ON SOME QUICK DECISION METHODS IN MULTIVARIATE AND UNIVARIATE ANALYSIS

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0. Introduction

0.1. A statistical procedure optimum in some classical sense amongst all procedures based on the same number of observations may not be an economic procedure when the cost of observation or of computations are taken into account. For instance, the optimum procedure (in some sense) for testing the equality of means of a number of normal populations with the same variance is the method of analysis of variance, but since the computational labour involved is heavy, quality control engineers prefer the method of control charts. If the cost function for various computational methods can be properly formulated, it may be possible to incorporate this in the general decision theory and optimum rules can be obtained. The difficulties involved in such procedures can be easily imagined. In such circumstances, methods that appear to be cheap and easy to apply have their use. The control chart method, for instance, has not yet been proved to be the most economic procedure, but its justification is that it works quite well.

0.2. Methods based on counting rather than measurements have sometimes been used in industrial problems. Stevens (1940) considered the problem of setting up of control charts by "gauging", that is by counting the number of items in a sample with quality characteristic above or falling short of specified limits. To study the errors of a gun, it may be more convenient to count for a round fired the number of shots hitting the different concentric rings with the bull's-eye as centre rather than to measure the co-ordinates of each shot with respect to orthogonal axes with the bull's-eye as origin. In this case, two measurable characteristics are involved. Methods based on counting may therefore be very useful in certain situations, even though they may not be the optimum in the classical sense.

0.3. In this paper to examine some hypothesis about the underlying distribution of some measurable characteristic, we develop methods that are based mostly on counting rather than on measurement. Counting sometimes is cheaper than measurement. Another property of the method is that no new distribution problem has to be solved.

1. A test of a simple hypothesis against a simple alternative based on the binomial distribution

1.1. Let x be a p-dimensional chance variable with a continuous probability density function f. The problem is to test the simple hypothesis H_0 that $f = f_0$ against the simple alternative H_1 that $f = f_1$ on the basis of a random sample $x_1, x_2, \ldots, x_n$

of the sign n. The most powerful test when the first kind of error is fixed at α is given by:

$$\text{reject } H_0 \text{ if } \prod_{i=1}^{n} f_0(x_i) < \lambda \prod_{i=1}^{n} f_1(x_i) \quad \ldots \quad (1.1)$$

$$\text{accept } H_0 \text{ otherwise}$$

where λ is a constant to be so chosen that the first kind of error is α. The crux of the problem therefore is to derive the sampling distribution of the likelihood—ratio statistic

$$T = \prod_{i=1}^{n} (f_0(x_i)/f_1(x_i)) \quad \ldots \quad (1.2)$$

when the hypothesis H_0 is true, from which λ may be determined to ensure

$$\text{Prob} \quad (T < \lambda | H_0) = \alpha. \quad \ldots \quad (1.3)$$

1.2. In many situations, however, the sampling distribution of T may be very complicated and the evaluation of the percentage point still more complicated, or the computation of the statistic T itself may be too difficult, or measurement of the p variables may be inconvenient or costly. Under such circumstances, one may not like to use the test based on the statistic T even though it is the most powerful one.

The alternative method that we suggest here, though less powerful than the classical method, has certain advantages. The method is based on counting and no new distribution problem has to be solved.

1.3. Let ω be a sub-set of the p-dimensional Euclidean space such that

$$\pi_1 > \pi_0 > 0 \quad \ldots \quad (1.4)$$

where

$$\pi_i = \text{Prob.} \quad (x\epsilon\omega | H_i) \quad \ldots \quad (1.5)$$

$$i = 0, 1.$$

Let a pseudo-variate y_i be defined this way

$$y_i = 1 \quad \text{if } x_i \epsilon \omega$$

$$= 0 \quad \text{otherwise} \quad \ldots \quad (1.6)$$

Let

$$d = \sum_{i=1}^{n} y_i. \quad \ldots \quad (1.7)$$

Then

$$\text{Prob.} \quad (d = x | H_i) = \binom{n}{x} \pi_i^x (1-\pi_i)^{n-x} \quad i = 0, 1. \quad \ldots \quad (1.8)$$

The statistic d can therefore be used to test the hypothesis H_0 against the alternative H_1.

