcal{V}$ is weakly stable if and only if $\tilde{\mathcal{V}}$ is so and that $\mathcal{V}$ is weakly stable if and only if $\tilde{P} = P \times P$, where P is the subspace of invariant elements in H and $\tilde{P}$ is defined similarly. Section 3 contains applications of the results of Section 2 to families of measure-preserving transformations on a probability space. These give generalizations of the 'mixing theorem' of ergodic theory.

To see the extent of our generalization in relation to other recent generalizations of this theorem, we recall that, for an invertible measure-preserving transformation T on a probability space $(\Omega, \mathcal{B}, m)$, the mixing theorem (Halmos, 1956) states the equivalence of the following three statements :

(I) For every pair A and B of measurable sets, the sequence $\frac{1}{n}\sum_{i=0}^{n-1} |m(T^{-i}A \cap B) - m(A)m(B)|$ tends to zero.

(II) Constants are the only eigen functions for U, the unitary operator on $\mathcal{L}_2(\Omega)$ induced by T.

(III) The Cartesian square $\tilde{T}$ of T defined on the product space $\hat{\Omega} = \Omega \times \Omega$ by $\tilde{T}(\omega, \omega') = (T\omega, T\omega')$ is ergodic.

A generalization of this theorem to amenable topological semigroups of measure-preserving transformations was obtained by Dye (1965). The consideration of such semigroups is necessitated by the need for replacing the convergence condition of statement (I) by a suitable one in more general cases. Leaving statement (I) Moore (1966) has given a generalization of the equivalence of statements (II) and (III) to Borel transformation groups. We get, in Section 3, results of the latter type for an arbitrary family of measure-preserving transformations. The generality of the results in Section 2 allows us to apply them to semigroups of Markov transition functions also.

In Section 4, we consider the case of a family of measure-preserving transformations with a probability distribution. Such a family induces a Markov transition function as well as a certain 'skew' transformation. We study the weak stability of the family in conjunction with that of the skew transformation and of the contraction induced by the transition function. We find that the weak stability of the latter two are equivalent and equivalent to a weaker condition on the family than its weak stability. This extends a result of Gladysz (1956).

2. Semigroups of contractions in a Hilbert space

Let H be a (complex) Hilbert space and $\tilde{H} = H \times H$, the tensor product of H with itself. A contraction V on H is an operator (i.e., a bounded linear transformation) on H with $\|V\| \leqslant 1$. If V is a contraction on H, the tensor product of V with V, denoted by $\tilde{V}(= V \times V)$ is a contraction on $\tilde{H}$. For a semigroup $\mathcal{V}$ of contractions on H, the set $\tilde{\mathcal{V}} = \{\tilde{V} : V \epsilon \mathcal{V}\}$ is again a semigroup of contractions (on $\tilde{H}$). We may call $\tilde{V}(\tilde{\mathcal{V}})$ the tensor square of $V(\mathcal{V})$.

Following Jacobs (1962-63), we shall introduce the notion of reversible and flight vectors under a given semigroup $\mathcal{V}$ of contractions. The orbit of an element $x \epsilon H$ under the semigroup $\mathcal{V}$ is the set $\{Vx : V \epsilon \mathcal{V}\} = (\mathcal{V}x)$. By $w(\mathcal{V}x)$ we denote its weak closure. The weak operator closure $w(\mathcal{V})$ of $\mathcal{V}$ is again a semigroup of contractions and $(w(\mathcal{V})x) = w(\mathcal{V}x)$ for any $x \epsilon H$.

Definition 2.1 : An element $x \epsilon H$ is *reversible* if for every $W \epsilon w(\mathcal{V})$, there exists a $W_0 \epsilon w(\mathcal{V})$ such that $W_0 W x = x$.

Definition 2.2 : An element $x \epsilon H$ is *flight* if $O \epsilon w(\mathcal{V}x)$.

Let R and F denote, respectively, the set of all reversible and flight vectors. We say that an element $x \epsilon H$ is invariant (under $\mathcal{V}$) if $Vx = x$ for every $V \epsilon \mathcal{V}$; a subspace S of H is invariant (under $\mathcal{V}$) if $V(S) \subset S$ for every $V \epsilon \mathcal{V}$. We shall be using the following two theorems of Jacobs (1962-63) in our work.

