CLASSIFICATION AND ANALYSIS OF LINKED BLOCK DESIGNS

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1. Introduction and summary

In comparative experiments involving a fairly large number of varieties, when for lack of homogeneous experimental units, complete block designs are not available, Balanced Incomplete Block (BIB) designs prove very convenient in two respects. Firstly, the analysis is very simple and secondly, comparison between any two varieties has the same precision. On the other hand, a BIB design requires a large number of experimental units, because balancing is not possible unless the number of blocks is atleast as large as the number of varieties. To obviate this difficulty, different types of incomplete block designs have been introduced of which the most notable is the Partially Balanced Incomplete Block (PBIB) design introduced by Bose and Nair (1939). Another class of incomplete block designs, called Linked Block (LB) designs was introduced by Youden (1951) which he obtained by dualising several BIB designs, that is by taking the varieties and blocks of the BIB design respectively as blocks and varieties in the LB design. The LB designs so constructed by Youden happen to be all PBIB designs, but this is not necessarily true. A great advantage of LB designs is that the analysis can be easily worked out.

In this paper, we show how the intra-block analysis of a LB design can be neatly carried out. The efficiency factor of a LB design is obtained. The methods are illustrated with a numerical example. An exhaustive list of all LB designs with ten or less plots per block and involving ten or less replications is given. These designs fall into three groups: (1) symmetrical BIB designs, (2) PBIB designs and (3) Irregular designs. For plans of LB designs belonging to group (2) reference is made to the serial number of the two-associate class PBIB designs enumerated by Bose, Clatworthy and Shrikhande (1954). Plans for other designs belonging to groups (2) and (3) are given in detail. Certain theorems are derived which are useful in determining from the parameters of a given two-associate class PBIB design whether the design is of the LB type or not.

2. Some definitions

An arrangement of v varieties in b blocks, each of k plots, $k < v$ such that each variety occurs atmost once in any block and altogether in r blocks will be called and equi-replicate incomplete block design. Such a design is completely characterized by its "incidence-matrix"

$$N = ((n_{ij}))$$

where $n_{ij} = 1$ if the j-th variety occurs in the i-th block and 0 otherwise $i = 1, 2, ..., b$; $j = 1, 2, ..., v$.

Let λ_{ij} denote the number of blocks in which the i-th and the j-th varieties occur together $i \neq j = 1, 2, \ldots, v$ and μ_{ij} the number of varieties which occur both in the i-th and j-th blocks $i \neq j = 1, 2, \ldots, b$.

Then
$$\lambda_{ij} = \sum_{t=1}^{b} n_{it} n_{jt}$$

$$\mu_{ij} = \sum_{u=1}^{v} n_{iu} n_{ju}.$$

For the sake of completeness write

$$\lambda_{ii} = \sum_{t=1}^{b} n_{it}^2 = r$$

$$\mu_{ii} = \sum_{u=1}^{v} n_{iu}^2 = k$$

We shall call the matrix $\Lambda \equiv ((\lambda_{ij}))$ the 'association matrix' of the design and $M \equiv ((\mu_{ij}))$ the 'block-characteristic matrix' of the design. Obviously $\Lambda = N'N$ and $M = NN'$.

An equi-replicate incomplete block design is called a Linked Block (LB) design if $\mu_{ij} = \mu$ for all $i \neq j = 1, 2, \ldots, b$ and it is said to be a Balanced Incomplete Block (BIB) design if $\lambda_{ij} = \lambda$ for all $i \neq j = 1, 2, \ldots, v$. It is well known that the necessary and sufficient condition for a BIB design to be a LB design and vice-versa is that $v = b$.

A design obtained from a given design by considering its blocks as varieties and varieties as blocks is said to be its dual. Obviously a LB design is the dual of some BIB design and vice-versa. Since a BIB design requires at least as many blocks as the number of varieties in a LB design the number of blocks can not exceed the number of varieties.

The following definition of a PBIB design from Bose and Shimamoto (1952) will be required in later sections:

A design is said to be a PBIB design with m associate classes if there exists a relationship of association between every pair of the v varieties satisfying the following conditions:

(a) Any two varieties are either first, second, ..., or m-th associates and any pair of varieties which are s-th associates occur together in λ_s blocks, $(s = 1, 2, \ldots, m)$.