Let c be the smallest integer to satisfy

$$\sum_{x=c+1}^{n} \binom{n}{x} \pi_0^x(1-\pi_0)^{n-x} \leqslant \alpha. \qquad \dots \quad (1.9)$$

Then the test procedure is:

$$\text{reject } H_0 \text{ if}: \quad d > c$$

otherwise accept H_0. ... (1.10)

The power of this test is given by

$$\begin{aligned} &\text{Prob}\,(d > c \mid H_1) \\ &= \sum_{x=c+1}^{n} \binom{n}{x} \pi_1^x (1-\pi_1)^{n-x} \qquad \dots \quad (1.11) \\ &= \beta \quad \text{(say)}. \end{aligned}$$

It should be noted that β is an increasing function of π_1 for

$$\frac{d\beta}{d\pi_1} = \frac{m - n\pi_1\beta_1}{\pi_1(1-\pi_1)}$$

where
$$m = \sum_{x=c+1}^{n} x\binom{n}{x}\pi_1^x(1-\pi_1)^{n-x} > n\pi_1\beta$$

and therefore
$$\frac{d\beta}{d\pi_1} > 0, \qquad \dots \quad (1.12)$$

The test is therefore unbiassed, and uniformly so far all simple alternatives H: for which

$$\text{Prob.} \quad (x\epsilon\omega \mid H) > \pi_0. \qquad \dots \quad (1.13)$$

2. The most powerful binomial test of a simple hypothesis against a simple alternative

2.1 The power of the binomial test discussed in § 1 depends on ω and the question that naturally arises is: How should ω be chosen so that the power is maximised? Below we give a partial solution to this problem which states that the opitmum ω must be a member of a particular class.

2.2 Theorem: *Let ω be a given sub-set of the p-dimensional Euclidean space satisfying* (1.4) *and* (1.5). *Then under certain simple condition it is possible to find a sub-set ω_0 belonging to the class:*

inside ω_0:
$$f_0(x) < k\, f_1(x) \qquad \dots \quad (2.1)$$

such that for the same sample size the binomial test based on ω_0 is at least as powerful as that based on ω.

Proof: Choose k to satisfy:

$$\int_{\omega_0} f_0(x)dx = \pi_0 \qquad \dots \quad (2.2)$$

Then from Neyman and Pearson's (1933) fundamental lemma it follows that

$$\pi_1^0 = \int_{\omega_0} f_1(x)dx \geqslant \int_{\omega} f_1(x)dx = \pi_1. \qquad \dots \quad (2.3)$$

But the power of the binomial test is an increasing function of π_1. Therefore the test based on ω_0 is atleast as powerful as that based on ω.

2.3. The problem therefore reduces to that of finding an optimum values for k. If one wishes to maximise the difference between π_1 and π_0 the solution is $k = 1$, but this by itself will not ensure maximum power. It has not been possible to get a general solution for a fixed sample size. However, if n is so large that the normal approximation to the binomial is satisfactory, the power of the test is approximately given by

$$\beta = \phi\left\{\tau_\alpha \sqrt{\frac{\pi_0(1-\pi_0)}{\pi_1(1-\pi_1)}} - \sqrt{n}\, \frac{\pi_1-\pi_0}{\sqrt{\pi_1(1-\pi_1)}}\right\} \qquad \dots \quad (2.4)$$

where

$$\phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt \qquad \dots \quad (2.5)$$

and τ_α is defined by

$$\phi(\tau_\alpha) = \alpha \qquad \dots \quad (2.6)$$

for $0 < \alpha < 1$.

Since β increase as the argument of ϕ decreases, for large values of n the problem is solved if k is chosen to maximise

$$\frac{\pi_1-\pi_0}{\sqrt{\pi_1(1-\pi_1)}}.$$

No general solution could be obtained. To maximise the numerator we may take $k = 1$. For any given value of n however, the optimum value of k may be determined numerically.

2.4. *Example* 1: To test the hypothesis that the mean of a normal population with a known standard deviation σ is μ_0 against the alternative hypothesis that it is μ_1.

