PUBLIC EXPENDITURE, ENVIRONMENTAL POLLUTION AND ENDOGENOUS ECONOMIC GROWTH

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Dedicated to my parents

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CONTENTS

CHAPTER 1

1. INTRODUCTION AND LITERATURE SURVEY

1.1 MODERN GROWTH THEORY

1.1.1 Sources of Economic Growth and the Definition of Steady-State Equilibrium

Economic growth is defined as a continuous increase in national income taking place over a time horizon. According to the neoclassical theory of economic growth there are three sources of economic growth: (i) capital accumulation, (ii) growth of labour force and (iii) technological progress.

The steady-state growth equilibrium is defined as a state where all major macro-economic variables grow at the same rate so that the ratios of these variables remain unchanged over time. For example, in the one sector aggregative model like that of Solow (1956), capital and labour grow at equal rates and hence capital-labour ratio remains time-independent. If this equilibrium is stable then the rate of growth in the steady-state equilibrium is the long run rate of growth of the economy. In a multi-sectoral dynamic model, steady-state equilibrium growth means balanced growth of all sectors at equal rate.

1.1.2 Old Growth Theory Versus Endogenous Growth Theory

In the old growth theory developed by Solow (1956) and extended by many others, steady-state equilibrium growth rate is exogenous because the rate of growth of labour force and the rate of technological progress are exogenous. This exogenous growth rate cannot be influenced by public policy. However, the rate of growth is endogenous in the old theory when the economy is on the transitional growth path. On the other hand, the long-run rate of growth or the steady-state equilibrium rate of growth is endogenously determined in a model of endogenous growth. In such a model, the rate of growth of labour force or the rate of technical progress is assumed to depend on some macro-economic variables.

1.1.3 Sources of Endogenous Growth

The strand of endogenous growth literature identifies externalities arising from productive inputs. These spillover effects compensate for diminishing returns to physical capital accumulation and make the endogenous growth rate positive.

The seed of the idea of endogenous growth can be found in Arrow (1962) where 'learning-by-doing' mechanism leads to endogenous technical change. The labourer can gain experience as aggregate physical capital is accumulated and this experience gain is called the process of 'learning–by-doing'. This leads to an improvement in the labour productivity; and the improvement is internal to the economy as a whole though external to the individual firm. Hence the economy grows because diminishing returns to capital is halted by the increase in labour productivity.

Lucas (1988) finds the source of endogenous economic growth in endogenous human capital accumulation; and, in his model, technological change is identical to the human capital accumulation. The rate of accumulation of human capital is endogenous because the consumer allocates his resources between production and human capital accumulation solving a lifetime utility maximization problem.

In Romer (1990) and Grossman and Helpman (1991), the technical progress takes the form of product development and this is made by the R & D sector that is the engine of growth. Endogenous allocation of resources between the production sector and the R & D sector makes the rate of technological progress endogenous.

However, Barro (1990) deviates from the idea of endogenous technical progress as a source of endogenous economic growth; and shows that endogenous growth is possible even without such technical progress if the system generates external economies arising from productive public inputs. Public inputs used by firms create externalities which cannot be internalized by an individual firm's decision making process. However, these halt the diminishing returns to physical capital on an aggregate scale and make the growth rate positive in the long run.

1.2 PUBLIC EXPENDITURE AND ENDOGENOUS GROWTH

1.2.1 Empirical Support

There is substantial empirical evidence of public expenditure having a positive impact on economic growth in empirical papers like Gregoriou and Ghosh (2009), Hulten (1996), Neill (1996), Tuijl, Groof and Kolnaar (1997), Khan and Kumar (1997), Rioja (1999), Shioji (2001), Kneller, Bleaney and Gemmell (2001), Ghartey (2008), Forni, Monteforte and Sessa (2009), etc.

1.2.2 Barro (1990) Model

Barro (1990) first shows that productive public input can outweigh the diminishing returns of private physical capital even without endogenous technological progress, and can be the driving force behind economic growth. The production function in Barro's (1990) model satisfies constant returns to scale in private capital and productive public expenditure, as shown below.

$$y = f(k, g) = Ak^{1-\alpha}g^{\alpha}. \qquad \text{… … (1)}$$

Here, y, k and g stand for output, private physical capital and productive public input respectively, all in per capita units. While private physical capital is a durable input, Barro treats productive public input as a perishable input. Public input is financed by public expenditure which is a flow variable. A is the constant productivity term.

Government finances expenditure on public input with a proportional income tax. The budget of the government is balanced. Hence, we have

$$g = \tau y = \tau Ak^{1-\alpha}g^{\alpha}. \qquad \text{… … (2)}$$

Here τ is the income tax rate.

The representative household's budget balance equation is given by

$$\dot{k} = y(1-\tau) - c. \qquad \text{… … (3)}$$

Here, c is the level of per capita consumption. The dynamic optimization problem of the representative household is to maximize the discounted present value of utility over the infinite time horizon, $\int_0^{\infty} e^{-\rho t}u(c)dt$, with respect to c, subject to equations (1) and (3). Here, ρ is the discount rate. The instantaneous utility function is given by

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma};\ \sigma > 1. \qquad \text{… … (4)}$$

In the steady-state growth equilibrium, c and k grow at equal rates. Barro (1990) first finds out the growth rate maximizing income tax rate at the steady-state equilibrium; and then shows that the growth rate maximizing solution is identical to the welfare maximizing solution of the representative household. The income tax rate is identical to the ratio of productive public spending to national income; and the optimum income tax rate is equal to the competitive output share of the productive public input, as given by

$$\tau = \alpha. \qquad \text{… … (5)}$$

However, the growth rate in the decentralized economy falls short of that in the planned economy. This is clearly due to private individual's inability to internalize the positive externality caused by the productive public input. The social planner can internalize this externality.

1.2.3 Futagami, Morita and Shibata (1993) Model

However, the Barro (1990) model fails to exhibit transitional dynamic properties due to the assumption that public spending is a flow variable. This makes it identical to an AK model where marginal productivity of capital is independent of capital accumulation. All macro variables, in his model, start at their initial values and jump to their steady-state equilibrium values.

Futagami *et al.* (hereafter known as FMS) (1993) question the validity of the assumption that public productive input is a flow variable. Futagami *et al.* (1993) extend Barro (1990) model assuming that productive public input is a stock variable like physical capital. Equations (1), (3) and (4) of Barro (1990) model remain unchanged here, but equation (2) is modified as follows.

$$\dot{g} = \tau y$$

Here, $\dot{g}$ is the net investment in public capital and g is the stock of public capital. Both g and k accumulate over time; and, in the steady-state equilibrium, g, k and c grow at equal rates.

Transitional dynamic properties come back to this extended model; and Barro (1990) result about the optimal income tax rate remains valid in the steady-state equilibrium but not in the transitional phase of economic growth.

1.2.4 Various Extensions of Barro (1990) Model

Both Barro (1990) and Futagami *et al.* (1993) models are extended and reanalyzed by various authors in various directions; and the literature includes the works of Aschauer (1988, 1989, 1990), Turnovsky (1997, 1996), Tsoukis and Miller (2003), Lansing (1998), Mourmouras and Lee (1999), Tanaka (2002), Dasgupta (1999, 2001), Varvarigos (2003), Ghosh and Roy (2004), Yakita (2004), Marrero and Novales (2005), Greiner and Hanusch (1998), Park and Phillippopoulos (2002), Hu, Ohdoi and Shimomura (2008), Burguet and Fernandez-Ruiz (1998), Ghosh and Mourmouras (2004), Park (2009), Baier and

Glomm (2001), Cazzavillan (1996), Chen (2006), Zhang (2000), Chang (1999), Ohdoi (2007), Greiner and Semmler (2000), Kalaitzidakis and Kalyvitis (2004), Shioji (2001), Tamai (2007), Raurich-Puigdevall (2000), Neill (1996), Chen and Lee (2007), etc.

Neither Barro (1990) nor Futagami *et al.* (1993) considers adjustment cost of investment. Turnovsky (1996) and Tsoukis and Miller (hereafter known as TM) (2003), incorporate convex adjustment costs of private capital investment in an endogenous growth model with productive public expenditure. Public expenditure affects adjustment cost in Turnovsky (1996). However, public services have no effect on the adjustment cost in TM (2003).

Lansing (1998) develops an endogenous growth model of business cycle with public capital and examines optimal fiscal policy when utility of the consumer is enhanced by consumption of public goods.

Mourmouras and Lee (hereafter referred to as ML) (1999) and Tanaka (2002) examine the effects of government spending on infrastructure within a Barro (1990) type endogenous growth model populated by individuals within finite horizon.

Dasgupta (1999) constructs a two sector model of endogenous growth with durable productive public infrastructure where this public infrastructure is used to produce the final good as well as new public infrastructure. Private capital is also used by these two sectors. Government imposes a proportional profit tax on the household's aggregate capital income and charges a price per unit of the infrastructural service to producers of the final good.

In a Barro-type model, Varvarigos (2007) shows how policy variability can affect the time varying growth rate when a productive public good is involved, either as a direct input in production or as an input in human capital accumulation.

Ghosh and Roy (2004) develop an endogenous growth model with both stock and flow varieties of public input.

Yakita (2004) examines the effects of fiscal policy on growth and welfare in a model of public capital driven growth where different varieties of final goods are produced and markets for final goods are characterized by monopolistic competition. The utility is a function of a composite consumption index consisting of all these varieties of final goods.

The implications of alternative tax policies are examined by Marrero and Novales (hereafter referred to as MN) (2005) in an endogenous growth model with productive public expenditure as well as public consumption expenditure. Aggregate private capital and public capital have positive externality effects on production. With full depreciation of private capital as well as of public capital, the dynamic equilibrium is shown to be devoid of any transitional dynamic properties. However, properties of alternative tax policies are analyzed when the government wants to maximize the growth rate in the steady-state equilibrium.

Greiner and Hanusch (1998) also analyze growth rate maximizing and welfare maximizing policies in a Futagami *et al.* (1993) type of model when various fiscal instruments vary.

The problem of moral hazard of redistributive transfers and its implication for growth and fiscal policy are examined in Barro (1990) kind of model by Park and Philippopoulos (2003). They consider heterogeneous capital endowments across individuals to capture wealth inequality and consider a utility function defined over final good consumption and consumption of public services.

Hu, Ohdoi and Shimomura (referred to as HOS hereafter) (2008) extend the Barro (1990) one-sector model to a two-sector endogenous growth model with an investment good sector which is more capital intensive than the consumption good sector. They show the steady-state growth equilibrium to be unique and the transition path to be indeterminate. Thus they bring back transitional dynamic properties in Barro (1990) model without introducing durable public capital.

Burguet and Fernandez-Ruiz (1998) develop an open economy growth model with public capital in production and show the existence of multiple steady-state equilibria when a proportional output tax finances the public capital investment. If debt financing with borrowing from the international market is possible, then the economy may be able to move out from the low level equilibrium trap. Ghosh and Mourmouras (2002) also extend Barro (1990) model in the direction of a two-country world with capital being perfectly mobile between the two countries and with production in both the countries enjoying positive externalities. These externalities are obtained through spillover effects which originate from the average capital stock of domestic and foreign firms, and through average government consumption expenditure that provide direct utility to households.

Park (2009) investigates Ramsey optimal fiscal policy in an endogenous growth model with productive public expenditure and with labour-leisure choice. Baier and Glomm (hereafter referred to as BG) (2001) introduce public services as a flow to enhance the representative consumer's utility while endogenous growth in this model is driven by the accumulation of productive public capital.

Several authors have explored the effects of public good externality on utility function in the endogenous growth framework. Cazzavillan (1996) develops an endogenous growth model where public good creates positive externalities on production as well as on utility of the consumer. If the economies of scale, which arise from the complementarity between private consumption and public expenditure, are strong enough to generate increasing returns in the representative agent's utility function, then unique steady-state growth equilibrium exists and the transitional path to this equilibrium is locally indeterminate in this model. Chen (2006) extends Cazzavillan's (1996) model by considering public input in production as a stock variable. He shows the existence of unique balanced growth equilibrium and quantifies the parameter space of the consumption externality of public expenditure for indeterminate, unique and unstable transitional growth paths. Zhang (2000) also comes to similar conclusions when he explores the possibility of increasing returns in a Barro (1990)-type production function. Utility of the representative consumer is enhanced by consumption and public good and exhibits increasing returns in these two arguments. Chang (1999) explores a planner's optimization problem in an endogenous growth model when utility is enhanced by public expenditure and production of the final good requires public capital as one of the inputs. The steady-state growth equilibrium is shown to be saddle path stable. Changes take place in the steady-state equilibrium and in the transitional growth path due to changes in public consumption expenditure and in public investment.

1.3 PROBLEM OF CONGESTION EFFECT ON PUBLIC CAPITAL

1.3.1 Nature of the Problem

Public goods are not necessarily non-rival. In this case an agent cannot be prohibited from using the public good, although her use lowers its availability to others. Breakdown of the non-rival characteristic of public good gives rise to congestion effect where the per-capita availability of the public good varies inversely with the number of agents using it.

In the endogenous growth literature with productive public input, congestion effect arises from the accumulation of private physical capital. Public infrastructure acts as a complement to private capital input. Factories need roads, power and water to operate. So wherever such infrastructure is abundant in supply private investment takes place in these regions and in the process congests public capital.

1.3.2 Existing Dynamic Models with Congestion Effect

Raurich and Puigdevall (hereafter referred to as RP) (2000) develop a model of endogenous growth with congestion effect on productive public capital and with leisure in the utility function. The model shows the existence of multiple balanced growth paths and the possibility of local and global indeterminacy due to the relationship between public capital accumulation and labour-leisure choice of individuals.

Turnovsky (1997) extends the model of Futagami *et al*. (1993) introducing congestion effect on productive public capital and explores the design of fiscal policy when growth rate and welfare are maximized at the steady-state equilibrium. When welfare is maximized in the planned economy, the optimal public expenditure-output ratio falls short of the growth rate maximizing public expenditure-output ratio which is equal to the elasticity of output with respect to public capital. Fisher and Turnovsky (hereafter known as FT) (1998) work out a very similar model with the difference that both types of capital, public and private, are subject to depreciation and their analyses are qualitatively similar to that of Turnovsky (1997). Turnovsky (1996) also deals with congestion effect on productive public input.

Eicher and Turnovsky (2000), on the other hand, focus on the distinction between relative and aggregate congestion effects of public capital due to private capital accumulation; and explore their implications on fiscal policy in their model. The steady-state equilibrium growth rate is shown to be a function of the congestion parameters, both absolute and relative; and an increase in either type of congestion reduces this growth rate. The optimal public expenditure-income ratio is shown to be equal to the output elasticity of public capital in the socially efficient steady-state equilibrium; and the optimal income tax rate which replicates the socially efficient solution in the market economy is shown to be an increasing function of the congestion parameters. This tax rate also achieves the first-best optimum in the transitional phase too, unlike a time-varying optimal tax rate derived in Turnovsky (1997).

Ott and Turnovsky (hereafter known as OT) (2006) develop a Barro (1990) type model of endogenous growth with productive public inputs where these public inputs are categorized as excludable and non-excludable and where both public inputs are subject to congestion effect. Government not only imposes a proportional income tax but also collects a user fee on the usage of the excludable public good and exercises monopoly power over this pricing. In their first model, the government does not act as a monopolist providing the productive public input; and growth rate maximizing and welfare maximizing policies appear to be identical in the case of the central planner. Optimal income shares of expenditure on excludable and non-excludable public inputs are equal to their corresponding production elasticities. The optimal income tax rate is shown to be a function of congestion-adjusted production elasticities of the two types of public input and is higher than the optimal public expenditure-income ratio on the non-excludable public good. The optimal user fee also appears to be a function of congestion-adjusted output elasticities of the two public inputs and is lower than optimal public expenditure-income ratio on the excludable good. In the second model, when government is allowed to act as a monopolist with respect to the provision of excludable public input, the optimal income tax rate remains unaffected by monopoly pricing and thus coincides with the competitive case. However, the optimal user fee in this case is shown to be higher than that in the competitive case and vary positively with the degree of monopoly power.

Gomez (2008), however, develops an endogenous growth model with absolute as well as relative congestion of productive public capital and with Lucas (1988) type of human capital accumulation. Steady-state equilibria in the decentralized and in the centralized economy are shown to coincide and to satisfy saddle point stability; and various fiscal parameters do not affect the long run equilibrium growth rate in the market economy although the steady-state levels of the ratio variables are affected by changes in these policy parameters. This is so because technology of the education sector is linear in effective labour time. It is shown that an increase in absolute congestion reduces the steady-state equilibrium growth rate of output though a change in relative congestion has no effect on it. This result is different from ET (2000) where both types of congestion reduce the equilibrium growth rate. The socially optimum income tax rate varies positively with the value of the congestion parameter.

Bougheas, Demetriades and Mamuneas (hereafter known as BDM) (2000) develop an endogenous growth model in the lines of Romer (1987) with congestion affected public infrastructure whose role is to reduce cost of production of imperfectly substitutable intermediate inputs. They show that there exists a positive relationship between the degree of specialization and the size of public infrastructure while the average output of intermediate good bears an inverse relationship with the size of public capital. There also exists unique income tax rate that maximizes the balanced growth rate. Results of this model are empirically tested using US census data.

1.4 ROLE OF HEALTH CAPITAL

1.4.1 Empirical Works

There are models using Barro's (1990) theoretical framework which carry out various empirical studies emphasizing the role of health on economic growth. For example, Miyakoshi *et al.* (2010) develop a gradient method in order to arrive at the optimal adjustment of fiscal spending components so as to maximize growth rate, starting from the present shares of components. Public spending is composed of expenditures on health, education, security and other miscellaneous public services in their theoretical model. In a sample consisting of both developing and industrial countries, Bloom *et al.* (2004) find that good health (proxied by life expectancy) has a significantly positive impact on economic growth. Sala-i-Martin *et al.* (2004) also find similar evidence of positive relationship between health and economic growth. Several other authors examine this relationship using data from specific countries. For example, Jamison *et al.* (2005) use a sample of 53 countries over the period of 1965-1990 and show that improvements in health account for approximately 11% of growth. Gyimah-Brempong and Wilson (2004) show that 22-30% of the transition growth rate of per capita income in Sub-Saharan Africa can be attributed to health factors. Weil (2007) use microeconomic data to show that a significant part of growth in per capita income can be explained by health factors.

1.4.2 Dynamic Models with Public Expenditure and Health

Hosoya (2003) considers public expenditure on health input that helps accumulation of health capital through a flow channel while a physical capital deepening externality helps accumulate it through a stock channel. In this two-sector endogenous growth model, the stock channel is shown to be more significant than the flow channel for determining the long-run growth rate maximizing tax rate.

In an endogenous growth model with public infrastructure services, Agenor (2008) distinguishes between flow and stock approaches to health as an input in production. Health also affects utility of the consumer. In the first model, health is treated as a flow variable which is produced by a Cobb-Douglas technology that uses government expenditure on public infrastructure and on health as inputs. The growth rate maximizing income tax rate is the sum of the elasticities of health and public infrastructure input. However, this tax rate is less than the welfare maximizing income tax rate. Moreover, the welfare maximizing share of spending on infrastructure is lower than the growth rate maximizing share; and hence the welfare maximizing share of spending on health is higher than its growth rate maximizing share. The second model uses the same Cobb-Douglas production technology with infrastructure and health as inputs in the accumulation of health input which is treated as a stock variable. The welfare maximizing tax rate and the welfare maximizing share of spending on public infrastructure service are shown to vary inversely along the balanced growth path.

Agenor and Moreno-Dodson (hereafter called AM) (2006) develop an endogenous growth model with public infrastructure and health services where public infrastructure and government expenditure on health services are used as inputs in the production of health services. The government allocates its tax revenue into investment in public infrastructure and in health services; and income tax is the only source of tax revenue. Production of final good uses public infrastructure, health services and private physical capital as inputs. AM (2006) show the steady-state growth equilibrium in the planned economy to be unique and saddle-path stable. They also show the steady-state growth rate to vary positively with the efficiency of the investment expenditure in the public infrastructure production technology. A revenue-neutral shift in expenditure share from health to infrastructure is shown to have a positive effect on the long-run growth rate if public infrastructure is sufficiently productive in the health production technology. Growth rate maximizing public expenditure allocation rule states that the spending share on public infrastructure varies positively with the elasticity of output of health services with respect to infrastructure capital. So it may be more effective to increase expenditure on public infrastructure rather than to directly increase expenditure on health.

Agenor and Neanidis (hereafter called AN) (2011) develop a model of endogenous growth with productive public capital and health services similar to AM (2006) though there are minor points of differences between these two models. Instead of a single proportional income tax, there is a consumption tax as well in this model, while in AM (2006), there is only an income tax. AN (2011) assumes no tax collection costs in their first benchmark model and then introduces both exogenous and endogenous collection costs in their second model while AM (2006) do not introduce tax collection cost. AN (2011) find the growth rate maximizing consumption tax rate to be zero but the income tax rate to be equal to the competitive output share of the two public inputs taken together. However, welfare maximizing consumption tax is not necessarily zero. Different combinations of distortionary tax rates can be used to achieve the optimum; and the optimal solutions are shown to depend inversely on the share of productive spending.

The interplay between health and environmental pollution is analyzed in the two period overlapping generation model of Gutierrez (2008). The savings rate is found out to be an increasing function of the total stock of pollution where the stock of pollution in the current period is proportional to the level of total output. The dynamic competitive equilibrium is suboptimal due to the narrow time horizon of the short-lived agents even without externalities; and with negative pollution externalities, this problem is even more aggravated. Gutierrez (2008) shows that the optimal tax rate varies inversely with the natural decay rate of pollution, and varies directly with the pollution-output coefficient. When pollution is the only cause of inefficiency, both generations receive transfers. However, only the younger generation pays taxes to transfer resources to the older generation when pollution is not the only cause of inefficiency.

1.5 DEPRECIATION OF PUBLIC CAPITAL

In the endogenous growth model of Funke and Strulik (hereafter called FS) (2000), public capital enters as an input in aggregate production function and it depreciates over time. However, public capital depreciates exogenously at the same rate as private capital and FS (2000) do not consider any maintenance expenditure in their model.

Rioja (2003 b) first shows the cost of ineffective public infrastructure. He numerically solves a general equilibrium model using data from seven Latin American countries; and shows that, in the long run, penalty of ineffective infrastructure is about 40% of per capita gross domestic product. Raising effectiveness has positive effects on per capita income, private investment, consumption and welfare. Rioja (2003 a) introduces the problem of public capital depreciation and the role of maintenance expenditure in a FMS (1993) type of open economy growth model. In this model, domestic income tax revenues finance maintenance expenditure and foreign aid finances new public investment. The optimal income tax rate varies inversely with the international aid-public capital ratio. However, in the absence of foreign aid, the optimal tax rate varies directly with the elasticity of output with respect to public capital.

Kalaitzidakis and Kalyvitis (hereafter called KK) (2004) extend Rioja's (2003 a) model in various directions. Income tax revenue is allocated between public investment and maintenance expenditure and foreign aid is not considered. A positive external effect of private capital on production is considered in the form of learning-by-doing effect and an adjustment cost of private investment is introduced. Moreover, a profit maximizing solution is considered instead of a utility maximizing solution. The income tax rate is identical to the income share of combined expenditure on public investment and maintenance; and the tax rate that maximizes steady-state equilibrium growth rate in that model exceeds the competitive output share of public capital. However, the public investment-output ratio is less than this competitive share. They also show that the unique steady-state equilibrium point is saddle-path stable.

Dioikitopoulos and Kalyvitis (hereafter called DK) (2008) introduce a negative congestion effect of public capital and the problem of depreciation of public capital with the role of maintenance expenditure in a FMS (1993) type of model. However, they do not consider the learning-by-doing effect of private capital accumulation and consider a utility maximizing solution instead of a profit maximizing solution. The transitional dynamic results and the properties of growth rate maximizing fiscal policy in the steady-state equilibrium in DK (2008) are similar to those in KK (2004).

In Agenor (2009), the maintenance expenditure plays a dual role of increasing the durability as well as the efficiency of public capital. The growth rate maximizing income tax rate in the steady-state equilibrium is found to be identical to that of Barro (1990); and the steady-state equilibrium is proved to be a saddle point.

1.6 INFORMAL SECTOR

1.6.1 Definition and Features with Empirical Support

The unorganized sector of an economy which is generally not monitored by the government is known as the informal sector. Typically the informal sector consists of unregistered firms who do not pay taxes and therefore, are legally not entitled to avail facilities of public services. The emergence of the informal sector is the result of various policies which increase transaction costs and thus create barriers to entry for formal firms. Formal sector firms may use expensive but less polluting technology as a legal requirement while firms in the informal sector often use cheaper and polluting technologies.

Various empirical works study features of informal sector firms in various countries. De Soto (1989) studies the informal sector in Peru. Chickering and Salahdine (1991) in their book present evidence from selected underdeveloped Asian countries. Tokman (1992) provide evidence from Latin American and Caribbean countries. Nippon (1991) and Alonzo (1991) study the informal sector in Thailand; and Mazumdar's (1976) study on informal sector is based on evidences from Bombay[1]. Huq and Sultan (1991) report evidences from Bangladesh. These empirical studies point out various causes of the growth of informal sector; and these include high corporate income taxes and bureaucratic controls on formal sector firms, existence of labour unions and labour legislation laws in the formal labour market, etc.

Various studies point out that informal sector firms adopt low cost and pollution generating technologies and the benefits of environmental policies of the government are largely restricted to formal sector firms. These studies include the works of Biller and Quintero (1995), Blackman and Bannister (1998), Blackman (2000), Kolstad (2000), Chaudhuri and Mukhopadhyay (2006), Kathuria (2007), etc.

[1] It is an industrial city of India presently known as Mumbai.

1.6.2 Dynamic Models with Public Expenditure and Informal Sector

There are a few theoretical works developing two sector dynamic models incorporating both the formal sector and the informal sector; and the literature includes works of Blackman and Bannister (1998), Gibson (2005), Antunes and Cavalcanti (2007), Saracoglu (2008), Loayza (1996), Penalosa and Turnovsky (2005), Turnovsky and Basher (2009), etc. Only a handful of them analyze the role of productive public expenditure on economic growth. This small set includes the works of Sarte (2000), Loayza (1996), Penalosa and Turnovsky (2005) and Turnovsky and Basher (2009); and the discussion is restricted to introduce only these four models because the present thesis also analyses the role of productive public expenditure on economic growth.

Sarte (2000) develops a small open economy model where final good production uses a range of intermediate goods and labour as inputs and each intermediate goods industry comprises of a number of formal and informal sector firms. The technologies which help use intermediate inputs in the final good production are learnt sequentially from abroad. Thus endogenous growth stem from domestic investments to adopt newer technologies. The intermediate goods industry is monopolistically competitive. The informal sector firms in each intermediate goods industry incurs a fixed cost of operating in that sector; it is the cost of non-availability of legal protection against theft or non-compliance of contracts. Similarly, the formal sector also incurs a fixed cost that is a tax paid to the government for the provision of public services like legal protection. The provision of this public service is subject to congestion. The steady-state equilibrium growth rate is derived as a function of the fixed cost of informal firms. It is shown that if the fixed cost to the informal firms is above a critical level then free entry in the formal sector rules out existence of informal firms and the growth rate is that of formal sector output only. If, on the other hand, the cost of informally operating falls below the threshold level then an informal sector comes to operate which raises the return of acquiring new technology. An extension of this model considers rent-seeking behavior of bureaucracy who can control entry in the formal sector. It is shown that the size of the informal sector is relatively larger in this case and the growth rate may also be lower than the previous case of free entry. Welfare is increasing in growth rate; therefore, in the case of unrestricted entry to the formal sector welfare is higher than that in the case of restricted-free entry.

Loayza (1996) develops a two-sector model with a formal sector and an informal sector to explore the implications of optimal fiscal policy on economic growth when taxes from the formal sector finances productive public services used by both the sectors. Formal sector pays proportional income tax which is used to finance all of public services and to partially finance the enforcement system. On the other hand, informal sector pays a penalty in order to operate illegally and also partially finances the enforcement system for the formal sector. Public services are fully funded by a fraction of the tax revenues from the formal sector; and this fraction varies directly with the quality of government institutions and inversely with the strength of enforcement. The penalty rate is an increasing function of the strength of enforcement and of the relative size of the informal sector. In the competitive equilibrium, the relative size of the informal sector is found to vary positively with the tax rate imposed on formal sector output. The steady-state equilibrium growth rate is shown to be decreasing in relative size of the informal sector. The optimal tax rate is shown to be lower than that in Barro and Sala-i-Martin (1992) who consider only the formal sector with public good congestion.

Penalosa and Turnovsky (hereafter referred to as PT) (2005) examine the implications of fiscal policy on the development of the informal sector when income only from the formal sector can be taxed. Production in the formal sector technology is more capital intensive than the technology used by the informal sector. Production is linear in average capital in both sectors. Moreover, formal sector needs public infrastructure to operate which is financed by the government by taxing income of the formal sector. However, this infrastructure does not affect productivity. If the government has no redistributive goals, then the socially efficient growth rate cannot be achieved with tax revenues only from the formal sector. In that case, for efficient sectoral allocation capital and wage income should be taxed equally at a rate equal to the infrastructure requirement rate; and in such a case the growth rate in the decentralized economy is less than the socially optimum growth rate. If the government has only a growth rate maximizing objective to fulfill, then capital and wage income should be taxed equally irrespective of how public expenditure is used. Otherwise, when welfare is to be maximized, then equal taxation is again optimal if public expenditure is used to create infrastructure. However, if redistribution is the goal, then labour income should be taxed at a rate less than the rate of taxation on capital income as long as the formal sector is more capital intensive.

Turnovsky and Basher (2009) also develop a growth model where informal sector uses more labour intensive technology than formal sector. Both sectors have requirements for public infrastructure, synonymous to fixed costs, and the rate of requirement is relatively more for the formal sector. Government can only audit a fraction of the informal sector and thus is able to impose a labour tax on the audited fraction only. Labour income and capital income in the formal sector are both taxed along with a lump sum tax collected from the representative consumer. The tax revenue and budget deficit go on to finance the public infrastructure requirement of the two sectors in the economy. The steady-state dynamic equilibrium is shown to satisfy saddle-path stability. The focus of the analysis is to examine whether existence of an informal sector hinders the government's revenue generating capacity in a developing country. It is shown that more auditing of the informal sector negatively affects the ability of tax policy to influence the size of that sector, but positively affects its impact on tax collection. On the other hand, higher tax rates enhance the ability of auditing to influence the size of the informal sector as well as its effectiveness to generate higher tax revenues.

1.6.3 Models with Informal Sector and Environmental Pollution

Informal sector is distinguished from formal sector by its pollution generating technology in Cassou and Hamilton (hereafter known as CH) (2004) model of endogenous growth. Both sectors use private capital, human capital adjusted effective labour and environmental quality as inputs. The formal sector produces a clean good but the informal sector produces a dirty good. Physical capital used in the informal sector is the source of environmental pollution whereas formal sector production technology uses physical capital that does not pollute. In each of these two sectors, production is augmented by accumulation of human capital that occurs through cumulative private investments in physical capital. Utility is enhanced by consumption and by the quality of environment and is reduced by work effort. CH (2004) show that growth rate in both sectors depends on the level of dirty capital. Also, when environmental externality on production is identical and fiscal policy does not discriminate between capital types then the output growth rate in the dirty sector exceeds that in the clean sector. The policy setting is shown to produce the Environmental Kuznet's Curve when dirty sector output is bounded. The clean sector grows endogenously and the growth in the dirty sector brings down growth in the clean sector.

1.7 HUMAN CAPITAL

1.7.1 Survey of Dynamic Models on Public Expenditure and Human Capital

Glomm and Ravikumar (2001) develop a simple overlapping-generations model of human capital accumulation. Human capital accumulation of an agent depends on human capital of the corresponding parent, quality of schooling and labour input. Income of each individual is assumed to be a linear function of his human capital. This income is proportionally taxed by the government and this tax revenue determines the quality of public schools. The existence and uniqueness of competitive equilibrium is proved with appropriate restrictions on preference parameters and parameters of the learning technology.

Chen and Lee (referred to as CL hereafter) (2007) develop a two sector model of endogenous growth with congestible public good where congestion effect comes from aggregate human capital in the economy to be used as input only in the final good production sector. A positive relationship is derived between the fraction of human capital and the fraction of physical capital employed in the final goods sector. CL (2007) proves the existence of unique balanced growth equilibrium and shows that the transition path to this equilibrium may be locally indeterminate.

Agenor (2008) develops an endogenous growth model with public infrastructure spending, public education expenditure and utility enhancing government services to examine the right composition of fiscal policy to finance all the above expenditures. He considers separable as well as non-separable utility functions and assumes the stock of educated labour force accumulation to be linear in the quality of education, which, in turn, is a concave function of the ratio of public expenditure on education to the educated labour force employed in the education sector. The rate of growth of total population is assumed to be equal to the rate of growth of the stock of educated labour in the steady-state equilibrium. The steady-state equilibrium growth rate maximizing income tax rate is equivalent to the sum of output elasticities of public infrastructure services and education input. The growth rate maximizing share of public expenditure on utility enhancing services is seen to be zero. With non-separable utility function, the steady-state growth rate maximizing tax rate is same as that obtained in the previous case. However, in the planned economy, the welfare maximizing tax rate and spending shares are not independent of each other and are determined simultaneously. With congestion of public educational infrastructure to be determined by the number of students, the growth rate maximizing share of public spending on infrastructure is higher than that without congestion effect.

Agenor (2011) develops a similar endogenous growth model with human capital and public infrastructure as inputs but does not consider utility enhancing public services. The growth rate maximizing tax rate is derived to be equal to the competitive output share of public infrastructure and human capital taken together. Also the growth rate maximizing shares of public expenditure on infrastructure and education depend not only on the output elasticities of public infrastructure and human capital but also on the productivity parameters of inputs in human capital formation. Agenor (2011) uses numerical techniques to examine the transitional as well as long-run effects of a budget-neutral shift in government spending from education to infrastructure for different values of parameters characterizing human capital accumulation technology. Under a plausible calibration for a low-income country, it is shown that reallocating funds from education to infrastructure may increase the growth rate even if public infrastructure only has a moderate effect on the production of human capital.

Cassou and Lansing (hereafter known as CS) (2006) analyze effects of tax reform in an endogenous growth model with human capital and with two types of public expenditures. The infinitely lived representative consumer derives utility from a public consumption good and suffers disutility from quality adjusted non-leisure activities. Aggregate human capital accumulation depends on its own stock in the previous period, private investment in human capital, government investment in human capital and time devoted to acquiring human capital. Government can finance expenditure on public consumption good and on public education by imposing either a pure income tax or a pure consumption tax or a hybrid between these two policy instruments. CS (2006) analyze the efficiency of these instruments in their model. CS (2006) show that the transitional path to the balanced growth equilibrium is unique. Then they analyze the optimal fiscal policy to determine the efficient size of the government, the efficient type of fiscal instrument and the optimal ratio of public to private expenditure on human capital.

1.8 ENVIRONMENTAL POLLUTION AND ECONOMIC GROWTH

1.8.1 Sources and Economic Effects of Pollution

That the production activity in an economy is a major cause of pollution is a well known and widely accepted fact. Running of factories leads to burning of fuel and the processing of raw materials leads to waste generation; and these, in turn, pollute the environment directly or indirectly. So majority of theoretical models available in the literature on environment treat production as the source of pollution. However, some models treat physical capital usage as the source of pollution. Burning of fuel is required mainly to run machineries. Intermediate goods can also be the source of pollution. For instance, the heating and melting of tar which is used to lay modern roads emits polluting fumes in the air. The level of emission also depends on the degree of cleanliness of production technology. For example, a leather industry may use chemicals which release fewer harmful pollutants to the water used to wash leather. Few theoretical models treat the level of consumption to be the source of pollution. For example, pollution takes place only when the services of the automobile are consumed by buyers.

Development of production activities with protection to the environment means sustainable development. Environmental pollution is a negative externality generating social cost and thus wasting the benefits of production in the long run. These social costs operate through various channels. Pollution can cause substantial damage to public infrastructure. For example, roads and bridges can be corroded due to harmful chemicals released in air and water; and this lowers longevity of such infrastructure. Degradation of environmental quality has health costs also. Air pollution is proven to increase cases of asthma, lung infections, skin diseases and cancer. Water pollution causes cholera, dysentery, ailment of the alimentary tract, etc. Usage of plastics, pesticides, fertilizers is documented to have widespread health risks. Thus all these health costs deteriorate the quality of human capital in an economy; and this, in turn, adversely affects efficient use of other productive factors.

1.8.2 Dynamic Models with Pollution

In Hartman and Kwon (hereafter referred to as HK) (2005), human capital accumulation is considered to be pollution free while physical capital is used to reduce pollution generated from final goods production. The representative agent allocates labour time between production and human capital accumulation. A reduction in the use of capital in production directly lowers the level of pollution through reduction in output and indirectly does so increasing the use of capital in abatement activities. Utility is a positive function of consumption and a negative function of pollution. In the long-run steady-state growth equilibrium, output, physical capital and consumption grow at the same rate and human capital grows faster than physical capital. Pollution may grow or decline in the long run depending upon the elasticity of intertemporal elasticity of marginal utility. The optimal allocation can be implemented in the competitive economy with a pollution tax imposed on the firm and this optimal tax rate is an increasing function of the pollution rate. HK (2005) also show that their model can consistently explain environmental Kuznets curve for realistic values of parameters.

There are few overlapping-generations models introducing environment as a public good in the utility function of the representative consumer; and the small literature consists of the works of Ono and Maeda (2002), Ono (2003) and Ono (2007). Ono and Maeda (2002) analyze the role of maintenance expenditure on investment but abstain from exploring its growth effects. In

Ono (2003), environmental quality accumulates over time depending upon its existing stock and maintenance expenditure of the consumer and depletes due to emissions caused by production. Government imposes taxes on emission to finance lump sum transfer to the elderly. The balanced growth rate maximizing pollution tax rate depends positively on the pollution parameter and negatively on the efficiency of maintenance expenditure. Ono (2007) develops a similar model where emission is also used as an input in production. It is shown that the competitive equilibrium allocation of emission input is time-independent and varies inversely with the pollution tax rate. However, none of these models analyze the role of productive public expenditure on economic growth.

In the endogenous growth model of Mohtadi (1996), environmental pollution, generated from capital stock used in production, negatively affects utility of the representative agent in the absence of abatement activities. He first shows that, when the elasticity of environmental degradation is high (low), the market economy growth rate falls short of (exceeds) the socially efficient growth rate. Also, when the rate of environmental degradation is low (high), maximization of the steady-state equilibrium growth rate justifies an output subsidy (tax) which is financed by a lump-sum tax (subsidy) on consumption. The saddle-path stability of the steady-state growth equilibrium is proved and the socially efficient income tax (subsidy) rate is found to be proportional to the elasticity of environmental degradation with respect to capital. In an extension to his first model, Mohtadi (1996) shows how capital and consumption grow at the same rate but not a constant one if environmental quality affects the productivity of capital in the production process. The socially efficient growth rate is even smaller than that in the previous case and thus the optimal subsidy (tax) rate prescribed is also smaller (greater).

Bovenberg and Smulders (hereafter referred to as BS) (1995) develop an endogenous growth model where environmental quality that affects utility deteriorates due to pollution generated as a by-product of production and is improved by its own natural regeneration process. In BS (1995), allocation of physical capital and pollution-generating inputs are considered between the production sector and the pollution-augmenting knowledge capital sector. In the steady-state equilibrium, physical capital, knowledge capital, output, consumption and the relative price of natural capital grow at the same rate while aggregate pollution and natural capital remain constant. They also show how balanced growth can be optimal if there are unitary elasticities of substitution between environmental quality and consumption in the utility function and between environmental quality and the other factors of production in the production function. The optimal pollution tax revenue used to finance research subsidies should grow at the rate equal to the rate of growth of knowledge capital.

Gradus and Smulders (hereafter referred to as GS) (1993) analyse two endogenous growth models which incorporate pollution in the utility function. The first model is an extension of Rebelo (1991) in which pollution is generated from physical capital use and the endogenous growth rate varies inversely with the increase in abatement expenditure. In their second model, GS (1993) follows Lucas (1988). Here the optimal growth rate remains unaffected by an increase in abatement activity when pollution does not influence agents' ability to learn. However, the optimal growth rate varies positively with abatement activity when pollution produces a negative effect on the ability to learn. In another model, Smulders and Gradus (hereafter referred to as SG) (1996) examines appropriate environmental policy and the institutional conditions where sustainable growth and preservation of environment are compatible and optimal. They consider pollution as an input in production and capital usage as the source of pollution. Utility is also adversely affected by pollution which can be countered by undertaking abatement activity. SG (1996) characterize appropriate forms of production function, utility function and environmental accumulation function so that a socially-efficient steady-state balanced growth equilibrium may be attained.

Ayong Le Kama (2001) follows previous authors closely to present an endogenous growth model with an environmental resource that affects utility and also enters as an input in the production function. Environment is self regenerative but is depleted by pollution originating from production. The existence of socially optimum steady-state equilibrium is shown along with its saddle-point stability property.

In Ligthart and van der Ploeg (hereafter known as LP) (1994), the consumer derives utility from public consumption expenditure and disutility from pollution when pollution is a by-product of production. If there is no productive public expenditure, then a greater concern for welfare raises optimal tax rate but lowers the long-run growth rate. If productive public expenditure is considered and if preferences are biased towards environmental quality then a reallocation of tax revenue takes place from productive public expenditure to public consumption expenditure and to abatement expenditure; and this lowers the long-run growth rate. In this case, they find improvement in environmental quality as well as in welfare. Withagen (1995), however, uses a pollution augmented Rebelo (1991) model where pollution generated from production causes disutility. He shows that growth may not be balanced in the long run and the negative externality of pollution on utility may affect the long-run growth rate.

Byrne (1997) develops a model of endogenous growth with pollution affecting the utility function of the consumer. However, he assumes technological progress to be a clean activity and pollution to be a stock variable that accumulates with labour and capital used in the final goods production and is reduced by an abatement process governed by a Cobb-Douglas technology. In the steady-state growth equilibrium, consumption, output and technology grow at the same rate but the stock of pollution grows at a different constant rate in the market economy. In the planned economy, the pollution growth rate is lower than that in the market economy while the sustainable growth rate exceeds the same in the decentralized economy when abatement activities are undertaken.

Oueslati (2002) develops a model with human capital driven endogenous growth where private physical capital is the source of pollution. Pollution that affects consumer's utility negatively, varies inversely with abatement activities undertaken by firms. At the steady-state growth equilibrium of the market economy, pollution stays time-independent while all other macro variables grow at the same rate. The optimal pollution tax revenue grows with the capital stock on the balanced growth path. Oueslati (2002) analyzes welfare costs when the economy transits from one steady state to another due to a change in pollution tax rate. In the transitional phase, an increase in the pollution tax rate raises the long-run growth rate. However, the level of welfare has a standard U-shaped relationship with this tax rate at the steady-state growth equilibrium and has an increasing relationship with it in the transitional phase.

In Pautrel (2006), economic growth is driven by human capital when it is affected by health in an overlapping-generations model with pollution reducing utility. Pollution affects health of agents which reduces their probability of survival; and thus reduces the aggregate human capital in the economy. Quality of health varies negatively with pollution and positively with public health expenditure. Pollution reduces optimal growth rate through the health channel. However, greener preferences are both health improving and welfare improving.

Other authors like Elbasha and Roe (1996) explore the welfare effects of environmental quality in a small open economy endogenous growth model when utility is enhanced by consumption of goods from two production sectors and by environmental quality. Jones and Manuelli (2001) assume pollution to cause disutility and to vary positively with capital used by different technologies in production; and show that the relationship between pollution and growth is not always monotonous. Liddle (2001) develops a simulation model to explore trade and environment in the context of development. He considers both production and consumption as sources of pollution and abatement investment is undertaken here to counter this pollution. Natural resource is an intermediate input which is traded. The benefit of trade can be either positive or negative, and it depends on the country-specific endowments. It is shown that pollution level is higher under free trade than under autarky.

However, results of this model do not support the pollution haven hypothesis which states that trade causes less pollution in developed countries and more pollution in developing ones. The model of Hart (2004) also considers final goods production as a source of pollution while damages to the environment caused by such emission adversely affect a consumer's utility. He illustrates the implications of pollution on the quality and quantity of research when technological change is of environment-friendly type. On the other hand, Itaya (2008) assumes capital stock and output as two alternative sources of pollution in two separate models and assumes pollution to be a public bad in the utility function. Production has learning-by-doing externality effect. The government imposes a pollution tax and transfers the revenue as a lump sum amount to the representative household. In the case of capital as the source of pollution, the transition path to the steady-state growth equilibrium is shown to be indeterminate (determinate) if the elasticity of substitution is less than unity (greater than unity); and the pollution tax varies positively (negatively) with the steady-state growth rate as well as with steady-state equilibrium employment level. In the case of pollution generated from output, an increase in the pollution tax rate may have a positive impact on the growth rate when the elasticity of substitution is less than unity and the steady-state growth equilibrium path is determinate. Itaya (2008) also considers the role of public abatement activities, financed by pollution tax, on the pollution generating function when capital use is the source of pollution. The transition path to the steady-state equilibrium is characterized; and the relationship between the balanced growth rate and the pollution tax rate is summarized under different assumptions of parameters of the utility function.

Di Vita (2008) establishes an inverse U shaped relationship between income and pollution in an endogenous growth framework by establishing a link between pollution dynamics and interest rate. Pollution is generated by production and is eliminated by abatement activities in which labour is used as the only input. Utility is assumed to be a positive function of consumption and a negative function of pollution stock. The model proves the existence of a saddle-point stable socially efficient steady-state equilibrium where consumption and capital are constant.

Bovenberg and Moorji (hereafter known as BM) (1998) deal with environmental tax reform in a static small open economy where they consider a polluting input in the production function. Public good consumption, environmental quality and final good consumption are assumed to raise utility while pollution reduces it.

Models like that of Jouvet, Michel and Pestieau (2000) and Inoue (1998) consider production to be the source of environmental pollution which is treated as a public bad by consumers. Inoue (1998) develops a two region model where environment is jointly polluted by two regions by their respective productive activities while at the same time use environmental quality as a common input in their production functions. Abatement is undertaken by the advanced region and the backward region receives technological aid for abatement from the former with a time lag. Inoue (1998) uses simulation results to emphasize the welfare effects of abatement policy using optimal control method. However, his analytical model fails to show the existence of steady-state balanced growth equilibrium.