Theorem A : *R and F are mutually orthogonal invariant subspaces of H and $H = R \oplus F$. The restriction of $w(\mathcal{V})$ to R is a group of unitary operators.*

Theorem B : *R is the closed linear span of an orthogonal system of minimal invariant subspaces of R. Every minimal invariant subspace of R is of finite dimension; of dimension one if $\mathcal{V}$ is abelian.*

It follows that if $\mathcal{V}$ is the semigroup $\{V^n\}$, $n = 0, 1, 2, \ldots$ generated by a single contraction V, then the subspace R is spanned by the eigen vectors of V corresponding to eigen values of modulus one.

Let P be the subspace of vectors invariant under a semigroup $\mathcal{V}$ of contractions. Then $P \subseteq R$. It is thus interesting to consider semigroups $\mathcal{V}$ for which $P = R$. Let us make the following definition.

Definition 2.3 : A semigroup $\mathcal{V}$ is *weakly stable* if every reversible vector is invariant.

Let us first consider $\mathcal{V}$ to be the semigroup generated by a contraction V. If $\mathcal{V}$ is weakly stable, we shall call V weakly stable. Clearly V is weakly stable if and only if every eigen vector of V corresponding to an eigen value of modulus one is invariant. The following theorem gives an interesting characterization of the weak stability of V. The motivation for this result comes from the spectral characterization theorem of Natarajan and Viswanath (1967). A lemma of Foguel (1963) is needed for its proof.

Lemma 2.1 : *Let* $K = \{x : \|V^n x\| = \|x\|, \|V^{*n} x\| = \|x\| \text{ for all } n\}$ *where* V^* *is the adjoint of* V. *Then* K *is a subspace of* H *invariant under* V *and* V^*. *The restriction of* V *to* K *is unitary. If* $x \perp K$, *the sequence* $V^n x$ *tends to* 0 *weakly.*

Theorem 2.1 : *A contraction* V *on* H *is weakly stable if and only if, for every pair* x *and* y *of elements in* H, *the sequence* $(V^n x, y)$ *is strong Cesaro convergent, i.e., there is a constant* $C_{x,y}$ *such that*

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} |(V^i x, y) - C_{x,y}| = 0.$$

Proof: Let V be weakly stable. By Theorem A, we have $H = P \oplus F$. We apply Lemma 2.1 to F and write $F = F_1 \oplus F_2$ where F_1 and F_2 are subspaces invariant under V, V on F_1 is unitary and $V^n x$ tends to 0 weakly for every x in F_2. It is immediate that the sequence $(V^n x, y)$ is strong Cesaro convergent for every x in the subspace $P \oplus F_2$ and every y in H. Let U be the restriction of V to F_1. Since U is unitary and every vector in F_1 is a flight vector, U has no point spectrum. If $E(\cdot)$ is the spectral measure associated with U and $U = \int \lambda dE$ in the usual notation, then, for any fixed $x \epsilon F_1$, the measure $\mu_x(A) = (E(A)x, x)$ is a nonatomic measure on the circle group. Hence the measure $\nu_x = \mu_x * \bar{\mu}_x$ where $\bar{\mu}_x(A) = \mu_x(\bar{A})$ is also nonatomic. A simple calculation shows that

$$\frac{1}{n} \sum_{i=0}^{n-1} |(U^i x, x)|^2 = \frac{1}{n} \sum_{i=0}^{n-1} \int \lambda^i \nu_x(d\lambda)$$

$$= \int \frac{1-\lambda^n}{n(1-\lambda)} \nu_x(d\lambda).$$

The integrand is bounded and tends to zero as $n \to \infty$. Thus $|(U^n x, x)|^2$ tends to zero in the Cesaro sense. Since

$$\frac{1}{n} \sum_{i=0}^{n-1} |(U^i x, x)| \leq \left\{ \frac{1}{n} \sum_{i=0}^{n-1} |(U^i x, x)|^2 \right\}^{1/2}$$

it follows that $(U^n x, x)$ is strong Cesaro convergent to zero as $n \to \infty$. This extends, by standard arguments, to the sequence $(U^n x, y)$ with $x, y \in F_1$. It is now clear that $(V^n x, y)$ is strong Cesaro convergent for any $x, y \in H$.