Here,

inside ω_0 :
$$\frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu_0)^2}}{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu_1)^2}} < k$$

or
$$x > \frac{\mu_0+\mu_1}{2}+l\sigma \qquad \text{if } \mu_1 > \mu_0 \qquad \dots \quad (2.7)$$

where l is a constant to be suitably determined.

Consequently
$$\pi_0 = \phi(l+\tfrac{1}{2}\,\delta) \qquad \dots \quad (2.8)$$

and
$$\pi_1 = \phi(l-\tfrac{1}{2}\,\delta) \qquad \dots \quad (2.9)$$

where
$$\delta = \frac{\mu_1-\mu_0}{\sigma} \qquad \dots \quad (2.10)$$

and $\phi(x)$ is as defined in (2.5).

As a numerical illustration we tabulate below the power β of the test for the case $n = 100$ and $\delta = 0.1$ with $\alpha = 0.05$ for different values of l. These were computed by using the approximate formula (2.4). The power of the optimum classical test is also presented.

TABLE 2.1 POWER OF DIFFERENT BINOMIAL TESTS AND OF THE CLASSICAL MOST POWERFUL TEST OF THE MEAN OF A NORMAL POPULATION WITH KNOWN STANDARD DEVIATION

$n = 100 \quad \alpha = 0.05 \quad \delta = 0.1$

test	power (normal approximation)
binomial test with $l =$	
−0.2	0.19
−0.1	0.20
0	0.20
+0.1	0.20
+0.2	0.19
most powerful classical test	0.26

2.6. *Example* 2: To test the hypothesis that the standard deviation of a normal population with a known mean μ is σ_0 against the alternative that it is σ_1.

Here inside ω_0 :

$$\frac{\frac{1}{\sigma_0\sqrt{2\pi}}e^{-\frac{1}{2\sigma_0^2}(x-\mu)^2}}{\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2\sigma_1^2}(x-\mu)^2}} < k$$

or

$$(x-\mu)^2 > l^2\sigma_0^2 \quad \text{if} \quad \sigma_1^2 > \sigma_1^2 \qquad \dots (2.11)$$

where l has to be suitably determined.

Then

$$\pi_0 = 2\phi(l) \qquad \dots (2.12)$$

$$\pi_1 = 2\phi(l\rho). \qquad \dots (2.13)$$

where

$$\rho = \frac{\sigma_0}{\sigma_1} .$$

The following table of values of β for $n = 100$, $\alpha = 0.05$ and $\rho = 0.9$ was computed for different values of l. The power of the classical most powerful test is also presented.

TABLE 2.2. POWER OF DIFFERENT BINOMIAL TESTS AND OF THE CLASSICAL MOST POWERFUL TEST FOR THE STANDARD DEVIATION OF A NORMAL POPULATION WITH A KNOWN MEAN

$n = 100 \quad \alpha = 0.05 \quad \rho = 0.9$

test	power (normal approximation)
binomial test with l =	
0.2	0.10
0.3	0.13
0.4	0.15
0.5	0.16
0.6	0.17
most powerful classical test	0.47

2.6. *Example* 3: To test the hypothesis that the vector of mean values of a p-variate normal distribution with known dispersion matrix Σ is μ_0 against the alternation that it is μ_1

Here, inside ω_0 :

$$\frac{\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}(x-\mu_0)\Sigma^{-1}(x-\mu_0)'}}{\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}}e^{-\frac{1}{2}(x-\mu_1)\Sigma^{-1}(x-\mu_1)'}} < k$$

or

$$(\mu_1-\mu_0)\Sigma^{-1}x' > l\Delta+\tfrac{1}{2}(\mu_1-\mu_0)\Sigma^{-1}(\mu_1+\mu_0)' \qquad \dots (2.14)$$

where $$\Delta^2 = (\mu_1 - \mu_0)\Sigma^{-1}(\mu_1 - \mu_0)' \qquad \ldots \quad (2.15)$$
and l is a constant to be suitably determined.

Therefore

$$\pi_0 = \phi(l+\tfrac{1}{2}\Delta) \qquad \ldots \quad (2.16)$$

and $$\pi_1 = \phi(l-\tfrac{1}{2}\Delta). \qquad \ldots \quad (2.17)$$

The power of this test depends only on l and Mahalanobis's distance Δ. If we consider the case $n = 100$, $\Delta = 0.1$, $\alpha = 0.05$, we have already tabulated the power of the binomial test for different values of l in Table 2.1.

