ON THE UNBOUNDEDNESS OF INFINITELY DIVISIBLE LAWS

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1. The object of this note is to prove that a bounded proper distribution can not be infinitely divisible (I.D.).

2. The following result of Polya (1949), will be used: A necessary and sufficient condition that a probability distribution should be bounded is that the definition of the characteristic function $f(t)$ can be extended to complex values of the variable and this extension shows that $f(t)$ is an entire function of exponential type. Moreover, if the distribution function is denoted by $F(x)$ then the right and left extremities are given respectively by

$$\overline{\lim_{r \to +\infty}} \; r^{-1} \log |f(-ir)| \quad \text{and} \quad -\overline{\lim_{r \to +\infty}} \; r^{-1} \log |f(ir)|$$

Theorem : *A proper bounded distribution can not be I.D.*

Proof : If possible, let $F(x)$ be a proper distribution bounded and I.D. In view of the fact that a random variable (r.v.) X is I.D. if and only if $X-a$ is I.D., where 'a' is any real number, we may take the bounds as 0 and $h(h > 0)$ To show that such a distribution is not possible we will prove that h is necessarily infinite. As it is I.D., its characteristic function (c.f.) $f(t)$ is represented by Gnedenko, B. and Kolmogorov, A. N. (1954).

$$\log f(t) = i\gamma t + \int \frac{e^{itu}-1-itu}{u^2} \, dK(u)$$

where γ is a constant and $K(u)[K(-\infty) = 0]$ is a non decreasing function of bounded variation.

Further
$$K(u) = \underset{n \to \infty}{Lt} \; K_n(u) \quad \text{for each } u$$

where
$$K_n(u) = n \int_{-\infty}^{u} z^2 dF_n(z)$$

$F_n(z)$ being the distribution function (d.f.) corresponding to the c.f. $[f(t)]^{1/n}$. That this is a c.f. follows from the fact that $F(x)$ is I.D. By Polya's Theorem $F_n(x)$ is a bounded d.f. with lower bound zero.

$$\therefore \quad K_n(u) = 0 \quad \textit{if } u \leqslant 0$$

$$= n \int_{0}^{u} z^2 dF_n(z) \text{ if } u > 0$$

$$\therefore \quad K(u) = 0 \quad \text{for } u \leqslant 0$$

Here
$$\log f(t) = i\gamma t + \int_{0}^{\infty} \frac{e^{itu}-1-itu}{u^2} \, dK(u)$$

The same representation goes over for $t = ir$, r "real" as can be easily proved by means of the proof in Gnedenko, B. and Kolmogorov, A. N. (1954), and using simple properties of entire functions.

Hence by Polya's Theorem

$$\lambda = \overline{\lim_{r \to +\infty}} \frac{1}{r} \log |f(-ir)|$$

$$= \overline{\lim_{r \to +\infty}} \left[\gamma + \frac{1}{r} \int_0^\infty \frac{e^{ru} - 1 - ru}{u^2} \, dK(u) \right]$$

Now,

$$\frac{1}{r} \int_0^\infty \frac{e^{ru} - 1 - ru}{u^2} \, dK(u)$$

$$> \frac{r}{2} \int_0^\infty dK(u)$$

$$> M.r$$

Therefore, $\lambda = \infty$ unless $K(u)$ is constant over $(0, \infty)$ in which case the law is improper.

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REFERENCES

POLYA, G. (1949): Remarks on characteristic functions, *First Berkeley Symposium*, 115-123.

GNEDENKO, B. AND KOLMOGOROV, A. N. (1954): Limit distributions of sums of independent random variables (Trans. by K. L. Chung).

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