Let now $(V^n x, y)$ be strong Cesaro convergent for every x and y in H. If V is not weakly stable, there exists a vector $x \in R$ such that $\|x\| = 1$, $Vx = \lambda x$, $|\lambda| = 1$ but $\lambda \neq 1$. Then $(V^n x, x) = \lambda^n$ which does not converge in the strong Cesaro sense. Thus V is weakly stable and the theorem is proved.

We shall now consider an arbitrary semigroup $\mathcal{V}$ of contractions on H and its tensor square $\tilde{\mathcal{V}}$. The reversible, flight and invariant subspaces for $\tilde{\mathcal{V}}$ will be denoted by $\tilde{R}$, $\tilde{F}$ and $\tilde{P}$ respectively. For notational convenience, we denote the tensor square of $w(\mathcal{V})$ by $\tilde{w}(\mathcal{V})$. It is easily seen that $\tilde{w}(\mathcal{V}) = w(\tilde{\mathcal{V}})$. To prove our main results on the weak stability of $\mathcal{V}$, we need the following lemma.

Lemma 2.2 : *$\tilde{R} = R \times R$ and $\tilde{F} = F \times R \oplus R \times F \oplus F \times F$.*

Proof : It is easy to see that $x \in F$, $y \in H$ imply that $x \cdot y$ and $y \cdot x$ are in $\tilde{F}$. Thus $F \times R$, $R \times F$ and $F \times F$ are contained in $\tilde{F}$. We have to show that $R \times R \subset \tilde{R}$. Let x and y be in R and let $W^* \in w(\tilde{\mathcal{V}})$ be given. Since $w(\tilde{\mathcal{V}}) = \tilde{w}(\mathcal{V})$, $W^* = \tilde{W}$ for some $W \in w(\mathcal{V})$. Since x and y are reversible and since $w(\mathcal{V})$ acts as a group of unitary operators on R, given $W \in w(\mathcal{V})$, there exists a $W_0 \in w(\mathcal{V})$ such that for all reversible vectors r, $W_0 W r = r$ and hence $\tilde{W}_0 \tilde{W} x \cdot y = x \cdot y$ and $\tilde{W}_0 \in \tilde{w}(\mathcal{V}) = w(\tilde{\mathcal{V}})$. This shows that $x \cdot y \in \tilde{R}$ and the lemma is proved.

Our first result shows that weak stability is preserved under passage to tensor squares if there exists at least one non-zero invariant vector.

Theorem 2.2 : *A semigroup $\mathcal{V}$ of contractions with $P \neq 0$ is weakly stable if and only if its tensor square $\tilde{\mathcal{V}}$ is so.*

Proof : Let $\mathcal{V}$ be weakly stable. Then $R = P$ and so $\tilde{R} = R \times R = P \times P \subset \tilde{P}$. Since $\tilde{P} \subset \tilde{R}$ always, it follows that $\tilde{R} = \tilde{P} (= P \times P)$. Thus $\tilde{\mathcal{V}}$ is weakly stable.

Conversely, suppose that $\tilde{\mathcal{V}}$ is weakly stable. Then $\tilde{R} = \tilde{P}$. If $\mathcal{V}$ is not weakly stable, there exists a non-invariant vector x in R. Let y be a non-zero vector in P. Then $x \cdot y \in R \times R = \tilde{R}$ and is not invariant under $\tilde{\mathcal{V}}$, a contradiction.

By a *conjugation* J on a Hilbert space H, we mean a one-one conjugate linear map of H onto H such that $J^2 = I$ and $(Jx, Jy) = (y, x)$ for all $x, y \in H$. An operator V on H is said to be *real* with respect to the conjugation J (Stone, 1932) if $VJ = JV$. Let us call a family $\mathcal{V}$ of operators *real* if there exists a conjugation J with respect to which every $V \in \mathcal{V}$ is real. Our next result is for real semigroups of contractions.