(b) Each variety has n_s s-th associates

(c) For any pair of varieties which are s-th associates, the number of varieties which are simultaneously the j-th associates of the first and u-th associates of the second is p^s_{ju} and this is independent of the pair of varieties with which we start. Furthermore

$$p^s_{ju} = p^s_{uj} \; (j \neq u, j, s, u, = 1, 2, \ldots, m).$$

It is known that the following conditions are satisfied by the parameters:

$$\sum_{s=1}^{m} n_s = v-1$$

$$\sum_{s=1}^{m} n_s\lambda_s = r(k-1)$$

$$\sum_{s=1}^{m} p^i_{js} = \begin{cases} n_i-1 & if \;\; i=j \\ n_i & if \;\; i \neq j \end{cases}$$

$$n_i p^i_{ju} = n_j p^j_{iu} = n_u p^u_{ij}.$$

3. Intra-block analysis of linked block designs

Consider a Linked Block (LB) design involving v varieties in b blocks, each of k plots, each variety replicated r times in which any two blocks have μ varieties in common. Obviously,

$$bk = rv, \qquad \mu(b-1) = k(r-1), \qquad b \leqslant v.$$

Let $N \equiv ((n_{ij}))$ be the incidence matrix of this design.

Then $$\sum_{i=1}^{b} n_{ij} = r, \qquad \sum_{j=1}^{v} n_{ij} = k, \qquad \sum_{j=1}^{v} n_{ij}\, n_{i'j} = \mu.$$

Let y_{ij} denote the yield from that plot of the i-th block which gets the j-th treatment. Then, denoting the effect of the i-th block by β_i and that of the j-th treatment by τ_j the normal equations for estimating these parameters from intra-block differences (under the usual assumption of additivity of block and treatment effects) are:

$$T_j = r\,\tau_j + \sum_{i=1}^{b} n_{ij}\,\beta_i \qquad (j = 1, 2, \ldots, v) \qquad \ldots \quad (3.1)$$

$$B_i = k\beta_i + \sum_{j=1}^{v} n_{ij}\,\tau_j \qquad (i = 1, 2, \ldots, b) \qquad \ldots \quad (3.2)$$

where T_j stands for the total yield of all plots getting the j-th variety and B_i that for all plots in the i-th block.

Eliminating τ_j's from (3.1) and (3.2) we get

$$\beta_i = \frac{r}{\mu b} P_i + \bar{\beta}$$

and finally $$\hat{\tau}_j = t_j - \bar{\beta}$$

where $$P_i = B_i - \frac{1}{r} \sum_{j=1}^{v} n_{ij} T_j. \qquad \dots \quad (3.3)$$

$$t_j = \frac{1}{r} T_j - \frac{1}{\mu b} \sum_{i=1}^{b} n_{ij} P_i \qquad \dots \quad (3.4)$$

Thus the parameters β_i and τ_j are estimable except for an additive indeterminate $\bar{\beta}$.

Let us take $$t_j$$

and $$b_i = \frac{1}{\mu b} P_i \qquad \dots \quad (3.5)$$

for a particular solution of the normal equations (3.1) and (3.2). Then any varietal contrast $\sum_{j=1}^{v} l_j \tau_j$ with $\sum_{j=1}^{v} l_j = 0$ is estimable and its best intra-block linear estimate is $\sum_{j=1}^{v} l_j t_j$ with variance given by

$$\sigma^2 \sum_{j,j'=1} \sum l_j l_{j'} C_{jj'} \qquad \dots \quad (3.6)$$

where $$C_{jj} = \frac{1}{r} + \frac{1}{\mu b}$$

and $$C_{jj'} = \frac{\lambda_{jj'}}{r \mu b} \qquad \dots \quad (3.7)$$

$\lambda_{jj'}$ denoting the number of blocks in which the j-th and j'-th varieties occur together and σ^2 is the intra-block error variance which can be estimated from the error component in the analysis of variance given below:

TABLE 1. ANALYSIS OF VARIANCE

variation due to	s.s.	d.f.	s.s.	variation due to
blocks (unadjusted)	$S_B^* = \frac{1}{k} \Sigma B_i^2 - CF$	$b-1$	$\frac{r}{\mu b} \Sigma P_i^2 = S_B$	blocks (adjusted)
varieties (adjusted)	$S_V = S_B + S_V^* - S_B^*$	$v-1$	$\frac{1}{r} \Sigma T_j^2 - CF = S_V^*$	varieties (unadjusted)
error	$S_E = S - S_V - S_B^*$	$bk-b-v+1$	$S - S_B - S_V^* = S_E$	error
total	$S = \Sigma\Sigma y_{ij}^2 - CF$	$bk-1$	$\Sigma\Sigma y_{ij}^2 - CF$	total

Grand total = G, $CF = G^2/bk$.