3. Minimisation of size of sample when both kinds of error are pre-assigned

3.1. Another problem in testing a simple hypothesis against a simple alternative is to determine the smallest sample size to ensure that the power of the test is a pre-assigned quantity β. For any given ω we can find the smallest integer n such that the power of the binomial test based on a sample of size n is at least β. Of course, n will depend on the pre-assigned values of α and β and the region ω. The problem is to so choose ω that n is minimised. Here again we need restrict ourselves to regions of the type:

$$f_0 < \lambda f_1$$

and try to determine λ to minimise n for fixed values of α and β.

3.2. Using the normal approximation to the binomial distribution, we get the following requirements on n and c to ensure that the first kind of error is α and that the power is β:

$$\frac{c-n\pi_0}{\sqrt{n\pi_0(1-\pi_0)}} \doteq \tau_\alpha$$

$$\frac{c-n\pi_1}{\sqrt{n\pi_1(1-\pi_1)}} \doteq \tau_\beta$$

with τ_α and τ_β defined by (2.6). From this we get

$$\sqrt{n} = \{\tau_\alpha\sqrt{\pi_0(1-\pi_0)} - \tau_\beta\sqrt{\pi_1(1-\pi_1)}\}/(\pi_1-\pi_0). \qquad \ldots \quad (3.1)$$

So ω has to be so chosen that this quantity is minimised.

3.3. *Example* 1: Suppose it is required to test that the mean of a normal population with a known standard deviation σ is μ_0 against the alternative that it is μ_1 at level of signficance α and power β. Then if $\mu_1 > \mu_0$ and we take

inside ω_0 : $$x > \frac{\mu_0+\mu_1}{2} + l\sigma \qquad \ldots \quad (3.2)$$

we have
$$\pi_0 = \phi(l+\tfrac{1}{2}\delta),$$
$$\pi_1 = \phi(l-\tfrac{1}{2}\delta)$$
where
$$\delta = \frac{\mu_1-\mu_0}{\sigma}.$$

The problem is to so choose l as to minimise n given in (3.1). This was done numerically for $\alpha = 0.05$ and $\beta = 0.90$ and 0.95 for $\delta = 0.25$, 0.30, 0.35 and 0.40. The values of l and n for the optimum binomial test and the sample size n_0 required for the optimum non-sequential test are presented in the table below:

TABLE 3.1. SAMPLE SIZE REQUIRED TO TEST THE MEAN OF A NORMAL POPULATION WITH KNOWN STANDARD DEVIATION AT 5% LEVEL OF SIGNIFICANCE

alternative	$\beta = 0.95$			$\beta = 0.90$		
	binomial test		classical test	binomial test		classical test
δ	l	n	n_0	l	n	n_0
0.25	0.275	271	174	0.325	214	138
0.30	0.300	188	121	0.350	148	96
0.35	0.375	137	89	0.375	100	70
0.40	0.400	103	68	0.450	83	64

It will be seen that the binomial test requires a sample size about 1.5 times that required for the classical non-sequential test. Consequently if the ratio of the cost per item sampled for the classical test to that for the binomial test is greater than 1.5 the method suggested here should prove more economic.

3.4. *Example 2*: Consider the problem of testing the vector of mean values of a p variate normal population with a known dispersion matrix. It immediately follows from 2.6 that the sample size required for the test at level of significance α to attain a power β for an alternative value of the vector of mean alues which is at a distance Δ (in Mahalanobis's sense) can be read off directly from Table 3.1 (with δ replaced by Δ) independently of the number of variates p involved so long as Δ is kept fixed.