Theorem 2.3 : *A real semigroup $\mathcal{V}$ of contractions is weakly stable if and only if $\tilde{P} = P \times P$.*

Proof: The necessity part is trivial and does not require the semigroup to be real. Suppose now that $\mathcal{V}$ is not weakly stable. Then, by Theorem B, there exists a minimal finite dimensional subspace R_1 of R which is invariant for $\mathcal{V}$ and does not consist of invariant vectors alone. Without loss of generality, we can assume that $R_1 \perp P$. Since $\mathcal{V}$ is real, $J(P) = P$, where J is the associated conjugation. Hence $J(R_1) \perp P$. If $\{x_1, x_2, \ldots, x_k\}$ is an orthonormal basis in R_1, then $\{Jx_1, Jx_2, \ldots, Jx_k\}$ is an orthonormal basis in $J(R_1)$. It can be shown that the non-zero vector $\sum_{i=1}^{k} x_i \cdot Jx_i$ is invariant under $\tilde{\mathcal{V}}$ (using the fact that the restriction of any $V \epsilon \mathcal{V}$ to R_1 is unitary), but is orthogonal to $P \times P$. This completes the proof.

3. Applications of the results in Section 2

In this section, we consider a probability space $(\Omega, \mathcal{B}, m)$ and its (complex) $\mathcal{L}_2$-space, $\mathcal{L}_2(\Omega)$. The tensor square of $\mathcal{L}_2(\Omega)$ is $\mathcal{L}_2(\tilde{\Omega})$ where $\tilde{\Omega}$ is the Cartesian square of $\Omega : \tilde{\Omega} = \Omega \times \Omega$. Let $\mathcal{V}$ be a semigroup of contractions in $\mathcal{L}_2(\Omega)$. All the definitions and results of Section 2 can be applied to $\mathcal{V}$. Besides, we shall consider the following two properties a semigroup may possess.

Definition 3.1 : A semigroup $\mathcal{V}$ is *ergodic* if every invariant function is a constant.

Definition 3.2 : A semigroup $\mathcal{V}$ is *weakly mixing* if it is weakly stable and ergodic, i.e., if every reversible function is a constant.

As immediate corollaries of Theorems 2.2 and 2.3, we get the following results.

Corollary 3.1 : *A semigroup $\mathcal{V}$ with $P \neq 0$ is weakly mixing if and only if its tensor square $\tilde{\mathcal{V}}$ is weakly mixing.*

Corollary 3.2 : *A real semigroup $\mathcal{V}$ with $P \neq 0$ is weakly mixing if and only if $\tilde{\mathcal{V}}$ is ergodic.*

In what follows, we shall consider the natural conjugation J in $\mathcal{L}_2(\Omega)$ which takes any element f to its complex conjugate $\bar{f}$. The condition of reality of a family $\mathcal{V}$ of operators then means that $V\bar{f} = \overline{Vf}$ for every f and every $V \epsilon \mathcal{V}$.

Let now $\mathcal{V}_0$ be an arbitrary family of isometries on $\mathcal{L}_2(\Omega)$. The motivation for defining the weak stability of $\mathcal{V}_0$ comes from the following lemma.

Lemma 3.1 : *A semigroup $\mathcal{V}$ of isometries on a Hilbert space H is weakly stable if and only if every finite dimensional invariant subspace of H consists only of invariant vectors.*

The proof of this lemma is done by showing that any finite dimensional invariant subspace of H is contained in R. The point of this lemma is that, for isometries, weak stability can be defined without any reference to reversible vectors.

With obvious definitions of invariant functions and subspaces for $\mathcal{V}_0$, we call $\mathcal{V}_0$ *ergodic* if every invariant function is a constant and *weakly stable* (resp. *weakly mixing*) if every finite dimensional invariant subspace of $\mathcal{L}_2(\Omega)$ consists only of invariant functions (resp. constants). It follows that the ergodicity, weak stability, weak mixing and reality of $\mathcal{V}_0$ and $\mathcal{V}$, the semigroup generated by $\mathcal{V}_0$ are respectively equivalent. Thus Theorem 2.2 and Corollary 3.1 hold for an arbitrary family $\mathcal{V}_0$ of isometries with $P_0 \neq 0$. If $\mathcal{V}_0$ is real and $P_0 \neq 0$, then Theorem 2.3 and Corollary 3.2 also hold for $\mathcal{V}_0$.