We have seen that the variance of the best estimate of a varietal difference say, $\tau_j - \tau'_j$ is

$$2\sigma^2\left\{\frac{1}{r}+\frac{1}{\mu b}-\frac{\lambda_{jj'}}{r\mu b}\right\}$$

so that the average variance of the estimate of a treatment difference is

$$\frac{2\sigma^2}{r}\left\{1+\frac{r-\bar{\lambda}}{\mu b}\right\}$$

where $$\bar{\lambda} = r(k-1)/(v-1).$$

In a randomised block design with the same intra-block error variance and the same number of replications, the variance of the estimate of any varietal difference is $2\sigma^2/r$. This bears to the former the ratio

$$E = 1\bigg/\left\{1+\frac{r-\bar{\lambda}}{\mu b}\right\} \quad \dots \quad (3.8)$$

which is defined to be the efficiency of the design. Since $r \geqq \bar{\lambda}$, $E \leqq 1$ and $E = 1$ implies $v = k$.

4. Numerical illustration

We give below the computational details of the intra-block analysis of a LB design involving 18 varieties in 9 blocks each containing 8 plots, in which each variety is replicated 4 times and the number of varieties common to any two blocks is 3. This is design number 24 in our list given in § 7. The peculiarity of this design is that pairs of varieties may occur together in 0, 1, 2 or 3 blocks, but it does not belong to any of the well known designs—e.g. the PBIB designs.

The plan and the yields are given in Table 2. Varieties are indicated by numbers in brackets and corresponding yields are written by their sides. In Table 3, the numerical details of estimating varietal effects are given and in Table 4 the intra-block analysis of variance is presented. Table 5 gives the estimated standard errors of estimates of different types of varietal differences.

TABLE 2. PLAN AND YIELD

blocks	numbers of varieties and corresponding yields							
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	(7) 57	(8) 53	(5) 54	(4) 58	(3) 51	(1) 54	(2) 53	(6) 48
2	(9) 52	(13) 63	(10) 54	(11) 56	(2) 56	(12) 60	(1) 56	(3) 48
3	(12) 65	(17) 57	(14) 60	(4) 62	(9) 52	(6) 53	(16) 66	(1) 60
4	(16) 63	(10) 60	(7) 62	(5) 57	(1) 59	(18) 58	(15) 57	(12) 65
5	(4) 63	(2) 60	(13) 56	(10) 57	(8) 54	(16) 61	(14) 60	(18) 58
6	(16) 59	(17) 55	(6) 47	(11) 62	(13) 47	(15) 54	(5) 58	(2) 53
7	(18) 54	(15) 55	(17) 52	(3) 51	(8) 51	(11) 54	(4) 50	(12) 59
8	(13) 52	(7) 62	(9) 60	(14) 57	(18) 53	(17) 58	(5) 60	(3) 51
9	(10) 46	(15) 61	(14) 54	(11) 60	(8) 48	(7) 62	(6) 39	(9) 54

TABLE 3. ESTIMATION OF VARIETAL EFFECTS

j	T_j	$\sum_i n_{ij}(rP_i)$	$\mu br\, t_j$	t_j	i	B_i	$\sum_j n_{ij}T_j$	rP_i
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	229	93	6080	56.30	1	428	1750	−44
2	222	−4	5998	55.54	2	436	1760	−16
3	201	−79	5506	50.98	3	481	1830	85
4	239	102	6351	58.81	4	481	1856	68
5	229	0	6183	57.25	5	478	1810	102
6	187	−135	5184	48.00	6	425	1746	−46
7	243	−84	6645	61.53	7	432	1769	−41
8	206	−113	5675	52.55	8	453	1790	22
9	218	−39	5925	54.86	9	404	1746	−130
10	217	24	5835	54.03	total	4017	16008	0
11	212	−233	5957	55.16				
12	249	96	6627	61.36				
13	208	62	5554	51.43				
14	246	79	6563	60.77				
15	217	−149	6008	55.63				
16	249	200	6514	60.31				
17	222	20	5974	55.31				
18	223	151	5870	54.35				
total	4017	0	108439	1004.20				

Computational checks:

$$\sum_i (rP_i) = 0$$

$$\sum_j \sum_i n_{ij}(rP_i) = 0$$

$$r \sum t_j = G$$

Here $v = 18$, $b = 9$, $r = 4$, $k = 8$, $\mu = 3$

Grand total $G = 4017$, $\quad$ CF $= G^2/bk$ $= 224115.12$

Total S.S. $= S = \Sigma y^2 - \text{CF}$ $= 2019.88$

Block S.S. (unadjusted) $= S_B^* = \frac{1}{k}\Sigma B_i^2 - \text{CF}$ $= 706.00$

Variety S.S. (unadjusted) $= S_V^* = \frac{1}{r}\Sigma T_j^2 - \text{CF}$ $= 1280.63$