Tahvonen and Kuuluvainen (referred to as TK hereafter) (1991) develop an endogenous growth model where pollution affects economic activities both as a stock and as a flow and where utility is reduced by stock pollution. In another similar model, TK (1993) first solves the planner's problem when production uses only capital and emission. Utility function and the evolution of the pollution stock are assumed to be similar to those in TK (1991). The existence and uniqueness of saddle-point stable steady-state equilibrium is proved where capital and pollution stock have zero growth rate. It is shown that, when pollution is not optimally controlled, then the steady-state levels of consumption and capital are higher than those when pollution is optimally controlled.

Butter and Hofkes (1995) concentrate on both stock and flow uses of environmental quality in production by considering its extractive and non-extractive uses. Environmental quality is also introduced as an argument in the utility function. They show that environmental quality remains time-independent on the steady-state balanced growth path and hence sustainable growth becomes feasible.

Chimeli (2003) explores the feasibility of attaining social optimum with pollution and consumption taxes in a dynamic model where environmental quality is degenerated by production of final output and is protected by its natural regeneration and by government's abatement expenditure. However, Chimeli (2003) does not study the role of environmental quality or of pollution in the utility function nor in the production function.

John and Pecchenino (1994) develop an endogenous growth model using overlapping-generations framework to illustrate the potential conflict between economic growth and the maintenance of the environmental quality when consumption degrades environmental quality. However, there is no productive public expenditure in this model.

Bertinelli, Strobl and Zou (2008) use a capital vintage model to show how environmental pollution decreases with the usage of capital of newer vintage in the production function. In Benarroch and Weder (2006) the usage of intermediate goods generates pollution.

Kempf and Rossignol (2007) use the median voter theorem to show how income inequality is harmful for the environment in a model of endogenous growth with productive public input and government expenditure on environmental protection. Given the abatement rate, the share of expenditure on public input used to maximize welfare also maximizes the growth rate. The choice of welfare maximizing abatement rate is then shown to be determined by majority voting according to the median voter theory. This abatement rate is shown to be an increasing function of the endowment parameter which implies that poorer individuals spend less on environmental protection and more on productive activities. This pollutes the environment.

1.9 GROWTH MODELS WITH ENVIRONMENTAL POLLUTION AND PRODUCTIVE PUBLIC EXPENDITURE

1.9.1 A Brief Survey of Existing Models

Only a few models on endogenous growth deal with the interaction between productive public expenditure and environmental pollution. Greiner (2005) develops a Futagami *et al.* (1993) type of model where he considers public expenditure as a stock variable and environmental pollution as a flow variable. Economides and Philippopoulos (hereafter called EP) (2008) extend the Barro (1990) model in this direction but treat environmental quality as a stock variable. In Greiner (2005) as well as in EP (2008), the level of production is the only source of pollution. However, both Greiner (2005) and EP (2008) introduce a negative external effect of environmental pollution only on the utility function of the representative household but not on the productivity of the inputs. In Greiner (2005), rate of abatement expenditure is treated as exogenous; and the properties of optimal income tax policy and optimal pollution tax policy are analyzed. The optimum share of investment to national income is identical to the optimum income tax rate because a separate pollution tax is introduced to finance abatement expenditure. The optimum share of investment to income is equal to the competitive output share of the public input and hence is independent of the rate of emission because the level of pollution affects only the utility function and not the production function. The optimal income tax rate in the case of welfare maximization in steady-state equilibrium is identical to the growth rate maximizing tax rate. However, this result is not true for the pollution tax because pollution directly affects the utility of the household in his model. Greiner (2005) also shows the steady-state equilibrium to be saddle-path stable.

Economides and Philippopoulos (2008) studies Ramsey optimal second-best fiscal policy in an endogenous growth model using the technique of an open-loop Stackleberg differential game in which government plays the role of a leader. Environmental quality being a stock variable is degraded through pollution and is improved through government's abatement activities. The government finances the infrastructural expenditure as well as the abatement expenditure with its tax revenue. The existence of unique steady-state equilibrium growth rate is proved and that equilibrium is shown to be socially efficient. The growth rate in the planned economy is higher than that obtained in the Ramsey second-best solution.

1.9.1 Existing Research Gap

Productive public expenditure is an important externality that can fuel economic growth. Sustainability of this growth process makes environment a very crucial source of externality, especially when the productivity of public expenditure is affected by the quality of the environment. Then maintenance of the environment in order to increase the durability and efficiency of other public goods should feature in a major way in government's tax policy if the goal is to boost economic growth. Thus, the tax structure to be imposed and the allocation of expenditure to be made to different budgetary heads must be designed to fulfill this objective.

In the context of endogenous growth models with productive public expenditure, there is no work exploring the policy implications when environmental pollution affects the productivity of an economy. Several models consider pollution to affect utility of the consumer. Although Greiner (2005) and EP (2008) deal with environmental pollution and public expenditure, no negative effect of pollution on productivity of public expenditure is considered in their models.

There are plenty of instances where degradation of environmental quality through various channels reduces the effective benefit of public investment expenditure. For example, deforestation reduces rainfall and thus lowers the efficiency of public irrigation programme by reducing canals' water supply and depleting the groundwater level. Poor quality of natural resources (coal) and the lack of current in the water flow of streams and rivers negatively affect the generation of electricity. Global warming leads to natural disasters like floods, earthquakes, cyclones, etc.; and these, in turn, cause severe damage to infrastructural capital like roads, electric lines, power plants, buildings, industrial plants, etc. Private capital goods like plants and equipments are also damaged by natural disasters. Water pollution and air pollution create a disease-friendly environment; and hence government's expenditure programme on public health cannot provide adequate security to the health capital of workers. When such a circular interaction exists, thus, designing fiscal instruments to maximize growth and welfare is a challenging exercise that has not been attempted before.

1.10 A SUMMARY OF THE PRESENT THESIS

1.10.1 The Basic Model

The chapter 2 is devoted to developing a model of endogenous growth with special consideration to the interaction between productive public input and environmental pollution in the presence of congestion effect on public input. Productive public expenditure is assumed to be a flow variable similar to that in Barro (1990). However, environmental quality is assumed to be a stock variable. Production of the final good uses private inputs and a productive public input financed by government's tax revenue. However, the productivity of this public input is positively affected by the quality of environment; and the average stock of private capital causes a negative congestion effect on this public input. Thus, production is subject to externalities. Environmental quality is degraded due to emission resulting from production. However, it improves if the government spends on abatement activities. The government allocates its income tax revenue between pollution abatement expenditure and productive public expenditure. The representative household maximizes the discounted present value of her instantaneous utility over infinite time horizon subject to the intertemporal budget constraint; and in section 2.2 the instantaneous utility is assumed to be a positive and concave function of the level of consumption of the final good only. Section 2.3 considers environmental quality as an additional argument in the utility function in an otherwise similar model of section 2.2 of chapter 2.

Greiner (2005) studies a similar model of endogenous growth but treats public input as a stock variable and environmental quality as a flow variable. Environmental pollution in this model is proportional to the level of output of the final good like that of ours. However, pollution affects the utility function and not the productivity of public expenditure in Greiner's (2005) model unlike that of ours. The government has two sources of revenue in the form of an income tax and a pollution tax which are used to finance the public input and the abatement activities respectively. On the other hand, in EP (2008) environmental quality is a stock variable and is renewable if the government spends on abatement activities. Otherwise it is similar to Greiner (2005).

We derive some interesting results analyzing the basic model in section 2.2 of this chapter. First, in the steady-state equilibrium, the optimum (growth rate maximizing) ratio of productive public expenditure - depicted as total tax revenue minus abatement expenditure - to national income is less than the competitive output share of the public input; and this ratio varies inversely with the magnitude of the emission-output coefficient. This is different from the corresponding result obtained in Barro (1990), Futagami *et al.* (1993), Greiner (2005), etc. The optimum ratio of productive public expenditure to national income in all these models is equal to the competitive output share of the public input. This is so because there is no environmental pollution in Barro (1990) model or in Futagami *et al.* (1993) model. If the emission-output coefficient is zero, the Barro-FMS result comes back to the present model. In Greiner (2005), there exists an alternative instrument for financing the abatement activities in the form of a pollution tax; and this makes the income tax rate identical to this optimum ratio and hence equivalent to the competitive share of the public input. Secondly, the market economy growth rate is not necessarily less than the socially efficient growth rate in the steady-state equilibrium due to the presence of conflicting externalities. Thirdly, transitional dynamic property comes back to this model even though it considers a flow public expenditure. Environmental quality, being a stock variable, protects the model from being trapped into an AK model.

In the model in section 2.3, we find that the steady-state growth equilibrium, if it exists, is either unique or multiple with two equilibria. In this model if the equilibrium is unique, then properties of optimum fiscal policy are similar to those obtained from the earlier model. Secondly, this model too shows transitional dynamic properties. But, the steady-state equilibrium is not necessarily a saddle-point in this model. In the case of multiple equilibria, we may have indeterminacy of the transitional growth path converging to one of the two equilibria. Fourthly, the planned economy steady-state equilibrium growth rate is not necessarily less than that of the market economy.

1.10.2 Extension of the Basic Model

1.10.2.1 Alternative Sources of Pollution

In chapter 3, the basic model is extended by introducing two alternative sources of pollution in two separate models. In one of these two models, we consider consumption as the source of pollution. In the other model, private capital usage is taken as the source of pollution. In both of these two extensions, the nature of optimal fiscal policy differs from that obtained in the basic model. The optimum ratio of the productive public expenditure to national income, in both the models, is equal to the competitive output share of the public input in the steady-state equilibrium and is independent of the rate of emission. Thus our result is similar to that obtained in Barro (1990) and in

Futagami *et al.* (1993). However, the income tax rate, the abatement expenditure rate and the growth rate in the steady-state equilibrium are determined simultaneously in these extended models unlike the basic model. Other results related to the transitional dynamic properties and to the social efficiency property of the steady-state equilibrium hold through.

1.10.2.2 Role of Health Expenditure

In chapter 4, we introduce health infrastructural capital as an input in the production function in addition to private capital and productive public input. Health capital, like environmental quality, also acts as an externality to the producer. Like environmental quality, health capital deteriorates due to pollution; and the government spends resources to augment this health capital. Thus tax revenue of the government is now allocated to three expenditure heads - public infrastructural expenditure, expenditure on health and pollution abatement expenditure. Agenor (2008) focuses on the allocation of government budget between health expenditure and public infrastructural expenditure but does not deal with the problem of environmental pollution. Following the basic model of chapter 2, we assume level of production as the only source of pollution in this extended model.

This addition of the productive role of health capital to the basic model generates some new results in the steady-state equilibrium. The optimal ratio of combined expenditure on public infrastructure and health capital to national income appears to be less than the competitive output share of infrastructure and health taken together; and it varies inversely with the pollution-output coefficient. This result is different from that obtained in the basic model and in the extended model of chapter 3. In the present model, the share of expenditure on health capital in national income is the fraction that accounts for the pollution induced depreciation of health capital plus the competitive share of health capital in total output adjusted for environmental pollution and health capital depreciation. Secondly, the steady-state equilibrium is never saddle-point stable in this extended model. There is a possibility of indeterminacy of the transitional growth path converging to the unique steady-state equilibrium. This result is different from the corresponding one obtained in the basic model.

1.10.2.3 Endogenous Depreciation of Public Capital

In all the previous models, productive public input is considered as a flow variable. In another extension to the basic model done in chapter 5, we assume public input to be a stock variable like that in Futagami *et al.* (1993). Public capital here is subject to depreciation; and this depreciation is endogenous. The depreciation of public capital can be slowed down by increasing expenditure on its maintenance and by an improvement in the environmental quality while it accelerates due to negative congestion effect of private capital. Here the government's problem is to allocate total expenditure among three heads - investment to augment public capital, maintenance expenditure and abatement expenditure. However, we do not consider health expenditure here and assume level of production to be the only source of pollution. A few endogenous growth models like Rioja (2003 a), Kalaitzidakis and Kalyvitis (2004), Dioikitopoulos and Kalyvitis (2008) also deal with the problem of endogenous depreciation of public capital and analyze the properties of optimal maintenance expenditure. However, they do not analyze the role of environmental pollution on the depreciation of public capital.

We obtain following results by analyzing this extended model. The optimum combined share of public investment expenditure and maintenance expenditure in national income can be less (greater) than the competitive output share of the public capital in the steady-state equilibrium if the emission-output coefficient is greater (less) than the optimum share of the maintenance expenditure. Secondly, the steady-state equilibrium is either unstable or the transitional path converging to the steady-state equilibrium is indeterminate. Lastly, we compare the planned economy solution to the market economy solution and find that the steady-state equilibrium growth rate in the former does not necessarily exceed the latter.

1.10.2.4 Formal and informal sector

All of the previous contributions are based on a one sector aggregative framework. In a two sector model, one of the two sectors that bears the full burden of taxation is called formal and the untaxed sector is called informal. In less developed countries like India, Bangladesh, etc., agriculture and urban unorganized small-scale sectors do not bear the burden of income taxation. There are many theoretical works developing dynamic models with informal sector; and the literature includes Emran and Stiglitz (2005), Enste and Schneider (2000), Turnovsky and Basher (2009), Gerxhani (2004), Saracoğlu (2008), Antunes and Cavalcanti (2007), Dessy and Pallage (2003), Sarté (2000), Amaral and Quintin (2006), Azuma and Grossman (2002), Rauch (1991), Auriol and Warlters (2005), etc. However, these models do not deal with the problem of environmental pollution. In the present extension developed in chapter 6, we develop a two sector endogenous growth model of an economy consisting of both formal sector and informal sector; and analyze the role of public infrastructural expenditure and environmental pollution. In the current extension to the basic model, the representative household allocates capital between the formal sector and the informal sector. There is no such allocation problem that the representative household faces in any model analyzed in the earlier chapters. Pollution is generated by production of both the sectors but the emission-output coefficients in the formal and informal sectors are different. Environmental quality and productive public expenditure are inputs in the production of both sectors. However, there is no congestion of public productive input. Government finances the abatement expenditure as well as public infrastructural expenditure from its tax revenue obtained from income of the formal sector. However, the informal sector derives the benefits of public expenditure without paying any tax.

Following results are derived from this model. First, we prove the existence of unique steady-state equilibrium growth path in the market economy with simultaneous existence of the formal and the informal sector. Secondly, the optimum income tax rate is dependent upon the emission-output coefficient of the formal sector only; and this result is independent of whether two sectors have identical production technologies or not. The congestion effect parameter does not enter into the expression for the optimum income tax rate unlike in Barro and Sala-i-Martin (1992) model with no informal sector. Thirdly, the optimum abatement expenditure rate as well as the optimum ratio of productive public expenditure to formal sector's output depends not only on the emission-output coefficient of the formal sector but also on that of the informal sector. Lastly, the relative size of the informal sector in the steady-state growth equilibrium of the competitive economy exceeds its socially efficient size when this sector pollutes the environment.

1.10.2.5 Human capital and pollution

In chapter 7, the basic model is extended through two separate models to analyze the role of human capital as an input in production. In the first model, human capital stock is jointly financed by public expenditure and private educational expenditure. Tax revenue is thus allocated to investment in human capital, expenditure on public input and abatement activity. Role of health expenditure and maintenance expenditure on public capital are ignored here. Uniqueness of the steady-state growth equilibrium is proved; and the growth rate maximizing ratio of combined net public expenditure on productive public input and on human capital to national income in the steady-state equilibrium is shown to be equal to the combined competitive unpolluted output share of public input and public expenditure financed human capital. Hence this optimum ratio varies inversely with the magnitude of the pollution-output coefficient and directly with the coefficient representing elasticity of human capital accumulation with respect to public expenditure on education. Agenor (2008, 2009) deals with similar models with public expenditure on human capital but does not include environmental pollution in these models.

The common thread running through the basic model and all of its extensions is the assumption that pollution-output coefficient is exogenous. However, the pollution-output coefficient may depend on one or more variables that are endogenous to the system. Under this assumption of an endogenous pollution rate, the second model analyses the allocation of tax revenue to different public goods. Pollution-output coefficient is assumed to be a positive function of the stock of private physical capital and is a negative function of the stock of human capital. Thus, any endogenous changes in the stocks of physical or human capital affect this pollution rate. However, human capital is solely funded by private expenditure in this model; and tax revenue finances public productive expenditure and abatement expenditure. The optimum ratio of net public expenditure on productive public input to national income in the steady-state growth equilibrium is shown to be equal to the competitive unpolluted output share of the public input; and hence this optimum ratio varies inversely with the magnitude of the parameters representing the pollution elasticity with respect to the ratio of human capital to physical capital.

CHAPTER 2

2. PUBLIC EXPENDITURE, ENVIRONMENT AND ECONOMIC GROWTH

2.1 INTRODUCTION

In this chapter, we develop a model of endogenous economic growth with special emphasis on the interaction of productive public expenditure and environmental pollution. In section 2.2 of the present chapter, we introduce environmental quality as an input in the production function. An improvement in environmental quality positively affects the productivity of a public input used to produce the final good. Environmental quality in turn is degraded by pollution generated as a by-product of final good production. Improvement in the environmental quality is caused by the increase in abatement expenditure. The government makes an allocation of its total revenue between creation of productive public input and financing abatement activity; and the government revenue is earned by imposing a proportional income tax. However, environmental quality does not affect the preference function of the consumer in this section. In section 2.3 of this chapter, we introduce environmental quality as an argument in the utility function, keeping the model of section 2.2 otherwise unchanged.

2.2 ENVIRONMENTAL QUALITY AFFECTING PRODUCTIVITY[2]

Our model developed in this section follows Greiner (2005) and EP (2008) to include productive public expenditure and environmental pollution. However, it is different from Greiner's (2005) model and EP's (2008) model on the following points: (i) Greiner (2005) introduces a negative external effect of environmental pollution only on the utility function of the representative household. So does EP (2008). We consider the negative effect of environmental pollution and the negative congestion effect of private capital accumulation on the effective production benefit derived from public input expenditure. Greiner (2005) and EP (2008) do not consider these. (ii) We consider flow public expenditure like Barro (1990) while Greiner (2005) assumes public expenditure as a stock variable like FMS (1993). It is more meaningful to consider public input as a stock variable because the effect of environmental quality on public input is relevant for stock public expenditure. For the sake of technical simplicity, we consider flow public expenditure in this model in this chapter. Chapter 5 of this thesis extends this basic model replacing flow public input by stock public input. (iii) We consider environmental quality as a stock variable which is upgraded through abatement activities and is degraded through emissions. In Greiner (2005) it is treated as a flow variable. (iv) Rate of abatement expenditure is treated exogenous in Greiner (2005) who analyzes the properties of optimal income tax and pollution tax policies. We do not consider a separate pollution tax here but make the allocation of income tax revenue between productive public expenditure and abatement expenditure endogenous to the analysis.

We obtain many interesting results analyzing this model. The optimum ratio of public input expenditure to national income is equal to the competitive share of the public input in the unpolluted output of the final good; and hence this optimum ratio varies inversely with the level of emission per unit of production. However, in Barro (1990) and in FMS (1993), there is no

[2] A related version of this section is published in Journal of Public Economic Theory.

environmental pollution; and hence this ratio is always equal to the competitive output share of the public input. In Greiner (2005), the optimum share of public investment to national income is also independent of the rate of emission because the level of pollution enters into the utility function as a negative argument but does not enter into the production function. Secondly, in our model, optimum income tax rate is higher than that predicted by Barro (1990) and FMS (1993); and this rate varies positively with the emission-output coefficient. This is so because a part of the income tax revenue is spent as abatement expenditure in this model. However, this is not necessarily true in Greiner (2005) because he considers pollution tax as an alternative instrument of financing abatement expenditure. Thirdly, our model exhibits transitional dynamic properties though it follows Barro (1990) to assume public expenditure to be a flow variable. Introducing environmental quality as an accumulable input in the production function and including the negative congestion effect of private capital, we protect our model from being an AK model and thus bring back transitional dynamic properties. In Greiner (2005), transitional dynamic properties are obtained because public expenditure is a stock variable there. Fourthly, like Barro (1990) and FMS (1993), we do not find any conflict between the growth rate maximizing solution and the social welfare maximizing solution along the steady-state equilibrium growth path. Greiner (2005) does not find such a conflict in the case of an income tax policy but finds it in the case of a pollution tax policy because pollution directly affects the utility of the household in his model. Fifthly, the competitive equilibrium growth rate in this model is not necessarily less than the socially efficient growth rate which is unlike in Barro (1990), FMS (1993), Greiner (2005), etc. This is so because we have two conflicting types of externalities on production - positive externalities arising from productive public expenditure and from the up-gradation of the environmental quality and negative externalities arising from the congestion effect of private capital and from environmental pollution. However, the welfare level is always higher in the planned economy where externalities are internalized. Barro (1990) and FMS (1993) consider only the positive externality from public expenditure. Greiner (2005) also does not consider any negative external effect on production.

Section 2.2.1 describes the basic model of the household economy. Section 2.2.2 elaborates on its dynamic equilibrium properties. Subsection 2.2.2.1 shows the possibility of the existence of unique steady-state equilibrium growth path in the market economy; and subsection 2.2.2.2 analyzes the properties of optimal fiscal policy along the steady-state equilibrium path. Section 2.2.3 shows transitional dynamic results and section 2.2.4 describes working of the command economy. Appendices 2.2A through 2.2E contain derivations related to results presented in sub-sections 2.2.2 to 2.2.4.

2.2.1 THE MODEL

The single production sector of the economy uses capital, labour, and public intermediate good as inputs in production. The production function is of Cobb-Douglas type satisfying increasing returns to scale in capital, public intermediate good and labour. However, it satisfies constant returns to scale in capital and public input and diminishing returns to each input.

The product market and the private input markets are competitive and every producer maximizes profit. The public intermediate good is treated as a flow variable like that in Barro (1990). The government imposes a proportional tax on income of the representative household who consumes a part of the post-tax income and invests the other part. The environmental quality is a stock variable. It deteriorates with emissions caused from production of the final good; and is improved by abatement activities of the government. Environmental quality is non-rival and is a free good. The budget of the government is balanced; and allocation of the tax revenue is made between expenditure on the public intermediate good and on the abatement of emission damage.

There is a negative congestion effect of private capital and a positive environmental effect on the efficiency of the public input; and so the effective benefit of public input expenditure received by the representative producer varies inversely with the average private capital stock of the society and directly with the environmental quality.

There is no population growth; and so labour endowment is normalized to unity. Every household maximizes her lifetime utility subject to the inter-temporal budget constraint. Lifetime utility is defined as the infinite integral of the discounted present value of instantaneous utility where instantaneous utility is a positive and concave function of the level of consumption and the rate of discount is constant. All variables are measured in terms of the final product.

Following equations describe the model.

$$Y = AK^{\alpha}\hat{G}^{1-\alpha} \text{ with } 0 < \alpha < 1 \text{ and } A > 0; \quad \dots\dots (2.2.1)$$

$$\hat{G} = G\bar{K}^{-\theta_1}E^{\theta_2} \text{ with } 0 < \theta_1, \theta_2 < 1; \quad \dots\dots (2.2.2)$$

$$G = (\tau - T)Y \text{ with } 0 < T < \tau < 1; \quad \dots\dots (2.2.3)$$

$$\dot{K} = (1-\tau)Y - C; \quad \dots\dots (2.2.4)$$

$$\dot{E} = \eta E + TY - \delta Y \text{ with } 0 < \eta, \delta < 1; \quad \dots\dots (2.2.5)$$

and

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma} \text{ with } \sigma > 0. \quad \dots\dots (2.2.6)$$

Equation (2.2.1) describes the Cobb-Douglas production function of the final good. Y is the level of output produced. K is the stock of capital and $\hat{G}$ is the effective benefit derived from the public intermediate good. Since labour endowment is normalized to unity, Y and K can be considered as per capita variables with the labour elasticity of output being $(1-\alpha)$. Elasticities of output with respect to capital and public intermediate good are denoted by α and

$(1-\alpha)$ respectively. A is the representative of all the exogenous fixed inputs affecting production in the final goods sector.

Equation (2.2.2) describes the nature of the combined effect of congestion and environment on the effectiveness of the public intermediate input. It shows that the effective production benefit of the public intermediate input varies inversely with the average capital stock of all private producers, $\bar{K}$, and positively with the environmental quality, E. θ_1 and θ_2 are two parameters governing the congestion effect of private capital and the environmental effect on productive public input respectively. For the sake of simplicity we ignore environmental effects on the efficiency of private capital, though these effects exist in reality.

Models focusing on the implications of the negative congestion effect[3] are available in the existing literature[4]. We assume that the negative congestion effect of the average capital stock of the society is not strong enough to outweigh the positive private technological contribution of capital of the representative producer. Hence, we assume $(\alpha-\theta_1\overline{1-\alpha})>0$ implying that the social marginal productivity of capital is positive. Here, $(\alpha-\theta_1\overline{1-\alpha})$ is the social elasticity of output with respect to capital.

Equation (2.2.3) describes the government budget constraint. The government finances public expenditure on the intermediate good and the abatement expenditure with its tax revenue. T is the abatement expenditure rate defined as the ratio of abatement expenditure to national income; and τ is the income tax rate. Using equations (2.2.1), (2.2.2) and (2.2.3), we have

$$Y=A^{\frac{1}{\alpha}}(\tau-T)^{\frac{1-\alpha}{\alpha}}K^{1-\frac{\theta_1(1-\alpha)}{\alpha}}E^{\frac{\theta_2(1-\alpha)}{\alpha}}. \qquad \ldots\ldots(2.2.1A)$$

This derived production function satisfies constant returns to scale in terms of K and E only if $\theta_1=\theta_2=\theta$, i.e., if the absolute value of the elasticity of

[3] When a number of industrial plants grow up, effective transportation service becomes slow given the availability of roads and streets. Demand for power consumption goes up leading to disruption in power supplies. Parks and footpaths of streets get occupied by informal sector businessmen.

[4] See the works of Ott and Soretz (2008), Van Tuijl *et. al* (1997), Raurich-Puigdevall (2000), Turnovsky (1996, 1997), etc.

efficiency of public input with respect to environmental quality is exactly equal to the absolute value of that elasticity with respect to aggregate capital. If the constant-returns-to-scale assumption is violated, then the average productivities and marginal productivities of different factors cannot be expressed as factor proportion ratios. As a result, the law of diminishing returns to the variable factor may not be ensured and the convergence to the steady-state growth equilibrium may not be guaranteed. So, for analytical convenience, henceforth we assume $\theta_1 = \theta_2 = \theta$ in the rest of the section as well as in the rest of the chapters.

If we ignore congestion effect of private capital but consider only the positive effect of environmental quality, then $\theta_1 = 0$ and $\theta_2 > 0$. In that case, equation (2.2.1A) is reduced to an AK production function similar to that in Barro (1990); and thus our model fails to show transitional dynamic properties.

Equation (2.2.4) describes the budget constraint of the household who allocates its post tax disposable income between consumption, C, and savings (investment), $\{Y(1-\tau) - C\}$; and there is no depreciation of private capital.

Equation (2.2.5) shows how environmental quality changes over time depending upon the magnitudes of emissions, δY and abatement expenditure, TY. Abatement activities bring improvements in environmental quality; and there exists a substantial theoretical and empirical literature dealing with the role of abatement activities and abatement policies of the government[5]. Here emission is assumed to be a flow variable being proportional to the level of production of the final good; and δ represents the constant emission-output coefficient. We also assume the existence of a dynamic process of natural regeneration of environmental quality. Here, $\eta > 0$ is the constant natural rate of regeneration.

[5] See the works of Liddle (2001), Managi (2006), Dinda (2005), Di Vita (2008), Smulders and Gradus (1996), Byrne (1997), etc.

Many models of environmental pollution assume the level of pollution to be a positive function of the level of production[6] of the final good. This is consistent with only one segment of the Environmental Kuznets curve[7], according to which, there exists an inverted U-shaped relationship between the pollution level and the income level.

Equation (2.2.6) describes the instantaneous utility function of the household. The utility is a positive and concave function of the level of consumption. σ represents the constant elasticity of marginal utility with respect to consumption. Many models assume utility to be a positive function of the environmental quality[8]. We ignore this in this model[9] for the sake of simplicity.

In the static equilibrium all markets clear. Stocks of E and K are exogenous in any particular point in time. E is a non rival stock and G is a non rival flow. Given the stocks of capital and environmental quality, and given the fiscal instrument rates, equations (2.2.1), (2.2.2) and (2.2.3) together determine Y and G at each point of time. Thus equation (2.2.5) determines the absolute rate of improvement in the environmental quality, denoted by $\dot{E}$. The household chooses C and this determines the absolute rate of private capital accumulation, $\dot{K}$, from equation (2.2.4).

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[6] For example, see the works of Liddle (2001), Oueslati (2002), Hartwick (1991), Smulders and Gradus (1996), Byrne (1997), Gruver (1976), Dinda (2005), etc.

[7] Analysis on this curve is available in Managi (2006), Dinda (2005), Di Vita (2008), Hartman and Kwon (2005), Selden and Song (1995), etc.

[8] See the works of Howarth (1996), Tahvonen and Kuuluvainen (1991), Smulders and Gradus (1996), D. Ayong Le Kama (2001), Greiner (2005), Gruver (1976), Itaya (2008), etc.

[9] The model with environmental quality as an argument in utility function is developed and analyzed in section 2.3 of this chapter.

2.2.2 DYNAMIC EQUILIBRIUM

The representative household maximizes $\int_0^\infty u(C)\, e^{-\rho t} dt$ with respect to C subject to equations (2.2.1), (2.2.4) and (2.2.6). The demand rate of growth[10] of consumption is derived from this maximizing problem as follows.

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right]. \qquad \ldots\ldots(2.2.7)$$

We consider a steady-state growth equilibrium where all macroeconomic variables grow at the same rate, g_m. Hence, we have

$$\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{E}}{E} = \frac{\dot{G}}{G} = g_m. \qquad \ldots\ldots(2.2.8)$$

2.2.2.1 Existence of Steady-State Growth Equilibrium

We now turn to show the existence of unique steady state equilibrium growth rate in the market economy; and so we use equations (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.5), (2.2.7) and (2.2.8) to obtain the following equations.

$$\frac{1}{\sigma}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right] = g_m; \qquad \ldots\ldots(2.2.9)$$

$$A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \frac{C}{K} = g_m; \qquad \ldots\ldots(2.2.10)$$

and

$$A^{\frac{1}{\alpha}}(T-\delta)(\tau - T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} = g_m - \eta. \qquad \ldots\ldots(2.2.11)$$

Using equations (2.2.9) and (2.2.11) we obtain the following equation[11] to solve for g_m.

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[10] The demand rate of growth of consumption is derived in Appendix 2.2A.

[11] The derivation of equation (2.2.12) is worked out in Appendix 2.2B.

$$(g_m - \eta)^{\theta(1-\alpha)}(\sigma g_m + \rho)^{\alpha-\theta(1-\alpha)}$$
$$= A\{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(T-\delta)^{\theta(1-\alpha)}(\tau - T)^{1-\alpha}. \quad \ldots\ldots (2.2.12)$$

The L.H.S. of equation (2.2.12) is an increasing function of g_m for all values of $g_m > \eta$ and its R.H.S. is constant, given the income tax rate, τ, and the abatement expenditure rate, T. Its R.H.S. is positive if $0 < \delta < T < \tau < 1$. Its L.H.S. is defined[12] only for $g_m \geq \eta$. So figure 2.2.1 shows the existence of unique positive value of g_m satisfying $g_m > \eta$ given that $0 < \delta < T < \tau < 1$. Then equations (2.2.10) and (2.2.11) show that equilibrium values of $\frac{E}{K}$ and $\frac{C}{K}$ are also unique in this case.

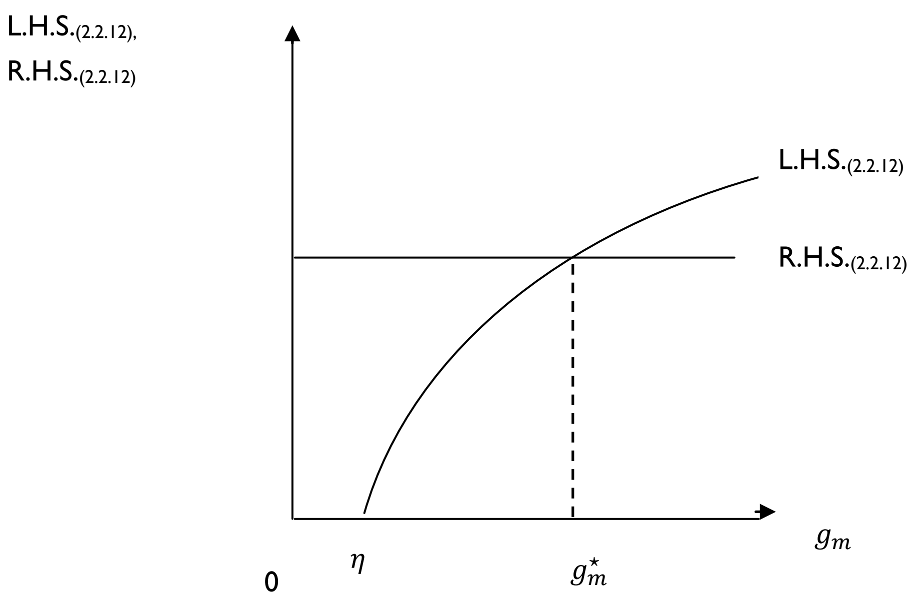

FIGURE 2.2.1

We can state the following proposition.

Proposition 2.2.1: There exists unique positive steady state equilibrium growth rate in the market economy only if the fiscal instruments

[12] $(g_m - \eta)^{\theta(1-\alpha)}$ is not a real number when $g_m < \eta$.

satisfy $0 < \delta < T < \tau < 1$; and the unique steady-state equilibrium growth rate exceeds the natural rate of regeneration of environmental quality.

If the chosen values of τ and T do not satisfy this chain of inequalities, then a steady state equilibrium growth path does not exist in this model. Here $T > \delta$ appears to be a strong assumption because then $\dot{E}$ varies positively with Y. If the abatement expenditure rate exceeds the pollution-output coefficient, then the pollution versus efficient public expenditure trade-off disappears[13]. However, it is necessary to assume $T > \delta$ to prove the existence of steady-state growth equilibrium. If $T = \delta$, then the R.H.S. of equation (2.2.12) is zero; and $T < \delta$ makes the R.H.S. to be an imaginary number. In this model, $g_m > \eta$; and in the steady-state growth equilibrium, $\frac{\dot{E}}{E} = g_m$. Also, the assumption that social elasticity of private physical capital is positive, i.e., $\alpha - \theta(1-\alpha) > 0$, is necessary for the existence of unique steady-state equilibrium growth rate. If private technological contribution cannot outweigh the negative congestion effect of physical capital, then effective marginal productivity of private physical capital is negative. We do not assume negative effective marginal productivity of private physical capital.

[13] When $T < \delta$, sufficiently high values of the natural regeneration coefficient, η, can also negate this trade-off.

2.2.2.2 Optimal Taxation

In this model, optimal taxation refers to the tax system designed to maximize the steady state equilibrium growth rate in a decentralized economy[14]. The government maximizes the steady-state equilibrium growth rate with respect to fiscal instruments, τ and T. The L.H.S. of equation (2.2.12) is a monotonically increasing function of g_m, because, by assumption, $\alpha > \theta(1-\alpha)$ and $g_m > \eta$. Since the L.H.S. is always equal to the R.H.S. in the steady-state growth equilibrium, maximization of g_m means maximization of the R.H.S. of equation (2.2.12).

We obtain following expressions of optimum tax rate and abatement expenditure rate[15].

$$\tau^\star = 1-(1-\delta)\{\alpha-\theta(1-\alpha)\}; \qquad \dots\dots(2.2.13)$$

and

$$T^\star = \delta+(1-\delta)\theta(1-\alpha). \qquad \dots\dots(2.2.14)$$

Using equations (2.2.13) and (2.2.14), we have

$$\tau^\star - T^\star = (1-\delta)(1-\alpha). \qquad \dots\dots(2.2.15)$$

Here $(\tau^\star - T^\star)$ is the optimum ratio of public expenditure on the intermediate good to the national income; and $(1-\delta)(1-\alpha)$ is the competitive unpolluted output share of the public intermediate good. So the optimum ratio is equal to the competitive share of the public intermediate good in the unpolluted output. In Barro (1990) and in FMS (1993), entire output is pollution free.

To ensure non-negativity of the decentralized growth rate, degradation of the accumulable indirect productive input, namely environmental quality, due to pollution is neutralized by allocating δ fraction of total output to abatement

[14] This tax system may not implement the social optimum in the decentralized economy. The problem of the social optimum will be analyzed in section 2.2.4.

[15] The derivation of equations (2.2.13) and (2.2.14) is worked out in Appendix 2.2C.

expenditure, TY. Optimum net expenditure rate is then, $T^{\star} - \delta$, which is equal to the competitive unpolluted output share of environmental quality.

The social welfare function is given by $W = \int_0^{\infty} e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt$; and assuming that the economy initially is on the steady-state equilibrium growth path, it can be shown[16] that

$$W = \alpha^{\sigma-1} \frac{K(0)^{1-\sigma}}{1-\sigma} \left[\frac{\rho-(\alpha-\sigma)g_m}{\rho-(1-\sigma)g_m}\right] [\rho - (\alpha - \sigma)g_m]^{-\sigma}. \qquad \text{......(2.2.16)}$$

Hence, W varies positively with g_m. Thus the level of social welfare is maximized when the steady-state equilibrium growth rate is maximized[17]. We can state the following proposition.

Proposition 2.2.2: (i) The optimum income tax rate and the optimum abatement expenditure rate in the steady state growth equilibrium are given by

$$\tau^{\star} = 1 - (1 - \delta)\{\alpha - \theta(1 - \alpha)\},$$

and

$$T^{\star} = \delta + (1 - \delta)\theta(1 - \alpha).$$

(ii) The optimum ratio of public input expenditure to national income in the steady-state equilibrium is equal to the competitive unpolluted output share of the public intermediate good; and hence this optimum ratio varies inversely with the magnitude of the emission-output coefficient[18].

The presence of congestion effect making $\theta > 0$ and the presence of environmental pollution causing $\delta > 0$ make our result different from those of Barro (1990) and FMS (1993). If we assume $\theta = \delta = 0$, we obtain $\tau^{\star} = 1 - \alpha$ and $T^{\star} = 0$; and these results are identical to those of Barro (1990) and FMS (1993). The optimum ratio of productive public expenditure to national income in this

[16] The derivation of equation (2.2.16) is shown in Appendix 2.2D.

[17] This is not true when the economy is off the steady-state growth path at the initial time point. In that case, we should include the welfare in the transitional phase too; and evaluating this analytically is a very hard technical work.

[18] The introduction of the constant natural regeneration rate of environmental quality will not alter the results about optimum fiscal policies in this model because the R.H.S. of equation (2.2.12) will remain independent of this regeneration rate.

model, with $0 < \delta < 1$ and $\theta > 0$, appears to be lower than that in Barro (1990) and in FMS (1993). This is obvious because production of the final good generates environmental pollution; and this, in turn, lowers the effective benefit derived from the public expenditure. However,

$$\tau^\star = 1 - \alpha + \alpha - (1 - \delta)\{\alpha - \theta(1 - \alpha)\}$$

$$= 1 - \alpha + \delta\alpha + (1 - \delta)\theta(1 - \alpha).$$

Here, $\tau^\star > 1 - \alpha$ because $0 < \delta < 1$, $0 < \alpha < 1$, and $\theta > 0$. So the optimum income tax rate in this model is higher than that in the Barro (1990) model and in the FMS (1993) model. This is so because income tax is the only source of revenue in this model and a part of the income tax revenue is used to finance abatement expenditure. This is not necessarily so in the Greiner (2005) model because pollution tax is an alternative instrument of financing abatement expenditure there. We do not consider a separate pollution tax.

In this model, not only the productive public expenditure, i.e., the excess of tax revenue over abatement expenditure, but also the level of emission is proportional to the level of income (production of the final good). So $(\tau^\star - T^\star)$ varies inversely with the emission-output coefficient, δ. If the level of emission is independent of the level of income and if it varies proportionally with C or K where δ is the relevant coefficient, then $(\tau^\star - T^\star)$ would be independent of δ and the Barro (1990) result would come back in that modified model[19].

2.2.3 TRANSITIONAL DYNAMICS

We define the following ratio variables.

$x = \frac{C}{K}$; and $y = \frac{E}{K}$.

Using equations (2.2.4), (2.2.5) and (2.2.7), we have

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[19] This is worked out in chapter 3 of this thesis.

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x - \frac{\rho}{\sigma}; \qquad \ldots\ldots (2.2.17)$$

and

$$\frac{\dot{y}}{y} = (T-\delta)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1} - (1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x + \eta.$$

$$\ldots\ldots (2.2.18)$$

These are the equations of motion of the dynamic system. The determinant of the Jacobian matrix[20] corresponding to differential equations (2.2.17) and (2.2.18) is given by

$$|J| = -\frac{\alpha - \theta(1-\alpha)}{\alpha}(T-\delta)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{1}{\sigma}\theta(1-\alpha)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

Here $\alpha - \theta(1-\alpha) > 0$; and $1 > \tau > T > \delta$ when τ and T are optimally chosen with $\theta > 0$. So $|J| < 0$ in this case; and hence the two latent roots of the Jacobian matrix must be real and of opposite signs. So the steady-state equilibrium is a saddle-point and there is only one transitional path converging to this point when $\theta > 0$. If $\theta = 0$, then equation (2.2.14) shows that optimum $T = \delta$; and hence, $|J| = 0$. Therefore we can state the following proposition.

Proposition 2.2.3: The unique steady-state equilibrium with optimally chosen values of fiscal instruments is saddle-point stable with unique saddle path converging to that equilibrium point when $\theta > 0$.

This result is important because Barro (1990) model, with a flow public expenditure, does not exhibit any transitional dynamic properties. FMS (1993) bring back transitional dynamic properties in Barro (1990) model introducing durable public input. We obtain the saddle-point property of the long run equilibrium in this model even with a flow public expenditure similar to that of Barro (1990). The environmental quality is a stock variable accumulating over time in this model; and this positively affects the productivity of the system

[20] The derivation of the determinant is worked out in Appendix 2.2E.

when $\theta > 0$. Also $\theta > 0$ implies the existence of a negative congestion effect of physical capital. Thus our flow public expenditure model is protected from being an AK model in this case. Greiner (2005) model exhibits transitional dynamic properties treating environmental pollution, in the form of emission from industrial production, as a flow variable in the utility function because public input is a stock variable there.

2.2.4 COMMAND ECONOMY

The market economy solution may be suboptimal due to the distortion caused by the proportional income tax and by the failure of the private individuals to internalize externalities in the system. The presence of two non-rival inputs - public good and environmental quality - in the production function causes positive externalities; and the congestion effect of physical capital and environmental pollution introduce negative externalities. The planner, who maximizes a social welfare function identical to that of the representative household's lifetime utility function, can internalize these externalities. Equations (2.2.1), (2.2.2) and (2.2.6) remain unchanged; and equations (2.2.3), (2.2.4), and (2.2.5) are modified as follows.

$$G = \Pi - \Omega; \qquad \text{…… (2.2.3.1)}$$

$$\dot{K} = Y - \Pi - C; \qquad \text{…… (2.2.4.1)}$$

and

$$\dot{E} = \eta E + \Omega - \delta Y. \qquad \text{…… (2.2.5.1)}$$

Here Π denotes planner's total lump sum expenditure on public intermediate input and abatement activities and Ω denotes the abatement expenditure.

The planner's problem is to maximize $\int_0^\infty e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt$ with respect to C, Π and Ω subject to equations (2.2.3.1), (2.2.4.1) and (2.2.5.1). We consider a steady-state growth equilibrium where the growth rate is denoted by g_c; and the following equation solves for the steady-state equilibrium growth rate[21] in the command (planned) economy.

$$(\rho + \sigma g_c)^{\alpha-\theta(1-\alpha)}(\rho + \sigma g_c - \eta)^{\theta(1-\alpha)}$$

$$= A(1-\delta)(1-\alpha)^{1-\alpha}\{\alpha - \theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \qquad \ldots\ldots (2.2.19)$$

The L.H.S. of equation (2.2.19) is an increasing function of g_c for all values of $\rho \geq \eta$[22] and the R.H.S. is a positive constant when $0 < \delta < 1$ and $\theta > 0$. So the existence of unique positive value of g_c is ensured when $0 < \delta < 1$, $\rho \geq \eta$ and $\theta > 0$. Here, g_c is the socially efficient growth rate. Figure 2.2.2 shows the determination of unique g_c.

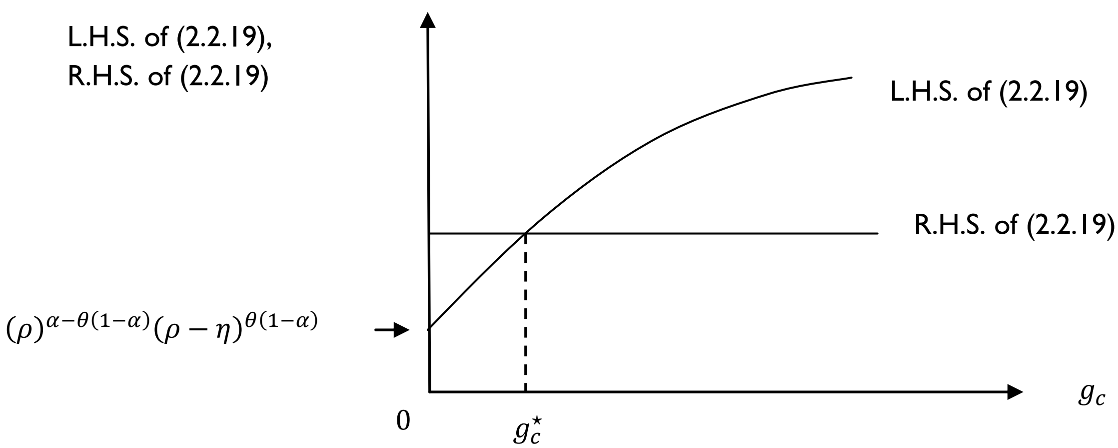

FIGURE 2.2.2

[21] Equation (2.2.19) is derived in the Appendix 2.2F.
[22] The L.H.S. of equation (2.2.19) is not real when $\rho < \eta$.