Next we consider an arbitrary family $\mathcal{T}_0$ of measure-preserving transformations on Ω. This induces a family $\mathcal{V}_0$ of isometries on $\mathcal{L}_2(\Omega)$. The Cartesian square $\tilde{T}$ of a transformation T is defined on $\tilde{\Omega}$ as $\tilde{T}(\omega, \omega') = (T\omega, T\omega')$ and is a measure-preserving transformation. Let $\tilde{\mathcal{T}}_0 = \{\tilde{T} : T \epsilon \mathcal{T}_0\}$. Clearly, the tensor square $\tilde{\mathcal{V}}_0$ of $\mathcal{V}_0$ is the family of isometries on $\mathcal{L}_2(\tilde{\Omega})$ induced by $\tilde{\mathcal{T}}_0$. A set $A \epsilon \mathcal{B}$ is called invariant if $m(A \Delta T^{-1}A) = 0$ for all $T \epsilon \mathcal{T}_0$. Let $\mathcal{I}_0$ be the σ-field of invariant sets. $\mathcal{T}_0$ is called *ergodic* if every set in $\mathcal{I}_0$ has measure zero or one. It follows that $\mathcal{T}_0$ is ergodic if and only if $\mathcal{V}_0$ is ergodic (in the sense of the preceding paragraph). We call $\mathcal{T}_0$ *weakly stable* (resp. *weakly mixing*) if $\mathcal{V}_0$ is weakly stable (resp. weakly mixing). The family $\mathcal{V}_0$ is obviously real and $P_0 \neq 0$ and so the preceding results hold for $\mathcal{V}_0$. Thus $\mathcal{T}_0$ is weakly stable if and only if $\tilde{\mathcal{T}}_0$ is so and also if and only if $\tilde{\mathcal{I}}_0 = \mathcal{I}_0 \times \mathcal{I}_0$ (which is equivalent to the relation $\tilde{P}_0 = P_0 \times P_0$). Besides, $\mathcal{T}_0$ is weakly mixing if and only if $\tilde{\mathcal{T}}_0$ is ergodic. This generalizes statements II and III (in the Introduction) of the mixing theorem to an arbitrary family of measure-preserving transformations.

The generality in which we have proved our results enables us to apply them to semigroups of Markov transition functions also. A function $P(\omega, A)$ on $\Omega \times \mathcal{B}$ to the unit interval such that (i) for fixed $A \epsilon \mathcal{B}$, $P(\cdot, A)$ is a measurable function, (ii) for fixed $\omega \epsilon \Omega$, $P(\omega, \cdot)$ is a probability measure and (iii) $\int P(\omega, A)\, m(d\omega) = m(A)$ is called a Markov transition function with invariant measure m. Any such function $P(\omega, A)$ induces a map V on $\mathcal{L}_2(\Omega)$ defined by $(Vf)(\omega) = \int f(\omega')\, P(\omega, d\omega')$ for $f \epsilon \mathcal{L}_2(\Omega)$. V is an operator with $\|V\| = 1$. The product of two transition functions P and Q is the transition function $R = PQ$ defined by $R(\omega, B) = \int P(\omega, d\omega')Q(\omega', B)$. This multiplication is associative in the set of all transition functions on $(\Omega, \mathcal{B}, m)$. Let $\mathcal{T}$ be a semigroup of transition functions and $\mathcal{V}$ the semigroup of contractions induced by $\mathcal{T}$. Clearly $\mathcal{V}$ is real and $P \neq 0$ and so Theorems 2.2 and 2.3 as well as Corollaries 3.1. and 3.2 are true for $\mathcal{V}$.

Before closing this section, we shall mention a generalization of the statements (I) and (II) of the mixing theorem to amenable topological semigroups. This is obtained as a simple corollary to the main theorem of Dye (1965) and so the proof is omitted.