Block S.S. (adjusted) $= S_B = \frac{1}{\mu br}(rP_i)^2$ $= 422.40$

Variety S.S. (adjusted) $= S_V = S_V^* + S_B - S_B^*$ $= 997.09$

Error S.S. $= S_E = S - S_V^* - S_B = S - S_B - S_V^*$ $= 316.79$

TABLE 4. ANALYSIS OF VARIANCE

variation due to	d.f.	s.s.	m.s.	F	m.s.	s.s.	d.f.	variation due to
blocks (unadjusted)	8	700.00	—	7.66	52.81	422.46	8	blocks (adjusted)
varieties (adjusted)	17	907.00	53.36	7.74	—	1280.63	17	varieties (unadjusted)
error	46	316.79	6.89		6.89	316.79	46	error
total	71	2019.88				2019.88		total

TABLE 5. STANDARD ERROR OF VARIETAL DIFFERENCES

number of blocks in which a given pair of varieties occur together λ	$f_\lambda = \frac{2}{r}\left\{1+\frac{r-\lambda}{\mu b}\right\}$	standard error of varietal difference $\left\{\frac{f_\lambda . S_E}{bk-b-v+1}\right\}^{\frac{1}{2}}$
(1)	(2)	(3)
0	0.574074	1.988
1	0.555556	1.936
2	0.537037	1.923
3	0.518519	1.890

Table 5. has to be used in carrying out a t-test to examine any particular varietal difference. For example, if we have to see if varieties 1 and 7 are different, we observe that they occur together in $\lambda = 2$ blocks. The estimate of the difference $\tau_1 - \tau_7$ is $t_1 - t_7 = 56.39 - 61.53 = -5.14$ obtained from Table 3 and its standard error is 1.923 as given in Table 5. Hence the statistic to use is

$$t = -5.14/1.923 = -2.673$$

which as a t-statistic with 46 degrees of freedom is significant at 1% level.

5 Classification of linked block designs

LB designs are duals of BIB designs and Shrikhande (1952) has shown that the dual of a BIB design is a PBIB design with two associate classes in the following cases:

(a) if in the BIB design $\lambda = 1 \quad (v \neq b)$

(b) if in the BIB design $\lambda = 2, \quad r = k+2, \quad k = k.$

Roy (1954) has shown further that the dual of any unreduced BIB design (that is a design obtained by forming a block with each possible combination of varieties subject to the condition that the number of plots in a block is fixed) is necessarily of the

PBIB type. Again, if a BIB design is symmetric, that is if in a BIB design the number of blocks is equal to the number of varieties, then its dual is also a BIB design. However, the dual of any BIB design is not necessarily a BIB or even PBIB design.

We may therefore classify all LB designs into the following three main groups: (1) Symmetrical BIB designs (2) PBIB designs and (3) Irregular designs not belonging to any of the known types. We shall omit the symmetrical BIB designs from our consideration as they do not present any new features.

In order to get the LB design of the PBIB type, one way is to start with the plans of all BIB designs, dualise them and then pick out from them the designs of the PBIB type. This, however, is a formidable task. On the other hand, Bose, Clatworthy and Shrikhande (1954) have prepared a list of useful two associate PBIB designs and enumerated their plans and parameters. It is considerably simpler to pick out from this list the LB designs with the use of the theorems derived in the next section which state the necessary and sufficient conditions on the parameters of a PBIB design so that it may be of the LB type.

In § 7 we give a detailed classified list of all LB designs with ten or less replications and ten or less plots per block.

6. Condition that a two-associate PBIB design may be of the LB type

It is a well known property of matrices that given any two matrices A, B such that the matrix products AB and BA are possible, the non-zero roots of the two determinantal equations

$$|AB-xI| = 0$$

$$|BA-xI| = 0$$

are identical. It follows therefore that for any design, the association matrix and the block characteristic martix have the same set of non-zero latent roots.

Now the block-characteristic matrix of a LB design has all diagonal elements equal to k and all off-diagonal elements equal to μ, k being the number of plots per block and μ the number of varieties common to any two blocks. Consequently the block-characteristic matrix of a LB design has only two distinct latent roots, namely $k+(b-1)\mu = rk$ and $k-\mu$ the latter of multiplicity $(b-1)$. On the other hand, for the block characteristic matrix, if rk is a latent root with $(1, 1, ..., 1)$ for latent vector, and $k-\mu$ is another latent root of multiplicity $(b-1)$, all other roots being zero, the design must be of the LB type. Again, it follows from the work of Connor and Clatworthy (1954) that latent roots other than rk of the association matrix of a two-associate PBIB design are except for repetitions the same as those of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where
$$\left.\begin{aligned} a_{11} &= r+\lambda_1 p^1_{11}+\lambda_2 p^1_{12}-\lambda_1 n_1 \\ a_{12} &= \lambda_1 p^2_{11}+\lambda_2 p^2_{12}-\lambda_1 n_1 \\ a_{21} &= \lambda_1 p^1_{21}+\lambda_2 p^1_{22}-\lambda_2 n_2 \\ a_{22} &= r+\lambda_1 p^2_{21}+\lambda_2 p^2_{22}-\lambda_2 n_2 \end{aligned}\right\} \quad \ldots \quad (6.1)$$

where the parameters have their usual significance. They have also shown that the matrix A can not have two equal latent roots. Hence we have the following