4. Test of a simple hypothesis about a single parameter against a class of simple alternatives

4.1. The tests considered in the previous sections do not necessarily possess optimum power properties against a sufficiently wide class of simple alternatives. But it is rather straight forward to build up such tests using general methods due to Neyman and Pearson whenever applicable. For instance, if it is required to build up a test which is uniformly unbiassed, we may proceed as follows: Suppose that the problem is to test the hypothesis H_0 that the value of the parameter θ involved in the probability density function of the chance variable x(not necessarily unidimensional) is θ_0. Then if we can find a region ω such that

$$\pi_{\theta_0} = \int_\omega f(x, \theta_0)dx < \int_\omega f(x, \theta)dx = \pi_\theta$$

for all $\theta \neq \theta_0$.

it is immediately seen that the binomial test based on ω must necessarily be uniformly unbiassed. If then we want to build up a binomial test that is uniformly unbiassed and most powerful for a particular alternative H_1 that specifies the value θ_1 for θ, by using a line of argument similar to that used in § 2 and Neyman and Pearson's (1933) fundamental lemma it is easy to show that under the usual regularity condition of differentiability within the intergral sign, we need restrict ourselves only to regions of the type:

inside ω : $$f(x, \theta_1) > \lambda_1 f(x, \theta_0) + \lambda_2 \left\{ \frac{d}{d\theta} f(x, \theta) \right\}_{\theta_0} \qquad \dots \quad (4.1)$$

where λ_1 and λ_2 are undetermined except for the stipulation that

$$\left[\frac{d}{d\theta} \int_{\omega} f(x, \theta) dx \right]_{\theta_0} = 0 \qquad \dots \quad (4.2)$$

This in general ensures $\pi_{\theta_0} < \pi_\theta$ only for values of θ in the neighbourhood of θ_0, but if it *happens to* ensure this uniformly for all θ, then only we get uniformly unbiassed tests.

This restriction (4.2) gives one of the two constants, the other one has to be determined numerically as in the previous section to maximise the power of the binomial test when the value of the parameter is θ_1 for a fixed sample size n, or for very large values of n.

4.2. *Example* 1: Suppose the problem is to examine the hypothesis that the mean of a normal population with known standard deviation σ is μ_0 against the alternative hypothesis that it is μ_1 with the stipulation that the search for the most powerful binomial test must be restricted amongst those that are uniformly unbiassed. From (4.1) we get after some simplification:

inside ω : $$e^{\delta(\tau - \frac{1}{2}\delta)} > \lambda_1 + \lambda_2 \tau \qquad \dots \quad (4.3)$$

where $$\tau = \frac{x - \mu_0}{\sigma},$$

$$\delta = \frac{\mu_1 - \mu_0}{\sigma}$$

and λ_1 and λ_2 are undetermined for the present. From the convexity property of the exponential function it follows that (4.3) may be written in the form

Outside ω: $$k_1 < \tau < k_2. \qquad \dots \quad (4.4)$$

The restriction (4.2) implies that $-k_1 = k_2$ and consequently (4.4) may be written in the final form:

inside ω: $$|\tau| > k$$

where k is a constant to be suitably determined.

It is now easy to see that

$$\pi_0 = \text{Prob.}\,(x\epsilon\omega|\mu_0) = 2\phi(k)$$

and

$$\pi_1 = \text{Prob}\,(x\epsilon w|\mu_1) = \phi(k-\delta)+\phi(k+\delta).$$

This incidentally brings out the symmetry of the power function of the binomial test.

For the special case $n = 100$, $\alpha = .05$, $\delta = 0.4$, the following table gives the power of the binomial test for different values of k as also that of the most powerful uniformly unbiassed classical test.

TABLE 4.1. POWER OF DIFFERENT UNIFORMLY UNBIASSED BINOMIAL TESTS AND OF THE CLASSICAL TEST FOR THE MEAN OF A NORMAL POPULATION WITH KNOWN STANDARD DEVIATION

($n = 100$, $\alpha = 0.05$, $\delta = 0.4$)

test	power (normal approximation)
binomial test with k	
0.5	0.14
1.0	0.21
1.5	0.25
2.0	0.25
uniformly unbiassed classical test	0.98

5. Asymptotically locally most powerful one sided test

5.1. An alternative approach in problems of testing a simple hypothesis about a single parameter is to try to maximise the rate of increase of the power function of a test in a neighbourhood (one sided) of the value of the parameter specified by the null hypothesis. A test for which this property holds may be called an one sided locally most powerful test. (Rao & Pati).

5.2. In this section we consider the problem of finding the locally most powerful binomial test when the sample size is large. In the illustrative example consideration is limited only to a very special class of the binomial tests, as a general solution to the problem could not be derived.