Now we turn to compare the market economy solution to the command economy solution. We modify equation (2.2.12) with $\tau = \tau^{\star}$ and $T = T^{\star}$ as follows.

$$\alpha^{-\alpha}\{\alpha(g_m - \eta)\}^{\theta(1-\alpha)}(\sigma g_m + \rho)^{\alpha-\theta(1-\alpha)}$$
$$= A(1-\delta)(1-\alpha)^{1-\alpha}\{\alpha - \theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \qquad (2.2.12.1)$$

The R.H.S. of equations (2.2.19) and (2.2.12.1) are identical. However, the L.H.S. of equation (2.2.12.1) is greater than that of equation (2.2.19) for all values of

$$g_m = g_c > \frac{\eta\left(1-\alpha^{\frac{\alpha}{\theta(1-\alpha)}}\right)+\rho\alpha^{\frac{\alpha}{\theta(1-\alpha)}}}{\alpha\left(1-\sigma\alpha^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}\right)}.$$

Hence we find that g_m exceeds (falls short of) g_c when the pollution parameter δ takes a high (low) value or when the technology parameter A, takes a low (high) value. For simplicity, we shall concentrate on variation of the value of the parameter δ. The analysis carries through as well when A varies. This is shown in figure 2.2.3. The L.H.S. of equations (2.2.12.1) and (2.2.19) are plotted as positively sloped curves and the common R.H.S. is depicted by horizontal straight lines for different exogenous values of δ. The L.H.S. of equation (2.2.12.1) curve starts from a point on the horizontal axis where $g_m = \eta$ but the L.H.S. of equation (2.2.19) curve starts from a point on the vertical axis. The intersection point of the two L.H.S. curves shows that

$$g_m = g_c = \frac{\eta\left(1-\alpha^{\frac{\alpha}{\theta(1-\alpha)}}\right)+\rho\alpha^{\frac{\alpha}{\theta(1-\alpha)}}}{\alpha\left(1-\sigma\alpha^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}\right)}.$$

When δ takes a high (low) value, points of intersection of the two L.H.S. curves with the lower (higher) horizontal line shows that $g_c^{\star}$ falls short of (exceeds) $g_m^{\star}$.

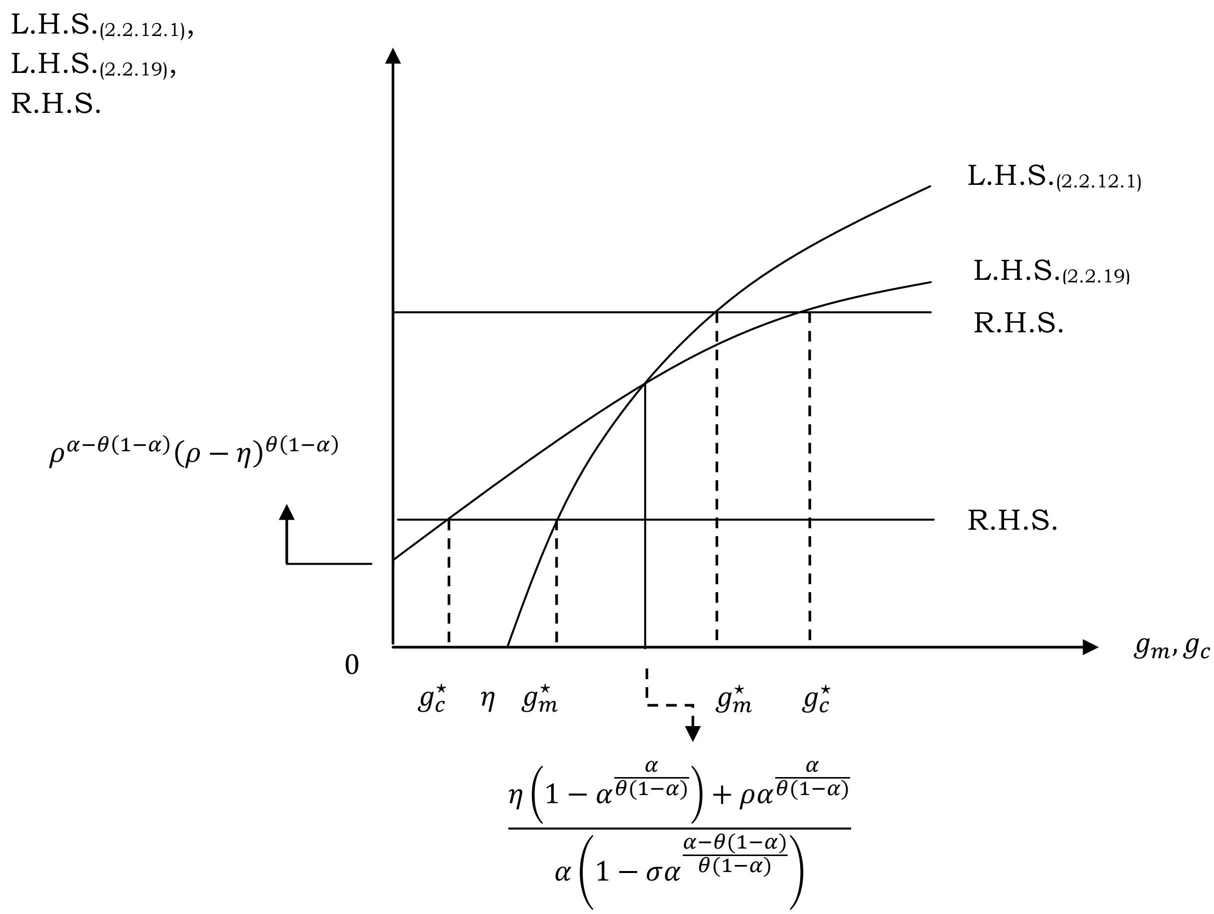

FIGURE 2.2.3

We can state the following proposition.

Proposition 2.2.4: If $\theta > 0$, then $(g_c - g_m)$ takes a positive (negative) sign when δ takes a low (high) value.

Barro (1990) and FMS (1993) show that the market economy growth rate in the steady-state equilibrium falls short of the socially efficient growth rate. Our result may be different from theirs'. The intuition behind this result may be explained as follows. The planner internalizes two conflicting types of externalities - the negative externalities arising from the congestion effect of physical capital accumulation and the environmental pollution, and the positive externalities caused by the presence of the public intermediate good and the environmental quality. So the net effect of internalization of externalities on the productivity of private inputs is ambiguous. Hence, the effect on the growth rate is also ambiguous. Socially efficient growth rate should exceed (fall short of) the competitive equilibrium growth rate when a positive (negative) externality is internalized. The negative externality on production does not exist in Barro (1990) model and in FMS (1993) model. In Greiner (2005), the negative externality of environmental pollution affects the utility function but does not affect the productivity of private inputs.

The equilibrium growth rate (in both the market economy and the command economy), in our model depends on the parameter, δ, which measures the magnitude of emission per unit of production. If the rate of emission is increased, then the level of unpolluted output should fall. This also has a negative effect on the rate of physical capital accumulation as well as on the up-gradation of the environmental quality. When δ takes a high (low) value, the negative externality of physical capital dominates (is dominated by) other positive externalities; and so the socially efficient rate of growth is lower (higher) than the market economy growth rate.

However, it is always beneficial to internalize externalities. What is important is the welfare of the agent and not the growth rate. It means that the agent's welfare is always smaller in the absence of internalization of externalities regardless of the relationship between the socially efficient growth rate and the competitive equilibrium growth rate. Moreover, it implies that the proposed scheme of proportional income tax and abatement expenditure is not sufficient to internalize the externalities though welfare maximization is attained through such a scheme. This is so because here the emission rate, δ, is beyond the control of the government. In the presence of alternative technologies, this emission rate is a variable; and the government can lower this rate of emission subsidizing the use of eco-friendly production technology. It is clear from figure 2.2.3 that, given other parameters, there exists a critical value[23] of δ that equates g_c to g_m; and the optimum subsidy rate should correspond to that critical value of δ.

2.3 ENVIRONMENTAL QUALITY AFFECTING UTILITY

Both Greiner (2005) and EP (2008) introduce pollution as an argument in the utility function. In this section, we also introduce environmental quality as an argument in the utility function in an otherwise identical model developed in section 2.2.

We obtain many interesting results analyzing this model. First, we find that the steady state growth equilibrium, if it exists, is either unique or multiple with two equilibria. If the equilibrium is unique, then properties of optimum fiscal policy derived in this model are similar to those obtained from the earlier model. Secondly, this model too shows transitional dynamic properties. The explanation of the presence of transitional dynamic properties is similar to that in section 2.2.3. But, unlike in section 2.2.3 steady-state equilibrium is not necessarily a saddle-point in this model. In the case of multiple equilibria, we may have indeterminacy of the transitional growth path converging to one of the two equilibria.

Section 2.3.1 presents the basic model of the household economy. Section 2.3.2 analyzes its dynamic equilibrium properties. Subsection 2.3.2.1 shows the problems related to the existence of unique equilibrium in the market economy; and subsection 2.3.2.2 analyzes the properties of optimal fiscal policy in the steady-state equilibrium. Section 2.3.3 derives transitional dynamic properties and section 2.3.4 analyzes the possibility of the existence of

[23] For that critical value of δ, the horizontal straight line passes through the point of intersection of the two L.H.S. curves.

a socially efficient growth rate. Appendices 2.3A to 2.3F contain derivations related to equations of the model developed in this section.

2.3.1 THE MODEL

We assume $A = 1$ in equation (2.2.1) and $\eta = 0$ in equation (2.2.5) in this section; the rest of the equations, (2.2.2) to (2.2.4), of section 2.2 remain unchanged. The major results of this section remain unaffected by the above assumptions. We assume utility to be enhanced by both consumption and environmental quality.

$$u(C, E) = C^{\beta\sigma} E^{\beta(1-\sigma)} \; with \; \beta > 0 \; and \; 0 < \sigma < 1. \qquad \dots\dots (2.3.1)$$

Equation (2.3.1) describes the instantaneous utility function of the household. The instantaneous utility is a positive function of the level of consumption as well as of the stock of environmental quality[24] and this function is homogeneous of degree β; $\beta\sigma$ and $\beta(1-\sigma)$ represent the constant elasticity of utility with respect to consumption and environmental quality respectively. We assume $1 - \beta\sigma > 0$ to satisfy diminishing marginal utility of consumption.

2.3.2 DYNAMIC EQUILIBRIUM

The representative household maximizes $\int_0^\infty u(C, E)\, e^{-\rho t} dt$ with respect to C subject to equations (2.2.1), (2.2.4) and (2.3.1); and the optimum solution leads to the following demand rate of growth[25] of consumption.

[24] See the works of Howarth (1996), Tahvonen and Kuuluvainen (1991), Smulders and Gradus (1996), Ayong Le Kama (2001), Greiner (2005), Gruver (1976), Itaya (2008), etc.

[25] The demand rate of growth of consumption is derived in Appendix 2.3A.

$$\frac{\dot{C}}{C} = \frac{1}{1-\beta\sigma}\left[\alpha(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} + \beta(1-\sigma)(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}-1} - \rho\right]. \quad \ldots\ldots (2.3.2)$$

Along a steady-state equilibrium growth path, all macroeconomic variables grow at the same rate, g_m. So equation (2.2.8) remains valid in this section.

2.3.2.1 Existence Of Steady-State Growth Equilibrium

To show the existence of unique growth rate in the steady-state equilibrium we use equations (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.5), (2.3.2) and (2.2.8) and then obtain following equations.

$$\frac{1}{1-\beta\sigma}\left[\alpha(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right] + \frac{\beta(1-\sigma)}{1-\beta\sigma}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}-1} = g_m; \quad \ldots\ldots (2.3.3)$$

$$(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \frac{C}{K} = g_m; \quad \ldots\ldots (2.3.4)$$

and

$$(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}-1} = g_m. \quad \ldots\ldots (2.3.5)$$

Equations (2.3.4) and (2.3.5) solve for unique values of $\left(\frac{E}{K}\right)$ and $\left(\frac{C}{K}\right)$ in terms of g_m. Using equations (2.3.3) and (2.3.5) we obtain the following equation[26] to solve for g_m.

$$g_m^{\theta(1-\alpha)}\{\rho-(\beta-1)g_m\}^{\alpha-\theta(1-\alpha)} = \{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(T-\delta)^{\theta(1-\alpha)}(\tau-T)^{1-\alpha}. \quad \ldots\ldots (2.3.6)$$

[26] The derivation of equation (2.3.6) is worked out in Appendix 2.3B.

For $\beta \leq 1$, the L.H.S. of equation (2.3.6) (hereafter denoted by $L.H.S._{(2.3.6)}(g_m)$) is an increasing function of g_m . However, for $\beta > 1$, $L.H.S._{(2.3.6)}(g_m)$ is not a monotonic function of g_m. When $\beta > 1$, $L.H.S._{(2.3.6)}(g_m)$ appears to have an inverted-U shaped curve against g_m plotted on the horizontal axis[27] starting from the origin, reaching a maxima at $g_m = \frac{\theta(1-\alpha)\rho}{\alpha(\beta-1)}$ and meeting the horizontal axis again at $g_m = \frac{\rho}{\beta-1}$. This is shown in figure 2.3.1. The R.H.S. of equation (2.3.6) is constant, given the income tax rate, τ, and the abatement expenditure rate, T. It is positive if $0 < \delta < T < \tau < 1$. Given that $0 < \delta < T < \tau < 1$, two possibilities exist. (i) Unique positive value of g_m exists if $\beta \leq 1$. (ii) When $\beta > 1$, either there are two steady-state growth equilibria or the equilibrium does not exist at all[28]. Figure 2.3.1 clearly shows that, in the case of multiple equilibria, the two equilibria are characterized by the values of g_m satisfying $0 < g_m < \frac{\theta(1-\alpha)\rho}{\alpha(\beta-1)}$ and $\frac{\theta(1-\alpha)\rho}{\alpha(\beta-1)} < g_m < \frac{\rho}{(\beta-1)}$ respectively.

[27] $L.H.S._{(2.3.6)}(g_m)$ is not real for $g_m > \frac{\rho}{\beta-1}$.

[28] Equilibrium may not exist if the value of the policy variables chosen are such that the R.H.S. of equation (2.3.6) exceeds $\left(\frac{\rho}{\alpha}\right)^{\alpha} \frac{\{\alpha-\theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}}{(\beta-1)^{\theta(1-\alpha)}}$.

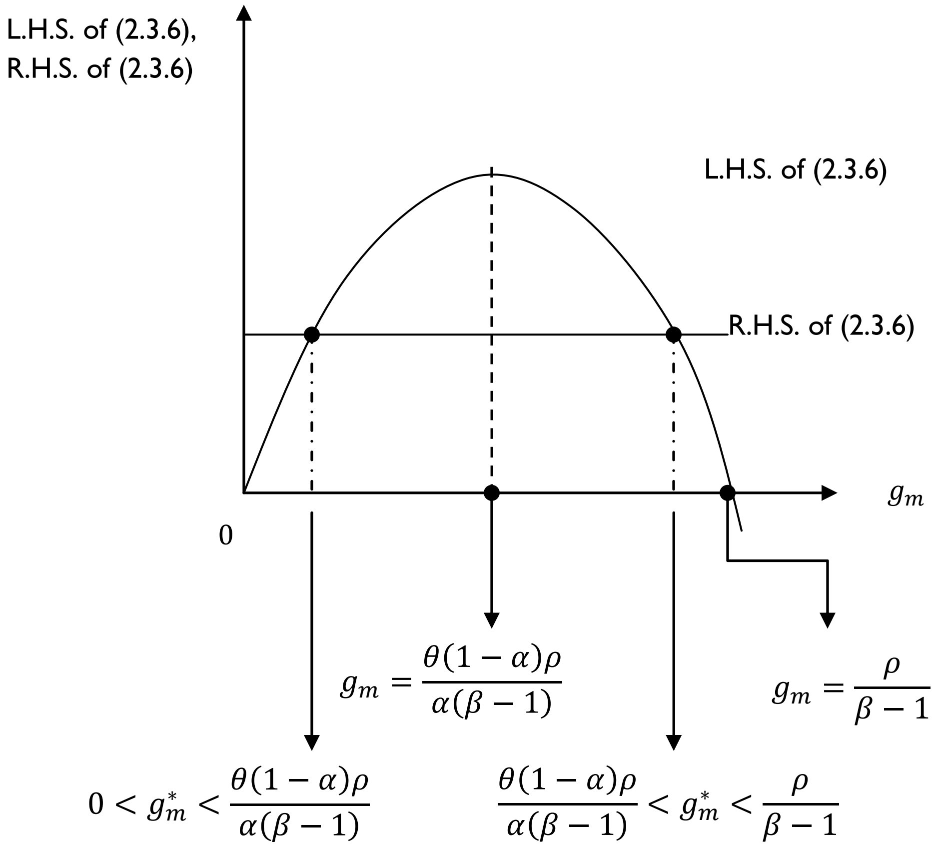

FIGURE 2.3.1

We can, therefore, state the following proposition.

Proposition 2.3.1: (i) Given that fiscal instruments satisfy $0 < \delta < T < \tau < 1$, there exists unique positive growth rate in the steady-state equilibrium of the market economy when $\beta \leq 1$.

(ii) When $\beta > 1$, either there exists two equilibria characterized by $0 < g_m < \frac{\theta(1-\alpha)\rho}{\alpha(\beta-1)}$ and $\frac{\theta(1-\alpha)\rho}{\alpha(\beta-1)} < g_m < \frac{\rho}{(\beta-1)}$ respectively, or the equilibrium does not exist at all.

This result is different from the corresponding one obtained in Barro (1990), FMS (1993) and Greiner (2005), etc. There is no possibility of the existence of multiple steady-state equilibria in any of those models. Our present result is different because, in this section, we consider a positive externality of environmental quality on the utility function of the representative consumer. Marginal utility of consumption varies positively with the quality of environment which is a stock variable in this model. That, the inclusion of a stock variable in the utility function leads to the possibility of multiple steady-state equilibria, is already established in the literature. Kurz (1968) and Liviatan and Samuelson (1969) show the possibility of multiple steady-state equilibria in the one-sector Ramsey-Solow model when the physical capital stock is introduced into the utility function. Barro (1990) and FMS (1993) do not include environment in the utility function. In Greiner (2005), flow pollution affects utility but the utility function is separable in terms of consumption and pollution.

If we assume $A = 1$, $\eta = 0$ and $\sigma = 1 - \beta$ in equation (2.2.12) of section 2.2 it becomes identical to equation (2.3.6). The existence of multiple equilibria crucially depends on the parameter determining the degree of homogeneity of the utility function and on the assumption of its range. In the specification of the utility function in this section, this parameter is β. If we assume away $\beta > 1$, there exists unique steady-state growth equilibrium, a result identical to that obtained in the model of section 2.2. The possibility of multiple steady-state growth equilibria never exists in the model of section 2.2 because the L.H.S. of equation (2.2.12) is a monotonically increasing function of g_m. Here the magnitude of β has no role to play because β does not enter into this equation. The importance of environmental quality entering into the utility function in this section lies in showing that it is the degree of homogeneity of the utility function which determines the existence of multiple equilibria and not the marginal utility of the environmental quality[29].

[29] An alternative specification of the utility function of the form $u(C,E) = \frac{(C^{\beta}E^{1-\beta})^{1-\sigma}}{1-\sigma}$ gives the steady-state equilibrium growth rate to be obtained as
$$g_m{}^{\theta(1-\alpha)}\{\rho + \sigma g_m\}^{\alpha-\theta(1-\alpha)} = \{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(T-\delta)^{\theta(1-\alpha)}(\tau - T)^{1-\alpha},$$

2.3.2.2 Optimal Policies

The government maximizes the steady-state equilibrium growth rate with respect to fiscal instruments, τ and T. The L.H.S. of equation (2.3.6) is a monotonically increasing function of g_m when $\beta \leq 1$ because $\alpha > \theta(1-\alpha)$. Since the L.H.S. is always equal to the R.H.S. in the steady-state growth equilibrium, maximization of g_m means maximization of the R.H.S. of equation (2.3.6).

We obtain following expressions of optimum income tax rate and abatement expenditure rate[30].

$$\tau^\star = 1 - (1-\delta)\{\alpha - \theta(1-\alpha)\}; \qquad \dots\dots(2.3.7)$$

and

$$T^\star = \delta + (1-\delta)\theta(1-\alpha). \qquad \dots\dots(2.3.8)$$

Using equations (2.3.7) and (2.3.8), we have

$$\tau^\star - T^\star = (1-\delta)(1-\alpha). \qquad \dots\dots(2.3.9)$$

Here $(\tau^\star - T^\star)$ is the optimum ratio of public expenditure on the intermediate good to the national income; and $(1-\delta)(1-\alpha)$ is the competitive unpolluted output share of the public intermediate good. So the optimum ratio of public expenditure on the intermediate good to national income is equal to the competitive share of the public intermediate good in the unpolluted output. In Barro (1990) and in FMS (1993), entire output is pollution free.

Using optimal values of the fiscal instruments as given by equations (2.3.7) and (2.3.8) in equation (2.3.6) we obtain

$$g_m{}^{\theta(1-\alpha)}\{\rho - (\beta-1)g_m\}^{\alpha-\theta(1-\alpha)} = (1-\delta)[\alpha\{\alpha-\theta(1-\alpha)\}]^{\alpha-\theta(1-\alpha)}$$
$$\{\theta(1-\alpha)\}^{\theta(1-\alpha)}(1-\alpha)^{1-\alpha}. \qquad \dots\dots(2.3.6.1)$$

In the special case when $\beta = 1$, equation (2.3.6.1) is modified as follows.

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which is identical to equation 2.2.12 in section 2.2 when $\eta = 0$. Since $\sigma > 0$, there is no possibility of the existence of multiple steady-state growth equilibria in this case.

[30] The derivation of equations (2.3.7) and (2.3.8) is worked out in Appendix 2.3C.

$$g_m = \left[(1-\delta)\left\{\frac{\alpha\{\alpha-\theta(1-\alpha)\}}{\rho}\right\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}(1-\alpha)^{1-\alpha}\right]^{\frac{1}{\theta(1-\alpha)}}.$$

…… (2.3.6.2)

The social welfare function is given by $W = \int_0^{\infty} e^{-\rho t} C^{\beta\sigma} E^{\beta(1-\sigma)}\, dt$; and assuming that the economy initially is on the steady-state equilibrium growth path and satisfying $\rho > \beta g_m$[31] we can show[32] that

$$W = \alpha^{-\beta\sigma}[K(0)]^{\beta\sigma}[E(0)]^{\beta(1-\sigma)}\left[\frac{\rho-\beta g_m+(1-\alpha)g_m}{\rho-\beta g_m}\right]^{\beta\sigma}[\rho-\beta g_m]^{\beta\sigma-1}. \quad \text{…… (2.3.10)}$$

Hence, W varies directly with g_m. Thus the level of social welfare is maximized when the steady-state equilibrium growth rate is maximized[33]. We can state the following proposition.

Proposition 2.3.2: (i) Given that $\beta \leq 1$, the optimum income tax rate and the optimum abatement expenditure rate in the steady-state growth equilibrium are given by

$$\tau^{\star} = 1-(1-\delta)\{\alpha-\theta(1-\alpha)\},$$

and

$$T^{\star} = \delta + (1-\delta)\theta(1-\alpha).$$

(ii) The optimum ratio of public input expenditure to national income in the steady-state equilibrium is equal to the competitive unpolluted output share of the public intermediate good; and hence this optimum ratio varies inversely with the magnitude of the emission-output coefficient.

These results are similar to those obtained in section 2.2 in which the utility function does not include environmental quality as an argument. Even if the marginal utility of environmental quality is positive with $\beta \leq 1$, our earlier

[31] When the fiscal instruments are optimally chosen and when $\beta = 1$, equation (2.3.6.1) shows that the steady-state equilibrium growth rate satisfies this inequality if $\left[\frac{\alpha}{\rho}\{\alpha-\theta(1-\alpha)\}\right]^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}\{\theta(1-\alpha)\}(1-\alpha)^{\frac{1-\alpha}{\theta(1-\alpha)}} < \rho$.

[32] The derivation of equation (2.3.10) is shown in Appendix 2.3D.

[33] This is not true when the economy is off the steady-state growth path at the initial time point. In that case, we should include the welfare in the transitional phase too; and evaluating this analytically is extremely difficult.

results remain valid. However, we cannot analyze the properties of optimal fiscal policy when $\beta > 1$ because the equilibrium growth rate is not unique.

The results in proposition 2.3.2 are unaltered from that in proposition 2.2.3 even though environmental quality is an argument in the utility function here. This is so because environmental quality and pollution-related damage to it affects productivity. So, when growth rate is maximized, preservation of environmental quality is also optimized from the welfare perspective. Thus, there arises no conflict between growth rate maximization and welfare maximization.

2.3.3 TRANSITIONAL DYNAMICS

The ratio variables defined in section 2.2.3 are relevant in this section too and hence, we keep them unchanged.

Using equations (2.2.4), (2.2.5) and (2.3.2), we have

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{1-\beta\sigma} - 1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + \frac{\beta(1-\sigma)}{1-\beta\sigma}(T-\delta)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1} + x - \frac{\rho}{1-\beta\sigma}; \quad \ldots\ldots (2.3.11)$$

and

$$\frac{\dot{y}}{y} = (T-\delta)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1} - (1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x. \quad \ldots\ldots (2.3.12)$$

These are the equations of motion of the dynamic system. The determinant of the Jacobian matrix[34] corresponding to differential equations (2.3.11) and (2.3.12) is given by

$$|J| = \left\{\frac{\alpha - \theta(1-\alpha)}{\alpha}\right\}\left\{\frac{\beta - 1}{1-\beta\sigma}\right\}(T-\delta)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2} - \frac{\theta(1-\alpha)}{1-\beta\sigma}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

[34] The derivation of the determinant is worked out in Appendix 2.3E.

Here $\alpha - \theta(1-\alpha) > 0;$ and $1 > \tau > T > \delta > 0$. When $\beta \leq 1$, $|J| < 0$ and the steady-state equilibrium is a saddle point. Also, when $\beta > 1$, $|J| < 0$ if the policy variables are chosen such that the second term dominates the first term; and the two latent roots of the Jacobian matrix will be real and of opposite signs. In this case, the steady-state equilibrium will be a saddle-point if

$$g_m^{\alpha} > (\tau - T)^{1-\alpha}(T-\delta)^{\theta(1-\alpha)} \left\{\frac{\alpha\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}\right\}^{\alpha-\theta(1-\alpha)} \left\{\frac{(1-\tau)}{\beta-1}\right\}^{\alpha-\theta(1-\alpha)}. \qquad \dots\dots(2.3.T.1)$$

The two latent roots of the Jacobian matrix will be real and negative in sign if the inequality (2.3.T.1) is reversed and the trace is negative. The transitional path converging to the steady-state equilibrium is indeterminate in that case. The trace is negative if

$$g_m \left[\{\alpha - \theta(1-\alpha)\}\left\{\frac{g_m}{(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}}\right\}^{\frac{1}{\alpha-\theta(1-\alpha)}} + \theta(1-\alpha)\frac{(1-\tau)}{(T-\delta)}\right] < \alpha. \qquad \dots\dots(2.3.T.2)$$

So, when $\beta > 1$, the low growth equilibrium characterized by $0 < g_m < \frac{\theta(1-\alpha)\rho}{\alpha(\beta-1)}$ is saddle-point stable if condition (2.3.T.1) is satisfied; and the transitional path to this equilibrium is indeterminate if the reverse of inequality (2.3.T.1) holds and inequality (2.3.T.2) is satisfied. The high growth equilibrium, characterized by $\frac{\theta(1-\alpha)\rho}{\alpha(\beta-1)} < g_m < \frac{\rho}{(\beta-1)}$, is also a saddle point or the transitional path is indeterminate for the same corresponding conditions mentioned above. Notably, each of these two equilibria is unstable when both the inequalities (2.3.T.1) and (2.3.T.2) are reversed.

Proposition 2.3.3: (i) When $\beta < 1$, the steady-state equilibrium is always saddle point stable.

(ii) When $\beta > 1$, either the two equilibria are unstable or they are saddle-point stable or the transitional path leading to each of those steady-state equilibria is indeterminate.

This result is important because Barro (1990) model, with a flow public expenditure, does not exhibit any transitional dynamic properties. FMS (1993) brings back transitional dynamic properties in Barro (1990) model introducing durable public input and shows that the equilibrium is necessarily a saddle point. But unlike in section 2.2.3, for different ranges of the growth rate we show here the possibility of indeterminacy and uniqueness in the transitional path converging to the long run equilibrium even with a flow public expenditure like that of Barro (1990). The environmental quality is a stock variable accumulating over time in this model; and this positively affects the utility of the representative household and the productivity of the system when $\theta > 0$. Also $\theta > 0$ implies the existence of a negative congestion effect of physical capital. Thus our flow public expenditure model is protected from being an AK model in this case too. Greiner (2005) model exhibits transitional dynamic properties treating environmental pollution as a flow variable entering the utility function and public input as a stock variable. However, the possibility of indeterminacy in the transitional path does not arise in his model because no stock variable enters the utility function there.

2.3.4 COMMAND ECONOMY

The market economy solution may be suboptimal due to the distortion caused by the proportional income tax and due to the failure of private individuals to internalize externalities in the system. The presence of two non-rival inputs, public good and environmental quality, cause positive externalities; and the congestion effect of physical capital and pollution introduce negative externalities. The planner, who maximizes a social welfare function identical to that of the representative household's lifetime utility function, can internalize these externalities. With a reminder that $A = 1$ and $\eta = 0$, equations (2.2.1), (2.2.2), (2.3.1), (2.2.3.1), (2.2.4.1) and (2.2.5.1) describe the model in the planned economy.

The planner's problem is to maximize $\int_0^\infty e^{-\rho t} C^{\beta\sigma} E^{\beta(1-\sigma)}\, dt$ with respect to C, Π and Ω subject to equations (2.2.1), (2.2.2), (2.2.3.1), (2.2.4.1) and (2.2.5.1). As in section 2.2.4, here too, the growth rate in the steady-state equilibrium, denoted by g_c, is the socially efficient growth rate. The following equation solves for the socially efficient growth rate[35].

$$\left\{\frac{\rho-(\beta-1)g_c}{\alpha-\theta(1-\alpha)}\right\}^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}} \left[\{\rho-(\beta-1)g_c\} + \frac{1-\sigma}{\sigma} g_c + \frac{1-\sigma}{\sigma} g_c \right.$$

$$\left. \left\{\frac{(1-\delta)^{\frac{1}{\alpha}}\overline{\alpha-\theta(1-\alpha)}(1-\alpha)^{\frac{1-\alpha}{\alpha}}}{\rho-(\beta-1)g_c}\right\}^{\frac{\alpha}{\theta(1-\alpha)}}\right] = (1-\alpha)^{\frac{1}{\theta}}(1-\delta)^{\frac{1}{\theta(1-\alpha)}}\left[\frac{\alpha-\sigma\{\alpha-\theta(1-\alpha)\}}{\sigma}\right].$$

…… (2.3.13)

The L.H.S. of equation (2.3.13) is a continuous function of g_c for all values of $g_c \in \left[\rho\left\{\frac{\rho}{\alpha-\theta(1-\alpha)}\right\}^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}, \frac{\rho}{\beta-1}\right]$; and its R.H.S. is a positive constant when $0<\delta<1$ and $\theta>0$. So the existence of at least one socially efficient growth rate is ensured when $0<\delta<1$, $\theta>0$, and when

$$\rho\left\{\frac{\rho}{\alpha-\theta(1-\alpha)}\right\}^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}} < (1-\alpha)^{\frac{1}{\theta}}(1-\delta)^{\frac{1}{\theta(1-\alpha)}}\left[\frac{\alpha-\sigma\{\alpha-\theta(1-\alpha)\}}{\sigma}\right] < \frac{\rho}{\beta-1}. \qquad \text{……} (2.3.T.3)$$

In the special case when $\beta=1$, equation (2.3.13) can be written as

$$g_c = \left(\frac{\sigma}{1-\sigma}\right)\left\{1+\left(\frac{(1-\delta)^{\frac{1}{\alpha}}(\alpha-\theta(1-\alpha))(1-\alpha)^{\frac{1-\alpha}{\alpha}}}{\rho}\right)^{\frac{\alpha}{\theta(1-\alpha)}}\right\}^{-1}$$

$$\left[(1-\alpha)^{\frac{1}{\theta}}(1-\delta)^{\frac{1}{\theta(1-\alpha)}}\left[\frac{\alpha-\sigma\{\alpha-\theta(1-\alpha)\}}{\sigma}\right]\left\{\frac{\alpha-\theta(1-\alpha)}{\rho}\right\}^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}} - \rho\right].$$

…… (2.3.13A)

Thus, in this special case, the existence of unique socially efficient growth rate is ensured. Now, we establish the following proposition.

[35] Equation (2.3.13) is derived in the Appendix 2.3F.

Proposition 2.3.4: If $\theta > 0$ and if $0 < \delta < 1$, then there exists at least one positive socially efficient growth rate when $(2.3.T.3)$ is satisfied. Further, if $\beta = 1$, this growth rate is unique.

Equations (2.3.6.2) and (2.3.13A) show that the steady-state equilibrium growth rate in the market economy as well as in the planned economy varies inversely with the magnitude of pollution output coefficient, δ. This is so because when production is highly polluting a larger fraction of the output is allocated to maintain efficiency of the productive public input. So less output is available for investment and for public input expenditure.

The relationship between market economy growth rate and the socially efficient growth rate involves ambiguity because the planner internalizes two conflicting types of externalities in this model. There is a negative externality whose root lies in the congestion effect of physical capital accumulation. However, positive externalities also result from the presence of the public intermediate good and the environmental quality. Negative externality on production and positive externality on utility does not exist in Barro (1990) model and in FMS (1993) model. In Greiner (2005), negative externality of environmental pollution affects the utility function but does not affect the productivity of private inputs.

APPENDIX 2.2A

DERIVATION OF EQUATION (2.2.7) IN SECTION 2.2.2:

The dynamic optimization problem of the representative household is to maximize $\int_0^\infty e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma}dt$ with respect to C subject to equation (2.2.4) and given $K(0)$. Here C is the control variable satisfying $0 \le C \le (1-\tau)Y$; and K is the state variable.

The Hamiltonian to be maximized at each point of time is given by

$$\mathcal{H} = e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma} + e^{-\rho t}\lambda_K[(1-\tau)Y - C].$$

Here λ_K is the co-state variable representing the shadow price of investment. Maximizing the Hamiltonian with respect to C and assuming an interior solution, we obtain

$$C^{-\sigma} = \lambda_K. \qquad \ldots\ldots(2.2A.1)$$

Also the optimum time path of λ_K satisfies the following.

$$\frac{\dot{\lambda_K}}{\lambda_K} = \rho - (1-\tau)\alpha A K^{\alpha-1}\hat{G}^{1-\alpha}. \qquad \ldots\ldots(2.2A.2)$$

Using equations (2.2.1), (2.2.2), (2.2.3) and (2.2A.2) we have

$$\frac{\dot{\lambda_K}}{\lambda_K} = \rho - \alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau - \mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}. \qquad \ldots\ldots(2.2A.3)$$

Using the two optimality conditions (2.2A.1) and (2.2A.3), we have

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau - \mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right]. \qquad \ldots\ldots(2.2A.4)$$

which is same as equation (2.2.7).

APPENDIX 2.2B

DERIVATION OF EQUATION (2.2.12) IN SECTION 2.2.2.1

Using equations (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.5), (2.2.7) and (2.2.8) we have the following equations.

$$g_m = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right]; \qquad \dots\dots(2.2B.1)$$

$$g_m = \frac{\dot{K}}{K} = A^{\frac{1}{\alpha}}(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \frac{C}{K}\,; \qquad \dots\dots(2.2B.2)$$

and

$$g_m = \frac{\dot{E}}{E} = A^{\frac{1}{\alpha}}(T-\delta)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} + \eta. \qquad \dots\dots(2.2B.3)$$

From equation (2.2B.1) we have,

$$\frac{E}{K} = \left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}(\sigma g_m+\rho)^{-1}\right]^{-\frac{\alpha}{\theta(1-\alpha)}}. \qquad \dots\dots(2.2B.4)$$

Using equations (2.2B.3) and (2.2B.4) we derive the following equation.

$$g_m = A^{\frac{1}{\alpha}}(T-\delta)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}(\sigma g_m+\rho)^{-1}\right]^{-\frac{\theta(1-\alpha)-\alpha}{\theta(1-\alpha)}} + \eta,$$

or,

$$g_m = A^{\frac{1}{\theta(1-\alpha)}}\alpha^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}(T-\delta)(1-\tau)^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}(\tau-\mathrm{T})^{\frac{1}{\theta}}(\sigma g_m+\rho)^{\frac{\theta(1-\alpha)-\alpha}{\theta(1-\alpha)}} + \eta,$$

or,

$$(g_m-\eta)(\sigma g_m+\rho)^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}} = \alpha^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}A^{\frac{1}{\theta(1-\alpha)}}(T-\delta)(1-\tau)^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}(\tau-\mathrm{T})^{\frac{1}{\theta}},$$

or,

$$(g_m-\eta)^{\theta(1-\alpha)}(\sigma g_m+\rho)^{\alpha-\theta(1-\alpha)}$$

$$= A\{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(T-\delta)^{\theta(1-\alpha)}(\tau-\mathrm{T})^{(1-\alpha)}. \qquad \dots\dots(2.2B.5)$$

This is same as equation (2.2.12).

APPENDIX 2.2C

DERIVATION OF EQUATIONS (2.2.13) AND (2.2.14) AND THE SECOND ORDER CONDITIONS IN SECTION 2.2.2.2:

We denote the L.H.S. and the R.H.S. of equation (2.2.12) by L.H.S.$_{(2.2.12)}$ and R.H.S.$_{(2.2.12)}$ respectively. Maximizing the R.H.S. of equation (2.2.12) with respect to τ, we obtain the following first order condition.

$$\text{R.H.S.}_{(2.2.12)}\,[(1-\alpha)(\tau-\mathrm{T})^{-1}-\{\alpha-\theta(1-\alpha)\}(1-\tau)^{-1}]=0;$$

or,

$$(1-\alpha)(1-\tau)=\{\alpha-\theta(1-\alpha)\}(\tau-\mathrm{T}). \qquad \dots\dots(2.2C.1)$$

Maximizing the R.H.S. of equation (2.2.12) with respect to T, we obtain the following first order condition.

$$\text{R.H.S.}_{(2.2.12)}\,[\theta(T-\delta)^{-1}-(\tau-\mathrm{T})^{-1}]=0;$$

or,

$$\theta(\tau-\mathrm{T})=(T-\delta). \qquad \dots\dots(2.2C.2)$$

Using equations (2.2C.1) and (2.2C.2) we obtain the following expressions.

$$\tau^{\star}=1-(1-\delta)\{\alpha-\theta(1-\alpha)\};$$

and

$$T^{\star}=\delta+(1-\delta)\theta(1-\alpha).$$

These are same as equations (2.2.13) and (2.2.14) in section 2.2.2.2.

To check the second order conditions for optimality we twice differentiate equation (2.2.12), with respect to τ and T respectively. At the equilibrium point $\text{L.H.S.}_{(2.2.12)}=\text{R.H.S.}_{(2.2.12)}$. We arrive at the following two second order conditions.

$$-[\theta(1-\alpha)(g_m-\eta)^{-2}+\{\alpha-\theta(1-\alpha)\}\sigma^2(\sigma g_m+\rho)^{-2}]\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+[\{\alpha-\theta(1-\alpha)\}\sigma(\sigma g_m+\rho)^{-1}+\theta(1-\alpha)(g_m-\eta)^{-1}]\frac{\partial^2 g_m}{\partial\tau^2}$$

$$= -\left[\frac{\{\alpha-\theta(1-\alpha)\}}{(1-\tau)^2} + \frac{(1-\alpha)}{(\tau-T)^2}\right];$$

and

$$-[\theta(1-\alpha)(g_m-\eta)^{-2} + \{\alpha-\theta(1-\alpha)\}\sigma^2(\sigma g_m+\rho)^{-2}]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+[\{\alpha-\theta(1-\alpha)\}\sigma(\sigma g_m+\rho)^{-1} + \theta(1-\alpha)(g_m-\eta)^{-1}]\frac{\partial^2 g_m}{\partial T^2}$$

$$= -\left[\frac{\theta}{(T-\delta)^2} + \frac{1}{(\tau-T)^2}\right].$$

Now we evaluate the above two second order conditions at $\tau = \tau^\star$ and $T = T^\star$ where $\frac{\partial g_m}{\partial \tau} = \frac{\partial g_m}{\partial T} = 0$ at the optimum. Hence we obtain the followings.

$$[\{\alpha-\theta(1-\alpha)\}\sigma(\sigma g_m+\rho)^{-1} + \theta(1-\alpha)(g_m-\eta)^{-1}]\frac{\partial^2 g_m}{\partial \tau^2}$$

$$= -\frac{\{\alpha-\theta(1-\alpha)\}+(1-\alpha)}{\theta\{\alpha-\theta(1-\alpha)\}[(1-\delta)(1-\alpha)]^2};$$

and

$$[\{\alpha-\theta(1-\alpha)\}\sigma(\sigma g_m+\rho)^{-1} + \theta(1-\alpha)(g_m-\eta)^{-1}]\frac{\partial^2 g_m}{\partial T^2} = -\frac{1+\theta}{\theta^2}.$$

$[\{\alpha-\theta(1-\alpha)\}\sigma(\sigma g_m+\rho)^{-1} + \theta(1-\alpha)(g_m-\eta)^{-1}] > 0$ and the R.H.S. of each of these two equations is negative. Hence the sign of both the second order derivatives is negative.

APPENDIX 2.2D

DERIVATION OF EQUATION (2.2.16) IN SECTION 2.2.2.2

Here the social welfare functional is given by $W = \int_0^\infty e^{-\rho t} u(C)\, dt$. From equation (2.2.9), we have

$$A^{\frac{1}{\alpha}}(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} = \frac{\sigma g_m+\rho}{\alpha}. \qquad \dots\dots(2.2D.1)$$

Using equations (2.2.10) and (2.2D.1), we have

$$\frac{C}{K} = \frac{\sigma g_m + \rho}{\alpha} - g_m = \frac{1}{\alpha}[\rho - (\alpha - \sigma)g_m]. \qquad \ldots\ldots(2.2D.2)$$

At the steady state equilibrium, $K = K(0)e^{g_m t}$; where $K(0)$ is the initial value of K. Thus equation (2.2D.2) can be written as

$$C = \frac{1}{\alpha}[\rho - (\alpha - \sigma)g_m]K(0)e^{g_m t}. \qquad \ldots\ldots(2.2D.3)$$

Using equations (2.2.6) and (2.2D.3) and the social welfare functional we have

$$W = \int_0^\infty e^{-\rho t} \frac{[\rho-(\alpha-\sigma)g_m]^{1-\sigma}[K(0)]^{1-\sigma}e^{g_m(1-\sigma)t}}{\alpha^{1-\sigma}(1-\sigma)} dt,$$

or,

$$W = \frac{[K(0)]^{1-\sigma}}{\alpha^{1-\sigma}(1-\sigma)}[\rho - (\alpha - \sigma)g_m]^{1-\sigma} \int_0^\infty e^{[g_m(1-\sigma)-\rho]t}\, dt.$$

For convergence we assume $\rho - g_m(1 - \sigma) > 0$. Thus,

$$W = \frac{[K(0)]^{1-\sigma}}{\alpha^{1-\sigma}(1-\sigma)}[\rho - (\alpha - \sigma)g_m]^{1-\sigma}[\rho - (1 - \sigma)g_m]^{-1},$$

or,

$$W = \alpha^{\sigma-1}\frac{[K(0)]^{1-\sigma}}{1-\sigma}\left[\frac{\rho-(\alpha-\sigma)g_m}{\rho-(1-\sigma)g_m}\right][\rho - (\alpha - \sigma)g_m]^{-\sigma}. \qquad \ldots\ldots(2.2D.4)$$

This is same as equation (2.2.16).

APPENDIX 2.2E

DERIVATION OF THE DETERMINANT OF THE JACOBIAN MATRIX IN SECTION 2.2.3

We consider the following equations from section 2.2.3.

$$\frac{\dot{x}}{x} = A^{\frac{1}{\alpha}}\left(\frac{\alpha}{\sigma} - 1\right)(1 - \tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x - \frac{\rho}{\sigma}; \qquad \ldots\ldots(2.2.17)$$

and

$$\frac{\dot{y}}{y} = (T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} - (1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x + \eta.$$

…… (2.2.18)

We obtain the following partial derivatives corresponding to these two equations.

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial x} = 1;$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial y} = \frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial x} = 1;$$

and

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial y} = -\frac{\alpha-\theta(1-\alpha)}{\alpha}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

So the determinant of the Jacobian matrix can be written as follows.

$$|J| = -\frac{\alpha-\theta(1-\alpha)}{\alpha}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$-\frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1};$$

or,

$$|J| = -\frac{\alpha-\theta(1-\alpha)}{\alpha}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\sigma}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

Here $\alpha > \theta(1-\alpha)$ and $1 > \tau > T > \delta$. Thus the determinant is negative in sign.

APPENDIX 2.2F

DERIVATION OF EQUATION (2.2.19) IN SECTION 2.2.4

The relevant Hamiltonian to be maximized by the planner at each point of time is given by

$$H = e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma} + e^{-\rho t}\lambda_K\left[A(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} - \Pi - C\right]$$
$$+e^{-\rho t}\lambda_E\left[\eta E + \Omega - \delta A(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)}\right].$$

The state variables are K and E. The three control variables are C, Π and Ω. λ_K and λ_E are two co-state variables.