Theorem 6.1. *The necessary and sufficient condition for a PBIB design with two associate classes to be of the LB type is that the matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *defined in* (6.1) *has only one non-zero root and this occurs* $(b-1)$ *times as a latent root of the association matrix of the PBIB design.*

Two associate PBIB designs have been classified by Bose, Clatworthy and Shrikhande (1954) as (1) Group Divisible (a) Singular (b) Semi-Regular (c) Regular (2) Triangular (3) Latin Square Type (4) Simple and (5) Cyclic. Theorem 6.1 gives the following conditions on the parameters of the first four of these designs which ensure that they belong to the LB type.

For a Group Divisible (GD) design

$$v = mn,\ n_1 = n-1,\ n_2 = m(n-1),\ p^1_{11} = n-2 \text{ and } r \geqslant \lambda_1,\ rk \geqslant v\lambda_2.$$

A GD design is singular if $r = \lambda_1$, semi-regular if $r > \lambda_1$, $rk = v\lambda_2$ and regular if $r > \lambda_1$, $rk > v\lambda_2$. For a singular GD design $b \geqslant m$ and for a semi-regular GD design $b \geqslant v-m+1$. These results are due to Bose and Connor (1952). Application of theorem 6.1 gives the following result. A regular GD is never of the LB type, the necessary and sufficient condition for a singular GD to be of the LB type is that $b = m$ and that for a semi-regular GD is that $b = v-m+1$.

In a Triangular design as defined by Bose and Shimamoto (1952) $v = \frac{1}{2}n(n-1)$ $n_1 = 2(n-2)$, $n_2 = \frac{1}{2}(n-2)(n-3)$, $p^1_{11} = (n-2)$. The necessary and sufficient condition for a Triangular design to be of the LB type is that either (*i*) $r = 2\lambda_1-\lambda_2$ and $b = n$ or (*ii*) $r = (n-3)\lambda_2-(n-4)\lambda_1$ and $b = \frac{1}{2}(n-1)(n-2)$.

For a Latin square type design with i constraints,

$$v = n^2,\ n_1 = i(n-1),\ n_2 = (n-1)(n-i+1) \text{ and } p^1_{11} = i(i-3)+n.$$

In order that a Latin Square type design may be of the LB type it is necessary and sufficient that either

(i) $r = (i-n)(\lambda_1-\lambda_2)+\lambda_2$ and $b = n(n-1)+i$

or (ii) $r = i(\lambda_1-\lambda_2)+\lambda_2$ and $b = i(n-1)+1$.

For a Simple design $\lambda_1 \neq 0$, $\lambda_2 = 0$ Bose and Clatworthy (1955) have shown that the complete class of simple designs with $\lambda_1 = 1$ and $k > r \geqslant 2$ is characterized by the parameters

$$v = k[(r-1)(k-1)+t]/t \qquad b = r[(r-1)(k-1)+t]/t$$

$$n_1 = r(k-1), \qquad n_2 = (r-1)(k-1)(k-t)/t$$

$$p_{11}^1 = (t-1)(r-1)+(k-2)$$

where $$1 \leqslant t \leqslant r.$$

The necessary and sufficient condition that this is of the LB type is that $t = r$.

Using these results, we have picked up all the LB designs amongst the two-associate PBIB designs tabulated by Bose, Clatworthy and Shrikhande (1954) and the designs are presented in § 7.

7. List and plan of all linked block disigns with r, $k \leqslant 10$

In this section we give a list of all LB designs (other than the symmetrical BIB designs) requiring atmost ten replications and blocks of atmost ten plots. The list is arranged in ascending order of the number of varieties, blocks and replications. The number of types of varietal differences estimable with different precisions is denoted by m, and the number of blocks in which a pair of varieties can occur together is denoted by λ_1. $\lambda_2, \ldots, \lambda_m$. The efficiency factor of the design is denoted by E. Column 3 gives the nature of the design, and in column 4 a reference is made to the serial number of the design (if it is a two-associate PBIB) in Bose, Clatworthy and Shrikhande (1954) where the plan of the design is given. The plans for designs other than two-associate PBIB are given in the following pages. It is interesting to observe that for all these designs, the efficiency factor is very high, being of the order of 90%.