5.3. Suppose that f involves a single parameter and the hypothesis H_0 to be tested is that the value of the parameter is θ_0. Then the power β of the binomial test based on the region ω when the value of the parameter is θ_1 is given approximately, for large values of n by

$$\beta = \phi(z) \qquad \ldots \quad (5.1)$$

where

$$z = \tau_\alpha \sqrt{\frac{\pi_0(1-\pi_0)}{\pi_1(1-\pi_1)}} - \sqrt{n}\left(\frac{\pi_1-\pi_0}{\sqrt{\pi_1(1-\pi_1)}}\right) \qquad \ldots \quad (5.2)$$

where

$$\pi_0 = \text{Prob.}\,(x\epsilon\omega|\theta_0)$$

$$\pi_1 = \text{Prob.}\,(x\epsilon\omega|\theta_1).$$

Therefore, the rate of increase of the power at the point θ_1 is

$$\frac{\partial\beta}{\partial\theta_1} = \frac{\partial\beta}{\partial z}\cdot\frac{\partial z}{\partial\pi_1}\cdot\frac{\partial\pi_1}{\partial\theta_1}$$

$$= \left(-\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}z^2}\right)\left\{\tau_a\,\frac{(\pi_1-\frac{1}{2})\sqrt{\pi_0(1-\pi_0)}}{\pi_1(1-\pi_1)\sqrt{\pi_1(1-\pi_1)}} - \right.$$

$$\left. -\sqrt{n}\left(\frac{(\pi_1-\pi_0)(\pi_1-\frac{1}{2})}{\pi_1(1-\pi_1)\sqrt{\pi_1(1-\pi_1)}} + \frac{1-\pi_0}{\sqrt{\pi_1(1-\pi_1)}}\right)\right\}\left(\frac{\partial\pi_1}{\partial\theta_1}\right).$$

Consequently

$$\left(\frac{\partial\beta}{\partial\theta_1}\right)_{\theta_1=\theta_0} = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\tau_a^2}\left\{\sqrt{n}\,\sqrt{\frac{1-\pi_0}{\pi_0}} - \tau_a\,\frac{\pi_0-\frac{1}{2}}{\pi_0(1-\pi_0)}\right\}\cdot\left(\frac{\partial\pi_1}{\partial\theta_1}\right)_{\theta_1=\theta_0} \quad \ldots \quad (5.3)$$

The dominant term in this expression is

$$\sqrt{n}.\,\frac{1}{\sqrt{2\pi}}\,.\,e^{-\frac{1}{2}\tau_a^2}\sqrt{\frac{1-\pi_0}{\pi_0}}\times\left(\frac{d\pi_1}{d\theta_1}\right)_{\theta_1=\theta_0} \quad \ldots \quad (5.4)$$

The problem therefore is to so choose ω that

$$\sqrt{\frac{1-\pi_0}{\pi_0}}\left(\frac{\partial\pi_1}{d\theta_1}\right)_{\theta_1=\theta_0} \quad \ldots \quad (5.5)$$

is maximised.

But a general solution to this problem could not be derived. However, if θ is a location parameter, that is if $f(x,\theta)\equiv f(x-\theta)$ and the range does not involve θ and consideration is limited only to regions of the type

$$\omega: \qquad x > \theta_0+k \quad \ldots \quad (5.6)$$

it is easy to see that

$$\left(\frac{\partial\pi_1}{d\theta_1}\right)_{\theta_1=\theta_0} = f(k).$$

Consequently k has to be chosen to maximise

$$U = \sqrt{\frac{1-\pi_0}{\pi_0}}\,f(k) \quad \ldots \quad (5.7)$$

and, here

$$\pi_0 = \int_k^\infty f(t)dt. \quad \ldots \quad (5.8)$$

The condition $\frac{dU}{dk} = 0$ gives

$$2\pi_0(1-\pi_0)f'(k)+\{f(k)\}^2 = 0. \qquad \text{... (5.9)}$$

It follows that for the problem of testing the mean of a normal population with unit standard deviation the value of k is a root of

$$2k = \frac{z}{\phi(1-\phi)} \qquad \text{... (5.10)}$$

where

$$z = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}k^2}$$

and

$$\phi = \int_k^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt.$$

References

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