Maximising H with respect to C, Π and Ω, we have

$$C^{-\sigma} = \lambda_K; \qquad \text{......} (2.2F.1)$$

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)(1-\alpha)A(\Pi-\Omega)^{-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} = \frac{\lambda_K}{\lambda_E}; \qquad \text{......} (2.2F.2)$$

and

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)(1-\alpha)A(\Pi-\Omega)^{-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} = 1. \qquad \text{......} (2.2F.3)$$

Using equations (2.2F.2) and (2.2F.3) we obtain

$$\frac{\lambda_K}{\lambda_E} = 1. \qquad \text{......} (2.2F.4)$$

Also, along the optimum path, time behaviour of the co-state variables satisfies the followings.

$$\left(1 - \delta\frac{\lambda_K}{\lambda_E}\right)\{\alpha - \theta(1-\alpha)\}A(\Pi-\Omega)^{1-\alpha}K^{\alpha-1-\theta(1-\alpha)}E^{\theta(1-\alpha)} = \rho - \frac{\dot{\lambda}_K}{\lambda_K};$$
$$\text{......} (2.2F.5)$$

and

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)\theta(1-\alpha)A(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)-1} + \eta = \rho - \frac{\dot{\lambda}_E}{\lambda_E}.$$
$$\text{......} (2.2F.6)$$

From equation (2.2F.1) we have,

$$-\sigma\frac{\dot{C}}{C}=\frac{\dot{\lambda_K}}{\lambda_K}. \qquad \ldots\ldots(2.2F.7)$$

Using equations (2.2F.3) and (2.2F.4) we obtain

$$\left(\frac{\Pi-\Omega}{K}\right)^{\alpha}=A(1-\delta)(1-\alpha)\left(\frac{E}{K}\right)^{\theta(1-\alpha)}. \qquad \ldots\ldots(2.2F.8)$$

Using equations (2.2F.4), (2.2F.5), (2.2F.6) and (2.2F.8) we obtain equations (2.2F.9) and (2.2F.10) as derived below.

$$\{A(1-\delta)\}^{\frac{1}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}}\{\alpha-\theta(1-\alpha)\}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}=\rho-\frac{\dot{\lambda_K}}{\lambda_K}; \qquad \ldots\ldots(2.2F.9)$$

and

$$\{A(1-\delta)\}^{\frac{1}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}\theta(1-\alpha)\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}-1}=\rho-\eta-\frac{\dot{\lambda_E}}{\lambda_E}; \qquad \ldots\ldots(2.2F.10)$$

From equations (2.2F.4), (2.2F.7) and (2.2F.9) we obtain

$$\frac{\dot{C}}{C}=\frac{1}{\sigma}\left[\{A(1-\delta)\}^{\frac{1}{\alpha}}(1-\alpha)^{\frac{1}{\alpha}}\{\alpha-\theta(1-\alpha)\}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}-\rho\right]. \qquad \ldots\ldots(2.2F.11)$$

In the steady-state growth equilibrium,

$$\frac{\dot{C}}{C}=\frac{1}{\sigma}\left[\{A(1-\delta)\}^{\frac{1}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}\{\alpha-\theta(1-\alpha)\}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}-\rho\right]=g_c; \quad \ldots\ldots(2.2F.12)$$

$$\frac{\dot{K}}{K}=A\left(\frac{K}{\Pi-\Omega}\right)^{\alpha-1}\left(\frac{E}{K}\right)^{\theta(1-\alpha)}-\frac{\Pi}{K}-\frac{C}{K}=g_c; \qquad \ldots\ldots(2.2F.13)$$

and

$$\frac{\dot{E}}{E}=\frac{\Omega}{E}-\delta A\left(\frac{K}{\Pi-\Omega}\right)^{\alpha-1}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-1}+\eta=g_c. \qquad \ldots\ldots(2.2F.14)$$

From equation (2.2F.12) we obtain

$$\frac{E}{K}=\{A(1-\delta)\}^{-\frac{1}{\theta(1-\alpha)}}(1-\alpha)^{-\frac{1-\alpha}{\theta(1-\alpha)}}\{\alpha-\theta(1-\alpha)\}^{-\frac{\alpha}{\theta(1-\alpha)}}(\rho+\sigma g_c)^{\frac{\alpha}{\theta(1-\alpha)}}.$$

$$\ldots\ldots(2.2F.15)$$

Now using equations (2.2F.4), (2.2F.7), (2.2F.10) and (2.2F.15) we obtain

$$(\rho+\sigma g_c)^{\alpha-\theta(1-\alpha)}(\rho+\sigma g_c-\eta)^{\theta(1-\alpha)}$$

$$= A(1-\delta)(1-\alpha)^{1-\alpha}\{\alpha-\theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \qquad \ldots\ldots (2.2F.16)$$

This is same as equation (2.2.19) in section 2.2.4.

APPENDIX 2.3A

DERIVATION OF EQUATION (2.3.2) IN SECTION 2.3.2

The dynamic optimization problem of the representative household is to maximize $\int_0^\infty e^{-\rho t} C^{\beta\sigma} E^{\beta(1-\sigma)}\, dt$ with respect to C subject to equation (2.2.4) and given $K(0)$. Here C is the control variable satisfying $0 \le C \le (1-\tau)Y$; and K is the state variable.

The Hamiltonian to be maximized at each point of time is given by

$$\mathcal{H} = e^{-\rho t} C^{\beta\sigma} E^{\beta(1-\sigma)} + e^{-\rho t}\lambda_K[(1-\tau)Y - C].$$

Here λ_K is the co-state variable representing the shadow price of investment. Maximizing the Hamiltonian with respect to C and assuming an interior solution, we obtain

$$\beta\sigma C^{\beta\sigma-1} E^{\beta(1-\sigma)} = \lambda_K. \qquad \ldots\ldots (2.3A.1)$$

Also the optimum time path of λ_K satisfies the following.

$$\frac{\dot{\lambda}_K}{\lambda_K} = \rho - \alpha(1-\tau)K^{\alpha-1}\hat{G}^{1-\alpha}. \qquad \ldots\ldots (2.3A.2)$$

Using equations (2.2.1), (2.2.2), (2.2.3) and (2.3A.2) we have

$$\frac{\dot{\lambda}_K}{\lambda_K} = \rho - \alpha(1-\tau)(\tau - \mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}. \qquad \ldots\ldots (2.3A.3)$$

Using equations (2.2.1) to (2.2.3), (2.2.5) and the two optimality conditions (2.3A.1) and (2.3A.3), we have

$$\frac{\dot{C}}{C}=\frac{1}{1-\beta\sigma}\Bigg[\alpha(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}$$

$$+\beta(1-\sigma)(T-\delta)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}-1}-\rho\Bigg]. \qquad \dots\dots(2.3A.4)$$

This is same as equation (2.3.2).

APPENDIX 2.3B

DERIVATION OF EQUATION (2.3.6) IN SECTION 2.3.2.1

Using equations (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.5), (2.3.2) and (2.2.8), we have

$$g_m=\frac{\dot{C}}{C}=\frac{1}{1-\beta\sigma}\Bigg[\alpha(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}$$

$$+\beta(1-\sigma)(T-\delta)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}-1}-\rho\Bigg]; \qquad \dots\dots(2.3B.1)$$

$$g_m=\frac{\dot{K}}{K}=(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}-\frac{C}{K}\ ; \qquad \dots\dots(2.3B.2)$$

and

$$g_m=\frac{\dot{E}}{E}=(T-\delta)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}}. \qquad \dots\dots(2.3B.3)$$

From equation (2.3B.3) we have,

$$\frac{E}{K}=\left[\frac{g_m}{(T-\delta)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}}\right]^{\frac{\alpha}{\theta(1-\alpha)-\alpha}}. \qquad \dots\dots(2.3B.4)$$

Using equations (2.3B.1) and (2.3B.4) we derive the following equation.

$$g_m{}^{\theta(1-\alpha)}\{\rho-(\beta-1)g_m\}^{\alpha-\theta(1-\alpha)}=\{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(T-\delta)^{\theta(1-\alpha)}(\tau-T)^{1-\alpha},$$

$$\dots\dots(2.3B.5)$$

This is same as equation (2.3.6) in section 2.3.2.1.

APPENDIX 2.3C

DERIVATION OF EQUATIONS (2.3.7) AND (2.3.8) AND THE SECOND ORDER CONDITIONS IN SECTION 2.3.2.2

We denote the L.H.S. and the R.H.S. of equation (2.3.6) by L.H.S.$_{(2.3.6)}$ and R.H.S.$_{(2.3.6)}$ respectively. Maximizing the R.H.S. of equation (2.3.6) with respect to τ, we obtain the following first order condition.

$$\text{R.H.S.}_{(2.3.6)}\,[(1-\alpha)(\tau-\mathrm{T})^{-1}-\{\alpha-\theta(1-\alpha)\}(1-\tau)^{-1}]=0;$$

or,

$$(1-\alpha)(1-\tau)=\{\alpha-\theta(1-\alpha)\}(\tau-\mathrm{T}). \qquad \ldots\ldots(2.3C.1)$$

Maximizing the R.H.S. of equation (2.3.6) with respect to T, we obtain the following first order condition.

$$\text{R.H.S.}_{(2.3.6)}\,[\theta(T-\delta)^{-1}-(\tau-\mathrm{T})^{-1}]=0;$$

or,

$$\theta(\tau-\mathrm{T})=(T-\delta). \qquad \ldots\ldots(2.3C.2)$$

Using equations (2.3C.1) and (2.3C.2) we obtain the following expressions.

$$\tau^{\star}=1-(1-\delta)\{\alpha-\theta(1-\alpha)\};$$

and

$$T^{\star}=\delta+(1-\delta)\theta(1-\alpha).$$

These are same as equations (2.3.7) and (2.3.8).

To check the second order conditions for optimality we twice differentiate equation (2.3.6), with respect to τ and T respectively. At the equilibrium point $\text{L.H.S.}_{(2.3.6)}=\text{R.H.S.}_{(2.3.6)}$. We arrive at the following two second order conditions.

$$-[\theta(1-\alpha)g_m{}^{-2}+\{\alpha-\theta(1-\alpha)\}(\beta-1)^2\{\rho-(\beta-1)g_m\}^{-2}]\left(\frac{\partial g_m}{\partial \tau}\right)^2$$

$$+[\theta(1-\alpha)g_m^{-1}-\{\alpha-\theta(1-\alpha)\}(\beta-1)\{\rho-(\beta-1)g_m\}^{-1}]\frac{\partial^2 g_m}{\partial \tau^2}$$

$$=-\left[\frac{\{\alpha-\theta(1-\alpha)\}}{(1-\tau)^2}+\frac{(1-\alpha)}{(\tau-T)^2}\right].$$

and

$$-[\theta(1-\alpha)g_m{}^{-2}+\{\alpha-\theta(1-\alpha)\}(\beta-1)^2\{\rho-(\beta-1)g_m\}^{-2}]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+[\theta(1-\alpha)g_m^{-1}-\{\alpha-\theta(1-\alpha)\}(\beta-1)\{\rho-(\beta-1)g_m\}^{-1}]\frac{\partial^2 g_m}{\partial T^2}$$

$$=-\left[\frac{\theta}{(T-\delta)^2}+\frac{1}{(\tau-T)^2}\right].$$

Now we evaluate the above two second order conditions at $\tau=\tau^\star$ and $T=T^\star$ where $\frac{\partial g_m}{\partial \tau}=\frac{\partial g_m}{\partial T}=0$, at the optimum. Hence we obtain the followings.

$$[\theta(1-\alpha)g_m^{-1}-\{\alpha-\theta(1-\alpha)\}(\beta-1)\{\rho-(\beta-1)g_m\}^{-1}]\frac{\partial^2 g_m}{\partial \tau^2}$$

$$=-\frac{[(1-\delta)\{\alpha-\theta(1-\alpha)\}]^2(1-\alpha)+[(1-\delta)(1-\alpha)]^2\{\alpha-\theta(1-\alpha)\}}{\theta(1-\alpha)[(1-\delta)\{\alpha-\theta(1-\alpha)\}]^2[(1-\delta)(1-\alpha)]^2};$$

and

$$[\theta(1-\alpha)g_m^{-1}-\{\alpha-\theta(1-\alpha)\}(\beta-1)\{\rho-(\beta-1)g_m\}^{-1}]\frac{\partial^2 g_m}{\partial T^2}$$

$$=-\frac{\theta[(1-\delta)(1-\alpha)]^2+[(1-\delta)\theta(1-\alpha)]^2}{\theta[(1-\delta)\theta(1-\alpha)]^2}.$$

When $\beta\le 1$, $[\theta(1-\alpha)g_m^{-1}-\{\alpha-\theta(1-\alpha)\}(\beta-1)\{\rho-(\beta-1)g_m\}^{-1}]$ is always positive; and the R.H.S. of each of these two equations is negative. Thus the sign of each of the two second order derivatives is negative.

APPENDIX 2.3D

DERIVATION OF EQUATION (2.3.10) IN SECTION 2.3.2.2

Here the social welfare functional is given by $W = \int_0^\infty e^{-\rho t} u(C, E)\, dt$. From equations (2.3.3) and (2.3.5), we have

$$(1-\tau)(\tau - \mathrm{T})^{\frac{1-\alpha}{\alpha}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} = \frac{\rho-(\beta-1)g_m}{\alpha}. \qquad \dots\dots(2.3D.1)$$

Using equations (2.3.4) and (2.3.5), we have

$$\frac{C}{K} = \frac{\rho-(\beta-1)g_m}{\alpha} - g_m = \frac{1}{\alpha}[\rho - (\alpha+\beta-1)g_m]. \qquad \dots\dots(2.3D.2)$$

Along the steady state equilibrium growth path, $K = K(0)e^{g_m t}$; where $K(0)$ is the initial value of K. Thus equation (2.3D.2) can be written as

$$C = \frac{1}{\alpha}[\rho - (\alpha+\beta-1)g_m]K(0)e^{g_m t}. \qquad \dots\dots(2.3D.3)$$

Using equations (2.3.1) and (2.3D.3) and the social welfare functional, we have

$$W = \int_0^\infty e^{-\rho t} \frac{[\rho-(\alpha+\beta-1)g_m]^{\beta\sigma}[K(0)]^{\beta\sigma}e^{\beta\sigma g_m t}}{\alpha^{\beta\sigma}} [E(0)]^{\beta(1-\sigma)} e^{\beta(1-\sigma)g_m t}\, dt,$$

or,

$$W = \frac{[K(0)]^{\beta\sigma}[E(0)]^{\beta(1-\sigma)}}{\alpha^{\beta\sigma}} [\rho - (\alpha+\beta-1)g_m]^{\beta\sigma} \int_0^\infty e^{[\beta g_m - \rho]t}\, dt.$$

For convergence we assume $\beta g_m - \rho < 0$. Thus,

$$W = \frac{[K(0)]^{\beta\sigma}[E(0)]^{\beta(1-\sigma)}}{\alpha^{\beta\sigma}} [\rho - (\alpha+\beta-1)g_m]^{\beta\sigma} [\rho - \beta g_m]^{-1},$$

or,

$$W = \frac{[K(0)]^{\beta\sigma}[E(0)]^{\beta(1-\sigma)}}{\alpha^{\beta\sigma}} \left[\frac{\rho-\beta g_m+(1-\alpha)g_m}{\rho-\beta g_m}\right]^{\beta\sigma} [\rho - \beta g_m]^{\beta\sigma-1}. \qquad \dots\dots(2.3D.4)$$

This is same as equation (2.3.10) in section 2.3.2.2.

APPENDIX 2.3E

DERIVATION OF THE DETERMINANT OF THE JACOBIAN MATRIX IN SECTION 2.3.3

We consider the following equations from the body of the paper.

$$\frac{\dot{x}}{x}=\left(\frac{\alpha}{1-\beta\sigma}-1\right)(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}}$$

$$+\frac{\beta(1-\sigma)}{1-\beta\sigma}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-1}+x-\frac{\rho}{1-\beta\sigma}; \qquad \ldots\ldots(2.3.11)$$

and

$$\frac{\dot{y}}{y}=(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)-\alpha}{\alpha}}-(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}}+x. \qquad \ldots\ldots(2.3.12)$$

We obtain the following partial derivatives corresponding to these two equations.

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial x}=1;$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial y}=\frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{1-\beta\sigma}-1\right)(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$+\left(\frac{\theta(1-\alpha)}{\alpha}-1\right)\frac{\beta(1-\sigma)}{1-\beta\sigma}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-2};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial x}=1;$$

and

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial y}=-\frac{\alpha-\theta(1-\alpha)}{\alpha}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

So the determinant of the Jacobian matrix can be written as follows.

$$|J|=\left\{\frac{\alpha-\theta(1-\alpha)}{\alpha}\right\}\left(\frac{\beta-1}{1-\beta\sigma}\right)(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{1-\beta\sigma}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

Here $\alpha > \theta(1-\alpha)$ and $1 > \tau > T > \delta$. So the determinant of the Jacobian matrix takes a positive (negative) sign if the following inequality is satisfied.

$$\left\{\frac{\alpha-\theta(1-\alpha)}{\alpha}\right\}\left(\frac{\beta-1}{1-\beta\sigma}\right)(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$> (<)\frac{\theta(1-\alpha)}{1-\beta\sigma}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-1},$$

or,

$$(1-\tau)y < (>)\frac{\alpha-\theta(1-\alpha)}{\alpha\theta(1-\alpha)}(\beta-1)(T-\delta).$$

Using the steady state equilibrium value of y given by equation (2.3B.4) in the inequality mentioned above we have,

$$g_m^{\alpha} < (>)(\tau-T)^{1-\alpha}(T-\delta)^{\theta(1-\alpha)}\left\{\frac{\alpha\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}\right\}^{\alpha-\theta(1-\alpha)}\left\{\frac{(1-\tau)}{\beta-1}\right\}^{\alpha-\theta(1-\alpha)}.$$

$$\ldots\ldots(2.3E.1)$$

The trace of the Jacobian matrix is given by

$$Tr\,J = 1-\left\{\frac{\alpha-\theta(1-\alpha)}{\alpha}\right\}(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

Again using equation (2.3B.4), we can show that the trace can take positive (negative) sign if

$$g_m\left[\{\alpha-\theta(1-\alpha)\}\left\{\frac{g_m}{(T-\delta)(\tau-T)^{\frac{1-\alpha}{\alpha}}}\right\}^{\frac{1}{\alpha-\theta(1-\alpha)}}+\theta(1-\alpha)\frac{(1-\tau)}{(T-\delta)}\right] < (>)\,\alpha.$$

APPENDIX 2.3F

DERIVATION OF EQUATION (2.3.13) IN SECTION 2.3.4

The relevant Hamiltonian to be maximized by the planner at each point of time is given by

$$H = e^{-\rho t}C^{\beta\sigma}E^{\beta(1-\sigma)} + e^{-\rho t}\lambda_K\left[(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} - \Pi - C\right]$$

$$+e^{-\rho t}\lambda_E\left[\Omega - \delta(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)}\right].$$

The state variables are K and E. The three control variables are C, Π and Ω. λ_K and λ_E are two co-state variables.

Maximising H with respect to C, Π and Ω, we have

$$\beta\sigma C^{\beta\sigma-1}E^{\beta(1-\sigma)} = \lambda_K,$$

or,

$$C^{\beta\sigma}E^{\beta(1-\sigma)} = \frac{C\lambda_K}{\beta\sigma}; \qquad \text{… …}(2.3F.1)$$

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)(1-\alpha)\left(\frac{\Pi-\Omega}{E}\right)^{-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha} = \frac{\lambda_K}{\lambda_E}; \qquad \text{… …}(2.3F.2)$$

and

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)(1-\alpha)\left(\frac{\Pi-\Omega}{E}\right)^{-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha} = 1. \qquad \text{… …}(2.3F.3)$$

Using equations (2.3F.2) and (2.3F.3) we obtain

$$\frac{\lambda_K}{\lambda_E} = 1. \qquad \text{… …}(2.3F.4)$$

Also, along the optimum path, time behaviour of the co-state variables satisfies the followings.

$$\left(1 - \delta\frac{\lambda_E}{\lambda_K}\right)\{\alpha - \theta(1-\alpha)\}\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)+(1-\alpha)} = \rho - \frac{\dot{\lambda}_K}{\lambda_K}; \qquad \text{… …}(2.3F.5)$$

and

$$\frac{\beta(1-\sigma)}{\lambda_E}C^{\beta\sigma}E^{\beta(1-\sigma)-1} + \left(\frac{\lambda_K}{\lambda_E} - \delta\right)\theta(1-\alpha)\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha}$$

$$= \rho - \frac{\dot{\lambda_E}}{\lambda_E}. \qquad \text{... ... } (2.3F.6)$$

Using equations (2.3F.3) and (2.3F.4), we have

$$\frac{\Pi - \Omega}{E} = \{(1-\alpha)(1-\delta)\}^{\frac{1}{\alpha}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}}. \qquad \text{... ... } (2.3F.7)$$

From equation (2.3F.1) we have,

$$(\beta\sigma - 1)\frac{\dot{C}}{C} + \beta(1-\sigma)\frac{\dot{E}}{E} = \frac{\dot{\lambda_K}}{\lambda_K}. \qquad \text{... ... } (2.3F.8)$$

Using equations (2.3F.1) and (2.3F.6) we obtain

$$\frac{\beta(1-\sigma)}{\beta\sigma}\left(\frac{C}{E}\right)\left(\frac{\lambda_K}{\lambda_E}\right) + \left(\frac{\lambda_K}{\lambda_E} - \delta\right)\theta(1-\alpha)\left(\frac{\Pi-\Omega}{\text{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha} = \rho - \frac{\dot{\lambda_E}}{\lambda_E}.$$

$$\text{... ... } (2.3F.9)$$

From equations (2.3F.4), (2.3F.5) and (2.3F.8) we obtain

$$\frac{\dot{C}}{C} = \frac{1}{1-\beta\sigma}\left[(1-\delta)\{\alpha - \theta(1-\alpha)\}\left(\frac{\Pi-\Omega}{\text{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)+(1-\alpha)} + \beta(1-\sigma)\frac{\dot{E}}{E} - \rho\right].$$

$$\text{... ... } (2.3F.10)$$

In the steady state growth equilibrium,

$$\frac{\dot{C}}{C} = \frac{1}{1-\beta\sigma}\left[(1-\delta)\{\alpha - \theta(1-\alpha)\}\left(\frac{\Pi-\Omega}{\text{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)+(1-\alpha)} + \beta(1-\sigma)\frac{\dot{E}}{E} - \rho\right]$$

$$= g_c; \qquad \text{... ... } (2.3F.11)$$

$$\frac{\dot{K}}{K} = \left(\frac{\Pi-\Omega}{\text{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)+1-\alpha} - \frac{\Pi}{K} - \frac{C}{K} = g_c; \qquad \text{... ... } (2.3F.12)$$

and

$$\frac{\dot{E}}{E} = \frac{\Omega}{E} - \delta\left(\frac{\Pi-\Omega}{\text{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha} = g_c. \qquad \text{... ... } (2.3F.13)$$

From equations (2.3F.11) and (2.3F.13), we have

$$(1-\beta\sigma)g_c = (1-\delta)\{\alpha - \theta(1-\alpha)\}\left(\frac{\Pi-\Omega}{\text{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)+(1-\alpha)}$$

$$-\rho + \beta(1-\sigma)g_c,$$

or,

$$(\beta-1)g_c=\rho-(1-\delta)\{\alpha-\theta(1-\alpha)\}\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)+(1-\alpha)}. \qquad \text{… … (2.3F.14)}$$

Manipulating equation (2.3F.12) we obtain

$$\left(\frac{E}{K}\right)^{-1}g_c=\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha}-\frac{\Pi}{E}-\frac{C}{E}. \qquad \text{… … (2.3F.15)}$$

Adding equations (2.3F.13) and (2.3F.15) we have

$$\left[\left(\frac{E}{K}\right)^{-1}+1\right]g_c=(1-\delta)\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha}-\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)-\frac{C}{E},$$

or,

$$\frac{C}{E}=(1-\delta)\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha}-\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)-\left[\left(\frac{E}{K}\right)^{-1}+1\right]g_c. \qquad \text{… … (2.3F.16)}$$

Using equations (2.3F.7) and (2.3F.16), we have

$$\frac{C}{E}=\{(1-\alpha)(1-\delta)\}^{\frac{1}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}}[(1-\alpha)^{-1}-1]-\left[\left(\frac{E}{K}\right)^{-1}+1\right]g_c. \qquad \text{… … (2.3F.17)}$$

Using equations (2.3F.7) and (2.3F.14), we obtain

$$(1-\delta)^{\frac{1}{\alpha}}\{\alpha-\theta(1-\alpha)\}(1-\alpha)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}=\rho-(\beta-1)g_c. \qquad \text{… … (2.3F.18)}$$

Thus from equation (2.3F.18), we have

$$\left(\frac{E}{K}\right)=\left[\frac{\rho-(\beta-1)g_c}{(1-\delta)^{\frac{1}{\alpha}}\{\alpha-\theta(1-\alpha)\}(1-\alpha)^{\frac{1-\alpha}{\alpha}}}\right]^{\frac{\alpha}{\theta(1-\alpha)}}. \qquad \text{… … (2.3F.19)}$$

Now, using equations (2.3F.4), (2.3F.8), (2.3F.11) and (2.3F.13) in equation (2.3F.9), we obtain

$$\frac{\beta(1-\sigma)}{\beta\sigma}\left(\frac{C}{E}\right)+(1-\delta)\theta(1-\alpha)\left(\frac{\Pi-\Omega}{\mathrm{E}}\right)^{1-\alpha}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-\alpha}=\rho-(\beta-1)g_c. \qquad \text{… … (2.3F.20)}$$

Again, using equations (2.3F.7), (2.3F.17), (2.3F.19) and (2.3F.20), we finally arrive at the following equation that solves for g_c.

$$\left\{\frac{\rho-(\beta-1)g_c}{\alpha-\theta(1-\alpha)}\right\}^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}}\left[\{\rho-(\beta-1)g_c\}+\frac{1-\sigma}{\sigma}g_c+\frac{1-\sigma}{\sigma}g_c\right.$$

$$\left.\left\{\frac{(1-\delta)^{\frac{1}{\alpha}}\overline{\alpha-\theta(1-\alpha)}(1-\alpha)^{\frac{1-\alpha}{\alpha}}}{\rho-(\beta-1)g_c}\right\}^{\frac{\alpha}{\theta(1-\alpha)}}\right]=(1-\alpha)^{\frac{1}{\theta}}(1-\delta)^{\frac{1}{\theta(1-\alpha)}}\left[\frac{\alpha-\sigma\{\alpha-\theta(1-\alpha)\}}{\sigma}\right].$$

$$\ldots\ldots\ (2.3F.21)$$

This is same as equation (2.3.13) in section 2.3.4.

CHAPTER 3

3. ALTERNATIVE SOURCES OF POLLUTION

3.1 INTRODUCTION

Models developed by Greiner (2005), EP (2008), etc., do not consider sources of pollution to be any other than output. Our basic model developed in chapter 2 also assumes level of production to be the only source of pollution. In this chapter, we develop a model of endogenous economic growth where pollution is generated by two alternative sources – consumption and capital usage. We explore the properties of optimal fiscal policy in the presence of productive public expenditure and environmental degradation caused by these two alternative sources of pollution.

3.2 CONSUMPTION AS THE SOURCE OF POLLUTION[36]

In this section we consider consumption to be the only source of pollution. Otherwise, the model developed in this section is identical to that developed in section 2.2 of chapter 2. Consumption of natural resources and consumption of energy-intensive luxury goods are important sources of pollution. Consumption of automobile services leads to air and sound pollution. Consumption of various electronic appliances leads to radiation and sound pollution. Household wastes and municipal sewages causing pollution of water bodies are by-products of consumption activities. Fossil fuels like coal, wood, kerosene oil, etc., are burnt for consumption in the rural areas. Some

[36] A related version of this model is published in Economic Modelling.

models in the existing literature adopt this consumption-caused-pollution hypothesis. Liddle (2001) develop a simulation model to explore trade and environment in the context of development where natural resource is an internationally traded good. He considers both production and consumption as sources of pollution; to counter this pollution abatement policy is explicitly modelled in an open economy thus allowing for differences in pollution-intensive technology across countries.

The many interesting results derived from this model are summarized as follows. The optimum ratio of productive public expenditure to national income is equal to the competitive share of the public input in the output of the final good; and this optimum ratio is independent of the rate of pollution. This result is different from that derived in chapter 2, section 2.2, where this optimum ratio is dependent on the rate of pollution. However, the optimum proportional income tax rate in this model is greater than this competitive output share of public input because a positive fraction of output is spent as abatement expenditure. Also, this optimum tax rate and the abatement expenditure rate depend on the rate of pollution. In Barro (1990) and in FMS (1993), there is no environmental pollution and abatement cost; and hence this ratio of productive public expenditure to national income is always equal to the proportional income tax rate, whose optimum value, in turn, is equal to the competitive output share of the public input. In Greiner (2005), this ratio depends on the pollution-output coefficient. Secondly, in this model, the optimal tax rate and the optimal abatement expenditure rate are functions of the steady-state equilibrium growth rate. So, in this model, the optimal values of the fiscal instruments and the steady-state equilibrium growth rate are determined simultaneously. However, in Barro (1990) and in FMS (1993), the optimum tax rate is determined independent of the growth rate and the same is true for Greiner (2005). Thirdly, our model exhibits transitional dynamic properties though it follows Barro (1990) to assume productive public expenditure to be a flow variable. Environmental quality like that in the basic model of section 2.2 is an accumulable input in this model; and this protects it from being trapped into an AK model. Fourthly, like Barro (1990) and FMS (1993), and like the basic model, there is no conflict between the growth rate maximizing solution and the social welfare maximizing solution in the steady-state growth equilibrium in our model. Greiner (2005) does not find such a conflict in the case of an income tax policy but finds it in the case of a pollution tax policy because pollution directly affects the utility of the household in his model. Fifthly, the competitive equilibrium growth rate in this model does not necessarily fall short of the socially efficient growth rate which is unlike in Barro (1990) or in FMS (1993). This is so because we consider externalities of conflicting nature on production - positive externalities resulting from the public expenditure and the environmental quality and negative externalities resulting from capital accumulation and environmental pollution. Barro (1990) as well as FMS (1993) consider only a positive externality in their models.

Section 3.2.1 describes the basic model of the market economy; and its steady-state equilibrium properties related to fiscal policies are presented in section 3.2.2. Section 3.2.3 shows transitional dynamic results and section 3.2.4 describes working of the command economy.

3.2.1 THE MODEL

We borrow equations (2.2.1) to (2.2.4) and (2.2.6) from the basic model, keeping in mind that $\theta_1 = \theta_2 = \theta$ in equation (2.2.2). Equation (2.2.5) of the basic model is modified as equation (3.2.1) because consumption of the final good is the only source of environmental pollution. Thus, the equations below together summarize this extended model.

$$Y = AK^{\alpha}\hat{G}^{1-\alpha} \; with \; 0 < \alpha < 1; \qquad \text{… … (2.2.1)}$$

$$\hat{G} = G\bar{K}^{-\theta}E^{\theta} \; with \; 0 < \theta < 1; \qquad \text{… … (2.2.2)}$$

$$G = (\tau - T)Y \; with \; 0 < T < \tau < 1; \qquad \text{… … (2.2.3)}$$

$$\dot{K} = (1 - \tau)Y - C; \qquad \text{… … (2.2.4)}$$

$$\dot{E} = TY - \delta C \; with \; 0 < \delta < 1; \qquad \text{......(3.2.1)}$$

and

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma} \; with \; \sigma > 0. \qquad \text{......(2.2.6)}$$

Equation (3.2.1) shows how environmental quality improves over time depending upon the magnitudes of pollution and abatement expenditure. Abatement activities bring improvements in environmental quality. TY is the abatement expenditure made by the government. Environmental pollution is assumed to be a flow variable and to be proportional to the level of consumption of the final good; and δ is the constant pollution-consumption coefficient.

Stocks of E and K are exogenous at a particular point of time. E is a non rival stock and G is a non rival flow. Given the stocks of capital and environmental quality, and given the fiscal instrument rates, equations (2.2.1), (2.2.2) and (2.2.3) together determine Y and G at each point of time. The household then chooses C and this determines the absolute rate of private capital accumulation, $\dot{K}$, given by equation (2.2.4). Then equation (3.2.1) determines the absolute rate of improvement in the environmental quality, denoted by $\dot{E}$.

3.2.2 DYNAMIC EQUILIBRIUM AND STEADY-STATE

The representative consumer's optimization problem in this extension is identical to that in the basic model in section 2.2.2. The demand rate of growth[37] of consumption is given by

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right]. \qquad \text{......(3.2.2)}$$

[37] Derivation of demand rate of growth of consumption here is identical to that of equation (2.2.7) in Appendix 2.2A of chapter 2.

A steady-state growth equilibrium is considered where all macroeconomic variables grow at the same rate, g_m. Thus, we again consider the steady-state condition given by equation 2.2.8 in chapter 2.

We now turn to analyze the existence of unique steady-state equilibrium growth rate in the market economy. Using equations (2.2.1), (2.2.2), (2.2.3), (2.2.4), (3.2.1), (3.2.2) and (2.2.8) we obtain following equations.

$$\frac{1}{\sigma}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}-\rho\right]=g_m; \qquad \dots\dots(3.2.3)$$

$$A^{\frac{1}{\alpha}}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}-\frac{C}{K}=g_m; \qquad \dots\dots(3.2.4)$$

and

$$A^{\frac{1}{\alpha}}T(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}}-\delta\left(\frac{C}{K}\right)\left(\frac{E}{K}\right)^{-1}=g_m. \qquad \dots\dots(3.2.5)$$

Using equations (3.2.3), (3.2.4) and (3.2.5) we obtain the following equation[38] to solve for g_m.

$$(\sigma g_m+\rho)^{\alpha}g_m^{\theta(1-\alpha)}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right]^{-\theta(1-\alpha)}$$

$$=A\alpha^{\alpha}(1-\tau)^{\alpha}(\tau-T)^{1-\alpha}. \qquad \dots\dots(3.2.6)$$

The solution is unique if $\frac{T}{(1-\tau)}>\delta$ because the L.H.S. of equation (3.2.6) is a positive function of g_m in that case; and its R.H.S. is a parametric constant, given the income tax rate and the abatement expenditure rate.

Equations (3.2A.4) and (3.2A.5) in Appendix 3.2A express $\frac{E}{K}$ and $\frac{C}{K}$ in terms of g_m; and thus we can prove the uniqueness of steady-state equilibrium.

We have the following proposition.

Proposition 3.2.1: There exists unique steady-state equilibrium growth rate in the market economy given the income tax rate and the abatement expenditure rate, if $\frac{T}{(1-\tau)}>\delta$.

[38] The derivation of equation (3.2.6) is worked out in Appendix 3.2A.

In equation (3.2.6), the existence of unique balanced growth path is ensured if $\frac{T}{(1-\tau)} > \delta$. Here $\delta(1-\tau)$ is the effective pollution-output coefficient when pollution arises from consumption. This is so because consumption depends on disposable income, $(1-\tau)Y$. Also, the abatement expenditure rate, T, must exceed the effective pollution-output coefficient.

3.2.2.1 Optimal Taxation

The government maximizes growth rate in the steady-state equilibrium with respect to fiscal instruments, τ and T. The L.H.S. of equation (3.2.6) is a monotonically increasing function of g_m, because, by assumption, $\frac{T}{(1-\tau)} > \delta$. Since the equality between the L.H.S. and the R.H.S. is always ensured in the steady-state growth equilibrium, maximization of g_m means maximization of the R.H.S. of equation (3.2.6). Maximizing the R.H.S. of equation (3.2.6) with respect to τ and T respectively, we obtain the following two equations[39].

$$\theta(1-\alpha)\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right]^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}(\tau-T)=\frac{\alpha(\tau-T)-(1-\alpha)(1-\tau)}{T}; \quad \text{......} (3.2.7)$$

and

$$\theta(1-\alpha)\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right]^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}(\tau-T)=1-\alpha. \quad \text{......} (3.2.8)$$

Using these two equations we obtain the optimum income tax rate and the optimum abatement expenditure rate.

$$\tau^{\star}=\frac{\{(1+\theta)(1-\alpha)\}(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}}{(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}}=\frac{[1-\{\alpha-\theta(1-\alpha)\}](\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}}{(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}}<1; \quad \text{......} (3.2.9)$$

and

$$T^{\star}=\frac{\theta(1-\alpha)(\sigma g_m+\rho)+\alpha\delta\{(\sigma-\alpha)g_m+\rho\}}{(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}}<1. \quad \text{......} (3.2.10)$$

[39] The derivation of equations (3.2.7) and (3.2.8) is worked out in Appendix 3.2B.

Using equations (3.2.9) and (3.2.10), the sufficient condition for the existence of unique steady-state equilibrium can be expressed as

$$\theta(1-\alpha)(\sigma g_m + \rho) + \alpha\delta\{(\sigma-\alpha)g_m + \rho\} > \delta\{\alpha - \theta(1-\alpha)\}(\sigma g_m + \rho);$$

or,

$$g_m > -\frac{(1+\delta)\theta(1-\alpha)\rho}{\theta(1-\alpha)\sigma + \alpha\delta(\sigma-\alpha) - \delta\{\alpha-\theta(1-\alpha)\}\sigma}.$$

So the optimum income tax rate and the optimum abatement expenditure rate are functions of the steady-state equilibrium growth rate, g_m; and equations (3.2.6), (3.2.9) and (3.2.10) solve for g_m, $\tau^\star$ and $T^\star$ simultaneously. In Barro (1990) and in FMS (1993), $T = 0$, by assumption. So, $\tau^\star = 1-\alpha$ and hence is independent of the growth rate.

It can easily be shown that

$$\frac{\partial\tau^\star}{\partial g_m} = -\frac{\{\alpha-\theta(1-\alpha)\}\alpha\delta\rho}{[(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}]^2} = \frac{\partial T^\star}{\partial g_m}$$

By assumption, $0 < \alpha - \theta(1-\alpha) < 1$. Hence, $0 < \tau^\star,\ T^\star < 1$. This ensures that $\tau^\star$ and $T^\star$ varies negatively with g_m. A higher steady-state equilibrium growth rate is associated with lower optimum values of τ and T.

Using equations (3.2.9) and (3.2.10) we have

$$\tau^\star - T^\star = 1 - \alpha. \qquad \text{… … (3.2.11)}$$

This result, illustrated by equation (3.2.11), is different from that in the basic model. The pollution rate, δ, does not affect the optimum ratio of productive public expenditure to national income when consumption is the source of pollution. This is so because the expenditure on public intermediate good is proportional to the level of income and not to the level of consumption. However, the optimum income tax rate and the optimum abatement expenditure rate are sensitive to the pollution rate.

The pollution rate, δ, does not affect the optimum ratio of productive public expenditure to national income when consumption is the source of pollution. This is so because the expenditure on public intermediate good is proportional to the level of income and not to the level of consumption.

However, the optimum income tax rate and the optimum abatement expenditure rate are sensitive to the pollution rate.

Using equations (3.2.6) and (3.2.8) we obtain following equation solving for the steady-state equilibrium growth rate.

$$g_m^{\theta(1-\alpha)}(\sigma g_m+\rho)^{\alpha-\theta(1-\alpha)} = A\{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(1-\alpha)^{1-\alpha}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \quad \ldots\ldots(3.2.6a)$$

Further, we use equations (2.2.14), (2.2.15) and (2.2.12) of chapter 2 and obtain

$${g_m}^{\theta(1-\alpha)}(\sigma g_m+\rho)^{\alpha-\theta(1-\alpha)} = A\{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(1-\alpha)^{1-\alpha}\{\theta(1-\alpha)\}^{\theta(1-\alpha)} (1-\delta)^{1-\{\alpha-\theta(1-\alpha)\}}. \quad \ldots\ldots(2.2.12a)$$

Both equations (3.2.6a) and (2.2.12a) are otherwise identical except the presence of additional multiplicative term $(1-\delta)^{1-\{\alpha-\theta(1-\alpha)\}}$ in the R.H.S. of equation (2.2.12a). Here $(1-\delta)^{1-\{\alpha-\theta(1-\alpha)\}}$ is a positive fraction. Thus comparing the two equations we can conclude that the steady-state equilibrium growth rate derived in section 2.2 is less than that derived in section 3.2, given the income tax rate τ. This is so because, it is not the entire output but only a part of that, which is consumed, is the source of pollution. Hence expenditure required to abate this pollution is also less in this case; and hence, a higher fraction of output is available to meet other productive expenditures. Therefore, the equilibrium growth rate is higher in this model compared to that obtained in section 2.2 of chapter 2.

The next exercise is to check whether there is any conflict between the growth rate maximizing tax rate and the social welfare maximizing tax rate in the steady-state equilibrium. The social welfare function is given by

$$W = \int_0^\infty e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma}dt. \quad \ldots\ldots(3.2.12)$$

Using equations (3.2.3) and (3.2.4) and assuming that the economy is on the steady-state equilibrium growth path, we can show that

$$C = \frac{1}{\alpha}[\rho-(\alpha-\sigma)g_m]K(0)e^{g_m t}. \quad \ldots\ldots(3.2.13)$$

Using equations (3.2.12) and (3.2.13) we have

$$W = \frac{K(0)^{1-\sigma}}{1-\sigma}\alpha^{\sigma-1}[\rho-(1-\sigma)g_m]^{-1}[\rho-(\alpha-\sigma)g_m]^{1-\sigma}, \qquad \dots\dots(3.2.14)$$

or,

$$W = \alpha^{\sigma-1}\frac{K(0)^{1-\sigma}}{1-\sigma}\left[\frac{\rho-(\alpha-\sigma)g_m}{\rho-(1-\sigma)g_m}\right][\rho-(\alpha-\sigma)g_m]^{-\sigma}.$$

W varies positively with g_m.

Thus the level of social welfare in the steady-state growth equilibrium is maximized when the steady-state equilibrium growth rate is maximized[40]. We now have the following proposition.

Proposition 3.2.2: (i) The optimum income tax rate and the optimum abatement expenditure rate in the steady-state growth equilibrium are given by

$$\tau^{\star} = \frac{\{(1+\theta)(1-\alpha)\}(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}}{(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}},$$

and

$$T^{\star} = \frac{\theta(1-\alpha)(\sigma g_m+\rho)+\alpha\delta\{(\sigma-\alpha)g_m+\rho\}}{(\sigma g_m+\rho)+\delta\{(\sigma-\alpha)g_m+\rho\}}.$$

These fiscal policy rates in the steady-state growth equilibrium are simultaneously determined with the growth rate; and a higher growth rate involves lower optimum values of the fiscal instruments.

(ii) The optimum ratio of productive public expenditure to national income is equal to the competitive output share of the public input; and hence this optimum ratio does not depend on the rate of pollution.

The presence of congestion effect making $\theta > 0$ and the presence of environmental pollution causing $\delta > 0$ make our result different from those of Barro (1990) and FMS (1993). If we assume $\theta = \delta = 0$, we obtain $\tau^{\star} = 1-\alpha$ and $T^{\star} = 0$; and these results are identical to those of Barro (1990) and FMS (1993). In Greiner (2005), pollution is proportional to the level of output of the final goods sector; and the optimum ratio of productive public investment to

[40] We do not analyze social welfare maximization including transitional dynamics. FMS (1993) does that.

national income is not independent of the value of the pollution-output coefficient.

3.2.3 STABILITY PROPERTY

We now analyze the transitional dynamic properties of this model. Equations of motion of the growth model are given by (2.2.4), (3.2.1) and (3.2.2). We consider the following ratio variables from the previous chapter.

$x = \frac{C}{K}$; and $y = \frac{E}{K}$.

Using equations (2.2.4), (3.2.1) and (3.2.2), we have

$$\frac{\dot{x}}{x} = A^{\frac{1}{\alpha}}\left(\frac{\alpha}{\sigma} - 1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x - \frac{\rho}{\sigma}; \qquad \text{… … (3.2.15)}$$

and

$$\frac{\dot{y}}{y} = A^{\frac{1}{\alpha}} T(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1} - \delta x y^{-1} - A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x.$$

… … (3.2.16)

The determinant of the Jacobian matrix[41] corresponding to the differential equations (3.2.15) and (3.2.16) is given by

$$|J| = -\frac{\alpha - \theta(1-\alpha)}{\alpha} A^{\frac{1}{\alpha}} T(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{1}{\sigma}\theta(1-\alpha) A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1} + \delta x y^{-2}$$

$$+\delta \frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma} - 1\right) A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}.$$

Here the social elasticity of output with respect to private capital is positive by assumption, i.e., $\alpha - \theta(1-\alpha) > 0$. Also $1 > \tau > T > \delta$ when τ and T are optimally chosen and when $\theta > 0$. The determinant is negative[42] if

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[41] Derivation of the determinant is worked out in Appendix 3.2C.

[42] Derivation of in-equation (3.2.T) is worked out in Appendix 3.2C. Condition (3.2.T) is sufficient but not necessary.

$$g_m > \frac{\rho[\alpha-\theta(1-\alpha)-\alpha^2\rho(1+\delta)\{\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)\}]}{\alpha^2\rho\{\sigma+\delta(\sigma-\alpha)\}\{\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)\}-\sigma\{\alpha-\theta(1-\alpha)\}}. \quad \ldots\ldots (3.2.T)$$

The two latent roots of the Jacobian matrix must be real and of opposite signs in that case; and the unique steady-state equilibrium is a saddle-point with only one transitional path converging to this point. Therefore we can state the following proposition.

Proposition 3.2.3: It is sufficient to show that

$$g_m > \frac{\rho[\alpha-\theta(1-\alpha)-\alpha^2\rho(1+\delta)\{\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)\}]}{\alpha^2\rho\{\sigma+\delta(\sigma-\alpha)\}\{\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)\}-\sigma\{\alpha-\theta(1-\alpha)\}}$$

for the unique steady-state growth equilibrium to be saddle-point stable with unique saddle path converging to that equilibrium point.

Figure 3.2.2 shows the saddle path converging to the steady-state equilibrium point in the special case with $\alpha = \sigma$. $\dot{x} = 0$ locus is itself the saddle path in this case. In-equation (3.2.T) satisfies $\frac{T}{1-\tau} > \delta$ for $\tau = \tau^\star$ and $T = T^\star$ when $\alpha = \sigma$. This can be easily understood comparing in-equations (3.2C.3) and (3.2C.4) in Appendix 3.2C.

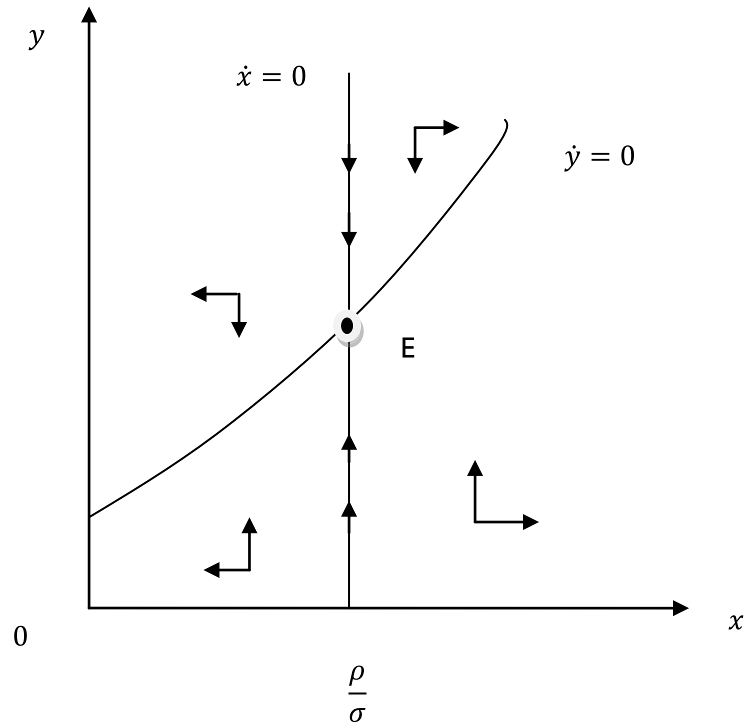

FIGURE 3.2.2

The transitional path to the steady-state equilibrium in this model is not unambiguously unique as is the case in the basic model in chapter 2, section 2.2.3. In this model, restriction on the value of the steady-state growth rate defines a sufficient condition for this transitional path to be unique.