It also follows easily from the results of Connor (1952) that

$$\frac{2\mu r}{k} - (r+\mu-k) \geqslant \lambda_j \geqslant r+\mu-k$$

for $$j = 1, 2, \ldots, m.$$

Hence the number $$m \leqslant \frac{2}{k}(k-r)(k-\mu)+1.$$

This is one of the Irregular Linked Block designs. In this design estimates of differences between pairs of treatments can be divided into four groups, each having a different precision. The peculiarity of this design which can be easily analysed and has an efficiency of 91% but is not partially balanced will be clear from the association matrix given below:

TABLE 6. LIST OF LINKED BLOCK DESIGNS WITH $r, k \leq 10$

design no.	parameters of the design											nature of the design	reference to plan
	v	b	r	k	μ	m	λ_1	λ_2	λ_3	λ_4	E		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	6	3	2	4	2	2	2	1	–	–	.88	singular G.D.	S 1
2	6	4	2	3	1	2	1	0	–	–	.77	semi regular G.D,	SR 1
3	8	4	3	6	4	2	3	2	–	–	.95	singular G.D.	S 7
4	9	3	2	6	3	2	2	1	–	–	.92	singular G.D.	S 12
5	10	5	2	4	1	2	1	0	–	–	.79	triangular	T 1
6	10	5	3	6	3	2	2	1	–	–	.92	triangular	T 15
7	10	5	4	8	6	2	4	3	–	–	.97	singular G. D.	S 18
8	10	6	3	5	2	2	2	1	–	–	.91	triangular	T 9
9	12	3	2	8	4	2	2	1	–	–	.94	singular G.D.	S 22
10	12	4	2	6	2	3	2	1	0	–	.88	three associate PBIB	** 1
11	12	4	3	9	6	2	3	2	–	–	.97	singular G.D.	S 24
12	12	6	5	10	8	2	5	4	–	–	.98	singular G.D.	S 28
13	12	9	3	4	1	2	1	0	–	–	.80	semi regular G.D.	SR 20
14	12	9	6	8	5	2	4	3	–	–	.95	semi regular G.D.	SR 26
15	14	7	3	6	2	2	3	1	–	–	.88	singular G.D.	S 40
16	14	7	4	8	4	2	4	2	–	–	.94	singular G.D.	S 41
17	14	8	4	7	3	2	2	0	–	–	.92	semiregular G.D.	SR32
18	15	3	2	10	5	2	2	1	–	–	.95	singular G.D.	S 46
19	15	6	2	5	1	2	1	0	–	–	.81	triangular	T 20

TABLE 6. (*Continued*)

design no.	parameters of the design											nature of the design	reference to plan
	v	b	r	k	n	m	λ_1	λ_2	λ_3	λ_4	E		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
20	15	6	4	10	6	2	3	2	–	–	.96	triangular	T 25
21	15	10	4	6	2	2	2	1	–	–	.89	triangular	T 22
22	15	10	6	9	5	2	4	3	–	–	.95	triangular	T 27
23	18	4	2	9	3	3	2	1	0	–	.92	three associate PBIB	** 2
24	18	9	4	8	3	4	3	2	1	0	.91	irregular	*
25	18	9	5	10	5	4	4	3	2	1	.95	irregular	*
26	18	10	5	9	4	4	4	3	2	1	.93	irregular	*
27	20	5	2	8	2	3	2	1	0	–	.89	three associate PBIB	** 3
28	20	6	3	10	4	3	3	2	1	–	.94	three associate PBIB	** 4
29	20	16	4	5	1	2	1	0	–	–	.84	semiregular G.D.	SR 51
30	21	7	2	6	1	2	1	0	–	–	.82	triangular	T 31
31	21	7	3	9	3	2	3	1	–	–	.92	singular G.D.	S 77
32	22	11	5	10	4	2	5	2	–	–	.94	singular G.D.	S 81
33	24	16	6	9	3	3	3	2	0	–	.92	three associate PBIB	*
34	26	13	3	6	1	2	1	0	–	–	.84	simple	S 4
35	26	13	4	8	2	2	4	1	–	–	.90	singular G.D.	S 80
36	28	8	2	7	1	2	1	0	–	–	.84	triangular	T 32
37	30	6	2	10	2	3	2	1	0	–	.90	three associate PBIB	** 5
38	30	10	3	9	2	3	2	1	0	–	.90	irregular	