Let G be a topological semigroup and $C(G)$ the supnorm algebra of all bounded complex-valued continuous functions on G. By a left mean M on G is meant a linear functional on $C(G)$ such that (i) $f \geqslant 0 \Longrightarrow M(f) \geqslant 0$ (ii) $M(1) = 1$ and (iii) $M({}_hf) = M(f)$ for all $f \epsilon C(G)$ and all $h \epsilon G$ where $({}_hf)(g) = f(hg)$. There is a similar notion of right mean using the functions $(f_h)(g) = f(gh)$. G is *amenable* if it admits both a left and a right mean. We say that a function $f \epsilon C(G)$ is *almost convergent* with limit s if $M(f) = s$ for each right mean and each left mean M on G.

Let G be an amenable topological semigroup. Suppose that to each element g of G, there corresponds a measure-preserving transformation T_g of Ω such that $T_{gh} = T_h T_g$ and the map $g \rightarrow m(T_g^{-1} A \bigcap B)$ is continuous for all $A, B \epsilon \mathcal{B}$. We call T_g a continuous measure-preserving anti-representation of G. For such semigroups we have the following result.

Corollary 3.3 : *Let T_g be a continuous measure-preserving anti-representation of an amenable topological semigroup G on Ω. Then T_g is weakly stable if and only if the function $| m(T_g^{-1} A \bigcap B) - \int_B P(A \mid \mathcal{J}) m(d\omega) |$ is almost convergent to zero for every $A, B \epsilon \mathcal{B}$, where $P(A \mid \mathcal{J})$ is the conditional probability of A with respect to $\mathcal{J}$, the σ-field of invariant sets.*

4. Family of transformations with a distribution

We shall assume in this section that we are given a family $\mathcal{T}$ of measure-preserving transformations on $(\Omega, \mathcal{B}, m)$ endowed with a probability distribution. For notational convenience, let us index the transformations in $\mathcal{T}$ by a set X and let X be a probability space $(X, \mathcal{A}, \mu)$. Various authors (Kakutani, 1951; Ryll-Nardzewski, 1954; Gladysz, 1956) have considered this set-up and proved 'random ergodic theorems' as well as discussed the ergodic and weak mixing properties of the family $\mathcal{T}$. Before stating our result in this direction, we need to introduce certain notations and definitions.

We shall assume hereafter that the family $\mathcal{T}$ satisfies the following property : $B \epsilon \mathcal{B}$ implies that $\{(\omega, x) : T_x \omega \epsilon B\} \epsilon \mathcal{B} \times \mathcal{A}$. The definitions of invariant sets, functions and subspaces used in Section 3 have to be modified slightly in accordance with the principle that sets of zero measure are 'negligible'; e.g., a subspace S of $\mathcal{L}_2(\Omega)$ is invariant if $V_x(S) \subset S$ for almost all $x \epsilon X$, where V_x is the isometry induced by T_x.

The family $\mathcal{T}$ induces a Markov transition function $P(\omega, B)$ on $\Omega \times \mathcal{B}$ defined as follows (Kakutani, 1951) :

$$P(\omega, B) = \mu\{x : T_x \omega \epsilon B\}. \quad \omega \epsilon \Omega, B \epsilon \mathcal{B}.$$

This transition function is invariant with respect to the measure m :

$$\int_\Omega P(\omega, B) m(d\omega) = m(B).$$

The contraction V on $\mathcal{L}_2(\Omega)$ induced by the transition function $P(\omega, B)$ (see Section 3) has the interesting and useful property :

$$(Vf)(\omega) = \int_X f(T_x \omega) \mu(dx).$$

Let $(X^*, \mathcal{A}^*, \mu^*)$ be the one-sided infinite product of $(X, \mathcal{A}, \mu)$ with itself. The n-th coordinate of a point $x^* \epsilon X^*$ is denoted by $x_n(x^*)$, $n = 0, 1, 2, \ldots$. The shift χ in X^* defined by $x_n(\chi x^*) = x_{n+1}(x^*)$ is a measure-preserving transformation on X^*. We also consider the product space $(\Omega \times X^*, \mathcal{B} \times \mathcal{A}^*, m \times \mu^*)$. The 'skew' transformation ϕ on $\Omega \times X^*$ defined by the equation

$$\phi(\omega, x^*) = (T_{x_0(x^*)}\omega, \chi x^*)$$

is measure-preserving. It is easy to see that

$$\phi^n(\omega, x^*) = (T_{x_{n-1}(x^*)} \ldots T_{x_0(x^*)}\omega, \chi^n x^*)$$

for $n = 1, 2, 3, \ldots$.