3.2.4 COMMAND ECONOMY

The command economy solution removes distortion of the market economy caused by proportional income tax and due to failure of the private individuals to internalize externalities in the system. Positive externality is caused by the presence of two non rival productive inputs - public intermediate good and environmental quality; and environmental pollution, caused due to private consumption, and congestion effect of private capital accumulation give rise to negative externality. So we solve the planner's problem to obtain the first best solution. The planner's social welfare function is identical to that of the representative household's lifetime utility function. Equations (2.2.1), (2.2.2) and (2.2.6) remain unchanged; equations (2.2.3) and (2.2.4) are re-written as (2.2.3.1) and (2.2.4.1) respectively, for the command economy. Equation (3.2.1) is modified as follows.

$$\dot{E} = \Omega - \delta C. \qquad \text{... ... (3.2.1.1)}$$

Π , as before, denotes planner's total expenditure including public intermediate input and abatement activities; and the abatement expenditure is denoted by Ω.

The planner wants to maximize $\int_0^{\infty} e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt$ with respect to C, Π and Ω subject to equations (2.2.3.1), (2.2.4.1) and (3.2.1.1). A steady-state growth equilibrium is considered and the growth rate is denoted by g_c. The equation that solves for the steady-state equilibrium growth rate[43] in the command (planned) economy is given by the following.

$$(\rho + \sigma g_c)^{\alpha} = A(1-\alpha)^{1-\alpha}\{\alpha - \theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \qquad \text{... ... (3.2.17)}$$

The L.H.S. of equation (3.2.17) is a positive function of g_c and the R.H.S. is a parametric constant. Figure 3.2.3 shows how the unique value of g_c is determined. It is worth noticing that the socially efficient growth rate determined from equation (3.2.17) is independent of the emission-consumption coefficient. Thus a higher value of this coefficient does not lower the socially efficient growth rate. This is because consumption of final output generates pollution in this extended model; and the social planner allocates resources for abatement so as to lower consumption and use resources from lowered consumption to negate pollution, in the process, keeping growth rate unaffected by pollution-consumption coefficient. However, the market economy

[43]Equation (3.2.17) is derived in the Appendix 3.2D.

growth rate, as is illustrated below, is not independent of this coefficient. In the decentralized economy, government has income tax as the only fiscal instrument to allocate resources for productive public expenditure and for abatement.

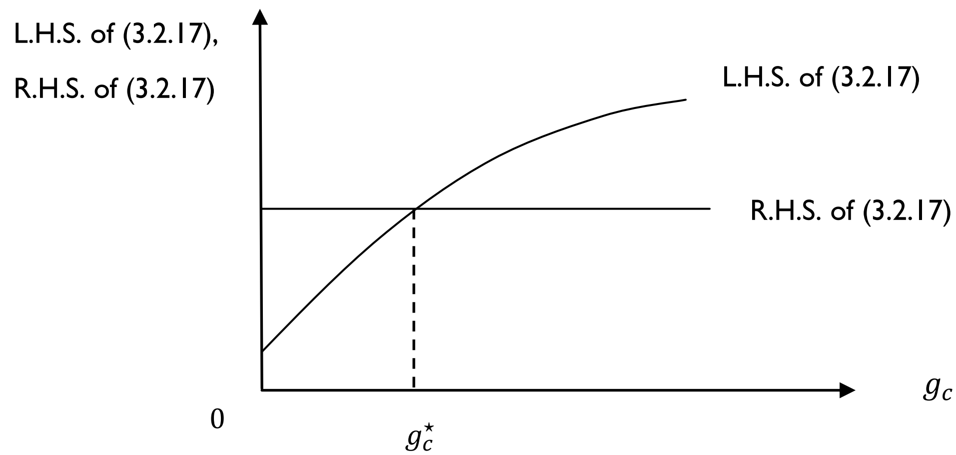

FIGURE 3.2.3

Now we turn to compare the market economy solution to the command economy solution by comparing equation (3.2.17) to equation (3.2.6) when $\tau = \tau^\star$ and $T = T^\star$. We modify equation (3.2.6) with $\tau = \tau^\star$ and $T = T^\star$ as follows.

$$g_m{}^{\theta(1-\alpha)}\left[\frac{1}{\alpha}(\sigma g_m + \rho) + \frac{\delta}{\alpha}\{(\sigma - \alpha)g_m + \rho\}\right]^{\alpha-\theta(1-\alpha)}$$

$$= A(1-\alpha)^{1-\alpha}\{\alpha - \theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \qquad \ldots\ldots(3.2.6.1)$$

To aide comparison we also modify equation (3.2.17) as follows.

$$(\rho + \sigma g_c)^{\theta(1-\alpha)}(\rho + \sigma g_c)^{\alpha-\theta(1-\alpha)}$$

$$= A(1-\alpha)^{1-\alpha}\{\alpha - \theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \qquad \ldots\ldots(3.2.17.1)$$

The R.H.S. of equations (3.2.17.1) and (3.2.6.1) are identical. Hence comparing equation (3.2.6.1) to equation (3.2.17.1) we find that g_m exceeds (falls short of) g_c when the technology parameter A takes a low (high) value.

This is shown in Figure 3.2.4. The L.H.S. of equations (3.2.6.1) and (3.2.17.1) are plotted as positively sloped curves and the R.H.S. is depicted by horizontal straight lines for exogenous values of the technology parameter A. The L.H.S. curve of equation (3.2.6.1) starts from origin but the L.H.S. curve of equation (3.2.17.1) starts from a point on the vertical axis. When A takes a very low value, the points of intersection of the two L.H.S. curves with the lower horizontal line show that $g_c^\star$ fall short of $g_m^\star$. When A takes a high value, similar mechanism shows that $g_c^\star > g_m^\star$.

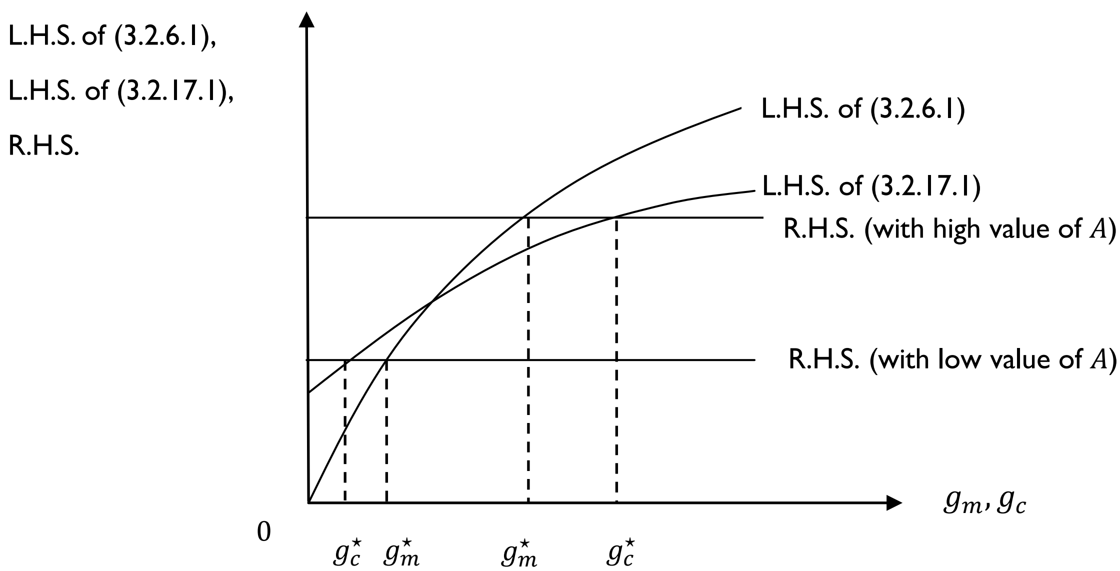

FIGURE 3.2.4

We can state the following proposition.

Proposition 3.2.4: $(g_c - g_m)$ may take a positive (negative) sign when the technology parameter A takes a high (low) value.

The result in proposition 3.2.4 when $\frac{T}{1-\tau} > \delta$, is similar to that in the basic model in section 2.2.4 of chapter 2. However, this result is valid only for $\frac{T}{1-\tau} > \delta$ unlike in the basic model, where no such restriction is required.

3.3 CAPITAL AS THE SOURCE OF POLLUTION

In this section, capital usage is treated as the only source of pollution. Pollution is not necessarily an unavoidable feature of production. There exist pollution-free production technologies. Pollution is generated when some special types of machineries, energy resources or chemical inputs are used in production. Several authors have considered physical capital or one of the intermediate inputs as the source of environmental pollution[44]. Mohtadi (1996) and Oueslati (2002) analyse the properties of environmental fiscal policy in endogenous growth model with level of pollution being an increasing function of the amount of capital used and a decreasing function of abatement activity. Itaya (2008) considers the effect of environmental taxation in a Romer type of learning-by-doing model with endogenous labour supply where pollution enters as a negative externality in utility; and the level of pollution varies positively with the amount of capital used. Smulders and Gradus (1996) and Byrne (1997) treat pollution as an accumulable by-product that varies positively with the level of capital use. Dirty varieties of intermediate inputs cause pollution in Benarroch and Weder (2006) and Elbasha and Roe (1996). Bertinelli, Strobl and Zou (2008) use a capital vintage model to show how environmental pollution decreases with the usage of capital of newer vintage in production. In their model investment in the production sector generates pollution, part of which is absorbed by nature's self regeneration capacity. The utility of consumers is enhanced by consumption of the final good as well as by

[44]See the works of Benarroch and Weder (2006), Itaya (2008), Oueslati (2002), Bertinelli, Strobl and Zou (2008), Smulders and Gradus (1996), Byrne (1997), Mohtadi (1996), Bovenberg and Smulders (1995), Elbasha and Roe (1996), Cassou and Hamilton (2004), Hart (2004), etc.

environmental quality. In the planning problem output, consumption and investment grow at a constant rate. At the social optimal environmental quality improves, however, not at a constant rate.

We treat environmental pollution to be a flow variable in this model and to be caused by the use of physical capital.

We derive following results from this model. The optimum ratio of productive public expenditure to national income is equal to the competitive share of the public input in the output of the final good and is independent of the rate of pollution. However, the optimum proportional income tax rate in this model is greater than this competitive output share of public input because a positive fraction of output is spent on abatement activity. Also this optimum tax rate and the optimum abatement expenditure rate depend on the rate of pollution. In Barro (1990) and in FMS (1993), there is no environmental pollution and abatement cost; and hence this ratio of productive public expenditure to national income is always equal to the proportional income tax rate whose optimum value is equal to the competitive output share of the public input. In Greiner (2005), the optimum public investment to national income ratio depends on the pollution-output coefficient. Secondly, in this model, the optimal tax rate and the optimal abatement expenditure rate are functions of the growth rate in the steady-state equilibrium. So, in this model, the optimal values of the fiscal instruments and the steady-state equilibrium growth rate are determined simultaneously. However, in Barro (1990) and in FMS (1993), the optimum tax rate is determined independently of the growth rate and the same is true for Greiner (2005). Thirdly, our model exhibits transitional dynamic properties though it follows Barro (1990) to assume productive public expenditure to be a flow variable. Environmental quality is an accumulable input in this model; and this protects our model from being trapped into an AK model. Fourthly, like Barro (1990) and FMS (1993), there is no conflict between the growth rate maximizing solution and the social welfare maximizing solution in the steady-state growth equilibrium in our model. Greiner (2005) does not find such a conflict in the case of an income tax policy but finds it in the case of a pollution tax policy because pollution directly affects the utility of the household in his model. Fifthly, the competitive equilibrium growth rate in this model does not necessarily fall short of the socially efficient growth rate which is unlike in Barro (1990) or in FMS (1993). This result is obtained because there are two conflicting types of externalities on production - a positive externality resulting from the public expenditure and technology and a negative externality resulting from capital accumulation and environmental pollution. Barro (1990) as well as FMS (1993) considers only a positive externality.

Following sections are organized in this way; the basic model of the market economy is described in section 3.3.1 and its steady-state equilibrium properties related to fiscal policies are presented in section 3.3.2. Section 3.3.3 shows transitional dynamic results; and section 3.3.4 describes the working of the command economy.

3.3.1 THE MODEL

Equations (2.2.1) to (2.2.4) and (2.2.6) are borrowed from section 2.2 in chapter 2 and equation (3.2.1) is modified as follows.

$$\dot{E} = TY - \delta K \text{ with } 0 < \delta < 1; \qquad \ldots\ldots(3.3.1)$$

Environmental quality improves over time depending upon the magnitudes of pollution and abatement expenditure. TY is the abatement expenditure made by the government. Here environmental pollution is assumed to be proportional to the use of capital stock. δ is the constant pollution-capital coefficient.

Stocks of E and K are exogenous at a particular point of time. E is a non rival stock and G is a non rival flow. Given the stocks of capital and environmental quality, and given the fiscal instrument rates, equations (2.2.1), (2.2.2) and (2.2.3) together determine Y and G at each point of time. Thus equation (3.3.1) determines the absolute rate of improvement in the environmental quality, denoted by $\dot{E}$, given abatement expenditure rate and capital stock. The household then chooses C and this determines the absolute rate of private capital accumulation, $\dot{K}$.

3.3.2 DYNAMIC EQUILIBRIUM AND STEADY-STATE

The dynamic analysis of this model and its steady-state growth equilibrium condition are identical to section 3.2.2 of the present chapter.

We, therefore, turn to show the existence of unique steady-state equilibrium growth rate in the market economy; and so we use equations (2.2.1) to (2.2.4), (3.3.1), (3.2.2) and the steady-state equilibrium condition given by equation (2.2.8) to obtain following equations.

$$\frac{1}{\sigma}\left[\alpha A^{\frac{1}{\alpha}}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}-\rho\right]=g_m; \qquad \ldots\ldots(3.3.2)$$

$$A^{\frac{1}{\alpha}}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}}-\frac{C}{K}=g_m; \qquad \ldots\ldots(3.3.3)$$

and

$$A^{\frac{1}{\alpha}}T(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}}-\delta\left(\frac{E}{K}\right)^{-1}=g_m. \qquad \ldots\ldots(3.3.4)$$

Here g_m, as before, denotes the steady-state equilibrium growth rate in the market economy. Using equations (3.3.2), (3.3.3) and (3.3.4) we obtain the following equation[45] to solve for g_m.

$$(\sigma g_m+\rho)^{\alpha}g_m^{\theta(1-\alpha)}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right]^{-\theta(1-\alpha)}=\alpha^{\alpha}A(1-\tau)^{\alpha}(\tau-T)^{1-\alpha}. \qquad \ldots\ldots(3.3.5)$$

The solution is unique when its L.H.S. is an increasing function of g_m if $\frac{T\rho}{\alpha(1-\tau)}>\delta$; and the R.H.S. is a positive parametric constant, given the income tax rate and the abatement expenditure rate satisfying $0<T<\tau<1$.

[45] The derivation of equation (3.3.5) is worked out in Appendix 3.3A.

We can state the following proposition.

Proposition 3.3.1: Unique steady-state equilibrium growth rate exists in the market economy given the income tax rate and the abatement expenditure rate if $\frac{T\rho}{\alpha(1-\tau)} > \delta$ and if $0 < T < \tau < 1$.

The existence of unique steady-state equilibrium growth rate depends on the condition $\frac{T\rho}{\alpha(1-\tau)} > \delta$, which means that the discount rate, ρ, must exceed the ratio of the rate of pollution, (generated from the fraction of disposable income going to physical capital accumulation), $\alpha\delta(1-\tau)$, to the abatement expenditure rate, T, for capital is the source of pollution here.

3.3.2.1 Optimal Taxation

Government maximizes growth rate in the steady-state equilibrium with respect to fiscal instruments, τ and T. The L.H.S. of equation (3.3.5) is a monotonically increasing function of g_m, because, by assumption, $\frac{T\rho}{\alpha(1-\tau)} > \delta$. Thus, solving the growth rate maximization problem of the government at the steady-state equilibrium subject to equation (3.3.5) with respect to fiscal instruments, τ and T, we obtain following two equations[46].

$$\theta(1-\alpha)\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right]^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}(\tau-T)=\frac{\alpha(\tau-T)-(1-\alpha)(1-\tau)}{T}; \quad \text{... ... (3.3.6)}$$

and

$$\theta(1-\alpha)\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right]^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}(\tau-T)=1-\alpha. \quad \text{... ... (3.3.7)}$$

Using these two equations we obtain following expressions of the optimum income tax rate and of the optimum abatement expenditure rate.

$$\tau^{\star}=\frac{(1+\theta)(1-\alpha)(\sigma g_m+\rho)+\alpha\delta}{(\sigma g_m+\rho)+\alpha\delta}=\frac{[1-\{\alpha-\theta(1-\alpha)\}](\sigma g_m+\rho)+\alpha\delta}{(\sigma g_m+\rho)+\alpha\delta}; \quad \text{... ... (3.3.8)}$$

[46]Derivation of equations (3.3.6) and (3.3.7) is worked out in Appendix 3.3B.

and

$$T^{\star} = \frac{\theta(1-\alpha)(\sigma g_m+\rho)+\alpha^2\delta}{(\sigma g_m+\rho)+\alpha\delta}. \qquad \text{......(3.3.9)}$$

So $\tau^{\star}$, $T^{\star}$ and g_m are simultaneously determined by equations (3.3.5), (3.3.8) and (3.3.9). It can be easily shown that

$$\frac{\partial\tau^{\star}}{\partial g_m} = -\frac{\{\alpha-\theta(1-\alpha)\}\alpha\delta\sigma}{[(\sigma g_m+\rho)+\alpha\delta]^2} = \frac{\partial T^{\star}}{\partial g_m}.$$

By assumption, $0 < \alpha - \theta(1-\alpha) < 1$; and hence $0 < 1 - \{\alpha - \theta(1-\alpha)\} = (1+\theta)(1-\alpha) < 1$. This ensures $0 < \tau^{\star}, T^{\star} < 1$. Also this ensures that $\tau^{\star}$ as well as $T^{\star}$ varies negatively with g_m. A higher steady-state equilibrium growth rate is associated with lower optimum values of τ and T.

Using equations (3.3.8) and (3.3.9) we have

$$\tau^{\star} - T^{\star} = 1 - \alpha. \qquad \text{......(3.3.10)}$$

$\tau^{\star} - T^{\star}$ is the optimum ratio of productive public expenditure to national income, which is equal to the competitive output share of productive public input, $1-\alpha$.

The pollution rate, δ, does not affect the optimum ratio of productive public expenditure to national income when capital use is the source of pollution. This is so because the expenditure on public intermediate good is proportional to the level of income and not to the level of capital use. However, the optimum income tax rate and the optimum abatement expenditure rate are sensitive to the pollution rate.

Using equations (3.3.5) and (3.3.7) of chapter 3 we obtain an equation exactly identical to (3.2.6a). Thus comparing equations (3.2.6a) to (2.2.12a) we can conclude that the steady-state equilibrium growth rate derived in section 2.2 is less than that derived in the present section, given the income tax rate τ. Here, the entire output is not the source of pollution but only a part which is used to create additional capital. Hence, expenditure required to abate this pollution is less than that required in the model in section 2.2; and, consequently, a higher fraction of output is available to meet other productive expenditures. This is why the equilibrium growth rate is found to be higher in this case compared to that obtained in section 2.2 of chapter 2.

The next exercise is to check whether there is any conflict between the growth rate maximizing tax rate and the social welfare maximizing tax rate. The social welfare function is identical to equation (3.2.12) in section (3.2.2.1) and its expression is identical to that in section 3.2.2.1 when expressed in terms of steady-state equilibrium growth rate. Hence W varies positively with g_m. Thus, in this case too, the level of social welfare in the steady-state growth equilibrium is maximized when the steady-state equilibrium growth rate is maximized. We now have the following proposition.

Proposition 3.3.2: (i) The optimum income tax rate and the optimum abatement expenditure rate in the steady-state growth equilibrium are simultaneously determined with the growth rate; and a higher growth rate involves lower optimum values of fiscal instruments.

(ii) Optimum ratio of productive public expenditure to national income is equal to the competitive output share of the public input in the final goods sector; and hence is independent of the pollution rate per unit of capital.

3.3.3 STABILITY PROPERTY

We now analyze the transitional dynamic properties of this model. Equations of motion of the growth model are given by (3.2.2), (2.2.4) and (3.3.1). We consider the ratio variables, x and y from section 3.2.3 and using equations (3.2.2), (2.2.4) and (3.3.1), we have

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x - \frac{\rho}{\sigma}; \qquad \text{......(3.3.11)}$$

and

$$\frac{\dot{y}}{y} = A^{\frac{1}{\alpha}} T(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1} - \delta y^{-1} - A^{\frac{1}{\alpha}}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x.$$

$$\text{......(3.3.12)}$$

The determinant of the Jacobian matrix[47] corresponding to the differential equations (3.3.11) and (3.3.12) is given by

$$|J| = -\frac{\alpha-\theta(1-\alpha)}{\alpha} A^{\frac{1}{\alpha}} T(\tau-T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)-2\alpha}{\alpha}}$$

$$-\frac{1}{\sigma}\theta(1-\alpha) A^{\frac{1}{\alpha}}(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}} y^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} + \delta y^{-2}.$$

Here, by assumption, $\alpha-\theta(1-\alpha)>0$. Also $1>\tau>T>\delta$ when τ and T are optimally chosen and when $\theta>0$. So the determinant is unrestricted in sign. It is negative[48] if

$$g_m > \frac{\{\alpha-\theta(1-\alpha)\}[\delta\{\alpha+\theta(1-\alpha)\}+\alpha\rho]\rho-\alpha^2\rho(\rho+\alpha\delta)}{\alpha\rho\sigma\theta(1-\alpha)+\sigma\delta\{\alpha-\theta(1-\alpha)\}\{\alpha+\theta(1-\alpha)\}}. \quad \ldots\ldots(3.3.T)$$

The two latent roots of the Jacobian matrix must be real and of opposite signs in that case; and the unique steady-state equilibrium is a saddle-point with only one transitional path converging to this point. Therefore, we can state the following proposition.

Proposition 3.3.3: The steady-state equilibrium is saddle point stable with unique saddle path converging to that equilibrium point if

$$g_m > \frac{\{\alpha-\theta(1-\alpha)\}[\delta\{\alpha+\theta(1-\alpha)\}+\alpha\rho]\rho-\alpha^2\rho(\rho+\alpha\delta)}{\alpha\rho\sigma\theta(1-\alpha)+\sigma\delta\{\alpha-\theta(1-\alpha)\}\{\alpha+\theta(1-\alpha)\}}.$$

Figure 3.3.1 shows the saddle path converging to the steady-state equilibrium point in the special case with $\alpha=\sigma$. $\dot{x}=0$ locus is itself the saddle path in this case. In-equation (3.3.T) satisfies $\frac{T\rho}{\alpha(1-\tau)}>\delta$ for $\tau=\tau^\star$ and $T=T^\star$ when $\alpha=\sigma$[49].

---

[47]Derivation of the determinant is worked out in Appendix 3.3C.

[48]Derivation of in-equation (3.3T) is worked out in Appendix 3.3C. Condition (3.3T) is sufficient but not necessary.

[49]This can be easily understood comparing in-equations (3.3C.3) and (3.3C.4) in Appendix 3.3C.

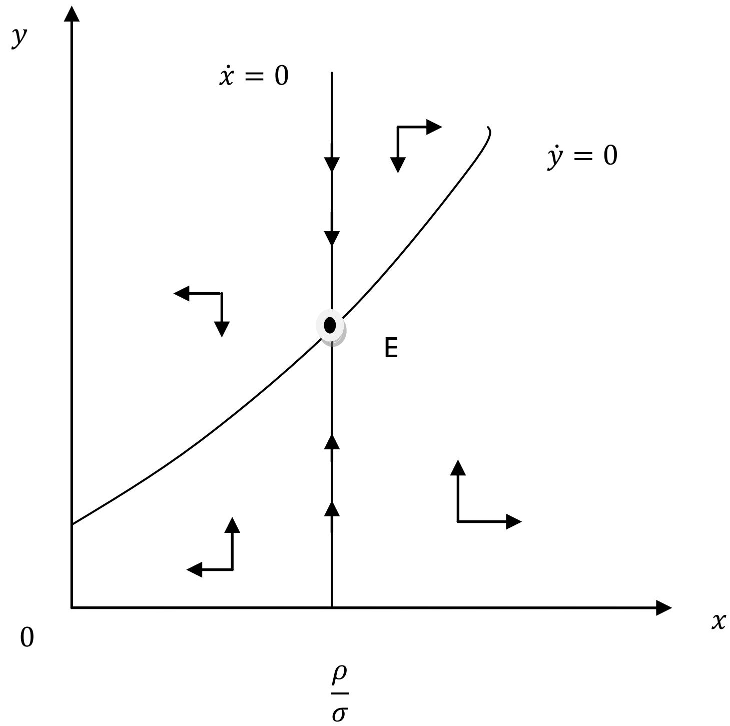

FIGURE 3.3.1

This result is similar to that in section 3.2.3 when consumption of final good is the source of pollution.

3.3.4 COMMAND ECONOMY

Sub-optimality of the market economy solution due to distortion caused by the proportional income tax and the failure of the private individuals to internalize externalities in the system are well known. Equation (3.3.1) is modified for the planned economy as follows.

$$\dot{E} = \Omega - \delta K. \qquad \text{… … (3.3.1.1)}$$

Here Ω denotes planner's expenditure on abatement activities while total expenditure including expenditure on public intermediate input and on abatement activities is denoted by Π, as before. Thus, equations (2.2.1), (2.2.2), (2.2.6), (2.2.3.1), (2.2.4.1) and (3.3.1.1) describe the model in the planned economy.

The planner wants to maximize $\int_0^\infty e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt$ with respect to C, Π and Ω subject to equations (2.2.3.1), (2.2.4.1) and (3.3.1.1). A steady-state growth equilibrium is considered and the growth rate is denoted by g_c. The equation for the steady-state equilibrium growth rate[50] in the planned economy is given by the following.

$$(\rho + \sigma g_c)^{\theta(1-\alpha)}(\rho + \sigma g_c + \delta)^{\alpha-\theta(1-\alpha)}$$
$$= A(1-\alpha)^{1-\alpha}\{\alpha - \theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \quad \ldots\ldots (3.3.13)$$

The L.H.S. of equation (3.3.13) is a positive function of g_c and the R.H.S. is a positive parametric constant because $1 > \alpha > \theta(1-\alpha)$.

Now we turn to compare the market economy solution to the command economy solution by comparing equation (3.3.13) to equation (3.3.5) when $\tau = \tau^\star$ and $T = T^\star$. We modify equation (3.3.5) with $\tau = \tau^\star$ and $T = T^\star$ as follows.

$${g_m}^{\theta(1-\alpha)} \left\{\frac{1}{\alpha}(\sigma g_m + \rho) + \delta\right\}^{\alpha-\theta(1-\alpha)} = A(1-\alpha)^{1-\alpha}\{\alpha - \theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}$$
$$\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \quad \ldots\ldots (3.3.5.1)$$

Comparing the L.H.S. of equation (3.3.5.1) to that of equation (3.3.13) we find that for all values of $g_m = g_c$, the term $\left\{\frac{1}{\alpha}(\sigma g_m + \rho) + \delta\right\}^{\alpha-\theta(1-\alpha)}$ is greater than the term $(\rho + \sigma g_c + \delta)^{\alpha-\theta(1-\alpha)}$. When $g_m = g_c > \frac{\rho}{1-\sigma}$, the term ${g_m}^{\theta(1-\alpha)}$ is greater than the term $(\rho + \sigma g_c)^{\theta(1-\alpha)}$. Thus for all values of $g_m = g_c \geq \frac{\rho}{1-\sigma}$, the L.H.S. of equation (3.3.5.1) exceeds that of equation (3.3.13). So for some value of $g_m = g_c < \frac{\rho}{1-\sigma}$, the L.H.S. of both the equations are equal; and, for all sufficiently small values of $g_m = g_c < \frac{\rho}{1-\sigma}$, the L.H.S. of equation (3.3.13) exceeds that of equation (3.3.5.1). The R.H.S. of both the equations is identical.

[50] Equation (3.3.13) is derived in the Appendix (3.3D).

The comparison is shown in Figure 3.3.2. The L.H.S. curve of equation (3.3.5.1) starts from the origin and the same of equation (3.3.13) starts from a point on the vertical axis. The R.H.S. curve of both the equations is denoted by the horizontal line. When the parameter A takes a sufficiently low value, the points of intersection of the two L.H.S. curves with the R.H.S. horizontal curve show that $g_c^\star < g_m^\star$. However, we find $g_c^\star > g_m^\star$ when A takes a high value. Hence comparing equation (3.3.5.1) to equation (3.3.13) we find that g_m exceeds (falls short of) g_c when the parameter A takes a low (high) value. We can state the following proposition.

Proposition 3.3.4: $(g_c - g_m)$ may take a positive (negative) sign when the technology parameter A takes a high (low) value.

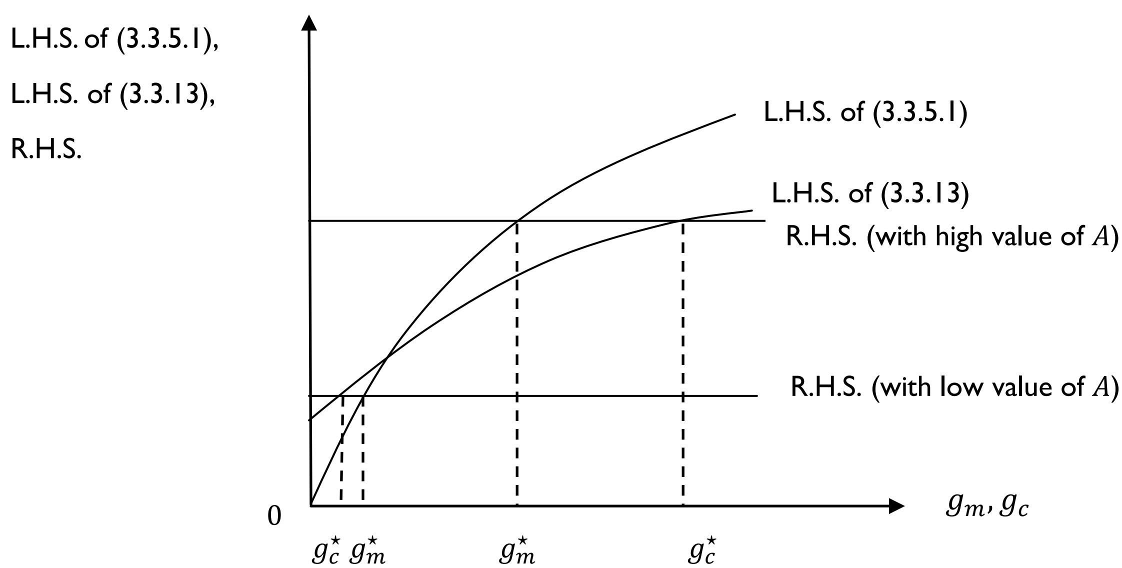

FIGURE 3.3.2

APPENDIX 3.2A

DERIVATION OF EQUATION (3.2.6) IN SECTION 3.2.2

Using equations (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.8), (3.2.1) and (3.2.2) we have the following equations.

$$g_m = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\alpha A^{\frac{1}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right]; \qquad \ldots\ldots(3.2A.1)$$

$$g_m = \frac{\dot{K}}{K} = A^{\frac{1}{\alpha}}(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \frac{C}{K}; \qquad \ldots\ldots(3.2A.2)$$

and

$$g_m = \frac{\dot{E}}{E} = T(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} - \delta\left(\frac{C}{K}\right)\left(\frac{E}{K}\right)^{-1}. \qquad \ldots\ldots(3.2A.3)$$

From equation (3.2A.1) we have,

$$\frac{E}{K} = \left[(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\alpha A^{\frac{1}{\alpha}}(\sigma g_m+\rho)^{-1}\right]^{-\frac{\alpha}{\theta(1-\alpha)}}. \qquad \ldots\ldots(3.2A.4)$$

Using equations (3.2A.1) and (3.2A.2), we have

$$\frac{C}{K} = \frac{1}{\alpha}[(\sigma-\alpha)g_m+\rho]. \qquad \ldots\ldots(3.2A.5)$$

Using equations (3.2A.3), (3.2A.4) and (3.2A.5) we derive the following equation.

$$g_m = \left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right]$$

$$(\sigma g_m+\rho)^{-\frac{\alpha}{\theta(1-\alpha)}}\{\alpha(1-\tau)\}^{\frac{\alpha}{\theta(1-\alpha)}}(\tau-\mathrm{T})^{\frac{1}{\theta}}A^{\frac{1}{\theta(1-\alpha)}},$$

or,

$$g_m(\sigma g_m+\rho)^{\frac{\alpha}{\theta(1-\alpha)}} = \left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right]$$

$$\{\alpha(1-\tau)\}^{\frac{\alpha}{\theta(1-\alpha)}}(\tau-\mathrm{T})^{\frac{1}{\theta}}A^{\frac{1}{\theta(1-\alpha)}},$$

or,

$$(\sigma g_m+\rho)^{\alpha}g_m^{\theta(1-\alpha)}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right]^{-\theta(1-\alpha)}$$

$$= \alpha^{\alpha} A (1-\tau)^{\alpha} (\tau - T)^{1-\alpha}. \qquad \ldots\ldots (3.2A.6)$$

This is same as equation (3.2.6) in section 3.2.2.

APPENDIX 3.2B

DERIVATION OF EQUATIONS (3.2.7) AND (3.2.8) AND THE SECOND ORDER CONDITIONS IN SECTION 3.2.2.1

We denote the L.H.S. and the R.H.S. of equation (3.2.6) as L.H.S. $_{(3.2.6)}$ and R.H.S. $_{(3.2.6)}$ respectively. Differentiating equation (3.2.6) with respect to τ, we obtain the following first order condition.

$$\text{L.H.S.}_{(3.2.6)} \left[\alpha\sigma(\sigma g_m + \rho)^{-1} + \theta(1-\alpha) g_m^{-1} \right.$$

$$\left. -\theta(1-\alpha) \left\{ T \frac{\sigma g_m + \rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}\{(\sigma - \alpha) g_m + \rho\} \right\}^{-1} \left\{ \frac{T\sigma}{\alpha(1-\tau)} - \frac{\delta(\sigma-\alpha)}{\alpha} \right\} \right] \frac{\partial g_m}{\partial \tau}$$

$$-\text{L.H.S.}_{(3.2.6)}\, \theta(1-\alpha) \left[T \frac{\sigma g_m + \rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}\{(\sigma - \alpha) g_m + \rho\} \right]^{-1} T \frac{\sigma g_m + \rho}{\alpha(1-\tau)^2}$$

$$= \text{R.H.S.}_{(3.2.6)} \left[-\alpha(1-\tau)^{-1} + (1-\alpha)(\tau - T)^{-1} \right]. \qquad \ldots\ldots (3.2B.1)$$

At the equilibrium point $\text{L.H.S.}_{(3.2.6)} = \text{R.H.S.}_{(3.2.6)}$; and $\frac{\partial g_m}{\partial \tau} = 0$ at the optimum.

Thus equation (3.2B.1) takes the following form.

$$\theta(1-\alpha) \left\{ T \frac{\sigma g_m + \rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}\{(\sigma - \alpha) g_m + \rho\} \right\}^{-1} T \frac{\sigma g_m + \rho}{\alpha(1-\tau)^2}$$

$$= \left[(1-\alpha)(\tau - T)^{-1} - \alpha(1-\tau)^{-1} \right],$$

or,

$$\theta(1-\alpha) \left[T \frac{\sigma g_m + \rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}\{(\sigma - \alpha) g_m + \rho\} \right]^{-1} \frac{\sigma g_m + \rho}{\alpha(1-\tau)} (\tau - T) = \frac{\alpha(\tau - T) - (1-\alpha)(1-\tau)}{T}.$$

$$\ldots\ldots (3.2B.2)$$

This is the same as equation (3.2.7) in section 3.2.2.1.

Again, differentiating the R.H.S. of equation (3.2.6) with respect to T, we obtain the following first order condition.

$$\text{L.H.S.}_{(3.2.6)}\,[\alpha\sigma(\sigma g_m+\rho)^{-1}+\theta(1-\alpha)g_m^{-1}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}\left\{\frac{T\sigma}{\alpha(1-\tau)}-\frac{\delta(\sigma-\alpha)}{\alpha}\right\}\Bigg]\frac{\partial g_m}{\partial T}$$

$$-\text{L.H.S.}_{(3.2.6)}\,\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}$$

$$=-\text{R.H.S.}_{(3.2.6)}\,(1-\alpha)(\tau-T)^{-1}. \qquad \ldots\ldots(3.2B.3)$$

Similar to Equation (3.2B.2), (3.2B.3) can be written as equation (3.2B.4).

$$\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}=(1-\alpha)(\tau-T)^{-1},$$

or,

$$\theta(1-\alpha)\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right]^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}(\tau-T)=1-\alpha. \quad \ldots\ldots(3.2B.4)$$

This is same as equation (3.2.8) in section 3.2.2.1.

To check the second order conditions for optimality we differentiate both sides of equation (3.2B.2) with respect to τ and both sides of equation (3.2B.4) with respect to T.

We arrive at the following two second order conditions.

$$-[\theta(1-\alpha){g_m}^{-2}+\alpha\sigma^2(\sigma g_m+\rho)^{-2}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}-\frac{\delta(\sigma-\alpha)}{\alpha}\right\}^{2}\Bigg]\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+[\alpha\sigma(\sigma g_m+\rho)^{-1}+\theta(1-\alpha)g_m^{-1}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}\left\{\frac{T\sigma}{\alpha(1-\tau)}-\frac{\delta(\sigma-\alpha)}{\alpha}\right\}\Bigg]\frac{\partial^2 g_m}{\partial\tau^2}$$

$$-2\theta(1-\alpha)[-\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}-\frac{\delta(\sigma-\alpha)}{\alpha}\right\}T\frac{\sigma g_m+\rho}{\alpha(1-\tau)^2}$$

$$+\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}\frac{T\sigma}{\alpha(1-\tau)^2}\Bigg]\frac{\partial g_m}{\partial\tau}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}T\frac{\sigma g_m+\rho}{\alpha(1-\tau)^3}$$

$$\left[2-\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}\right]=-\left[\frac{\alpha}{(1-\tau)^2}+\frac{(1-\alpha)}{(\tau-T)^2}\right].$$

and

$$-[\theta(1-\alpha){g_m}^{-2}+\alpha\sigma^2(\sigma g_m+\rho)^{-2}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}-\frac{\delta(\sigma-\alpha)}{\alpha}\right\}^2\Bigg]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+[\alpha\sigma(\sigma g_m+\rho)^{-1}+\theta(1-\alpha)g_m^{-1}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}\left\{\frac{T\sigma}{\alpha(1-\tau)}-\frac{\delta(\sigma-\alpha)}{\alpha}\right\}\Bigg]\frac{\partial^2 g_m}{\partial T^2}$$

$$-2\theta(1-\alpha)[-\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}-\frac{\delta(\sigma-\alpha)}{\alpha}\right\}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}$$

$$+\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-1}\frac{\sigma}{\alpha(1-\tau)}\Bigg]\frac{\partial g_m}{\partial T}$$

$$+\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}\{(\sigma-\alpha)g_m+\rho\}\right\}^{-2}\left\{\frac{\sigma g_m+\rho}{\alpha(1-\tau)}\right\}^2=-\frac{(1-\alpha)}{(\tau-T)^2}.$$

Now we evaluate the above two second order conditions at $\tau=\tau^\star$ and $T=T^\star$ where $\frac{\partial g_m}{\partial\tau}=\frac{\partial g_m}{\partial T}=0$. Hence we obtain the followings.

$$\frac{\partial^2 g_m}{\partial\tau^2}=-\frac{(1-\alpha)^{-1}+\left[\frac{\theta(1-\alpha)\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}^2+\delta^2\{(\overline{\sigma-\alpha})g_m+\rho\}^2}{\theta(1-\alpha)\{\alpha-\theta(1-\alpha)\}(\sigma g_m+\rho)}\right]}{\frac{\{\alpha-\theta(1-\alpha)\}\{(\sigma+\delta\overline{\sigma-\alpha})g_m\}+\theta(1-\alpha)\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}}{g_m\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}}}$$

and

$$\frac{\partial^2 g_m}{\partial T^2} = -\frac{\frac{(1+\theta)}{\theta(1-\alpha)}}{\frac{\{\alpha-\theta(1-\alpha)\}\{(\sigma+\delta\overline{\sigma-\alpha})g_m\}+\theta(1-\alpha)\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}}{g_m\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}}}.$$

The R.H.S. of each of these two equations is negative. Thus the sign of both the second order derivatives are negative.

APPENDIX 3.2C

DERIVATION OF THE DETERMINANT AND IN-EQUATION (3.2.T) IN SECTION 3.2.3

We consider the following equations from section 3.2.3.

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + x - \frac{\rho}{\sigma}; \qquad \ldots\ldots (3.2.15)$$

and

$$\frac{\dot{y}}{y} = T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} + x - \delta x y^{-1} - (1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}}. \qquad \ldots\ldots (3.2.16)$$

We obtain the following partial derivatives corresponding to the above two equations.

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial x} = 1;$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial y} = \frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma} - 1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial x} = 1 - \delta y^{-1};$$

and

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial y} = -\frac{\alpha-\theta(1-\alpha)}{\alpha} T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2} - \frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$+\delta x y^{-2}.$$

So the determinant of the Jacobian matrix can be written as follows.

$$|J| = \delta x y^{-2} - \frac{\alpha-\theta(1-\alpha)}{\alpha} T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$-\frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$+\delta \frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2},$$

or,

$$|J| = \delta x y^{-2} - \frac{\alpha-\theta(1-\alpha)}{\alpha} T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\sigma}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$+\delta \frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}.$$

Now, we use equations (3.2.15) and (3.2.16) and the steady-state equilibrium condition, $\dot{x} = \dot{y} = 0$, to obtain the following equation.

$$\frac{\alpha}{\sigma}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} = T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} - \delta x y^{-1} + \frac{\rho}{\sigma}. \quad \ldots\ldots(3.2C.1)$$

Thus, using equation (3.2C.1) in the R.H.S. expression of the determinant, we find that $|J|$ is positive (negative) if

$$\left[\frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}y^{-1} - 1\right] T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}}$$

$$+\left[1 - \frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}y^{-1} + \frac{\theta(1-\alpha)}{\alpha}\right]\delta x$$

$$+\left[\frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}y^{-1} - \frac{\theta(1-\alpha)}{\alpha}\right]\frac{\rho}{\sigma}y > (<)0. \quad \ldots\ldots(3.2C.2)$$

Now using equations (3.2A.4), (3.2.6) and the value of $(\tau^\star - T^\star)$ given by (3.2.11), we find that the in-equation (3.2C.2) takes the following form.

$$\left[\frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}g_m\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}^{-1} - 1\right]$$

$$\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\} + \frac{\theta(1-\alpha)}{\alpha}\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)$$

$$\left[\frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}g_m\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}^{-1}-\frac{\theta(1-\alpha)}{\alpha}\right]$$

$$\frac{\rho}{\sigma}g_m{}^{-1}\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}>(<)0$$

or,

$$\left[\frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}g_m\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}^{-1}-1\right]$$

$$\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}+\frac{\theta(1-\alpha)}{\alpha}\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)$$

$$-\frac{\theta(1-\alpha)}{\alpha}\frac{\rho}{\sigma}g_m{}^{-1}\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}$$

$$+\frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}>(<)0,$$

or,

$$\left[\frac{\theta(1-\alpha)}{\alpha}\delta\left(\frac{\alpha}{\sigma}-1\right)\frac{\sigma}{\alpha}g_m\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}^{-1}-1-\frac{\theta(1-\alpha)}{\alpha}\frac{\rho}{\sigma}g_m{}^{-1}\right]$$

$$\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}+\frac{\theta(1-\alpha)}{\alpha}\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)$$

$$-\frac{\theta(1-\alpha)}{\alpha}\delta\frac{(\sigma-\alpha)}{\alpha}>(<)0,$$

or,

$$-\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha)g_m-\alpha\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}$$

$$-\theta(1-\alpha)\frac{\rho}{\sigma}g_m{}^{-1}\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}>(<)\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha)$$

$$-\theta(1-\alpha)\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho),$$

or,

$$\theta(1-\alpha)\frac{\delta}{\alpha}\rho-\alpha\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}$$

$$>(<)\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha)+\theta(1-\alpha)\frac{\rho}{\sigma}g_m{}^{-1}\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}.$$

Here,

$$L.H.S.=\theta(1-\alpha)\frac{\delta}{\alpha}\rho-\alpha\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}\rightarrow\theta(1-\alpha)\frac{\delta}{\alpha}\rho$$

$$-T\frac{\rho}{\alpha(1-\tau)}+\delta\rho;$$

and

$$R.H.S.=\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha)+\theta(1-\alpha)\frac{\rho}{\sigma}{g_m}^{-1}\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}\to\infty$$

as

$$g_m\to 0.$$

Again,

$$L.H.S.=\theta(1-\alpha)\frac{\delta}{\alpha}\rho-\alpha\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}\to-\infty;$$

and

$$R.H.S.=\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha)+\theta(1-\alpha)\frac{\rho}{\sigma}{g_m}^{-1}\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\frac{\delta}{\alpha}(\overline{\sigma-\alpha}g_m+\rho)\right\}$$

$$\to\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha)$$

as

$$g_m\to\infty.$$

Hence, the upper bound of the L.H.S. of in-equation (3.2.T) is $\theta(1-\alpha)\frac{\delta}{\alpha}\rho-T\frac{\rho}{\alpha(1-\tau)}+\delta\rho$ and the lower bound of its R.H.S. is $\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha)$.

Thus the determinant will be negative in sign if

$$\theta(1-\alpha)\frac{\delta}{\alpha}\rho-T\frac{\rho}{\alpha(1-\tau)}+\delta\rho<\frac{\theta(1-\alpha)}{\alpha}\delta(\sigma-\alpha),$$

or,

$$\frac{T}{(1-\tau)}>\delta\frac{\theta(1-\alpha)}{\alpha}\frac{(\rho-\sigma+\alpha)}{\rho}+\delta, \qquad \dots\dots(3.2C.3)$$

or,

$$\frac{1-(\tau-T)}{(1-\tau)}>\delta\frac{\theta(1-\alpha)}{\alpha}\frac{(\rho-\sigma+\alpha)}{\rho}+\delta+1,$$

or,

$$\frac{\alpha}{(1-\tau)}>\frac{\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)}{\alpha\rho},$$

or,

$$\tau>1-\alpha^2\rho[\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)]^{-1}. \qquad \dots\dots(3.2C.4)$$

Using equation (3.2.9) and in-equation (3.2C.3) we obtain the following in-equation.

$$g_m > \frac{\rho[\alpha-\theta(1-\alpha)-\alpha^2\rho(1+\delta)\{\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)\}]}{\alpha^2\rho\{\sigma+\delta(\sigma-\alpha)\}\{\delta\theta(1-\alpha)(\rho-\sigma+\alpha)+\alpha\rho(1+\delta)\}-\sigma\{\alpha-\theta(1-\alpha)\}}$$

This is the same as in-equation (3.2.T) in section 3.2.3.