TABLE 6. (*Continued*)

design no.	parameters of the design										nature of the design	reference to plan
	v	b	r	k	μ	m	λ_1	λ_2	λ_3	λ_4	E		
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
39	30	21	7	10	3	4	3	2	1	0	.93	irregular	
40	30	25	5	6	1	2	1	0	–	–	.86	semi regular G.D.	SR 70
41	35	15	3	7	1	2	1	0	–	–	.86	simple	S 9
42	36	9	2	8	1	2	1	0	–	–	.85	triangular	T 33
43	36	28	7	9	2	2	2	1	–	–	.91	triangular	T 34
44	42	21	5	10	2	2	3	1	–	–	.91	singular G.D.	S 111
45	45	10	2	9	1	2	1	0	–	–	.86	triangular	T 35
46	50	25	4	8	1	2	1	0	–	–	.88	simple	S 17
47	55	11	2	10	1	2	1	0	–	–	.87	triangular	T 36
48	56	49	7	8	1	2	1	0	–	–	.89	semiregular G.D.	SR 85
49	57	19	3	9	1	2	1	0	–	–	.88	simple	S 18
50	63	28	4	9	1	2	1	0	–	–	.89	simple	S 21
51	70	21	3	10	1	2	1	0	–	–	.89	simple	S 22
52	72	64	8	9	1	2	1	0	–	–	.90	semiregular G. D.	SR 89
53	82	41	5	10	1	2	1	0	–	–	.90	simple	S 25

*For the plan of designs asterisked see the following pages.

**1 Replace each variety in S1 by two varieties to get the plan for Design no 10

**2 Replace each variety in S1 by three varieties to get the plan for Design no 23

**3 Replace each variety in T1 by two varieties to get the plan for Desgin no 27.

**4 Replace each variety in T9 by two varieties to get the plan for Design no. 28

**5 Replace each variety in T20 by two varieties to get the plan for Design no. 37

TABLE 7. DESIGN NO. 24 (Irregular)

$v = 18$, $b = 9$, $r = 4$, $k = 8$, $\rho = 3$, $E = 0.91$.
$m = 4$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, $\lambda_4 = 0$.

blocks			plan		plan			
1	1	2	3	4	5	6	7	8
2	1	2	3	9	10	11	12	13
3	1	4	6	9	12	14	16	17
4	1	5	7	10	12	15	16	18
5	2	4	8	10	13	14	16	18
6	2	5	6	11	13	15	16	17
7	3	4	8	11	12	15	17	18
8	3	5	7	9	13	14	17	18
9	6	7	8	9	10	11	14	15

TABLE 8. ASSOCIATION MATRIX OF THE DESIGN NO. 24

number of blocks in which the pair of treatments (i, j) occur together

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1		2	2	2	2	2	2	1	2	2	1	3	1	1	1	2	1	1
2			2	2	2	2	1	2	1	2	2	1	3	1	1	2	1	1
3				2	2	1	2	2	2	1	2	2	2	1	1	0	2	2
4					1	2	1	3	1	1	1	2	1	2	1	2	2	2
5						2	3	1	1	1	1	1	2	1	2	2	2	2
6							2	2	2	1	2	1	1	2	2	2	2	0
7								2	2	2	1	1	1	2	2	1	1	2
8									1	2	2	1	1	2	2	1	1	2
9										2	2	2	2	3	1	1	2	1
10											2	2	2	2	2	2	0	2
11												2	2	1	3	1	2	1
12													1	1	2	2	2	2
13														2	1	2	2	2
14															1	2	2	2
15																2	2	2
16																	2	2
17																		2
18																		

TABLE 9. DESIGN NO. 25 (Irregular)

$v = 18,\ b = 9,\ r = 5\ \ k = 10\ \ \mu = 5\ \ E = 0.95$
$m = 4\ \ \lambda_1 = 4\ \ \lambda_2 = 3\ \ \lambda_3 = 2\ \ \lambda_4 = 1$

blocks	plan									
1	1	7	8	9	10	13	14	15	17	18
2	2	3	6	8	9	10	12	14	16	17
3	1	2	5	7	8	9	11	12	13	16
4	2	3	5	6	7	10	11	13	14	15
5	3	4	5	9	10	12	13	15	16	18
6	1	2	4	5	6	8	12	14	15	18
7	1	3	4	7	11	12	14	15	16	17
8	1	2	3	4	6	9	11	13	17	18
9	4	5	6	7	8	10	11	16	17	18

TABLE 10. DESIGN NO. 26 (Irregular)