The ergodicity of ϕ on $\Omega \times X^*$ is defined as usual. That of $\mathcal{T}$ and of V have been defined in Section 3. (Note the remark above regarding null sets.) The main result of Kakutani (1951) is that the ergodicities of $\mathcal{T}$, V and ϕ are equivalent.

The weak stability and the weak mixing of the family $\mathcal{T}$, the transformation ϕ and the operator V have already been defined in Sections 2 and 3. Kakutani (1951) posed the interesting problem of discussing the equivalence of the weak mixing (hence also the weak stability) of $\mathcal{T}$, ϕ and V. We shall prove that the weak stability of ϕ and V are equivalent and equivalent to the following property of the family $\mathcal{T}$, which is (shown to be) weaker than the weak stability of $\mathcal{T}$.

Definition 4.1 : The family $\mathcal{T}$ is called *weakly G-stable* if $|\lambda| = 1$, $f \epsilon \mathcal{L}_2(\Omega)$ and $V_x f = \lambda f$ for almost all x imply that $\lambda = 1$.

This definition corresponds to the definition of weak mixing (called weak G-mixing in what follows) of the family $\mathcal{T}$ according to Gladysz (1956). Gladysz has proved the equivalence of the weak G-mixing of $\mathcal{T}$ and the weak mixing of ϕ. Here, we go a step further and bring the operator V into the picture. Besides we do not assume the invertibility of the transformations in $\mathcal{T}$. First of all, let us establish that the weak G-stability of $\mathcal{T}$ is weaker than its weak stability.

If $\mathcal{T}$ is weakly stable and if there is an $f \epsilon \mathcal{L}_2(\Omega)$ and λ of modulus one such that $V_x f = \lambda f$ for almost all x, then the subspace spanned by f is a one-dimensional invariant subspace and so consists entirely of invariant functions, i.e., f is invariant. Hence $\mathcal{T}$ is weakly G-stable. But the converse is not true in general. Consider $\Omega = X =$ the circle group with Lebesgue measure and $T_x\omega = \omega x$ for all $x \epsilon X$, $\omega \epsilon \Omega$. The family $\mathcal{T}$ is then weakly G-mixing and hence weakly G-stable. For, if $f \epsilon \mathcal{L}_2(\Omega)$ with $V_x f = \lambda f$ for almost all x, $|\lambda| = 1$, let $f(\omega) = \Sigma c_n \omega^n$. (We know that the functions $f_n(\omega) = \omega^n$, $n = 0, \pm 1, \pm 2, \ldots$ form a complete orthonormal basis for $\mathcal{L}_2(\Omega)$.) Then $(V_x f)(\omega) = \Sigma\, c_n x^n \omega^n = \Sigma\, \lambda c_n \omega^n$ and hence $c_n x^n = \lambda c_n$ for all n. If f is not a constant, some c_n for $n \neq 0$ is non-zero, say $c_{n_0} \neq 0$. Then $x^{n_0} = \lambda$, i.e., x is a root of λ. Thus $V_x f = \lambda f$ can hold only for at most a countable number of values of x (those x which are roots of λ), a contradiction. Hence f must be a constant. However, the family $\mathcal{T}$ is not weakly stable, since the one-dimensional subspace of $\mathcal{L}_2(\Omega)$ generated by the function $f(\omega) = \omega^n$, $n \neq 0$, is invariant, but contains the non-invariant function $f(\omega) = \omega^n$.

We are now in a position to prove our main result. We need the following lemma due to Gladysz (1960).