APPENDIX 3.2D

DERIVATION OF EQUATION (3.2.17) IN SECTION 3.2.4

The relevant Hamiltonian to be maximized by the planner at each point of time is given by

$$\mathcal{H} = e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma} + e^{-\rho t}\lambda_K\left[A(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} - \Pi - C\right]$$

$$+e^{-\rho t}\lambda_E[\Omega-\delta C].$$

The state variables, as before, are K and E. The control variables are C, Π and Ω. λ_K and λ_E are the two co-state variables.

Maximising H with respect to C, Π and Ω we have

$$C^{-\sigma} = \lambda_K + \delta\lambda_E; \qquad \ldots\ldots(3.2D.1)$$

$$(1-\alpha)A(\Pi-\Omega)^{-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} = 1; \qquad \ldots\ldots(3.2D.2)$$

and

$$(1-\alpha)A(\Pi-\Omega)^{-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} = \frac{\lambda_E}{\lambda_K}. \qquad \ldots\ldots(3.2D.3)$$

Using equations (3.2D.2) and (3.2D.3) we obtain

$$\frac{\lambda_E}{\lambda_K} = 1. \qquad \ldots\ldots(3.2D.4)$$

Also, along the optimum path, time behaviour of co-state variables satisfies equations (3.2D.5) and (3.2D.6) as defined below.

$$\{\alpha-\theta(1-\alpha)\}A(\Pi-\Omega)^{1-\alpha}K^{\alpha-1-\theta(1-\alpha)}E^{\theta(1-\alpha)} = \rho - \frac{\dot{\lambda}_K}{\lambda_K}; \qquad \ldots\ldots(3.2D.5)$$

and

$$\frac{\lambda_K}{\lambda_E}\theta(1-\alpha)A(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)-1} = \rho - \frac{\dot{\lambda_E}}{\lambda_E}. \quad \dots\dots(3.2D.6)$$

From equations (3.2D.4), (3.2D.5) and (3.2D.6) we obtain the following equation.

$$\frac{E}{K} = \frac{\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}. \quad \dots\dots(3.2D.7)$$

Using equations (3.2D.1) and (3.2D.4) we obtain the following equation.

$$C^{-\sigma} = \lambda_K[1+\delta],$$

or,

$$-\sigma\frac{\dot{C}}{C} = \frac{\dot{\lambda_K}}{\lambda_K}. \quad \dots\dots(3.2D.8)$$

Using equations (3.2D.5), (3.2D.7) and (3.2D.8) we obtain

$$\rho + \sigma g_c = \{\alpha-\theta(1-\alpha)\}A\left(\frac{K}{\Pi-\Omega}\right)^{\alpha-1}\left\{\frac{\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}\right\}^{\theta(1-\alpha)}. \quad \dots\dots(3.2D.9)$$

Using equations (3.2D.2) and (3.2D.7) we have

$$\frac{K}{\Pi-\Omega} = A^{-\frac{1}{\alpha}}(1-\alpha)^{-\frac{1}{\alpha}}\left\{\frac{\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}\right\}^{-\frac{\theta(1-\alpha)}{\alpha}}. \quad \dots\dots(3.2D.10)$$

Using equations (3.2D.9) and (3.2D.10) we obtain the following equation.

$$(\rho+\sigma g_c) = A^{\frac{1}{\alpha}}(1-\alpha)^{\frac{(1-\alpha)}{\alpha}}\{\alpha-\theta(1-\alpha)\}^{\frac{\{\alpha-\theta(1-\alpha)\}}{\alpha}}\{\theta(1-\alpha)\}^{\frac{\theta(1-\alpha)}{\alpha}},$$

or,

$$(\rho+\sigma g_c)^{\alpha} = A(1-\alpha)^{1-\alpha}\{\alpha-\theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \quad \dots\dots(3.2D.11)$$

This is same as equation (3.2.17) in section 3.2.4.

APPENDIX 3.3A

DERIVATION OF EQUATION (3.3.5) IN SECTION 3.3.2

Using equations (2.2.1), (2.2.2), (2.2.3), (2.2.4), (3.3.1), (3.2.2) and (2.2.8) we obtain the following equations.

$$g_m = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\alpha A^{\frac{1}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \rho\right]; \quad \ldots\ldots(3.3A.1)$$

$$g_m = \frac{\dot{K}}{K} = A^{\frac{1}{\alpha}}(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)}{\alpha}} - \frac{C}{K}; \quad \ldots\ldots(3.3A.2)$$

and

$$g_m = \frac{\dot{E}}{E} = T(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} - \delta\left(\frac{E}{K}\right)^{-1}. \quad \ldots\ldots(3.3A.3)$$

From equation (3.3A.1) we have

$$\frac{E}{K} = \left[(1-\tau)(\tau-\mathrm{T})^{\frac{1-\alpha}{\alpha}}\alpha A^{\frac{1}{\alpha}}(\sigma g_m+\rho)^{-1}\right]^{-\frac{\alpha}{\theta(1-\alpha)}}. \quad \ldots\ldots(3.3A.4)$$

Using equations (3.3A.3) and (3.3A.4) we derive the following equation.

$$g_m = \left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right](\sigma g_m+\rho)^{-\frac{\alpha}{\theta(1-\alpha)}}\{\alpha(1-\tau)\}^{\frac{\alpha}{\theta(1-\alpha)}}(\tau-\mathrm{T})^{\frac{1}{\theta}}A^{\frac{1}{\theta(1-\alpha)}},$$

or,

$$g_m(\sigma g_m+\rho)^{\frac{\alpha}{\theta(1-\alpha)}} = \left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right]\{\alpha(1-\tau)\}^{\frac{\alpha}{\theta(1-\alpha)}}(\tau-\mathrm{T})^{\frac{1}{\theta}}A^{\frac{1}{\theta(1-\alpha)}},$$

or,

$$(\sigma g_m+\rho)^{\alpha}g_m^{\theta(1-\alpha)}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right]^{-\theta(1-\alpha)} = A\alpha^{\alpha}(1-\tau)^{\alpha}(\tau-T)^{1-\alpha}.$$

$$\ldots\ldots(3.3A.5)$$

This is same as equation (3.3.5) in section 3.3.2.

APPENDIX 3.3B

DERIVATION OF EQUATIONS (3.3.6) AND (3.3.7) IN SECTION 3.3.2.1

We denote the L.H.S. and the R.H.S. of equation (3.3.5) as L.H.S. $_{(3.3.5)}$ and R.H.S. $_{(3.3.5)}$ respectively. Differentiating equation (3.3.5) with respect to τ, we obtain the following first order condition.

$$\mathrm{L.H.S.}_{(3.3.5)}\, [\alpha\sigma(\sigma g_m + \rho)^{-1} + \theta(1-\alpha)g_m^{-1}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right\}^{-1}\frac{T\sigma}{\alpha(1-\tau)}\Bigg]\frac{\partial g_m}{\partial\tau}$$

$$-\mathrm{L.H.S.}_{(3.3.5)}\,\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right\}^{-1} T\frac{\sigma g_m+\rho}{\alpha(1-\tau)^2}$$

$$= \mathrm{R.H.S.}_{(3.3.5)}\,[-\alpha(1-\tau)^{-1} + (1-\alpha)(\tau-T)^{-1}]. \qquad \ldots\ldots(3.3B.1)$$

In equilibrium, $\mathrm{L.H.S.}_{(3.3.5)} = \mathrm{R.H.S.}_{(3.3.5)}$; and $\frac{\partial g_m}{\partial\tau} = 0$ at the optimum.

Thus equation (3.3B.1) takes the following form.

$$\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right\}^{-1} T\frac{\sigma g_m+\rho}{\alpha(1-\tau)^2} = [(1-\alpha)(\tau-T)^{-1} - \alpha(1-\tau)^{-1}];$$

or,

$$\theta(1-\alpha)\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right]^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}(\tau-T) = \frac{\alpha(\tau-T)-(1-\alpha)(1-\tau)}{T}. \qquad \ldots\ldots(3.3B.2)$$

This is same as equation (3.3.6) in the body of the paper.

Again, differentiating the R.H.S. of equation (3.3.5) with respect to T, we obtain the following first order condition.

$$\mathrm{L.H.S.}_{(3.3.5)}\, [\alpha\sigma(\sigma g_m + \rho)^{-1} + \theta(1-\alpha)g_m^{-1}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right\}^{-1}\frac{T\sigma}{\alpha(1-\tau)}\Bigg]\frac{\partial g_m}{\partial T}$$

$$-\mathrm{L.H.S.}_{(3.3.5)}\,\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right\}^{-1}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}$$

$$= -\mathrm{R.H.S.}_{(3.3.5)}\,(1-\alpha)(\tau-T)^{-1}. \qquad \ldots\ldots(3.3B.3)$$

Equation (3.3B.3) can be written as

$$\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-1} T\frac{\sigma g_m+\rho}{\alpha(1-\tau)^2} = (1-\alpha)(\tau - T)^{-1};$$

or,

$$\theta(1-\alpha)\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right]^{-1} \frac{\sigma g_m+\rho}{\alpha(1-\tau)}(\tau - T) = 1-\alpha. \qquad \ldots\ldots(3.3B.4)$$

This is same as equation (3.3.7).

To check the second order conditions for optimality we differentiate equation (3.3B.1) with respect to τ and equation (3.3B.3) with respect to T.

We arrive at the following two second order conditions.

$$-[{g_m}^{-2} + \alpha\sigma^2(\sigma g_m+\rho)^{-2} - \theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}\right\}^2\Bigg]\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+[\alpha\sigma(\sigma g_m+\rho)^{-1} + \theta(1-\alpha)g_m^{-1} - \theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}\right\}^{-1}\frac{T\sigma}{\alpha(1-\tau)}\Bigg]\frac{\partial^2 g_m}{\partial\tau^2}$$

$$-2\theta(1-\alpha)\left[\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-1}\frac{T\sigma}{\alpha(1-\tau)^2} - \left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}\right\}T\frac{\sigma g_m+\rho}{\alpha(1-\tau)^2}\right]\frac{\partial g_m}{\partial\tau}$$

$$-\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-1} T\frac{\sigma g_m+\rho}{\alpha(1-\tau)^3}$$

$$\left[2-\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-1} T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}\right] = -\left[\frac{\alpha}{(1-\tau)^2}+\frac{(1-\alpha)}{(\tau-T)^2}\right];$$

and

$$-[{g_m}^{-2} + \alpha\sigma^2(\sigma g_m+\rho)^{-2} - \theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}\right\}^2\Bigg]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+[\alpha\sigma(\sigma g_m+\rho)^{-1} + \theta(1-\alpha)g_m^{-1} - \theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-1}\left\{\frac{T\sigma}{\alpha(1-\tau)}\right\}\Bigg]\frac{\partial^2 g_m}{\partial T^2}$$

$$-2\theta(1-\alpha)\left[\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-1}\frac{\sigma}{\alpha(1-\tau)} - \left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-2}\left\{\frac{T\sigma}{\alpha(1-\tau)}\right\}\frac{\sigma g_m+\rho}{\alpha(1-\tau)}\right]\frac{\partial g_m}{\partial T}$$

$$+\theta(1-\alpha)\left\{T\frac{\sigma g_m+\rho}{\alpha(1-\tau)}-\delta\right\}^{-2}\left\{\frac{\sigma g_m+\rho}{\alpha(1-\tau)}\right\}^2 = -\frac{(1-\alpha)}{(\tau-T)^2}.$$

Now we evaluate the above two second order conditions at $\tau=\tau^\star$ and $T=T^\star$ where $\frac{\partial g_m}{\partial\tau} = \frac{\partial g_m}{\partial T} = 0$. Hence we obtain the followings.

$$\frac{\partial^2 g_m}{\partial \tau^2} = -\frac{(1-\alpha)^{-1}+\left[\frac{\theta(1-\alpha)\{\sigma g_m+\rho+\alpha\delta\}^2+(\alpha\delta)^2\{\alpha-\theta(1-\alpha)\}}{\theta(1-\alpha)\{\alpha-\theta(1-\alpha)\}(\sigma g_m+\rho)^2}\right]}{\frac{\{\alpha-\theta(1-\alpha)\}\{(\sigma+\delta\overline{\sigma-\alpha})g_m\}+\theta(1-\alpha)\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}}{g_m(\sigma g_m+\rho)}}$$

and

$$\frac{\partial^2 g_m}{\partial T^2} = -\frac{\frac{(!+\theta)}{\theta(1-\alpha)}}{\frac{\{\alpha-\theta(1-\alpha)\}\{(\sigma+\delta\overline{\sigma-\alpha})g_m\}+\theta(1-\alpha)\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}}{g_m\{(\sigma g_m+\rho)+\delta((\overline{\sigma-\alpha})g_m+\rho)\}}}.$$

The R.H.S. of each of these two equations is negative. Thus second order conditions are also satisfied.

APPENDIX 3.3C

DERIVATION OF THE DETERMINANT AND IN-EQUATION (3.3.T) IN SECTION 3.3.3

We consider following equations from section 3.3.3.

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}} + x - \frac{\rho}{\sigma}; \qquad \dots\dots(3.3.11)$$

and

$$\frac{\dot{y}}{y} = T(\tau-T)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}y^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} + x - \delta y^{-1} - (1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}}.$$

$$\dots\dots(3.3.12)$$

We obtain the following partial derivatives corresponding to the above two equations.

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial x} = 1;$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial y} = \frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau-T)^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}y^{\frac{\theta(1-\alpha)}{\alpha}-1};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial x} = 1;$$

and

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial y} = \delta y^{-2} - \frac{\alpha-\theta(1-\alpha)}{\alpha} T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}.$$

So the determinant of the Jacobian matrix can be written as follows.

$$|J| = \delta y^{-2} - \frac{\alpha-\theta(1-\alpha)}{\alpha} T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2}$$

$$-\frac{\theta(1-\alpha)}{\alpha}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$-\frac{\theta(1-\alpha)}{\alpha}\left(\frac{\alpha}{\sigma}-1\right)(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1},$$

or,

$$|J| = -\frac{\alpha-\theta(1-\alpha)}{\alpha} T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-2} - \frac{\theta(1-\alpha)}{\sigma}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}-1}$$

$$+\delta y^{-2}.$$

Now, at the steady-state equilibrium point, we use equations (3.3.11) and (3.3.12) with $\dot{x} = \dot{y} = 0$ and obtain the following equation.

$$\frac{\alpha}{\sigma}(1-\tau)(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} = T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)-\alpha}{\alpha}} - \delta y^{-1} + \frac{\rho}{\sigma}. \quad \ldots\ldots (3.3C.1)$$

Thus, using equation (3.3C.1) in the determinant, we find that it is positive (negative) if

$$\frac{\alpha+\theta(1-\alpha)}{\alpha}\delta > (<) T(\tau - T)^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} y^{\frac{\theta(1-\alpha)}{\alpha}} + \frac{\rho}{\sigma}\frac{\theta(1-\alpha)}{\alpha} y. \quad \ldots\ldots (3.3C.2)$$

Now using equations (3.3A.4), (3.3.5) and the value of $(\tau^\star - T^\star)$ given by (3.3.10), we simplify in-equation (3.3C.2) into the following form.

$$\frac{\alpha+\theta(1-\alpha)}{\alpha}\delta > (<) T\frac{\sigma g_m+\rho}{(1-\tau)} + \frac{\rho}{\sigma}\frac{\theta(1-\alpha)}{\alpha}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right] {g_m}^{-1},$$

or,

$$\frac{\alpha+\theta(1-\alpha)}{\alpha}\delta - \frac{\rho}{\sigma}\frac{\theta(1-\alpha)}{\alpha}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right] {g_m}^{-1} > (<) T\frac{\sigma g_m+\rho}{(1-\tau)}. \quad \ldots\ldots (3.3C.3)$$

Here,

$$L.H.S. = \frac{\alpha+\theta(1-\alpha)}{\alpha}\delta - \frac{\rho}{\sigma}\frac{\theta(1-\alpha)}{\alpha}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right]{g_m}^{-1} \to -\infty;$$

and

$$R.H.S. = T\frac{\sigma g_m+\rho}{(1-\tau)} \to \frac{T\rho}{(1-\tau)}$$

as

$$g_m \to 0.$$

Again,

$$L.H.S. = \frac{\alpha+\theta(1-\alpha)}{\alpha}\delta - \frac{\rho}{\sigma}\frac{\theta(1-\alpha)}{\alpha}\left[T\frac{\sigma g_m+\rho}{\alpha(1-\tau)} - \delta\right]{g_m}^{-1} \to \frac{\alpha+\theta(1-\alpha)}{\alpha}\delta;$$

and

$$R.H.S. = T\frac{\sigma g_m+\rho}{(1-\tau)} \to \infty$$

as

$$g_m \to \infty.$$

Hence, the upper bound of the L.H.S. of in-equation (3.3C.3) is $\frac{\alpha+\theta(1-\alpha)}{\alpha}\delta$ and the lower bound of its R.H.S. is $\frac{T\rho}{(1-\tau)}$.

Thus the determinant will be negative in sign if

$$\frac{\alpha+\theta(1-\alpha)}{\alpha}\delta < \frac{T\rho}{(1-\tau)},$$

or,

$$\frac{T}{(1-\tau)} > \frac{\alpha+\theta(1-\alpha)}{\alpha\rho}\delta,$$

or,

$$\frac{1-(\tau-T)}{(1-\tau)} > \frac{\alpha+\theta(1-\alpha)}{\alpha\rho}\delta + 1,$$

or,

$$\frac{\alpha}{(1-\tau)} > \frac{\delta\{\alpha+\theta(1-\alpha)\}+\alpha\rho}{\alpha\rho},$$

or,

$$\tau > 1 - \alpha^2\rho[\delta\{\alpha+\theta(1-\alpha)\}+\alpha\rho]^{-1}. \qquad \ldots\ldots(3.3C.4)$$

Using equation (3.3.8) and in-equation (3.3C.4) we obtain the following in-equation.

$$g_m > \frac{\{\alpha-\theta(1-\alpha)\}[\delta\{\alpha+\theta(1-\alpha)\}+\alpha\rho]\rho-\alpha^2\rho(\rho+\alpha\delta)}{\alpha\rho\sigma\theta(1-\alpha)+\sigma\delta\{\alpha-\theta(1-\alpha)\}\{\alpha+\theta(1-\alpha)\}}.$$

This is same as in-equation (3.3.T).

APPENDIX 3.3D

DERIVATION OF EQUATION (3.3.13) IN SECTION 3.3.4

The relevant Hamiltonian to be maximized by the planner at each point of time is given by

$$\mathcal{H} = e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma} + e^{-\rho t}\lambda_K\left[A(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} - \Pi - C\right]$$
$$+e^{-\rho t}\lambda_E[\Omega - \delta K].$$

The state variables are K and E. The control variables are C, Π, and Ω. λ_K and λ_E are two co-state variables.

Maximising H with respect to C, Π, and Ω we have

$$C^{-\sigma} = \lambda_K; \qquad \ldots\ldots(3.3D.1)$$

$$(1-\alpha)A(\Pi-\Omega)^{-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} = 1; \qquad \ldots\ldots(3.3D.2)$$

and

$$(1-\alpha)A(\Pi-\Omega)^{-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)} = \frac{\lambda_E}{\lambda_K}. \qquad \ldots\ldots(3.3D.3)$$

Using equations (3.3D.2) and (3.3D.3) we obtain

$$\frac{\lambda_E}{\lambda_K} = 1. \qquad \ldots\ldots(3.3D.4)$$

Also, along the optimum path, time behaviour of the co-state variables satisfies the followings.

$$\{\alpha-\theta(1-\alpha)\}A(\Pi-\Omega)^{1-\alpha}K^{\alpha-1-\theta(1-\alpha)}E^{\theta(1-\alpha)} - \delta\frac{\lambda_E}{\lambda_K} = \rho - \frac{\dot{\lambda}_K}{\lambda_K}; \qquad \ldots\ldots(3.3D.5)$$

and

$$\frac{\lambda_K}{\lambda_E}\theta(1-\alpha)A(\Pi-\Omega)^{1-\alpha}K^{\alpha-\theta(1-\alpha)}E^{\theta(1-\alpha)-1} = \rho - \frac{\dot{\lambda}_E}{\lambda_E}. \qquad \ldots\ldots(3.3D.6)$$

From equation (3.3D.1) we obtain the following equation.

$$-\sigma\frac{\dot{C}}{C}=\frac{\dot{\lambda_K}}{\lambda_K}. \qquad \dots\dots(3.3D.7)$$

From equation (3.3D.2) we obtain

$$\frac{E}{K}=\left[\frac{\left\{\frac{K}{\Pi-\Omega}\right\}^{-\alpha}}{A(1-\alpha)}\right]^{\frac{1}{\theta(1-\alpha)}}. \qquad \dots\dots(3.3D.8)$$

Using equations (3.3D.4), (3.3D.7), (3.3D.6) and (3.3D.8) we obtain

$$(\rho+\sigma g_c)=\theta(1-\alpha)A^{\frac{1}{\theta(1-\alpha)}}(1-\alpha)^{\frac{1-\theta(1-\alpha)}{\theta(1-\alpha)}}\left(\frac{K}{\Pi-\Omega}\right)^{\frac{\alpha-\theta(1-\alpha)}{\theta(1-\alpha)}},$$

or,

$$\frac{K}{\Pi-\Omega}=(\rho+\sigma g_c)^{\frac{\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}}\{\theta(1-\alpha)\}^{-\frac{\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}}A^{-\frac{1}{\alpha-\theta(1-\alpha)}}(1-\alpha)^{-\frac{1-\theta(1-\alpha)}{\alpha-\theta(1-\alpha)}}.$$

$$\dots\dots(3.3D.9)$$

Again, from equations (3.3D.4), (3.3D.7), (3.3D.5) and (3.3D.8) we obtain

$$(\rho+\sigma g_c)=\{\alpha-\theta(1-\alpha)\}(1-\alpha)^{-1}\left(\frac{K}{\Pi-\Omega}\right)^{-1}-\delta. \qquad \dots\dots(3.3D.10)$$

Using equations (3.3D.9) and (3.3D.10) we obtain the following equation.

$$(\rho+\sigma g_c)^{\theta(1-\alpha)}(\rho+\sigma g_c+\delta)^{\alpha-\theta(1-\alpha)}=A(1-\alpha)^{1-\alpha}$$

$$\{\alpha-\theta(1-\alpha)\}^{\alpha-\theta(1-\alpha)}\{\theta(1-\alpha)\}^{\theta(1-\alpha)}. \qquad \dots\dots(3.3D.11)$$

This is same as equation (3.3.13) in section 3.3.4.

CHAPTER 4

4. HEALTH INFRASTRUCTURE AND ENVIRONMENTAL POLLUTION

4.1 INTRODUCTION[51]

We extend the basic model of section 2.2 in chapter 2 to include health infrastructure as an additional productive public input which is adversely affected by environmental pollution. AM (2006) and Agenor (2008) extend Barro (1990) model introducing productive health expenditure in addition to the infrastructural expenditure, where financing of both types of expenditure is made by the allocation of income tax-revenue. However, neither AM (2006) nor Agenor (2008) deals with environmental pollution in their models. Greiner (2005) and EP (2008), who deal with the interaction between economic growth and environmental pollution when public expenditure is the engine of economic growth, do not introduce health as a productive public input in their models.

We follow AM (2006) and Agenor (2008) to introduce health capital as an input in the production function. However, we assume health capital to be an accumulable input in the production function following the second model of Agenor (2008); in his first model, it is in the form of a flow variable. We also consider the negative role of environmental pollution on the depreciation of public health capital.

We obtain interesting results analyzing this model. The optimum ratio of combined public expenditure on infrastructure and health to national income is equal to the sum of competitive shares of public infrastructural input and health capital in the unpolluted output of the final good; and hence this

[51] A related version of this model is published in Journal of Macroeconomics.

optimum ratio varies inversely with the rate of pollution per unit of production. However, in Barro (1990) and in FMS (1993), there is neither any environmental pollution nor any productive health capital; hence this ratio is always equal to the competitive output share of the public infrastructural input. In Greiner (2005), the optimum share of investment to national income is also independent of the rate of pollution per unit of production because pollution, being a flow variable in his model, enters the utility function and it is countered by a separate pollution tax. However, neither environmental quality nor health infrastructure enters production function as an accumulable input in his model. Secondly, in the model in this chapter, optimum income tax rate is higher than that predicted by Barro (1990) and FMS (1993); and this rate varies positively with the pollution-output coefficient. This is so because a part of the income tax revenue is spent as abatement expenditure and health expenditure in this model. However, this is not necessarily so in Greiner (2005) who considers pollution tax as an alternative instrument of financing abatement expenditure. In both AM (2006) and Agenor (2008), the optimum tax rate is lower than that in our model but is higher than the Barro-FMS optimum tax rate because both these models have a tax financed health expenditure but no abatement expenditure. Thirdly, this extended model exhibits transitional dynamic properties though it follows Barro (1990) to assume public expenditure to be a flow variable. By introducing environmental quality and health capital as accumulable inputs in the production function, we protect this model, from being an AK model and thus get back transitional dynamic properties. AM (2006) show the balanced growth path to be unique in their model; however, the model of Agenor (2008) shows (does not show) transitional dynamic properties when health expenditure is a stock (flow) variable. However, steady-state equilibrium is a saddle-point when health expenditure is a stock variable in his model. In our model, with both health capital and environmental quality being stock variables, steady-state equilibrium never satisfies saddle-point stability. But we find a possibility of indeterminacy of the transitional growth path, that which neither AM (2006) nor Agenor (2008) find in their models. FMS (1993) and Greiner (2005) also find the saddle-point stability property of the steady-state equilibrium in their models. Fourthly, like in the basic model, we do not find any conflict between the growth rate maximizing solution and the social welfare maximizing solution along the steady-state equilibrium growth path because neither health nor pollution affects utility in this model. Agenor (2008) finds a conflict between these two goals because health affects the utility function of the household in his model. Greiner (2005) also finds a similar conflict because environmental pollution affects the utility function in his model. Fifthly, the competitive equilibrium growth rate in this model is not necessarily less than the socially efficient growth rate which is similar to the basic model of section 2.2 in chapter 2. This is so because, here too, we have two conflicting types of externalities on production - positive externality arising from the gross public expenditure and negative externality arising from capital accumulation and environmental pollution. Market economy growth rate may exceed socially efficient growth rate when the pollution-output coefficient takes a high value. Barro (1990) and FMS (1993) consider only the positive externality of public expenditure. Agenor (2008) also considers two sources of positive externality from health expenditure and infrastructural expenditure. So market economy growth rate falls short of the socially efficient growth rate in their models.

Following sections are organized as follows. Section 4.2 describes the basic model of the household economy. Section 4.3 analyses its dynamic equilibrium properties. Subsection 4.3.1 shows the existence of unique steady-state equilibrium growth rate in the market economy and subsection 4.3.2 analyses the properties of optimal fiscal policy along the steady-state equilibrium path. Section 4.4 shows transitional dynamic results; and section 4.5 compares the market economy steady-state equilibrium growth rate to the command economy steady-state equilibrium growth rate.

4.2 THE MODEL

Health capital, a productive input, is a stock variable in this model. The basic model does not consider health infrastructure as a productive input. The government imposes a proportional tax on income of the representative household who consumes a part of the post-tax income and saves (invests) the other part. Government allocates a part of the tax revenue to build up the infrastructure on health and provides it free of charge to the representative household. However, health capital deteriorates with pollution. Environmental quality is also considered a stock variable; and it deteriorates with pollution and is improved by abatement activities undertaken by the government. Environmental quality is non-rival and is a free good. The budget of the government is again, balanced; and the allocation of tax revenue in this extension is made among three expenditure heads - public infrastructural expenditure, health expenditure and abatement expenditure.

Following equations describe this model. Here equations (2.2.4) and (2.2.6) are borrowed from the model developed in section 2.2 in chapter 2.

$$Y = K^{\alpha}\hat{G}_I^{1-\alpha-\beta}H^{\beta} \text{ with } 0 < \alpha, \beta < 1; \quad \ldots\ldots(4.1)$$

$$\hat{G}_I = G_I\overline{K}^{-\theta}E^{\theta} \text{ with } 0 < \theta < 1; \quad \ldots\ldots(4.2)$$

$$\dot{K} = (1-\tau)Y - C; \quad \ldots\ldots(2.2.4)$$

$$\dot{E} = TY - \delta Y \text{ with } 0 < \delta < 1; \quad \ldots\ldots(4.3)$$

$$\dot{H} = G_H - \eta\delta Y \text{ with } 0 < \eta, \delta < 1; \quad \ldots\ldots(4.4)$$

$$G = G_I + G_H = (\tau - T)Y \text{ with } 0 < T < \tau < 1; \quad \ldots\ldots(4.5)$$

$$G_i = v_i(\tau - T)Y \text{ with } i = I, H; \quad \ldots\ldots(4.6)$$

and

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma} \text{ with } \sigma > 0. \quad \ldots\ldots(2.2.6)$$

Equation (4.1) describes the Cobb-Douglas production function in the final good sector. It satisfies constant returns to scale in terms of K, G and H. Y

is the level of output produced, K is the stock of physical capital, and $\hat{G}_I$ is the congestion effect adjusted effective benefit derived from the public infrastructural input. H is the stock of health capital; and this is the new input whose presence makes equation (4.1) different from equation (2.2.1) of chapter 2. Elasticities of output with respect to physical capital, public infrastructural input, and health capital are denoted by α, $(1-\alpha-\beta)$ and β respectively. If $\beta = 0$, equation (4.1) is identical to (2.2.1) with $A = 1$ and $\hat{G} = \hat{G}_I$.

Equation (4.2) is identical to equation (2.2.2) of chapter 2 when $\theta_1 = \theta_2$.

Equation (4.3) shows how environmental quality changes over time depending upon the magnitudes of pollution and abatement activity. It is identical to equation (2.2.5) when $\eta = 0$, i.e., there is no self-regeneration of environmental quality.

Accumulation of the stock of health capital, H, is given by equation (4.4). The government spends an amount G_H on health infrastructure. Pollution causes depreciation of this stock; and this relationship is assumed to be proportional for the sake of simplicity. η is the resulting depreciation of health capital per unit of pollution.

Besides causing damage to public infrastructure as previously discussed in chapter 2, degradation of environmental quality also reduces the effective benefit of health expenditure in various ways. For example, water and air pollution create a disease-friendly environment and hence public health expenditure programme cannot provide the maximum benefit to workers. This, in turn, lowers the efficiency of the workers.

Equation (4.5) describes government budget constraint. The government finances the public infrastructural expenditure and health expenditure with its tax revenue after meeting its expenditure on abatement. T is the ratio of abatement expenditure to income; and τ is the income tax rate. A fraction v_I of the tax revenue net of abatement expenditure is used to finance infrastructural expenditure, G_I; v_H is the fraction of the net tax revenue, net of abatement expenditure, allocated to health expenditure. Public expenditure allocation ratios are given by equation (4.6).

Equation (2.2.6) shows that neither health nor environment is considered as an argument in the utility function. AM (2006), AN (2011) and Agenor (2008) introduce health as an argument in the utility function and Greiner (2005) introduces pollution as an argument in the utility function. We ignore these complications in this model for the sake of simplicity.

4.3 DYNAMIC EQUILIBRIUM

The representative household maximizes $\int_0^\infty u(C)\,e^{-\rho t}dt$ with respect to C subject to equations (4.1), (2.2.4) and (2.2.6). The demand rate of growth[52] of consumption is derived from this maximizing problem as follows.

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]. \qquad \dots\dots(4.7)$$

A steady-state growth equilibrium is again considered where all macroeconomic variables grow at the same rate, g_m. Hence, the steady-state condition is given by

$$\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{E}}{E} = \frac{\dot{H}}{H} = \frac{\dot{G}}{G} = g_m. \qquad \dots\dots(4.8)$$

4.3.1 Existence of Steady-State Growth Equilibrium

Using equations (4.1) to (4.6), (2.2.4), (4.7) and (4.8), we obtain the following equations.

$$\frac{1}{\sigma}\left[\alpha(1-\tau)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right] = g_m; \qquad \dots\dots(4.9)$$

$$(1-\tau)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}} - \frac{C}{K} = g_m; \qquad \dots\dots(4.10)$$

[52] The demand rate of growth of consumption is derived in Appendix 4A.

$$(T-\delta)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}}=g_m; \qquad \text{... ... (4.11)}$$

and

$$\{v_H(\tau-T)-\eta\delta\}\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}\left(\frac{H}{E}\right)^{-\frac{\alpha}{\alpha+\beta}}=g_m. \qquad \text{... ... (4.12)}$$

We obtain equation (4.13) below[53] that solves for g_m by using equations (4.9) to (4.12).

$$g_m{}^{\beta+\theta(1-\alpha-\beta)}(\sigma g_m+\rho)^{\alpha-\theta(1-\alpha-\beta)}=\alpha^{\alpha-\theta(1-\alpha-\beta)}(1-\tau)^{\alpha-\theta(1-\alpha-\beta)}$$
$$\{v_H(\tau-T)-\eta\delta\}^{\beta}\{v_I(\tau-T)\}^{1-\alpha-\beta}(T-\delta)^{\theta(1-\alpha-\beta)} \qquad \text{... ... (4.13)}$$

The L.H.S. of equation (4.13) is an increasing function of g_m since $\alpha-\theta(1-\alpha-\beta)>0$ (social elasticity of private physical capital is positive) and its R.H.S. is a constant term, given τ, T and v_i. Figure 4.1 diagrammatically shows the existence of unique value of g_m.

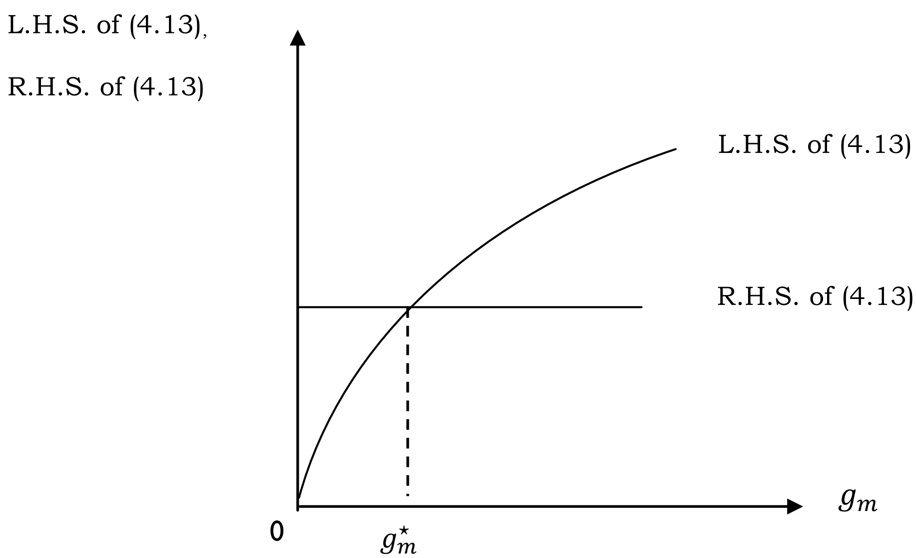

FIGURE 4.1

[53] Derivation of equation (4.13) is worked out in Appendix 4B.

We have the following proposition.

Proposition 4.1: There exists unique steady-state equilibrium growth rate in the market economy given the income tax rate, the abatement expenditure rate and the public expenditure allocation ratio.

Equations (4B.7), (4B.8) and (4B.9) in Appendix 4B show that $\frac{E}{K}$, $\frac{H}{E}$ and $\frac{C}{K}$ in the steady-state equilibrium are functions of g_m. This proves that the steady-state equilibrium is also unique. The existence of unique steady state equilibrium growth rate is guaranteed under two conditions given $0 < \delta < T < \tau < 1$. These are (i) $\alpha - \theta(1 - \alpha - \beta) > 0$, which means that the social elasticity of private physical capital is positive and (ii) $v_H(\tau - T) - \eta\delta > 0$, which means that public expenditure on health capital must exceed the magnitude of damage on health caused by pollution.

4.3.2 Optimal Taxation

In this section, government's growth maximization fiscal policy at the steady-state equilibrium is examined vis-à-vis the welfare maximization policy at the same equilibrium. The government maximizes the steady-state equilibrium growth rate with respect to fiscal instruments, τ, T and v_I. The L.H.S. of equation (4.13) is a monotonically increasing function of g_m, because, by assumption, $\alpha > \theta(1 - \alpha - \beta)$. Since the L.H.S. is always equal to the R.H.S. in the steady-state growth equilibrium, maximization of g_m implies maximization of the R.H.S. of equation (4.13).

Maximizing the R.H.S. of equation (4.13) with respect to τ, T and v_I respectively, we obtain following expressions of their optimum values[54].

$$\tau^{\star} = 1 - (1 - \delta - \eta\delta)\{\alpha - \theta(1 - \alpha - \beta)\}; \qquad \ldots\ldots(4.14)$$

[54] The derivation of equations (4.14), (4.15) and (4.16) is worked out in Appendix 4C.

$$T^{\star} = \delta + (1 - \delta - \eta\delta)\theta(1 - \alpha - \beta); \qquad \dots\dots(4.15)$$

and

$$v_I^{\star} = \frac{(1-\delta-\eta\delta)(1-\alpha-\beta)}{\eta\delta+(1-\delta-\eta\delta)(1-\alpha)}. \qquad \dots\dots(4.16)$$

Using equations (4.14) and (4.15), we have

$$\tau^{\star} - T^{\star} = \eta\delta + (1 - \delta - \eta\delta)(1 - \alpha). \qquad \dots\dots(4.17)$$

To ensure that the growth rate is non-negative deterioration of the two accumulable inputs - environmental quality and health infrastructure - due to pollution is neutralized by allocating δ and $\eta\delta$ fractions of the total output to abatement expenditure, TY and aggregate productive public expenditure, $(\tau - T)Y$, respectively. The optimum net abatement expenditure rate is then $(T^{\star} - \delta)$ and $(1 - \delta - \eta\delta)\theta(1 - \alpha - \beta)$ is the competitive unpolluted output share of environmental quality. So the net optimum ratio is equal to the competitive share of environmental input in the unpolluted output. Analyzing, similarly, $(\tau^{\star} - T^{\star} - \eta\delta)$ is the optimum ratio of net aggregate of public expenditure on the intermediate public good and health infrastructure to the national income; and $(1 - \delta - \eta\delta)(1 - \alpha)$ is the net competitive unpolluted output share of the two inputs taken together which is financed by government's tax revenue. So the net optimum ratio is equal to the competitive share of the public intermediate good in the unpolluted output. In Barro (1990), FMS (1993), and in AN (2011) entire output is pollution free and this ratio is equal to the competitive share of the public input in the total output.

We now examine whether the growth rate maximizing solution is consistent with the social welfare maximizing solution in the steady-state equilibrium. The social welfare function is given by

$$W = \int_0^{\infty} e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt. \qquad \dots\dots(4.18)$$

Using equations (4.9) and (4.10) and assuming that the economy is on the steady-state equilibrium growth path, it can be shown that

$$C = \frac{1}{\alpha}[\rho - (\alpha - \sigma)g_m]K(0)e^{g_m t}. \qquad \dots\dots(4.19)$$

Using equations (4.18) and (4.19) we arrive at the same equation as (2.2.16) from the basic model in chapter 2, which shows W to vary positively with g_m.

Thus the level of social welfare in the steady-state equilibrium is maximized when the steady-state equilibrium growth rate is maximized[55]. We now can state the following proposition.

Proposition 4.2: (i) The optimum income tax rate, the optimum abatement expenditure rate and the optimum public infrastructural expenditure allocation ratio in the steady-state growth equilibrium are given by

$$\tau^\star = 1 - (1 - \delta - \eta\delta)\{\alpha - \theta(1 - \alpha - \beta)\},$$

$$T^\star = \delta + (1 - \delta)\theta(1 - \alpha),$$

and

$$v_I^\star = \frac{(1-\delta-\eta\delta)(1-\alpha-\beta)}{\eta\delta+(1-\delta-\eta\delta)(1-\alpha)}.$$

(ii) The net optimum ratio of combined public expenditure on infrastructure and health to national income in the steady-state equilibrium is equal to the combined competitive share of these two inputs in the unpolluted output of the final good; and hence this optimum ratio varies inversely with the magnitude of the pollution-output coefficient.

The presence of three different effects makes our result different from those available in the existing literature. These are (i) congestion effect on public expenditure that makes $\theta > 0$, (ii) environmental pollution effect causing $\delta > 0$ and (iii) effect of pollution on health capital causing $\eta > 0$. Health capital and damage to it caused by pollution are what differentiates this model from the basic model. Tax revenue, net of abatement expenditure, now has two productive public inputs to finance – public infrastructure and public health expenditure. If we assume $\theta = \delta = 0$, we obtain $\tau^\star = 1 - \alpha$ and $T^\star = 0$; and these

[55] As with analysis of the basic model in chapter 2, here too, we abstain from analyzing social welfare maximization along the transitional path.

results are identical to those of Barro (1990) and FMS (1993). The net optimum ratio of combined public expenditure on infrastructure and health to national income in this model, with $0 < \delta, \eta < 1$ and $\theta > 0$, appears to be lower than that obtained in Barro (1990) and in FMS (1993). This is obvious because production of the final good generates environmental pollution. This, in turn, lowers the rate of accumulation of environmental quality and of health capital. Thus producer's effective benefit derived from public expenditure is reduced. So it is optimal for the government to allocate a smaller fraction of tax revenue to meet this expenditure. However,

$$\tau^{\star} = 1 - \alpha + \alpha\delta + \alpha\eta\delta + (1 - \delta - \eta\delta)\theta(1 - \alpha - \beta).$$

Here, $\tau^{\star} > 1 - \alpha$ because $0 < \delta, \eta < 1$, $0 < \alpha < 1$, and $\theta > 0$. So the optimum income tax rate in the present model is higher than the corresponding rate obtained in the models like Barro (1990), FMS (1993), AM (2006) and Agenor (2008). This is so because income tax is the only source of public revenue in this model and a part of that revenue is used to meet the abatement expenditure. This is not so in the models of Barro (1990), FMS (1993), AM (2006), Agenor (2008), etc., because there is no environmental pollution in those models. Moreover, in AN (2011) optimum tax rate is higher than the growth maximizing tax rate as one of the productive public inputs affects household's utility. In the present model the optimum fiscal policy is identical to the growth rate maximizing fiscal policy. This is because utility is a function of only consumption.

In this model too, aggregate productive public expenditure, i.e., excess of tax revenue over abatement expenditure, as well as level of environmental pollution is proportional to the level of income. So $(\tau^{\star} - T^{\star})$ varies inversely with the pollution-output coefficient, δ. If $\delta = 0$, then Barro (1990) - FMS (1993) - Agenor (2008) result comes back in this model in this special case.

Comparing the values of the growth rate maximizing fiscal instruments of this section to those in section 2.2 of chapter 2, we find that the abatement expenditure rates are identical in both the models. However, the share of expenditure on public intermediate good in chapter 4, given by

$$v_I^*(\tau^\star - T^\star) = (1-\delta-\eta\delta)(1-\alpha-\beta),$$

is less than the corresponding share obtained in section 2.2 of chapter 2 which is given by

$$(\tau^\star - T^\star) = (1-\delta)(1-\alpha).$$

This is so because income tax revenue net of pollution abatement expenditure is now spent not only on public intermediate good but also on health capital.

If $\eta\delta\{\alpha-\theta(1-\alpha-\beta)\} > (=)(<)\theta\beta(1-\delta)$, then the steady-state equilibrium growth rate maximizing income tax rate derived here is greater than (equal to) (less than) that derived in section 2.2 of chapter 2. Now, comparing equation (2.2.12) with $A = 1$ to equation (4.13) of chapter 4, we find that L.H.S. of equation (4.13) is greater than that of equation (2.2.12) with $\theta, \beta > 0$. Here,

$$g_m{}^{\beta+\theta(1-\alpha-\beta)}(\sigma g_m+\rho)^{\alpha-\theta(1-\alpha-\beta)} = \alpha^{\alpha-\theta(1-\alpha-\beta)}(1-\tau)^{\alpha-\theta(1-\alpha-\beta)}$$
$$\{v_H(\tau-T)-\eta\delta\}^{\beta}\{v_I(\tau-T)\}^{1-\alpha-\beta}(T-\delta)^{\theta(1-\alpha-\beta)}. \quad \ldots\ldots (4.13)$$

However, such conclusions cannot be unambiguously drawn about the R.H.S. of these two equations. Only if we assume that $\beta = 0$ and $v_I = 1$ (which, in turn, implies $v_H = 0$) , then this equation (4.13) is identical to equation (2.2.12) in the special case with $A = 1$.

4.4 TRANSITIONAL DYNAMICS

We now turn to investigate the local stability properties of the unique steady-state equilibrium point in the market economy. Equations of motion of the dynamic system are given by (2.2.4), (4.3), (4.4) and (4.7). We use the ratio variables, x and y, from previous chapters and define a new one, $z = \frac{H}{E}$.