$v = 18\ \ b = 10\ \ r = 5\ \ k = 9\ \ \mu = 4\ \ E = 0.93$
$m = 4\ \ \lambda_1 = 4\ \ \lambda_2 = 3\ \ \lambda_3 = 2\ \ \lambda_4 = 1$

blocks	plan								
1	1	2	3	4	5	6	7	8	9
2	1	2	3	4	10	11	12	13	14
3	1	2	5	6	10	11	15	16	17
4	1	3	7	8	10	12	15	16	18
5	1	4	7	9	11	13	16	17	18
6	2	3	5	7	13	14	15	17	18
7	2	4	6	9	12	14	15	16	18
8	4	5	6	8	10	12	13	17	18
9	3	5	8	9	11	12	14	16	17
10	6	7	8	9	10	11	13	14	15

TABLE 11. DESIGN NO 33

(Partially Balanced with Three Associate-Classes)

$v = 24,\quad b = 16,\quad r = 6,\quad k = 9,\quad \mu = 3,\quad E = 0.92$

$m = 3,\quad \lambda_1 = 3,\quad \lambda_2 = 2,\quad \lambda_3 = 0,\quad n_1 = 4,\quad n_2 = 18,\quad n_3 = 1$

blocks	plan								
1	1	2	3	4	5	6	7	8	9
2	1	2	3	10	11	12	13	14	15
3	4	5	6	10	11	12	16	17	18
4	7	8	9	13	14	15	16	17	18
5	1	4	7	10	13	16	19	20	21
6	1	5	8	11	14	16	19	22	23
7	2	4	8	11	13	17	20	22	24
8	2	5	7	10	14	17	21	23	24
9	3	6	9	11	14	17	19	20	21
10	3	4	7	12	15	17	19	22	23
11	1	6	7	12	14	18	20	23	24
12	1	4	9	11	15	18	21	23	24
13	2	5	8	12	15	18	19	20	21
14	2	6	9	10	13	18	19	22	23
15	3	5	9	10	15	16	20	22	24
16	3	6	8	12	13	16	21	23	24

This is a partially balanced design with three-associate classes. The values of the p^i_{jk} parameters are given below:

$$\begin{bmatrix} p^1_{11} = 2 & p^1_{12} = 0 & p^1_{13} = 1 \\ & p^1_{22} = 18 & p^1_{23} = 0 \\ & & p^1_{33} = 0 \end{bmatrix}$$

$$\begin{bmatrix} p^2_{11} = 0 & p^2_{12} = 4 & p^2_{13} = 0 \\ & p^2_{22} = 12 & p^2_{23} = 1 \\ & & p^2_{33} = 0 \end{bmatrix}$$

$$\begin{bmatrix} p^3_{11} = 4 & p^3_{12} = 0 & p^3_{13} = 0 \\ & p^3_{22} = 18 & p^3_{23} = 0 \\ & & p^3_{33} = 0 \end{bmatrix}$$

TABLE 12. DESIGN NO. 38 (Irregular)

$v = 30,\ b = 10,\ r = 3,\ k = 9,\ \mu = 2,\ E = 0.90$
$m = 3,\ \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 0.$

blocks	plan								
1	1	2	3	4	5	6	7	8	9
2	1	2	10	11	12	13	14	15	16
3	1	3	10	17	18	19	20	21	22
4	2	4	11	17	18	23	24	25	26
5	3	5	12	13	19	23	24	27	28
6	4	6	10	14	19	25	27	29	30
7	5	7	14	15	17	20	26	28	29
8	6	8	12	16	18	21	26	28	30
9	7	9	13	15	21	22	23	25	30
10	8	9	11	16	20	22	24	27	29

TABLE 13. DESIGN NO. 39 (Irregular)

$v = 30,\ b = 21,\ r = 7,\ k = 10\ \ \mu = 3,\ E = 0.93,$
$m = 4,\ \lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = 0.$

blocks	plan									
1	1	2	4	8	9	11	15	16	18	28
2	2	3	5	9	10	12	16	17	19	22
3	1	3	7	8	10	14	15	17	21	27
4	2	6	7	9	13	14	16	20	21	26
5	1	5	6	8	12	13	15	19	20	25
6	4	5	7	11	12	14	18	19	21	24
7	3	4	6	10	11	13	17	18	20	23
8	4	5	9	14	17	20	25	27	28	30
9	3	4	8	13	16	19	24	26	27	30
10	2	3	12	14	15	18	23	25	26	30
11	1	2	11	13	17	21	22	24	25	30
12	1	7	10	12	16	20	23	24	28	30
13	6	7	9	11	15	19	22	23	27	30
14	5	6	8	10	18	21	22	26	28	30
15	3	6	11	12	16	21	25	27	28	29
16	2	5	10	11	15	20	24	26	27	29
17	1	4	9	10	19	21	23	25	26	29
18	3	7	8	9	18	20	22	24	25	29
19	2	6	8	14	17	19	23	24	28	29
20	1	5	13	14	16	18	22	23	27	29
21	4	7	12	13	15	17	22	26	28	29

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