Lemma 4.1 : *If $f(\omega, x^*) \epsilon \mathcal{L}_2(\Omega \times X^*)$ is such that*

$$f(\phi(\omega, x^*)) = af(\omega, x^*) \text{ a.e.}$$

for some constant a of modulus one, then there exists a $\mathcal{B}$-measurable function $g(\omega)$ such that

$$f(\omega, x^*) = g(\omega) \quad a.e.$$

Theorem 4.1 : *The following statements are equivalent :*

(i) *$\mathcal{T}$ is weakly O-stable*

(ii) *V is weakly stable*

(iii) *ϕ is weakly stable.*

Proof : We shall prove that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii).

If ϕ is weakly stable, then, by Theorem 2.1, the sequence $(f^*(\phi^k), g^*)$ is strong Cesaro convergent for $f^*, g^* \epsilon \mathcal{L}_2(\Omega \times X^*)$. Taking now $f, g \epsilon \mathcal{L}_2(\Omega)$ and putting $f^*(\omega, x^*) = f(\omega)$, $g^*(\omega, x^*) = g(\omega)$, we have

$$f^*(\phi^k(\omega, x^*)) = f\left(T_{x_{k-1}(x^*)} \cdots T_{x_0(x^*)}\, \omega\right)$$

and so

$$(f^*(\phi^k), g^*) = \int_{\Omega \times X^*} f\left(T_{x_{k-1}(x^*)} \cdots T_{x_0(x^*)}\, \omega\right)\overline{g(\omega)}\, m(d\omega)\mu^*(dx^*)$$

$$= \int_{\Omega} \int_{X_{k-1}} \cdots \int_{X_0} f\left(T_{x_{k-1}} \cdots T_{x_0}\, \omega\right)\overline{g(\omega)} m(d\omega)\mu(dx_{k-1}) \ldots \mu(dx_0)$$

$$= (V^k f, g)$$

and hence the sequence $(V^k f, g)$ is strong Cesaro convergent. By Theorem 2.1, V is weakly stable.

Let V be weakly stable. If λ is a complex number of modulus one such that, for some $f \epsilon \mathcal{L}_2(\Omega)$, $V_x f = \lambda f$ for almost all x, then,

$$(Vf)(\omega) = \int_X (V_x f)(\omega)\mu(dx)$$

$$= \lambda f(\omega).$$

The weak stability of V now implies that $\lambda = 1$.

Let now $\mathcal{T}$ be weakly O-stable. If ϕ is not weakly stable, then there is a $\lambda \neq 1$, $|\lambda| = 1$ and an $f(\omega, x^*) \epsilon \mathcal{L}_2(\Omega \times X^*)$ such that

$$f(\phi(\omega, x^*)) = \lambda f(\omega, x^*) \quad \text{a.e.}$$

By Lemma 4.1 there is a function $g(\omega) \epsilon \mathcal{L}_2(\Omega)$ such that

$$f(\omega, x^*) = g(\omega) \quad \text{a.e.}$$

Hence we have

$$g(T_{x_0(x^*)}\omega) = \lambda g(\omega) \quad \text{a.e.}$$

It is now easy to see that for almost all x,

$$g(T_x\, \omega) = \lambda g(\omega) \text{ a.e. } (\omega).$$

Since $\mathcal{T}$ is weakly G-stable, $\lambda = 1$, a contradiction.

Corollary 4.1 : *The following statements are equivalent* :

(i) $\mathcal{T}$ *is weakly G-mixing*

(ii) V *is weakly mixing*

(iii) ϕ *is weakly mixing.*

In connection with the results of this section, the referee has posed the following question. Let $(X, \mathcal{A}, \mu)$ be a probability space and for each $x \epsilon X$, let T_x be a contraction on a Hilbert space H. The weak G-stability of the family $\mathcal{T} = \{T_x\}$ is defined in the obvious way. The equation

$$(Vy, z) = \int (T_x\, y, z)\mu(dx)$$

for $y, z \epsilon H$ yields a new contraction V (under suitable measurability assumptions). Is the weak stability of V equivalent to the weak G-stability of the family $\mathcal{T}$? It is easy to see that if V is weakly stable, then $\mathcal{T}$ is weakly G-stable. The answer to the other part of the question is not known.

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