Using equations (2.2.4), (4.3), (4.4) and (4.7), we have

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma}-1\right)(1-\tau)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} y^{\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}} z^{\frac{\beta}{\alpha+\beta}} + x - \frac{\rho}{\sigma}; \quad \ldots\ldots (4.20)$$

$$\frac{\dot{y}}{y} = (T-\delta)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}} z^{\frac{\beta}{\alpha+\beta}}$$

$$-(1-\tau)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} y^{\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}} z^{\frac{\beta}{\alpha+\beta}} + x; \qquad \dots\dots(4.21)$$

and

$$\frac{\dot{z}}{z} = \{v_H(\tau - T) - \eta\delta\}\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}} z^{-\frac{\alpha}{\alpha+\beta}}$$

$$-(T-\delta)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}} z^{\frac{\beta}{\alpha+\beta}}. \qquad \dots\dots(4.22)$$

The determinant of the Jacobian matrix [56] corresponding to the differential equations (4.20), (4.21) and (4.22) is given by

$$|J| = \frac{\alpha-\theta(1-\alpha-\beta)}{\alpha+\beta}(T-\delta)\{v_I(\tau - T)\}^{2\frac{1-\alpha-\beta}{\alpha+\beta}}\{v_H(\tau - T) - \eta\delta\} y^{2\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}-1} z^{-2\frac{\alpha}{\alpha+\beta}}$$

$$+\frac{\alpha}{\sigma}\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}(1-\tau)\{v_I(\tau - T)\}^{2\frac{1-\alpha-\beta}{\alpha+\beta}}\{v_H(\tau - T) - \eta\delta\} y^{2\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}} z^{-2\frac{\alpha}{\alpha+\beta}}$$

$$+\frac{\alpha}{\sigma}\frac{\beta}{\alpha+\beta}(1-\tau)(T-\delta)\{v_I(\tau - T)\}^{2\frac{1-\alpha-\beta}{\alpha+\beta}} y^{2\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}} z^{2\frac{\beta}{\alpha+\beta}-1}.$$

Here $\alpha - \theta(1-\alpha-\beta) > 0$. Also $1 > \tau > T > \delta$ when τ and T are optimally chosen and when $\theta > 0$. Also $v_H(\tau - T) - \eta\delta > 0$. So, $|J| > 0$ in this case when it is evaluated at the steady-state equilibrium point. So either all the three latent roots of J matrix are positive or two of them are negative with the third one being positive. Hence the steady-state equilibrium cannot be a saddle point. Either it is unstable with all latent roots being positive or there exists indeterminacy in the transitional growth path converging to the equilibrium point.

Trace of the Jacobian matrix is given by

$$Tr\,J = 1 + \frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}(T-\delta)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}-1} z^{\frac{\beta}{\alpha+\beta}}$$

$$-\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}(1-\tau)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}} z^{\frac{\beta}{\alpha+\beta}}$$

[56] Derivation of the determinant is worked out in Appendix 4D.

$$-\frac{\alpha}{\alpha+\beta}\{v_H(\tau-T)-\eta\delta\}\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}z^{-\frac{\alpha}{\alpha+\beta}-1}$$

$$-\frac{\beta}{\alpha+\beta}(T-\delta)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}z^{-\frac{\alpha}{\alpha+\beta}}.$$

Using equations (4.14), (4.15), (4.16) and the expression of the steady-state equilibrium values of x, y and z in terms of g_m from equations (4B.7), (4B.8) and (4B.9) in Appendix 4B, we find that the trace of the Jacobian matrix is negative[57] if

$$1+\frac{\rho}{\sigma}\frac{\{\alpha-\theta(1-\alpha-\beta)\}^2}{\alpha+\beta}\frac{\alpha}{\theta(1-\alpha-\beta)}\frac{g_m}{(\sigma g_m+\rho)}+\frac{\rho}{\sigma}\frac{\theta(1-\alpha-\beta)}{\beta}<\left[\frac{\alpha}{\sigma}\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}+\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}\right.$$

$$\left.\frac{1}{\sigma}\frac{\{\theta(1-\alpha-\beta)\}^2}{\beta\{\alpha-\theta(1-\alpha-\beta)\}}\frac{(\sigma g_m+\rho)}{g_m}\right](1-\delta-\eta\delta)^{\frac{1}{\alpha+\beta}}(1-\alpha-\beta)^{\frac{(1-\alpha-\beta)}{\alpha+\beta}}$$

$$\left\{\frac{\beta}{\theta(1-\alpha-\beta)}\right\}^{\frac{\beta}{\alpha+\beta}}\{\alpha-\theta(1-\alpha-\beta)\}^{1+\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}}\left\{\frac{\alpha}{\theta(1-\alpha-\beta)}\right\}^{\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}}$$

$$\left\{\frac{g_m}{(\sigma g_m+\rho)}\right\}^{\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}}. \qquad \ldots\ldots(4.T)$$

If the determinant of the Jacobian matrix takes a positive sign and its trace takes a negative sign, then there are one positive and two negative latent roots of this matrix[58]. It means that there exists indeterminacy in the transitional growth path converging to the unique equilibrium point. So we have the following proposition.

Proposition 4.3: The unique steady-state equilibrium point never satisfies saddle-point stability; but there exists indeterminacy in the transitional growth path converging to the steady-state equilibrium point if the steady-state equilibrium growth rate satisfies the following condition:

$$1+\frac{\rho}{\sigma}\frac{\{\alpha-\theta(1-\alpha-\beta)\}^2}{\alpha+\beta}\frac{\alpha}{\theta(1-\alpha-\beta)}\frac{g_m}{(\sigma g_m+\rho)}+\frac{\rho}{\sigma}\frac{\theta(1-\alpha-\beta)}{\beta}<\left[\frac{\alpha}{\sigma}\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}+\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}\right.$$

[57] This derivation is worked out in Appendix 4D.
[58] It is a sufficient condition but not a necessary one. There may be one positive and two negative roots even if the trace takes a positive sign. However, all the roots may also be positive implying that no trajectory converges to the equilibrium point. See Benhabib and Perili (1994).

$$\frac{1}{\sigma}\frac{\{\theta(1-\alpha-\beta)\}^2}{\beta\{\alpha-\theta(1-\alpha-\beta)\}}\frac{(\sigma g_m+\rho)}{g_m}\Bigg](1-\delta-\eta\delta)^{\frac{1}{\alpha+\beta}}(1-\alpha-\beta)^{\frac{(1-\alpha-\beta)}{\alpha+\beta}}$$

$$\left\{\frac{\beta}{\theta(1-\alpha-\beta)}\right\}^{\frac{\beta}{\alpha+\beta}}\{\alpha-\theta(1-\alpha-\beta)\}^{1+\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}}\left\{\frac{\alpha}{\theta(1-\alpha-\beta)}\right\}^{\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}}$$

$$\left\{\frac{g_m}{(\sigma g_m+\rho)}\right\}^{\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{\alpha+\beta}}.$$

This sufficient condition is always satisfied for low values of $\frac{g_m}{(\sigma g_m+\rho)}$; and the value of g_m is determined by the exogenous values of the parameters. Here very low values of δ and η will ensure that $(1-\delta-\eta\delta)$ is positive; and this is necessary for the inequality to be satisfied. Note that $\frac{g_m}{(\sigma g_m+\rho)}$ is low when g_m is high; and figure 4.2 shows that g_m is high when δ and η take very low values[59].

This is an important result. Barro (1990) model, with a flow public expenditure, does not exhibit any transitional dynamic properties. FMS (1993) brings back transitional dynamic properties in Barro (1990) model introducing durable public input but shows saddle-point stability property of the unique steady-state equilibrium. AM (2006) also find the steady-state growth equilibrium to be saddle-point stable. Agenor (2008) shows saddle-point stability property of the steady-state equilibrium when health expenditure is a stock variable but does not exhibit transitional dynamic properties when health expenditure is a flow variable. AN (2011), however, find the steady-state equilibrium in their benchmark model to be unstable. Greiner (2005), Dasgupta (1999), etc., also prove saddle-point stability property of the long-run equilibrium in their models. However, we show that saddle-point stability property of the steady-state equilibrium is never satisfied in our model. On the contrary, we find a possibility of indeterminacy of the transitional growth path without introducing physical capital stock or public expenditure into the utility

[59] Since it is a sufficient condition and not a necessary one, a low value of g_m does not rule out the possibility of indeterminacy.

function[60]. This is so because both the environmental quality and health infrastructure are stock variables in our model generating externalities in the productivity of the system. Also, physical capital stock generates a negative externality through congestion effect. That the externality of physical capital may generate indeterminacy in the transitional growth path has been explained by Benhabib and Farmer (1993), Chen and Lee (2006), Mino (2001), Benhabib, Meng and Nishimura (2000); and the interaction between conflicting type of externalities may generate indeterminacy in the transitional growth path.

4.5 COMMAND ECONOMY

The presence of three non rival inputs in the production function - public infrastructure, health capital and environmental quality - causes positive externalities in this extension to the basic model. Also, physical capital generates negative externality through congestion effect as considered in chapters 2 and 3. Moreover, environmental pollution now degrades health capital and hence introduces an additional source of negative externality. Therefore, we next turn to solve the planner's problem in order to obtain the first best solution. The planner, who maximizes a social welfare function identical to that of the representative household's lifetime utility function, can internalize the externalities. Equations (4.1), (4.2), (4.4) and (2.2.6) remain unchanged; equations (2.2.4) and (4.3) are replaced by equations (2.2.4.1) and (4.3.1). Equations (4.5) and (4.6) are modified as follows.

$$\dot{K} = Y - \Pi - C; \qquad \text{… … (2.2.4.1)}$$

$$\dot{E} = \Omega - \delta Y; \qquad \text{… … (4.3.1)}$$

$$G = G_I + G_H = \Pi - \Omega; \qquad \text{… … (4.5.1)}$$

and

[60] Cazzavillan (1996), Chang (1999), Chen (2006), Zhang (2000), Raurich-Puigdevall (2000), etc., explain indeterminacy when public expenditure enters as an argument in the utility function.

$$G_i = v_i(\Pi - \Omega) with\ i = I, H. \qquad \ldots\ldots (4.6.1)$$

Here Π now denotes planner's combined lump sum expenditure on public intermediate input, health infrastructure and abatement activities; and the abatement expenditure, as before, is denoted by Ω.

The planner maximizes $\int_0^\infty e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma}dt$ with respect to C, Π, Ω and v_I subject to equations (2.2.4.1), (4.3.1), (4.4), (4.5.1) and (4.6.1). We again consider a steady-state growth equilibrium with the growth rate being denoted by g_c; and the following equation solves for the steady-state equilibrium growth rate[61] in the command (planned) economy.

$$(\rho + \sigma g_c)^{\alpha+\beta} = (1 - \delta - \eta\delta)\beta^{\beta}(1 - \alpha - \beta)^{1-\alpha-\beta}\{\theta(1 - \alpha - \beta)\}^{\theta(1-\alpha-\beta)}$$

$$\{\alpha - \theta(1 - \alpha - \beta)\}^{\alpha-\theta(1-\alpha-\beta)} \qquad \ldots\ldots (4.23)$$

The L.H.S. of equation (4.23) is an increasing function of g_c and the R.H.S. is a parametric constant.

We compare the market economy solution to the socially efficient solution by comparing equation (4.23) to equation (4.13) when $\tau = \tau^\star$, $T = T^\star$ and $v_I = v_I^\star$. We modify equation (4.13) with $\tau = \tau^\star$, $T = T^\star$ and $v_I = v_I^\star$ as follows.

$$\alpha^{-\alpha-\beta}\left[\frac{\alpha g_m}{(\sigma g_m+\rho)}\right]^{\beta+\theta(1-\alpha-\beta)}(\sigma g_m + \rho)^{\alpha+\beta} = (1 - \delta - \eta\delta)\beta^{\beta}(1 - \alpha - \beta)^{1-\alpha-\beta}$$

$$\{\theta(1 - \alpha - \beta)\}^{\theta(1-\alpha-\beta)}\{\alpha - \theta(1 - \alpha - \beta)\}^{\alpha-\theta(1-\alpha-\beta)}. \qquad \ldots\ldots (4.13.1)$$

The R.H.S. of equations (4.23) and (4.13.1) are identical. However, the L.H.S. of equation (4.13.1) is greater than that of equation (4.23) for all values of

$g_m = g_c > \frac{\rho}{\alpha^{\frac{\theta(1-\alpha-\beta)-\alpha}{\theta(1-\alpha-\beta)+\beta}}-\sigma}$. Hence comparing equation (4.13.1) to equation (4.23) we find that g_m exceeds (falls short of) g_c when the parametric term $(1 - \delta - \eta\delta)$ takes a low (high) value. This is shown in figure 4.2. The L.H.S. of equations (4.13.1) and (4.23) are plotted as positively sloped curves and the R.H.S. is depicted by horizontal straight lines for exogenous values of the parameters.

[61] Equation (4.23) is derived in the Appendix 4E.

The L.H.S. curve obtained from equation (4.13.1) starts from the origin but the L.H.S. curve obtained from equation (4.23) starts from a point on the vertical axis. The intersection point of the two L.H.S. curves shows that $g_m = g_c = \frac{\rho}{\alpha^{\frac{\theta(1-\alpha-\beta)-\alpha}{\theta(1-\alpha-\beta)+\beta}}-\sigma}$. The points of intersection of the two L.H.S. curves with the lower horizontal line in figure 4.2 show that $g_c^{\star}$ falls short of $g_m^{\star}$ when $(1-\delta-\eta\delta)$ takes a very low value. When $(1-\delta-\eta\delta)$ takes a high value we find that $\widehat{g_c} > \widehat{g_m}$.

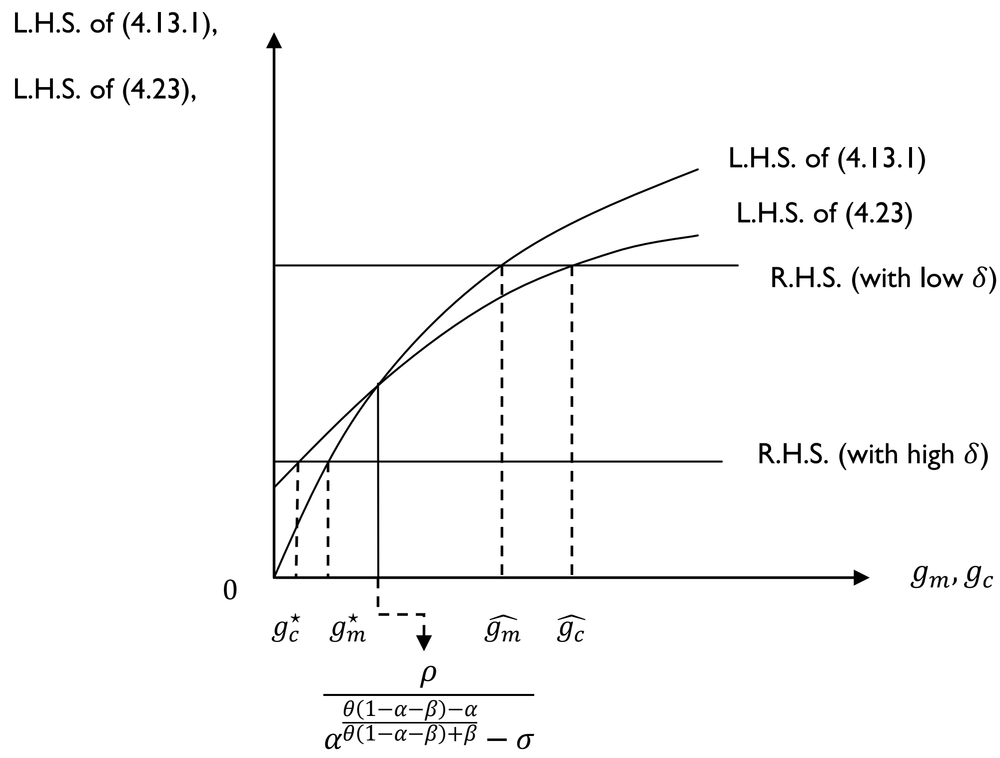

FIGURE 4.2

We can state the following proposition.

Proposition 4.4: If $\theta > 0$, then $(g_c - g_m)$ takes a positive (negative) sign when $(1 - \delta - \eta\delta)$ takes a high (low) value[62].

The market economy growth rate in the steady-state equilibrium falls short of the socially efficient growth rate in Barro (1990) and FMS (1993). Agenor (2008) does not find out the socially efficient solution but the implication should be same as those of Barro (1990) and FMS (1993). Each of them considers the role of a positive externality. The result obtained from the present model in this chapter, may be different from theirs'. Since the planner internalizes two conflicting externalities - negative externality arising due to pollution of the environment as well as due to congestion effect of capital accumulation and positive externality caused by the presence of public infrastructure, health capital and environmental quality - the net benefit of internalization of externalities is ambiguous. Socially efficient growth rate should exceed (fall short of) the competitive equilibrium growth rate when the positive (negative) externality dominates.

The relationship between the market economy equilibrium growth rate and the socially efficient growth rate in this model depends on the value $(1 - \delta - \eta\delta)$. This term takes a high (low) value if the pollution-output coefficient, δ, takes a low (high) value or if the pollution produces a weak (strong) negative effect on the depreciation of health capital. When $(1 - \delta - \eta\delta)$ takes a low value, the negative externality of environmental pollution dominates all other positive externalities; and the opposite happens when $(1 - \delta - \eta\delta)$ takes a high value.

[62] We assume the existence of unique point of intersection of two L.H.S. curves in figure 4.2. If they never intersect, g_m is always greater than g_c. If they intersect twice, $(g_c - g_m)$ takes a positive sign for very low and very high values of $(1 - \delta - \eta\delta)$ but is negative for its intermediate values.

APPENDIX 4A

DERIVATION OF EQUATION (4.7) IN SECTION 4.3

The dynamic optimization problem of the representative household is identical to that discussed in the basic model in appendix 2.2A of chapter 2.

The Hamiltonian to be maximized at each point of time is given by

$$H = e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma} + e^{-\rho t}\lambda_K[(1-\tau)Y - C].$$

Here λ_K is the co-state variable representing the shadow price of investment. Maximizing the Hamiltonian with respect to C and assuming an interior solution, we obtain

$$C^{-\sigma} = \lambda_K. \qquad \text{......}(4A.1)$$

Also the optimum time path of λ_K satisfies the following.

$$\frac{\dot{\lambda_K}}{\lambda_K} = \rho - (1-\tau)\alpha K^{\alpha-1}\hat{G}^{1-\alpha-\beta}H^{\beta}. \qquad \text{......}(4A.2)$$

Using equations (4.1), (4.2), (4.5), (4.6) and (4A.2) we have

$$\frac{\dot{\lambda_K}}{\lambda_K} = \rho - \alpha(1-\tau)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}}. \qquad \text{......}(4A.3)$$

Using the two optimality conditions (4A.1) and (4A.3), we have

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]. \qquad \text{......}(4A.4)$$

This is same as equation (4.7).

APPENDIX 4B

DERIVATION OF EQUATION (4.13) IN SECTION 4.3.1

Using equations (4.1) to (4.6), (2.2.4), (4.7) and (4.8) we have the following equations.

$$g_m = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}}-\rho\right]; \qquad \dots\dots(4B.1)$$

$$g_m = \frac{\dot{K}}{K} = (1-\tau)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}}-\frac{C}{K}; \qquad \dots\dots(4B.2)$$

$$g_m = \frac{\dot{E}}{E} = (T-\delta)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}}; \qquad \dots\dots(4B.3)$$

and

$$g_m = \frac{\dot{H}}{H} = \{v_H(\tau-T)-\eta\delta\}\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}\left(\frac{H}{E}\right)^{-\frac{\alpha}{\alpha+\beta}}; \qquad \dots\dots(4B.4)$$

From equation (4B.1) we have,

$$\frac{E}{K} = \left[\{v_I(\tau-T)\}^{1-\alpha-\beta}\left\{\frac{(\sigma g_m+\rho)}{\alpha(1-\tau)}\right\}^{-\alpha-\beta}\left(\frac{H}{E}\right)^{\beta}\right]^{-\frac{1}{\theta(1-\alpha-\beta)+\beta}}. \qquad \dots\dots(4B.5)$$

Again, from equation (4B.3) we have,

$$\frac{E}{K} = \left[\{v_I(\tau-T)\}^{1-\alpha-\beta}\left(\frac{g_m}{T-\delta}\right)^{-\alpha-\beta}\left(\frac{H}{E}\right)^{\beta}\right]^{\frac{1}{\alpha-\theta(1-\alpha-\beta)}}. \qquad \dots\dots(4B.6)$$

Using equations (4B.5) and (4B.6) we derive the following equation.

$$\frac{H}{E} = \left[\{v_I(\tau-T)\}^{1-\alpha-\beta}\left(\frac{T-\delta}{g_m}\right)^{\beta+\theta(1-\alpha-\beta)}\left\{\frac{\alpha(1-\tau)}{(\sigma g_m+\rho)}\right\}^{\alpha-\theta(1-\alpha-\beta)}\right]^{-\frac{1}{\beta}}. \qquad \dots\dots(4B.7)$$

Using equations (4B.6) and (4B.7) we obtain the following equation.

$$\frac{E}{K} = \frac{(\sigma g_m+\rho)}{g_m}\frac{(T-\delta)}{\alpha(1-\tau)}. \qquad \dots\dots(4B.8)$$

Similarly using equations (4B.1) and (4B.2) we can show that

$$\frac{C}{K} = \frac{(\sigma-\alpha)g_m+\rho}{\alpha}. \qquad \dots\dots(4B.9)$$

Now, using equations (4B.4), (4B.7) and (4B.8) we derive the following equation.

$$g_m = \{v_H(\tau - T) - \eta\delta\}\{v_I(\tau - T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left[\frac{(\sigma g_m + \rho)}{g_m} \frac{(T-\delta)}{\alpha(1-\tau)}\right]^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}$$

$$\left[\{v_I(\tau - T)\}^{1-\alpha-\beta} \left(\frac{T-\delta}{g_m}\right)^{\beta+\theta(1-\alpha-\beta)} \left\{\frac{\alpha(1-\tau)}{(\sigma g_m + \rho)}\right\}^{\alpha-\theta(1-\alpha-\beta)}\right]^{\frac{\alpha}{\beta(\alpha+\beta)}},$$

or,

$$g_m^{\beta+\theta(1-\alpha-\beta)}(\sigma g_m + \rho)^{\alpha-\theta(1-\alpha-\beta)} = \alpha^{\alpha-\theta(1-\alpha-\beta)}(1-\tau)^{\alpha-\theta(1-\alpha-\beta)}$$

$$\{v_H(\tau - T) - \eta\delta\}^{\beta}\{v_I(\tau - T)\}^{1-\alpha-\beta}(T-\delta)^{\theta(1-\alpha-\beta)}. \qquad \ldots\ldots(4B.10)$$

This is same as equation (4.13).

APPENDIX 4C

DERIVATION OF EQUATIONS (4.14), (4.15) AND (4.16) AND THE SECOND ORDER CONDITIONS IN SECTION 4.3.2

Maximizing the R.H.S. of equation (4.13) with respect to τ, we obtain the following first order condition.

$$\alpha^{\alpha-\theta(1-\alpha-\beta)}(1-\tau)^{\alpha-\theta(1-\alpha-\beta)}\{v_H(\tau - T) - \eta\delta\}^{\beta}\{v_I(\tau - T)\}^{1-\alpha-\beta}(T-\delta)^{\theta(1-\alpha-\beta)}$$

$$\begin{bmatrix}\beta(1-v_I)\{v_H(\tau - T) - \eta\delta\}^{-1} - \{\alpha - \theta(1-\alpha-\beta)\}(1-\tau)^{-1} \\ +(1-\alpha-\beta)(\tau - T)^{-1}\end{bmatrix} = 0;$$

or,

$$\beta(1-v_I)\{v_H(\tau - T) - \eta\delta\}^{-1} - \{\alpha - \theta(1-\alpha-\beta)\}(1-\tau)^{-1}$$

$$+(1-\alpha-\beta)(\tau - T)^{-1} = 0. \qquad \ldots\ldots(4C.1)$$

Maximizing the R.H.S. of equation (4.13) with respect to T, we obtain the following first order condition.

$$\alpha^{\alpha-\theta(1-\alpha-\beta)}(1-\tau)^{\alpha-\theta(1-\alpha-\beta)}\{v_H(\tau - T) - \eta\delta\}^{\beta}\{v_I(\tau - T)\}^{1-\alpha-\beta}(T-\delta)^{\theta(1-\alpha-\beta)}$$

$$\begin{bmatrix} -\beta(1-v_I)\{v_H(\tau-T)-\eta\delta\}^{-1}+\theta(1-\alpha-\beta)(T-\delta)^{-1} \\ -(1-\alpha-\beta)(\tau-T)^{-1} \end{bmatrix}=0;$$

or,

$$-\beta(1-v_I)\{v_H(\tau-T)-\eta\delta\}^{-1}+\theta(1-\alpha-\beta)(T-\delta)^{-1}$$

$$-(1-\alpha-\beta)(\tau-T)^{-1}=0. \qquad \ldots\ldots(4C.2)$$

Maximizing the R.H.S. of equation (4.13) with respect to v_I, we obtain the following first order condition.

$$\alpha^{\alpha-\theta(1-\alpha-\beta)}(1-\tau)^{\alpha-\theta(1-\alpha-\beta)}\{v_H(\tau-T)-\eta\delta\}^{\beta}\{v_I(\tau-T)\}^{1-\alpha-\beta}(T-\delta)^{\theta(1-\alpha-\beta)}$$

$$[-\beta(\tau-T)\{v_H(\tau-T)-\eta\delta\}^{-1}+(1-\alpha-\beta)v_I^{-1}]=0. \qquad \ldots\ldots(4C.3)$$

Using equations (4C.1), (4C.2) and (4C.3) we arrive at the following expressions for the optimal tax rate, optimal abatement expenditure rate and the optimal public expenditure allocation ratio.

$$\tau^\star=1-(1-\delta-\eta\delta)\{\alpha-\theta(1-\alpha-\beta)\};$$

$$T^\star=\delta+(1-\delta-\eta\delta)\theta(1-\alpha-\beta);$$

and

$$v_I^\star=\frac{(1-\delta-\eta\delta)(1-\alpha-\beta)}{\eta\delta+(1-\delta-\eta\delta)(1-\alpha)}.$$

These are same as equations (4.14), (4.15) and (4.16) in section 4.3.2.

To check the second order conditions for optimality we twice differentiate equation (4.13), with respect to τ, T and v_I respectively and arrive at the following three second order conditions.

$$-\left[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-2}+\sigma^2\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-1}+\sigma\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-1}]\frac{\partial^2 g_m}{\partial\tau^2}$$

$$=-[(1-\alpha-\beta)(\tau-\mathrm{T})^{-2}+\{\alpha-\theta(1-\alpha-\beta)\}(1-\tau)^{-2}$$

$$+\beta\{v_H(\tau-T)-\eta\delta\}^{-2}v_H^2]; \qquad \ldots\ldots(4C.4)$$

$$-\left[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-2}+\sigma^2\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-1}+\sigma\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-1}]\frac{\partial^2 g_m}{\partial T^2}$$

$$=-[(1-\alpha-\beta)(\tau-\mathrm{T})^{-2}+\theta(1-\alpha-\beta)(T-\delta)^{-2}$$

$$+\beta\{v_H(\tau-T)-\eta\delta\}^{-2}v_H^2]; \qquad \ldots\ldots(4C.5)$$

and

$$-\left[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-2}+\sigma^2\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial v_I}\right)^2$$

$$+[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-1}+\sigma\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-1}]\frac{\partial^2 g_m}{\partial v_I^2}$$

$$=-[\beta(\tau-\mathrm{T})^2\{v_H(\tau-T)-\eta\delta\}^{-2}+(1-\alpha-\beta)v_I^{-2}]; \qquad \ldots\ldots(4C.6)$$

Now we evaluate the above three second order conditions at $\tau=\tau^\star$, $T=T^\star$ and $v_I = v_I^\star$ where $\frac{\partial g_m}{\partial \tau}=\frac{\partial g_m}{\partial T}=\frac{\partial g_m}{\partial v_I}=0$. Hence we obtain the followings.

$$\frac{\partial^2 g_m}{\partial \tau^2}=-\frac{(1-\alpha-\beta)(\tau^\star-T^\star)^{-2}+\{\alpha-\theta(1-\alpha-\beta)\}(1-\tau^\star)^{-2}+\beta\{v_H^\star(\tau^\star-T^\star)-\eta\delta\}^{-2}{v_H^\star}^2}{[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-1}+\sigma\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-1}]};$$

$$\frac{\partial^2 g_m}{\partial T^2}=-\frac{(1-\alpha-\beta)(\tau^\star-T^\star)^{-2}+\theta(1-\alpha-\beta)(T^\star-\delta)^{-2}+\beta\{v_H^\star(\tau^\star-T^\star)-\eta\delta\}^{-2}{v_H^\star}^2}{[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-1}+\sigma\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-1}]};$$

and

$$\frac{\partial^2 g_m}{\partial v_I^2}=-\frac{\left[\beta(\tau^\star-T^\star)^2\{v_H(\tau^\star-T^\star)-\eta\delta\}^{-2}+(1-\alpha-\beta){v_I^\star}^{-2}\right]}{[\{\beta+\theta(1-\alpha-\beta)\}{g_m}^{-1}+\sigma\{\alpha-\theta(1-\alpha-\beta)\}(\sigma g_m+\rho)^{-1}]}.$$

The R.H.S. of each of these three equations is negative. Thus the second order conditions are also satisfied.

APPENDIX 4D

DERIVATION OF THE DETERMINANT AND THE TRACE OF THE JACOBIAN MATRIX IN SECTION 4.4

We define the following variables.

$$M=(1-\tau)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}y^{\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}}z^{\frac{\beta}{\alpha+\beta}}; \qquad \ldots\ldots(4D.1)$$

$$N=(T-\delta)\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}z^{\frac{\beta}{\alpha+\beta}}; \qquad \ldots\ldots(4D.2)$$

and

$$Q=\{v_H(\tau-T)-\eta\delta\}\{v_I(\tau-T)\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}y^{\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}z^{-\frac{\alpha}{\alpha+\beta}}. \qquad \ldots\ldots(4D.3)$$

Thus, using equations (4D.1), (4D.2) and (4D.3) we modify equations (4.20), (4.21) and (4.22) in the section 4.4 as follows.

$$\frac{\dot{x}}{x}=\left(\frac{\alpha}{\sigma}-1\right)M+x-\frac{\rho}{\sigma}; \qquad \ldots\ldots(4D.4)$$

$$\frac{\dot{y}}{y}=N-M+x; \qquad \ldots\ldots(4D.5)$$

and

$$\frac{\dot{z}}{z}=Q-N. \qquad \ldots\ldots(4D.6)$$

We obtain the following partial derivatives corresponding to three modified differential equations.

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial x}=1;$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial y}=\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}\left(\frac{\alpha}{\sigma}-1\right)\frac{M}{y};$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial z}=\frac{\beta}{\alpha+\beta}\left(\frac{\alpha}{\sigma}-1\right)\frac{M}{z};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial x}=1;$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial y}=\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}\frac{N}{y}-\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}\frac{M}{y};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial z}=\frac{\beta}{\alpha+\beta}\frac{N}{z}-\frac{\beta}{\alpha+\beta}\frac{M}{z};$$

$$\frac{\partial\left(\frac{\dot{z}}{z}\right)}{\partial x}=0;$$

$$\frac{\partial\left(\frac{\dot{z}}{z}\right)}{\partial y}=\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}\frac{Q}{y}-\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}\frac{N}{y};$$

and

$$\frac{\partial\left(\frac{\dot{z}}{z}\right)}{\partial z} = -\frac{\alpha}{\alpha+\beta}\frac{Q}{z} - \frac{\beta}{\alpha+\beta}\frac{N}{z}.$$

So the determinant of the Jacobian matrix can be written as follows.

$$|J| = \left[\frac{\{\theta(1-\alpha-\beta)-\alpha\}}{(\alpha+\beta)}\frac{N}{y} - \frac{\{\theta(1-\alpha-\beta)+\beta\}}{(\alpha+\beta)}\frac{M}{y}\right]\left[-\frac{\alpha}{(\alpha+\beta)}\frac{Q}{z} - \frac{\beta}{(\alpha+\beta)}\frac{N}{z}\right]$$

$$-\left[\frac{\{\theta(1-\alpha-\beta)-\alpha\}}{(\alpha+\beta)}\frac{Q}{y} - \frac{\{\theta(1-\alpha-\beta)-\alpha\}}{(\alpha+\beta)}\frac{N}{y}\right]\left[\frac{\beta}{(\alpha+\beta)}\frac{N}{z} - \frac{\beta}{(\alpha+\beta)}\frac{M}{z}\right]$$

$$-\frac{\{\theta(1-\alpha-\beta)+\beta\}}{(\alpha+\beta)}\left(\frac{\alpha}{\sigma}-1\right)\frac{M}{y}\left[-\frac{\alpha}{(\alpha+\beta)}\frac{Q}{z} - \frac{\beta}{(\alpha+\beta)}\frac{N}{z}\right]$$

$$+\frac{\beta}{(\alpha+\beta)}\left(\frac{\alpha}{\sigma}-1\right)\frac{M}{z}\left[\frac{\{\theta(1-\alpha-\beta)-\alpha\}}{(\alpha+\beta)}\frac{Q}{y} - \frac{\{\theta(1-\alpha-\beta)-\alpha\}}{(\alpha+\beta)}\frac{N}{y}\right];$$

or,

$$|J| = \frac{\{\alpha-\theta(1-\alpha-\beta)\}}{(\alpha+\beta)}\frac{N}{y}\frac{Q}{z} + \frac{\alpha}{\sigma}\frac{\theta(1-\alpha-\beta)}{(\alpha+\beta)}\frac{M}{y}\frac{Q}{z} + \frac{\alpha}{\sigma}\frac{\beta}{(\alpha+\beta)}\frac{M}{y}\frac{N}{z};$$

or,

$$|J| = \frac{\alpha-\theta(1-\alpha-\beta)}{\alpha+\beta}(T-\delta)\{v_I(\tau-T)\}^{2\frac{1-\alpha-\beta}{\alpha+\beta}}\{v_H(\tau-T)-\eta\delta\}y^{2\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}-1}z^{-2\frac{\alpha}{\alpha+\beta}}$$

$$+\frac{\alpha}{\sigma}\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}(1-\tau)\{v_I(\tau-T)\}^{2\frac{1-\alpha-\beta}{\alpha+\beta}}\{v_H(\tau-T)-\eta\delta\}y^{2\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}z^{-2\frac{\alpha}{\alpha+\beta}}$$

$$+\frac{\alpha}{\sigma}\frac{\beta}{\alpha+\beta}(1-\tau)(T-\delta)\{v_I(\tau-T)\}^{2\frac{1-\alpha-\beta}{\alpha+\beta}}y^{2\frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}}z^{2\frac{\beta}{\alpha+\beta}-1}.$$

Here $\alpha-\theta(1-\alpha-\beta)>0$. Also, $1>\tau>T>\delta$ and $v_H(\tau-T)-\eta\delta>0$ when τ and T are optimally chosen and when $\theta>0$. So $|J|>0$ in this case, when evaluated at the balanced growth equilibrium point. So either all the three latent roots of J matrix are positive or, two of them are negative with the third one being positive.

The trace of the Jacobian matrix is given by,

$$Tr\,J = 1 + \frac{\theta(1-\alpha-\beta)-\alpha}{\alpha+\beta}\frac{N}{y} - \frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}\frac{M}{y} - \frac{\alpha}{\alpha+\beta}\frac{Q}{z} - \frac{\beta}{\alpha+\beta}\frac{N}{z}.$$

Using equations (4D.4), (4D.5) and (4D.6) the trace can be written as follows.

$$Tr\,J = 1 + \left[\frac{\{\theta(1-\alpha-\beta)-\alpha\}}{(\alpha+\beta)}\frac{\alpha}{\sigma}\frac{1}{y} - \frac{\{\theta(1-\alpha-\beta)+\beta\}}{(\alpha+\beta)}\frac{1}{y} - \frac{\alpha}{\sigma}\frac{1}{z}\right]M - \frac{\{\theta(1-\alpha-\beta)-\alpha\}}{(\alpha+\beta)}\frac{\rho}{\sigma}\frac{1}{y} + \frac{\rho}{\sigma}\frac{1}{z}.$$

Now, $Tr\,J < 0$ if

$$1+\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{(\alpha+\beta)}\frac{\rho}{\sigma}\frac{1}{y}+\frac{\rho}{\sigma}\frac{1}{z}<\left[\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{(\alpha+\beta)}\frac{\alpha}{\sigma}\frac{1}{y}+\frac{\{\theta(1-\alpha-\beta)+\beta\}}{(\alpha+\beta)}\frac{1}{y}+\frac{\alpha}{\sigma}\frac{1}{z}\right]M$$

At the balanced growth equilibrium, $\frac{\dot{x}}{x}=\frac{\dot{y}}{y}=\frac{\dot{z}}{z}=0$. Using $\frac{\dot{z}}{z}=0$ and the optimal values of the policy variables given by equations (4.14), (4.15) and (4.16) we have,

$$z=\frac{v_H(\tau-T)-\eta\delta}{T-\delta}=\frac{\beta}{\theta(1-\alpha-\beta)}. \qquad \ldots\ldots(4D.7)$$

Now using equations (4B.5), (4D.7) and the optimal values of the policy variables, the condition for the trace of the Jacobian matrix to be negative can be written as

$$1+\frac{\alpha\{\alpha-\theta(1-\alpha-\beta)\}^2}{\theta(1-\alpha-\beta)(\alpha+\beta)}\frac{\rho}{\sigma}\frac{g_m}{(\sigma g_m+\rho)}+\frac{\rho}{\sigma}\frac{\theta(1-\alpha-\beta)}{\beta}$$

$$<\left[\frac{\{\alpha-\theta(1-\alpha-\beta)\}}{(\alpha+\beta)}\frac{\alpha}{\sigma}+\frac{\{\theta(1-\alpha-\beta)+\beta\}}{(\alpha+\beta)}+\frac{\alpha}{\sigma}\frac{\{\theta(1-\alpha-\beta)\}^2}{\alpha\beta\{\alpha-\theta(1-\alpha-\beta)\}}\frac{(\sigma g_m+\rho)}{g_m}\right]$$

$$(1-\delta-\eta\delta)^{\frac{1}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}}\alpha^{\frac{\alpha-\theta(1-\alpha-\beta)}{\alpha+\beta}}\{1-\alpha-\beta\}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\{\theta(1-\alpha-\beta)\}^{\frac{\alpha-\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}}$$

$$\{\alpha-\theta(1-\alpha-\beta)\}^{\frac{\beta+\theta(1-\alpha-\beta)}{\alpha+\beta}}\left\{\frac{g_m}{(\sigma g_m+\rho)}\right\}^{\frac{\alpha-\theta(1-\alpha-\beta)}{\alpha+\beta}}.$$

This is same as condition (4.T) in section 4.4. If the trace of the Jacobian matrix is negative, then all the latent roots cannot be positive.

APPENDIX 4E

DERIVATION OF EQUATION (4.23) IN SECTION 4.5

The relevant Hamiltonian to be maximized by the planner at each point of time is given by

$$\mathcal{H}=e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma}+e^{-\rho t}\lambda_K\left[\{v_I(\Pi-\Omega)\}^{1-\alpha-\beta}K^{\alpha-\theta(1-\alpha-\beta)}E^{\theta(1-\alpha-\beta)}H^{\beta}-\Pi-C\right]$$

$$+e^{-\rho t}\lambda_E\left[\Omega - \delta\{v_I(\Pi - \Omega)\}^{1-\alpha-\beta}K^{\alpha-\theta(1-\alpha-\beta)}E^{\theta(1-\alpha-\beta)}H^{\beta}\right]$$

$$+e^{-\rho t}\lambda_H\left[(1 - v_I)(\Pi - \Omega) - \eta\delta\{v_I(\Pi - \Omega)\}^{1-\alpha-\beta}K^{\alpha-\theta(1-\alpha-\beta)}E^{\theta(1-\alpha-\beta)}H^{\beta}\right].$$

The state variables are K, H and E. The control variables are C, Π, Ω and v_I; and λ_K, λ_H, and λ_E are three co-state variables.

Maximising L with respect to C, Π, Ω and v_I we have

$$C^{-\sigma} = \lambda_K; \qquad \text{......}(4E.1)$$

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)(1 - \alpha - \beta)\frac{Y}{\Pi-\Omega} + \frac{\lambda_H}{\lambda_E}\left[(1 - v_I) - \eta\delta(1 - \alpha - \beta)\frac{Y}{\Pi-\Omega}\right] = \frac{\lambda_K}{\lambda_E}; \qquad \text{......}(4E.2)$$

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)(1 - \alpha - \beta)\frac{Y}{\Pi-\Omega} + \frac{\lambda_H}{\lambda_E}\left[(1 - v_I) - \eta\delta(1 - \alpha - \beta)\frac{Y}{\Pi-\Omega}\right] = 1; \qquad \text{......}(4E.3)$$

and

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)(1 - \alpha - \beta)\frac{Y}{v_I} = \frac{\lambda_H}{\lambda_E}\left[(\Pi - \Omega) + \eta\delta(1 - \alpha - \beta)\frac{Y}{v_I}\right]. \qquad \text{......}(4E.4)$$

Using equations (4E.2) and (4E.3) we find that

$$\frac{\lambda_K}{\lambda_E} = 1. \qquad \text{......}(4E.5)$$

Using equations (4E.3) and (4E.5) we obtain the following.

$$\frac{\lambda_H}{\lambda_E} = \frac{1-(1-\delta)(1-\alpha-\beta)\frac{Y}{v_I}}{(1-v_I)-\eta\delta(1-\alpha-\beta)\frac{Y}{\Pi-\Omega}}. \qquad \text{......}(4E.6)$$

Using equations (4E.4), (4E.5) and (4E.6) we obtain the following equation.

$$(1 - \delta - \eta\delta)(1 - \alpha - \beta)\frac{Y}{v_I} = \Pi - \Omega. \qquad \text{......}(4E.7)$$

Now, using equations (4E.6) and (4E.7) we find that,

$$\frac{\lambda_H}{\lambda_E} = 1. \qquad \text{......}(4E.8)$$

Also, along the optimum path, time behaviour of the co-state variables satisfies the followings.

$$\left(1 - \delta\frac{\lambda_E}{\lambda_K}\right)\{\alpha - \theta(1 - \alpha - \beta)\}\frac{Y}{K} - \frac{\lambda_H}{\lambda_K}\eta\delta\{\alpha - \theta(1 - \alpha - \beta)\}\frac{Y}{K} = \rho - \frac{\dot{\lambda}_K}{\lambda_K}; \qquad \text{......}(4E.9)$$

$$\left(\frac{\lambda_K}{\lambda_E} - \delta\right)\theta(1-\alpha-\beta)\frac{Y}{E} - \frac{\lambda_H}{\lambda_E}\eta\delta\theta(1-\alpha-\beta)\frac{Y}{E} = \rho - \frac{\dot{\lambda_E}}{\lambda_E}; \qquad \ldots\ldots(4E.10)$$

and

$$\frac{\lambda_K}{\lambda_H}\beta\frac{Y}{H} - \delta\left(\frac{\lambda_E}{\lambda_H} + \eta\right)\beta\frac{Y}{H} = \rho - \frac{\dot{\lambda_H}}{\lambda_H}. \qquad \ldots\ldots(4E.11)$$

Equations (4E.5) and (4E.8) imply that $\frac{\dot{\lambda_K}}{\lambda_K} = \frac{\dot{\lambda_H}}{\lambda_H} = \frac{\dot{\lambda_E}}{\lambda_E}$. Thus using equations (4E.5), (4E.8), (4E.9) and (4E.10) we obtain the following equation.

$$\frac{E}{K} = \frac{\theta(1-\alpha-\beta)}{\alpha-\theta(1-\alpha-\beta)}. \qquad \ldots\ldots(4E.12)$$

Again using, equations (4E.5), (4E.8), (4E.10) and (4E.11) we obtain,

$$\frac{H}{E} = \frac{\beta}{\theta(1-\alpha-\beta)}. \qquad \ldots\ldots(4E.13)$$

Using equations (4.1), (4.2) and (4.6.1) we have

$$\frac{Y}{v_I(\Pi-\Omega)}\{v_I(\Pi-\Omega)\}^{\alpha+\beta} = K^{\alpha-\theta(1-\alpha-\beta)}E^{\theta(1-\alpha-\beta)}H^{\beta}. \qquad \ldots\ldots(4E.14)$$

Now, using equations (4E.7) and (4E.14) we derive the following.

$$\{v_I(\Pi-\Omega)\} = \left[(1-\delta-\eta\delta)(1-\alpha-\beta)K^{\alpha-\theta(1-\alpha-\beta)}E^{\theta(1-\alpha-\beta)}H^{\beta}\right]^{\frac{1}{\alpha+\beta}}. \ldots\ldots(4E.15)$$

From equation (4E.1), we have

$$\sigma\frac{\dot{C}}{C} = \frac{\dot{\lambda_K}}{\lambda_K}. \qquad \ldots\ldots(4E.16)$$

Using equations (4E.5), (4E.8), (4E.9), (4E.15) and (4E.16) we obtain the following

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\Bigg[\{\alpha-\theta(1-\alpha-\beta)\}(1-\delta-\eta\delta)^{\frac{1}{\alpha+\beta}}$$

$$(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\Bigg]. \qquad \ldots\ldots(4E.17)$$

In the steady state growth equilibrium,

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\Bigg[\{\alpha-\theta(1-\alpha-\beta)\}(1-\delta-\eta\delta)^{\frac{1}{\alpha+\beta}}$$

$$(1-\alpha-\beta)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)+\beta}{\alpha+\beta}}\left(\frac{H}{E}\right)^{\frac{\beta}{\alpha+\beta}}-\rho\Bigg]=g_c; \qquad \ldots\ldots(4E.18)$$

$$\frac{\dot{K}}{K}=\left(\frac{K}{\Pi-\Omega}\right)^{\alpha-1}\left(\frac{E}{K}\right)^{\theta(1-\alpha)}-\frac{\Pi}{K}-\frac{C}{K}=g_c; \qquad \ldots\ldots(4E.19)$$

$$\frac{\dot{H}}{H}=v_H\left(\frac{\Pi-\Omega}{\mathrm{H}}\right)-\eta\delta\left\{v_I\left(\frac{\Pi-\Omega}{\mathrm{H}}\right)\right\}^{1-\alpha-\beta}\left(\frac{E}{K}\right)^{\theta(1-\alpha-\beta)-\alpha}\left(\frac{H}{E}\right)^{-\alpha}=g_c; \qquad \ldots\ldots(4E.20)$$

and

$$\frac{\dot{E}}{E}=\frac{\Omega}{E}-\delta\left(\frac{K}{\Pi-\Omega}\right)^{\alpha-1}\left(\frac{E}{K}\right)^{\theta(1-\alpha)-1}=g_c. \qquad \ldots\ldots(4E.21)$$

Therefore, using equations (4E.12), (4E.13) and (4E.19) we obtain the following equation.

$$(\rho+\sigma g_c)^{\alpha}=(1-\delta-\eta\delta)\beta^{\beta}(1-\alpha-\beta)^{1-\alpha-\beta}\{\theta(1-\alpha-\beta)\}^{\theta(1-\alpha-\beta)}$$

$$\{\alpha-\theta(1-\alpha-\beta)\}^{\alpha-\theta(1-\alpha-\beta)}. \qquad \ldots\ldots(4E.22)$$

This is same as equation (4.23) in section 4.5.

CHAPTER 5

5. DEPRECIATION OF PUBLIC CAPITAL AND MAINTENANCE EXPENDITURE

5.1 INTRODUCTION

The basic model developed in section 2.2 of chapter 2 is extended in this chapter to include maintenance of public infrastructure capital that depreciates due to pollution and usage. So in this chapter we make a departure from the Barro (1990) assumption that public infrastructure is a flow variable and follow FMS (1993) to introduce durable public capital in place of perishable public input. The special feature of this extended model is that the public capital depreciates over time and the rate of depreciation is endogenous. This rate of depreciation can be reduced by increasing the maintenance expenditure; and hence the government faces an additional problem of allocating the budget, net of abatement expenditure, between public investment and maintenance expenditure.

Greiner (2005) as well as EP (2008), though deals with the interaction between economic growth and environmental pollution using the Barro-FMS framework, he does not analyze the problem of endogenous depreciation of public capital and the role of maintenance expenditure.

In this chapter, the problem of depreciation is worsened by environmental pollution caused by industrial production. In DK (2008) and KK (2004), depreciation rate of public capital is endogenous; and is a positive function of the level of production and a negative function of the level of maintenance expenditure. In Agenor (2009) it is a negative function of the ratio of maintenance expenditure to the stock of public capital. Neither of these models considers environmental pollution. In our model, the level of net public capital investment, defined as the gross investment minus depreciation, is assumed to vary inversely with the size of private capital and directly with the level of maintenance expenditure and with the environmental quality. Environmental quality enters as an additional argument in the depreciation function in the present model. Environmental quality accumulates over time through abatement activities of the government and degrades through pollution generated as a by-product of industrial production. Income tax revenue is used to finance public investment, maintenance expenditure and abatement activities.

We derive following results analyzing our model. The optimum income tax rate and the abatement expenditure rate depend on the pollution-output coefficient in the steady-state equilibrium. However, the share of maintenance expenditure in the budget is independent of the pollution-output coefficient. In DK (2008), KK (2004) and also in Agenor (2009), there is no environmental pollution; and hence the proportional income tax rate and the ratio of public investment to national income do not depend on pollution-output coefficient. Secondly, optimum ratio of combined expenditure on net public investment and maintenance expenditure to national income is not unambiguously greater than the competitive output share of the public capital in this steady-state equilibrium. Moreover, this optimum ratio is dependent on the pollution-output coefficient. Both DK (2008) and KK (2004) show that the optimum ratio of combined expenditure on net public investment and maintenance expenditure to national income to be always greater than the competitive output share of public capital, while it is equal to the latter in Agenor (2009). This is so because the abatement expenditure is an additional expenditure of the government in this model but not in the other models. Thirdly, we find a possibility of indeterminacy of the transitional growth path converging to the unique steady-state equilibrium point in our model. However, the possibility of saddle-point stability of the steady-state equilibrium never arises here. In models of Greiner (2005), DK (2008), KK (2004) and Agenor (2009) the saddle-point stability of the unique steady-state equilibrium point is always ensured. Fourthly, we compare the decentralized solution to the socially optimum solution and find that the competitive equilibrium growth rate is not necessarily higher than the socially efficient growth rate.

The rest of the chapter is organized as follows. Section 5.2 presents the basic model of the market economy. Properties of the steady-state equilibrium and optimal fiscal policies are analyzed in section 5.3. Stability properties of the steady-state equilibrium are analyzed in section 5.4 and section 5.5 presents the planned economy solution.

5.2 THE MODEL

Following equations describe the model.

$$Y = K^{\alpha}G^{1-\alpha} \ with \ 0 < \alpha < 1; \quad \text{… … (5.1)}$$

$$\dot{K} = (1-\tau)Y - C; \quad \text{… … (2.2.4)}$$

$$I = \mu(\tau - T)Y; \quad \text{… … (5.2)}$$

$$M = (1-\mu)(\tau - T)Y; \quad \text{… … (5.3)}$$

$$\dot{G} = \frac{I}{m}; \quad \text{… … (5.4)}$$

$$m = K^{\psi+\eta}E^{-\psi}M^{-\eta} \ with \ 0 < \psi, \eta < 1; \quad \text{… … (5.5)}$$

$$\dot{E} = TY - \delta Y \ with \ 0 < \delta < 1; \quad \text{… … (4.3)}$$

and

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma} \ with \ \sigma > 0. \quad \text{… … (2.2.6)}$$

Equation (5.1) describes the Cobb-Douglas production function of the final good. Congestion effect of private capital and positive externality effect of environmental quality on public capital considered in earlier chapters are not considered here. Instead, we consider these effects on the depreciation of public capital, as is described by equation (5.5). So equation (5.1) looks marginally different from equation (2.2.1) of chapter 2 and equation (2.2.2) of chapter 2 does not exist here. However, we get back to equation (5.1) from equations (2.2.1) and (2.2.2) using $\theta_1 = \theta_2 = 0$.

Equations (2.2.4) and (2.2.6) are borrowed from chapter 2; and equation (4.3) is taken from chapter 4.

Equations (5.2) to (5.5) describe the public sector. Government finances investment in new public capital with its tax revenue net of the abatement expenditure. Equation (5.2) shows the fraction of non-abatement expenditure used to finance public investment. τ and T hold same interpretation as described in previous chapters; and μ is the fraction of non-abatement expenditure used to finance public investment. Here I stands for the level of gross public investment. Equation (5.3) shows fraction of non-abatement expenditure going to maintenance of public capital.

Accumulation of public capital takes place according to equation (5.4). Here $\dot{G}$ is the net public capital investment. $I - \dot{G}$ is the level of depreciation of public capital. Thus using equation (5.4) we have

$$I - \dot{G} = I\left(1 - \frac{1}{m}\right).$$

Hence this equation combined with equation (5.5) shows that the level of depreciation of public capital varies positively with the stock of private capital and inversely with the level of maintenance expenditure and the stock of environmental quality[63]. Here E stands for the stock of environmental quality and M stands for the level of maintenance expenditure.

Increased usage of public infrastructure made by private firms lowers the durability of public capital; and thus the depreciation of public capital varies positively with the scale of operation of the private economy. Maintenance of public investment goods raises its durability and thus lowers the depreciation rate. Degradation of environmental quality hastens the depreciation process in

[63]Total depreciation of public capital may not be positive always. Here, depending upon the ratios of maintenance expenditure to private capital and environmental quality to private capital, total depreciation can take a negative value. This can be interpreted as an efficiency gain or a virtual expansion of the existing public capital stock brought about by maintenance expenditure as well as the stock of environmental quality.

public capital. For example, public irrigation programme uses canal and river water to irrigate fields of crops. With pollutants in the water government has to bear the cost to treat and cleanse it before it can release the water to the fields. Industrial pollutants, emitted as smoke react with air forming oxides, which precipitate in the form of acid rain. This causes severe damage to heritage buildings as well as other public properties increasing their maintenance cost. Industrial effluents also contaminate water posing serious health hazards to workers. In turn such loss of health takes a heavy toll on public health insurance payments; and thus government has to spend more to maintain proper health among the population. Global warming leads to natural disasters like floods, earthquakes, cyclones, etc.; and these, in turn, cause severe damages to infrastructural capital like roads, electric lines, power plants, buildings, industrial plants, etc.

5.3 THE DYNAMICS

5.3.1 Steady-State Equilibrium

The representative household's problem is to maximize $\int_0^\infty u(C)\, e^{-\rho t} dt$ subject to equations (5.1), (2.2.4) and (2.2.6).

The demand rate of growth [64] of consumption is derived from this maximizing problem as follows.

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \rho\right]. \qquad \ldots\ldots(5.6)$$

The growth rates of the three state variables, K, G and E, can be expressed as follows.

$$\frac{\dot{K}}{K} = (1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \frac{C}{K}; \qquad \ldots\ldots(5.7)$$

[64] The derivation is shown in Appendix 5A.

$$\frac{\dot{G}}{G} = \mu(1-\mu)^{\eta}(\tau - T)^{1+\eta}\left(\frac{E}{K}\right)^{(1+\eta)(1-\alpha)-1+\psi}\left(\frac{G}{E}\right)^{(1+\eta)(1-\alpha)-1}; \qquad \ldots\ldots(5.8)$$

and

$$\frac{\dot{E}}{E} = (T - \delta)\left(\frac{E}{K}\right)^{-\alpha}\left(\frac{G}{E}\right)^{1-\alpha}. \qquad \ldots\ldots(5.9)$$

We again consider steady-state growth equilibrium, hence, we have condition (2.2.8).

$$\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{E}}{E} = \frac{\dot{G}}{G} = g_m. \qquad \ldots\ldots(2.2.8)$$

Using equations (5.6) to (5.9) and (2.2.8) we arrive at the following equation[65] to solve for g_m.

$$g_m{}^{(1+\psi)(1-\alpha)}(\sigma g_m + \rho)^{\alpha-(\psi+\eta)(1-\alpha)} = \alpha^{\alpha-(\psi+\eta)(1-\alpha)}\mu^{1-\alpha}(1-\mu)^{\eta(1-\alpha)}$$

$$(T-\delta)^{\psi(1-\alpha)}(1-\tau)^{\alpha-(\psi+\eta)(1-\alpha)}(\tau - T)^{(1+\eta)(1-\alpha)}. \qquad \ldots\ldots(5.10)$$

We assume that $\alpha - (\psi + \eta)(1-\alpha) > 0$. This implies that the positive marginal technological contribution of private capital on output exceeds its negative marginal external effect that works through depreciation of public capital. Thus the L.H.S. of equation (5.10) is a monotonically increasing function of g_m; and the R.H.S. is independent of g_m. Hence the existence of unique steady-state equilibrium growth rate is guaranteed when $0 < \mu < 1$ and $\delta < T < \tau < 1$.

We summarize the result analyzed above in the following proposition.

Proposition 5.1: There exists unique steady-state equilibrium growth rate in the decentralized economy given the interior values of the income tax rate, the abatement expenditure rate and the public investment allocation share when $0 < \delta < T < \tau$.

The existence of unique steady-state equilibrium growth rate is ensured if the condition $\alpha - (\psi + \eta)(1-\alpha) > 0$ holds. This implies that the positive

[65] Equation (5.10) is derived in Appendix 5B.

marginal technological contribution of private capital on output exceeds its negative marginal effect that works through depreciation of public capital.

5.3.1 Optimal Fiscal Policy

At first, we assume that government maximizes the steady-state equilibrium growth rate with respect to fiscal instruments. The L.H.S. of equation (5.10) is a monotonically increasing function of g_m. Thus maximizing the growth rate with respect to the policy variables subject to equation (5.10) is synonymous to maximizing the R.H.S. of equation (5.10) with respect to those policy variables.

Maximizing the R.H.S. of equation (5.10) with respect to τ, T and μ, we obtain the following equations.

$$\tau^\star = 1 - (1-\delta)\{\alpha - (\eta + \psi)(1-\alpha)\}; \qquad \ldots\ldots(5.11)$$

$$T^\star = \delta + (1-\delta)\psi(1-\alpha); \qquad \ldots\ldots(5.12)$$

and

$$\mu^\star = \frac{1}{1+\eta}. \qquad \ldots\ldots(5.13)$$

Using equations (5.11) and (5.12), we have

$$\tau^\star - T^\star = (1-\delta)(1-\alpha)(1+\eta) = (1-\alpha) + (1-\alpha)(\eta - \delta - \eta\delta). \qquad \ldots\ldots(5.14)$$

Here $\tau^\star - T^\star$ is the optimum combined share of public investment expenditure and maintenance expenditure in the total output; and it is greater (less) than the competitive output share of the public capital when $\eta - \delta - \eta\delta >$ $(<)0$. Here, $1 - \mu^\star = \frac{\eta}{1+\eta}$; and hence $\eta - \delta - \eta\delta > 0$ implies that the pollution-output coefficient, δ, is smaller than the share of maintenance expenditure, $(1-\mu^\star)$. This result is obtained because depreciation of public capital is now negatively affected by environmental quality; and pollution resulting from production degrades environment. δ fraction of final output is polluted. So $(1-\delta)$ fraction of the output is used to meet gross public investment expenditure. Hence, if the effect of pollution is strong, i.e., if δ takes a high value, then it will cause the optimum combined share of public investment and maintenance expenditure in national income to scale down sufficiently so that it falls short of the competitive output share of the public capital. This result is different from that found in KK (2004) and DK (2008) where this optimum combined share is unambiguously greater than the competitive output share of public capital. This is so because $\delta = 0$ in these models. The result also differs from that found in Agenor (2009) where this optimum share is equal to the competitive share of public capital in total output. This is so in Agenor (2009) because of absence of pollution as well as the dual role of maintenance expenditure. In Greiner (2005), however, the growth rate maximizing income tax rate is equal to the competitive output share of the public capital because there is no depreciation of public capital. If $\eta = \delta = 0$, then $\tau^\star - T^\star = (1 - \alpha)$; and thus, in this model we get back the result of Barro (1990) and of Futagami *et al.* (1993). Here, $\eta = 0$ implies that depreciation is independent of maintenance expenditure and $\delta = 0$ implies the absence of environmental pollution.

Here equations (5.11) and (5.13) imply that the income tax rate, $\tau^\star$, and the share of maintenance expenditure, $1 - \mu^\star$, are positively related. Such a positive relationship is also found in KK (2004), DK (2008) and in Agenor (2009). However, the optimum abatement expenditure rate, $T^\star$, is independent of the share of maintenance expenditure in this model.

$(1 - \delta)\{\alpha - (\eta + \psi)(1 - \alpha)\}$ is the social elasticity of unpolluted output with respect to private capital; and equation (5.11) implies that the optimum income tax rate is equal to one less of this social elasticity. In FMS (1993), $\eta + \psi = \delta = 0$ because there is neither any external effect of private capital accumulation on depreciation of public capital nor any environmental pollution resulting from industrial production. So there is no difference between social elasticity of unpolluted output and private elasticity of total output each with respect to private capital. So, optimum income tax rate is equal to the competitive output share of public capital in that model. The same is true in Greiner (2005). In the present model, private elasticity of total output with respect to private capital always exceeds the corresponding social elasticity of unpolluted output. Hence the optimum income tax rate exceeds the competitive output share of public capital in this model. In KK (2004), the optimal income tax rate exceeds the social elasticity of output with respect to public capital due to the positive external learning-by-doing effect of private capital accumulation.

The exercise to analyze whether growth rate maximizing tax rate maximizes the level of social welfare in the steady-state equilibrium is identical to the same exercise carried out in the basic model in section 2.2 of chapter 2. Therefore, by similar intuition, here too, W varies positively with g_m.

Thus the level of social welfare along the steady-state equilibrium growth path is maximized when the steady-state equilibrium growth rate is maximized. We, therefore, state the following proposition.

Proposition 5.2: (i) Optimum income tax rate, optimum abatement expenditure rate and optimum public investment allocation share in the steady-state growth equilibrium are given by

$$\tau^{\star} = 1 - (1-\delta)\{\alpha - (\eta + \psi)(1-\alpha)\},$$

$$T^{\star} = \delta + (1-\delta)\psi(1-\alpha),$$

and

$$\mu^{\star} = \frac{1}{1+\eta}.$$

(ii) Optimum ratio of combined expenditure on net public investment and maintenance of public capital to national income in the steady-state growth equilibrium varies inversely with the magnitude of the pollution-output coefficient. It is greater (less) than the competitive output share of public capital if the pollution-output coefficient is smaller (greater) than the share of maintenance expenditure.

Using equations (5.10) to (5.13) we have

$$g_m{}^{(1+\psi)(1-\alpha)}(\sigma g_m + \rho)^{\alpha-(\psi+\eta)(1-\alpha)} = (1-\delta)[\alpha\{\alpha - (\psi+\eta)(1-\alpha)\}]^{\alpha-(\psi+\eta)(1-\alpha)}$$

$$\{(1-\alpha)(1+\eta)\}^{(1-\alpha)(1+\eta)}\{\psi(1-\alpha)\}^{\psi(1-\alpha)}. \qquad \dots\dots(5.10.1)$$

This equation solves for the unique value of g_m when the fiscal instruments are optimally chosen.

Comparing optimum values of fiscal instruments of this section to those in section 2.2 and without imposing any additional parametric restrictions we can conclude that the growth rate maximizing expenditure share on public intermediate good given by

$$\tau^\star - T^\star = (1-\delta)(1-\alpha)(1+\eta) > (1-\alpha)(1-\delta); \qquad \dots\dots(5.14)$$

is unambiguously greater than that given by equation (2.2.15) in section 2.2 of chapter 2. This is so because public intermediate good here is a stock variable with a positive depreciation rate; and, therefore, income tax revenue net of abatement expenditure now finances not only new investment but also maintenance of existing public capital.

Further, if we assume that $\theta > (=)(<)\eta + \psi$, then the balanced growth rate maximizing income tax rate given by equation (5.11) is greater than (equal to) (less than) that given by equation (2.2.13) of chapter 2. In all these three cases, the abatement expenditure rate given by equation (5.12) is less than that given by equation (2.2.14) in chapter 2 as long as $\theta > \psi$. Here,

$$\tau^\star = 1 - (1-\delta)\{\alpha - (\eta+\psi)(1-\alpha)\}; \qquad \dots\dots(5.11)$$

and

$$T^\star = \delta + (1-\delta)\psi(1-\alpha). \qquad \dots\dots(5.12)$$

In the special case with $\eta = 0$, $\theta = \psi$ and $\mu = 1$, equation (5.10) takes the following form.

$$g_m{}^{(1+\theta)(1-\alpha)}(\sigma g_m + \rho)^{\alpha-\theta(1-\alpha)} = \{\alpha(1-\tau)\}^{\alpha-\theta(1-\alpha)}(T-\delta)^{\theta(1-\alpha)}(\tau - T)^{(1-\alpha)}. \qquad \dots\dots(5.10a)$$

Now, comparing equation (5.10a) to equation (2.2.12) with $A = 1$, we find that the R.H.S. of both these equations are identical. However, the L.H.S. of equation (5.10a) is greater than that of equation (2.2.12) of chapter 2. Thus, it can be said that, under the assumptions that $\eta = 0$, $\theta = \psi$, $\mu = 1$ and $A = 1$, the steady-state equilibrium growth rate in the present model is greater than that in section 2.2.

5.4 TRANSITIONAL DYNAMICS

We now turn to investigate transitional dynamic properties of the model. Equations of motion of the dynamic system are given by (5.6), (5.7), (5.8) and (5.9). We reconsider the ratio variables x and y from the previous chapters while redefine the ratio variable z as follows.

$$z = \frac{G}{E}.$$

Using equations (5.6) to (5.9), we have

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right)(1-\tau)y^{1-\alpha}z^{1-\alpha} + x - \frac{\rho}{\sigma}; \qquad \ldots\ldots(5.15)$$

$$\frac{\dot{y}}{y} = (T-\delta)y^{-\alpha}z^{1-\alpha} - (1-\tau)y^{1-\alpha}z^{1-\alpha} + x; \qquad \ldots\ldots(5.16)$$

and

$$\frac{\dot{z}}{z} = \mu(1-\mu)^{\eta}(\tau - T)^{1+\eta}y^{(1+\eta)(1-\alpha)-1+\psi}z^{(1+\eta)(1-\alpha)-1} - (T-\delta)y^{-\alpha}z^{1-\alpha}. \ldots\ldots(5.17)$$

The determinant of the Jacobian matrix [66] corresponding to the differential equations (5.15), (5.16) and (5.17) is given by

$$|J| = \{\alpha - (1-\alpha)(\eta+\psi)\}\mu(1-\mu)^{\eta}(T-\delta)(\tau-T)^{1+\eta}$$
$$y^{(1+\eta)(1-\alpha)-\alpha-2+\psi}z^{(1+\eta)(1-\alpha)-\alpha-1} + \frac{\alpha}{\sigma}\psi(1-\alpha)\mu(1-\mu)^{\eta}(1-\tau)(\tau-T)^{1+\eta}$$
$$y^{(1+\eta)(1-\alpha)-\alpha-1+\psi}z^{(1+\eta)(1-\alpha)-\alpha-1} + \frac{\alpha}{\sigma}(1-\alpha)(1-\tau)(T-\delta)y^{-2\alpha}z^{1-2\alpha}.$$

Here $\alpha - (1-\alpha)(\eta+\psi) > 0$, by assumption. Also $1 > \tau > T > \delta$ and $0 < \mu < 1$ when τ, T and μ are optimally chosen. So $|J| > 0$ in this case. So either

[66] The derivation of the determinant is worked out in Appendix 5D.

all the three latent roots of J matrix are positive or two of them are negative with the third one being positive. Hence, the steady-state equilibrium cannot be a saddle point. Either it is unstable with all positive latent roots or there exists indeterminacy of the transitional growth path converging to the equilibrium point.

The trace of the Jacobian matrix is given by

$$Tr\,J = 1 - \alpha(T-\delta)y^{-\alpha-1}z^{1-\alpha} - (1-\alpha)(1-\tau)y^{-\alpha}z^{1-\alpha}$$

$$-\{\alpha - \eta(1-\alpha)\}\mu(1-\mu)^{\eta}(\tau - T)^{1+\eta}y^{(1+\eta)(1-\alpha)-1+\psi}z^{(1+\eta)(1-\alpha)-2}$$

$$-(1-\alpha)(T-\delta)y^{-\alpha}.$$

Using equations (5.15) to (5.17) and using the expressions of steady-state equilibrium values of x, y and z in terms of g_m obtained from equations (5B.7), (5B.8) and (5B.9) in Appendix 5B, we find that the trace of the Jacobian matrix is negative[67] if

$$1 < g_m^{1-\alpha}\frac{\{\alpha-(1-\alpha)(\eta+\psi)\}}{\psi}\{\psi(1-\alpha)(1-\delta)\}^{\alpha}$$

and if

$$\rho < \frac{(\sigma g_m+\rho)}{g_m^{\alpha}}\{\psi(1-\alpha)(1-\delta)\}^{\alpha}.$$

Both these inequalities are likely to be satisfied for high values of g_m.

If the determinant of the Jacobian matrix takes a positive sign and its trace takes a negative sign, then there are one positive and two negative latent roots of this Jacobian matrix[68]. It means that there may exist indeterminacy of the transitional growth path converging to the unique steady-state growth equilibrium when the rate of growth is sufficiently high. So we have the following proposition.

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[67]The derivation is worked out in Appendix 5D.

[68]It is a sufficient condition but not a necessary one. There may be one positive and two negative roots even if the trace takes a positive sign. However, all the roots may also be positive in that case implying that no trajectory converges to the equilibrium point. See Benhabib and Perili (1994).

Proposition 5.3: The unique steady-state growth equilibrium never satisfies saddle-point stability. Either there exists indeterminacy of the transitional growth path converging to the steady-state growth equilibrium or the steady-state growth equilibrium is unstable.

This result is different from what is found in FMS (1993), Greiner (2005), KK (2004), DK (2008), Agenor (2009) models and in the basic model in section 2.2 of chapter 2. All these models prove the saddle-point stability of the steady-state growth equilibrium. Here we show that saddle point stability of the steady-state growth equilibrium point can never be satisfied in this extended model. On the contrary there is a possibility of indeterminacy of the transitional growth path converging to the unique steady-state growth equilibrium. That externality of private capital accumulation explains indeterminacy of transitional growth path is well established in the literature of growth theory[69]. A negative external effect of private capital enters into the depreciation function in this model. However, the external effect of environmental quality in the depreciation function also explains indeterminacy in this model. Such externality effects do not appear in the endogenous depreciation functions in KK (2004), DK (2008) and Agenor (2009) models.

5.5 PLANNED ECONOMY

The market economy solution may not coincide with the socially efficient solution in the steady-state growth equilibrium due to distortions caused by proportional income tax and by the failure of private individuals to internalize externalities. The presence of two non rival inputs in the production function - public capital and environmental quality - causes positive externalities. This is no different from the basic model. Also, private capital accumulation generates negative externalities through congestion effects leading to the increase in the

[69] See, for example, Benhabib and Farmer (1994, 1996), Chen and Lee (2007), Mino (2001), etc.

depreciation of public capital. Congestion externality now drives an additional wedge between the market economy solution and the planned economy solution through a route which is intuitively similar but analytically different from that in the basic model. Here, this negative externality reduces total available public capital stock for productive usage while in the basic model it reduces effective benefit derived from public input expenditure. Environmental pollution, as a by-product of production, degrades environmental quality which in turn raises depreciation of public capital. Pollution externality too, acts through the same route as congestion externality in this extended model. The internalization of negative externalities is not within the scope of individual firms. Therefore, we next turn to solve the centralized planner's problem in order to obtain the first best solution. The centralized planner maximizes a social welfare function identical to that of the representative household's lifetime utility function, and internalizes these externalities. Equations (5.1), (5.4), (5.5) and (2.2.6) remain unchanged. Equations (2.2.4.1) and (4.3.1) are planned economy versions of equations (2.2.4) and (4.3) respectively. Equations (5.2) and (5.3) are modified as follows.

$$\mathrm{I} = \mu(\Pi - \Omega); \qquad \ldots\ldots(5.2.1)$$

and

$$\mathrm{M} = (1 - \mu)(\Pi - \Omega); \qquad \ldots\ldots(5.3.1)$$

Here Π denotes planner's combined lump sum expenditure on investment of public capital, maintenance of existing public capital and abatement activities. Abatement expenditure is, as usual, denoted by Ω.

The planner's problem is to maximize $\int_0^\infty e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma}dt$ with respect to C, Π, Ω and μ subject to equations (2.2.4.1), (5.4), (5.5) (5.2.1), (5.3.1) and (4.3.1). We again consider a steady-state growth equilibrium and the growth rate is denoted by g_c; the following equation solves for the steady-state equilibrium growth rate[70] in the command economy.

[70]Equation (5.18) is derived in the Appendix 5E.

$$(\rho+\sigma g_c)^{1+\alpha-\eta(1-\alpha)}\{\alpha(\rho+\sigma g_c)-(\eta+\psi)(1-\alpha)g_c\}^{-\alpha+(\eta+\psi)(1-\alpha)}$$

$$= g_c{}^{(\eta+\psi)(1-\alpha)}(1-\delta)\left\{\frac{\eta^{\eta}}{(1+\eta)^{1+\eta}}\right\}^{1-\alpha}\{(1-\alpha)(1+\eta)\}^{(1-\alpha)(1+\eta)}\{\psi(1-\alpha)\}^{\psi(1-\alpha)}. \quad \ldots\ldots (5.18)$$

However, we cannot make any analytical comparison of g_c and g_m using equations (5.18) and (5.10.1) when $\rho > 0$. We consider a special case[71] where $\rho = 0$. In this special case, we obtain the following modified version of equation (5.10.1) given by

$$g_m{}^{1-\eta(1-\alpha)}\sigma^{\alpha-(\psi+\eta)(1-\alpha)} = \alpha^{\alpha-(\psi+\eta)(1-\alpha)}(1-\delta)\left\{\frac{\eta^{\eta}}{(1+\eta)^{1+\eta}}\right\}^{1-\alpha}$$

$$\{(1-\alpha)(1+\eta)\}^{(1-\alpha)(1+\eta)}\{\psi(1-\alpha)\}^{\psi(1-\alpha)}\{\alpha-(\eta+\psi)(1-\alpha)\}^{\alpha-(\psi+\eta)(1-\alpha)} \quad \ldots\ldots (5.10.2).$$

To facilitate comparison we modify equation (5.18) with $\rho = 0$ as follows.

$$g_c{}^{1-\eta(1-\alpha)}\sigma^{\alpha-(\psi+\eta)(1-\alpha)} = \alpha^{\alpha-(\psi+\eta)(1-\alpha)}(1-\delta)\left\{\frac{\eta^{\eta}}{(1+\eta)^{1+\eta}}\right\}^{1-\alpha}$$

$$\{(1-\alpha)(1+\eta)\}^{(1-\alpha)(1+\eta)}\{\psi(1-\alpha)\}^{\psi(1-\alpha)}\frac{\{\alpha\sigma-(\eta+\psi)(1-\alpha)\}^{\alpha-(\psi+\eta)(1-\alpha)}}{\left\{\sigma^{(1-\alpha)(1-\psi)}\alpha^{\alpha-(\psi+\eta)(1-\alpha)}\right\}} \quad \ldots\ldots (5.18.1)$$

Thus comparing equations (5.10.2) and (5.18.1) we find that $g_m \gtrless g_c$ if

$$\{\alpha-(\eta+\psi)(1-\alpha)\}^{\alpha-(\psi+\eta)(1-\alpha)} \gtrless \frac{\{\alpha\sigma-(\eta+\psi)(1-\alpha)\}^{\alpha-(\psi+\eta)(1-\alpha)}}{\left\{\sigma^{(1-\alpha)(1-\psi)}\alpha^{\alpha-(\psi+\eta)(1-\alpha)}\right\}}.$$

So the relationship between the competitive equilibrium growth rate, g_m, and the socially efficient growth rate, g_c, is conditional on the exogenously given values of parameters of the model. If $\sigma = 1$, then $g_m < g_c$ because $0 < \alpha < 1$. If $\sigma \neq 1$, then socially efficient growth rate may fall short of or may exceed the competitive equilibrium growth rate.

Here, external effects of environmental quality and government expenditure on maintenance of public capital positively affect the rate of accumulation of public capital reducing its depreciation rate. However, accumulation of private capital and environmental pollution have negative

[71] We do not assume that $\rho = 0$. We are aware of the problem that, with $\rho = 0$, the integral $\int_0^{\infty} u(C)e^{-\rho t}\,dt$ may not converge. We want to mean that the result valid with $\rho = 0$ is likely to be valid with a very low but positive value of ρ.

external effects on the durability of public capital and hence on its rate of accumulation. These two sets of external effects act in opposite directions. Whenever the positive externality outweighs the negative externality, g_c exceeds g_m; but falls short in the reverse case. In Barro (1990), FMS (1993), etc., only public expenditure generates a positive externality on production and there is no negative externality. In DK (2008), KK (2004) there is only a positive externality on the durability of public capital arising from the public expenditure on its maintenance. So the competitive equilibrium growth rate always falls short of the socially efficient growth rate.

APPENDIX 5A

DERIVATION OF EQUATION (5.6) IN SECTION 5.3.1

The dynamic optimization problem of the representative household is to maximize $\int_0^\infty e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt$ with respect to C subject to equation (2.2.4) and given $K(0)$. Here C is the control variable satisfying $0 \leq C \leq (1-\tau)Y$; and K is the state variable.

The Hamiltonian to be maximized at each point of time is given by

$$\mathcal{H} = e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} + e^{-\rho t} \lambda_K [(1-\tau)Y - C].$$

Here λ_K is the co-state variable representing the shadow price of investment. Maximizing the Hamiltonian with respect to C and assuming an interior solution, we obtain

$$C^{-\sigma} = \lambda_K. \qquad \ldots\ldots(5A.1)$$

Also the optimum time path of λ_K satisfies the following.

$$\frac{\dot{\lambda_K}}{\lambda_K} = \rho - (1-\tau)\alpha K^{\alpha-1} G^{1-\alpha}. \qquad \ldots\ldots(5A.2)$$

Using equations (5.1) and (5A.2) we have

$$\frac{\dot{\lambda_K}}{\lambda_K} = \rho - \alpha(1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha}. \qquad \ldots\ldots(5A.3)$$

Using the two optimality conditions (5A.1) and (5A.3), we have

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \rho\right]. \qquad \dots\dots(5A.4)$$

This is same as equation (5.6) in section 5.3.1.

APPENDIX 5B

DERIVATION OF EQUATION (5.10) IN SECTION 5.3.1

Using equations (5.1) to (5.6), (2.2.4), (4.3) and (2.2.8) we have the following equations.

$$g_m = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \rho\right]; \qquad \dots\dots(5B.1)$$

$$g_m = \frac{\dot{K}}{K} = (1-\tau)\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \frac{C}{K}; \qquad \dots\dots(5B.2)$$

$$g_m = \frac{\dot{E}}{E} = (T-\delta)\left(\frac{E}{K}\right)^{-\alpha}\left(\frac{G}{E}\right)^{1-\alpha}; \qquad \dots\dots(5B.3)$$

and

$$g_m = \frac{\dot{G}}{G} = \mu(1-\mu)^{\eta}(\tau - T)^{1+\eta}\left(\frac{E}{K}\right)^{(1+\eta)(1-\alpha)-1+\psi}\left(\frac{G}{E}\right)^{(1+\eta)(1-\alpha)-1}; \qquad \dots\dots(5B.4)$$

From equation (5B.1) we have,

$$\frac{E}{K} = \left\{\frac{(\sigma g_m + \rho)}{\alpha(1-\tau)}\right\}^{\frac{1}{1-\alpha}}\left(\frac{G}{E}\right)^{-1}. \qquad \dots\dots(5B.5)$$

Again, from equation (5B.3) we have,

$$\frac{E}{K} = \left(\frac{g_m}{T-\delta}\right)^{-\frac{1}{\alpha}}\left(\frac{G}{E}\right)^{\frac{1-\alpha}{\alpha}}. \qquad \dots\dots(5B.6)$$

Using equations (5B.5) and (5B.6) we derive the following equation.

$$\frac{G}{E} = \left(\frac{g_m}{T-\delta}\right)\left\{\frac{(\sigma g_m + \rho)}{\alpha(1-\tau)}\right\}^{\frac{\alpha}{1-\alpha}}. \qquad \dots\dots(5B.7)$$

Using equations (5B.6) and (5B.7) we obtain the following equation.

$$\frac{E}{K} = \frac{(\sigma g_m + \rho)}{g_m} \frac{(T-\delta)}{\alpha(1-\tau)}. \qquad \ldots\ldots(5B.8)$$

Similarly using equations (5B.1) and (5B.2) we can show that

$$\frac{C}{K} = \frac{(\sigma-\alpha)g_m+\rho}{\alpha}. \qquad \ldots\ldots(5B.9)$$

Now, using equations (5B.4), (5B.7) and (5B.8) we derive the following equation.

$$g_m = \mu(1-\mu)^{\eta}(\tau-T)^{1+\eta}\left[\frac{(\sigma g_m+\rho)}{g_m}\frac{(T-\delta)}{\alpha(1-\tau)}\right]^{(1+\eta)(1-\alpha)-1+\psi}$$
$$\left[\left(\frac{g_m}{T-\delta}\right)\left\{\frac{(\sigma g_m+\rho)}{\alpha(1-\tau)}\right\}^{\frac{\alpha}{1-\alpha}}\right]^{(1+\eta)(1-\alpha)-1},$$

or,

$$g_m{}^{(1+\psi)(1-\alpha)}(\sigma g_m+\rho)^{\alpha-(\psi+\eta)(1-\alpha)} = \alpha^{\alpha-(\psi+\eta)(1-\alpha)}\mu^{1-\alpha}(1-\mu)^{\eta(1-\alpha)}$$
$$(T-\delta)^{\psi(1-\alpha)}(1-\tau)^{\alpha-(\psi+\eta)(1-\alpha)}(\tau-T)^{(1+\eta)(1-\alpha)}. \qquad \ldots\ldots(5B.10)$$

This is same as equation (5.10).

APPENDIX 5C

DERIVATION OF EQUATIONS (5.11), (5.12) AND (5.13) AND THE SECOND ORDER CONDITIONS IN SECTION 5.3.2

We denote the L.H.S. and the R.H.S. of equation (5.10) by L.H.S.$_{(5.10)}$ and R.H.S.$_{(5.10)}$ respectively. Maximizing the R.H.S. of equation (5.10) with respect to τ, we obtain the following first order condition.

$$\text{R.H.S.}_{(5.10)}\left[-\{\alpha-(\eta+\psi)(1-\alpha)\}(1-\tau)^{-1}+(1+\eta)(1-\alpha)(\tau-T)^{-1}\right]=0;$$

or,

$$(1+\eta)(1-\alpha)(\tau-T)^{-1}-\{\alpha-(\eta+\psi)(1-\alpha)\}(1-\tau)^{-1}=0. \qquad \ldots\ldots(5C.1)$$

Maximizing the R.H.S. of equation (5.10) with respect to T, we obtain the following first order condition.

$$\mathrm{R.H.S.}_{(5.10)}\,[\psi(1-\alpha)(T-\delta)^{-1}-(1+\eta)(1-\alpha)(\tau-T)^{-1}]=0;$$

or,

$$[\psi(1-\alpha)(T-\delta)^{-1}-(1+\eta)(1-\alpha)(\tau-T)^{-1}]=0. \qquad \dots\dots(5C.2)$$

Maximizing the R.H.S. of equation (5.10) with respect to μ, we obtain the following first order condition.

$$\mathrm{R.H.S.}_{(5.10)}\,[(1-\alpha)\mu^{-1}-\eta(1-\alpha)(1-\mu)^{-1}]=0;$$

or,

$$[(1-\alpha)\mu^{-1}-\eta(1-\alpha)(1-\mu)^{-1}]=0. \qquad \dots(5C.3)$$

Using equations (5C.1), (5C.2) and (5C.3) we obtain the following expressions.

$$\tau^{\star}=1-(1-\delta)\{\alpha-(1-\alpha)(\eta+\psi)\};$$

$$T^{\star}=\delta+(1-\delta)\psi(1-\alpha);$$

and

$$\mu^{\star}=\frac{1}{1+\eta}.$$

These are same as equations (5.11), (5.12) and (5.13) in section 5.3.2.

To check the second order conditions for optimality we twice differentiate equation (5.10), with respect to τ, T and μ respectively and arrive at the following three second order conditions.

$$-\left[(1+\psi)(1-\alpha){g_m}^{-2}+\sigma^2\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+\left[(1+\psi)(1-\alpha){g_m}^{-1}+\sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial\tau^2}$$

$$=-[(1+\eta)(1-\alpha)(\tau-\mathrm{T})^{-2}+\{\alpha-(1-\alpha)(\eta+\psi)\}(1-\tau)^{-2}]; \qquad \dots\dots(5C.4)$$

$$-\left[(1+\psi)(1-\alpha){g_m}^{-2}+\sigma^2\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+\left[(1+\psi)(1-\alpha){g_m}^{-1}+\sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial T^2}$$

$$= -[\psi(1-\alpha)(T-\delta)^{-2} + (1+\eta)(1-\alpha)(\tau-\mathrm{T})^{-2}]; \qquad \ldots\ldots(5C.5)$$

and

$$-\left[(1+\psi)(1-\alpha){g_m}^{-2} + \sigma^2\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial \mu}\right)^2$$

$$+\left[(1+\psi)(1-\alpha){g_m}^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial \mu^2}$$

$$= -[(1-\alpha)\mu^{-2} + \eta(1-\alpha)(1-\mu)^{-2}]; \qquad \ldots\ldots(5C.6)$$

Now we evaluate the three second order conditions mentioned above at $\tau = \tau^\star$, $T = T^\star$ and $\mu = \mu^\star$ where $\frac{\partial g_m}{\partial \tau} = \frac{\partial g_m}{\partial T} = \frac{\partial g_m}{\partial \mu} = 0$. Hence we obtain the followings.

$$\frac{\partial^2 g_m}{\partial \tau^2} = -\frac{(1+\eta)(1-\alpha)(\tau^\star-T^\star)^{-2} + \{\alpha-(1-\alpha)(\eta+\psi)\}(1-\tau^\star)^{-2}}{\left[(1+\psi)(1-\alpha){g_m}^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-1}\right]} < 0;$$

$$\frac{\partial^2 g_m}{\partial T^2} = -\frac{(1+\eta)(1-\alpha)(\tau^\star-T^\star)^{-2} + \psi(1-\alpha)(T^\star-\delta)^{-2}}{\left[(1+\psi)(1-\alpha){g_m}^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-1}\right]} < 0;$$

and

$$\frac{\partial^2 g_m}{\partial \mu^2} = -\frac{\left[(1-\alpha){\mu^\star}^{-2} + \eta(1-\alpha)(1-\mu^\star)^{-2}\right]}{\left[(1+\psi)(1-\alpha){g_m}^{-1} + \sigma\{\alpha-(1-\alpha)(\eta+\psi)\}(\sigma g_m+\rho)^{-1}\right]} < 0.$$

The R.H.S. of each of these three equations is negative. Thus the second order conditions are also satisfied.

APPENDIX 5D

DERIVATION OF THE DETERMINANT AND THE TRACE OF THE JACOBIAN MATRIX IN SECTION 5.4

We define the following variables.

$$M = (1-\tau)y^{1-\alpha}z^{1-\alpha}; \qquad \ldots\ldots(5D.1)$$

$$N = (T-\delta)y^{-\alpha}z^{1-\alpha}; \qquad \ldots\ldots(5D.2)$$

and

$$Q = \mu(1-\mu)^{\eta}(\tau - T)^{1+\eta} y^{(1+\eta)(1-\alpha)-1+\psi} z^{(1+\eta)(1-\alpha)-1}. \qquad \ldots\ldots(5D.3)$$

Now we consider following equations from section 5.4.

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right)(1-\tau) y^{1-\alpha} z^{1-\alpha} + x - \frac{\rho}{\sigma}; \qquad \ldots\ldots(5.15)$$

$$\frac{\dot{y}}{y} = (T-\delta) y^{-\alpha} z^{1-\alpha} - (1-\tau) y^{1-\alpha} z^{1-\alpha} + x; \qquad \ldots\ldots(5.16)$$

and

$$\frac{\dot{z}}{z} = \mu(1-\mu)^{\eta}(\tau - T)^{1+\eta} y^{(1+\eta)(1-\alpha)-1+\psi} z^{(1+\eta)(1-\alpha)-1} (T-\delta) y^{-\alpha} z^{1-\alpha}.$$

$$\ldots\ldots(5.17)$$

Thus using equations (5D.1), (5D.2) and (5D.3) we modify equations (5.15), (5.16) and (5.17) as follows.

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right) M + x - \frac{\rho}{\sigma}; \qquad \ldots\ldots(5D.4)$$

$$\frac{\dot{y}}{y} = N - M + x; \qquad \ldots\ldots(5D.5)$$

and

$$\frac{\dot{z}}{z} = Q - N. \qquad \ldots\ldots(5D.6)$$

We obtain the following partial derivatives corresponding to three modified differential equations.

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial x} = 1;$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial y} = (1-\alpha)\left(\frac{\alpha}{\sigma} - 1\right)\frac{M}{y};$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial z} = (1-\alpha)\left(\frac{\alpha}{\sigma} - 1\right)\frac{M}{z};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial x} = 1;$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial y} = -\alpha\frac{N}{y} - (1-\alpha)\frac{M}{y};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial z} = (1-\alpha)\frac{N}{z} - (1-\alpha)\frac{M}{z};$$

$$\frac{\partial\left(\frac{\dot{z}}{z}\right)}{\partial x} = 0;$$

$$\frac{\partial\left(\frac{\dot{z}}{z}\right)}{\partial y} = \{(1+\eta)(1-\alpha)+\psi-1\}\frac{Q}{y} + \alpha\frac{N}{y};$$

and

$$\frac{\partial\left(\frac{\dot{z}}{z}\right)}{\partial z} = -\{1-(1+\eta)(1-\alpha)\}\frac{Q}{z} - (1-\alpha)\frac{N}{z}.$$

So the determinant of the Jacobian matrix can be written as follows.

$$|J| = \left[-\alpha\frac{N}{y} - (1-\alpha)\frac{M}{y}\right]\left[-\{1-(1+\eta)(1-\alpha)\}\frac{Q}{z} - (1-\alpha)\frac{N}{z}\right]$$

$$-\left[\{(1+\eta)(1-\alpha)+\psi-1\}\frac{Q}{y} + \alpha\frac{N}{y}\right]\left[(1-\alpha)\frac{N}{z} - (1-\alpha)\frac{M}{z}\right]$$

$$-(1-\alpha)\left(\frac{\alpha}{\sigma}-1\right)\frac{M}{y}\left[-\{1-(1+\eta)(1-\alpha)\}\frac{Q}{z} - (1-\alpha)\frac{N}{z}\right]$$

$$+(1-\alpha)\left(\frac{\alpha}{\sigma}-1\right)\frac{M}{z}\left[\{(1+\eta)(1-\alpha)+\psi-1\}\frac{Q}{y} + \alpha\frac{N}{y}\right],$$

or,

$$|J| = \{\alpha-(1-\alpha)(\eta+\psi)\}\frac{N}{y}\frac{Q}{z} + \frac{\alpha}{\sigma}\psi(1-\alpha)\frac{M}{y}\frac{Q}{z} + \frac{\alpha}{\sigma}(1-\alpha)\frac{M}{y}\frac{N}{z},$$

or,

$$|J| = \{\alpha-(1-\alpha)(\eta+\psi)\}\lambda(1-\lambda)^{\eta}(T-\delta)(\tau-T)^{1+\eta}$$

$$y^{(1+\eta)(1-\alpha)-\alpha-2+\psi}z^{(1+\eta)(1-\alpha)-\alpha-1} + \frac{\alpha}{\sigma}\psi(1-\alpha)\lambda(1-\lambda)^{\eta}(1-\tau)(\tau-T)^{1+\eta}$$

$$y^{(1+\eta)(1-\alpha)-\alpha-1+\psi}z^{(1+\eta)(1-\alpha)-\alpha-1} + \frac{\alpha}{\sigma}(1-\alpha)(1-\tau)(T-\delta)y^{-2\alpha}z^{1-2\alpha}.$$

Here $\alpha > (1-\alpha)(\eta+\psi)$, $1 > \tau > T > \delta$ and $1-\lambda > 0$. Thus the determinant is positive in sign.

The trace of the Jacobian matrix is given by,

$$Tr\,J = 1 - \alpha\frac{N}{y} - (1-\alpha)\frac{M}{y} - \{1-(1+\eta)(1-\alpha)\}\frac{Q}{z} - (1-\alpha)\frac{N}{z}.$$

At the steady-state equilibrium, $\frac{\dot{x}}{x} = \frac{\dot{y}}{y} = \frac{\dot{z}}{z} = 0$. Using this condition and equations (5D.4), (5D.5) and (5D.6) the trace can be written as follows.

$$Tr\,J = 1 - \left[\alpha\frac{1}{y} - \{1-(1+\eta)(1-\alpha)\}\frac{1}{z} - (1-\alpha)\frac{1}{z}\right]\left(\frac{\alpha}{\sigma}M - \frac{\rho}{\sigma}\right) - (1-\alpha)M\frac{1}{y}.$$

Now, $Tr\,J < 0$ if

$$1 + \alpha\frac{\rho}{\sigma}\frac{1}{y} + \{1-\eta(1-\alpha)\}\frac{\rho}{\sigma}\frac{1}{z} < \left[\alpha\frac{\alpha}{\sigma}\frac{1}{y} + (1-\alpha)\frac{1}{y} + \{1-\eta(1-\alpha)\}\frac{\alpha}{\sigma}\frac{1}{z}\right]M.$$

Using the optimal values of the policy variables given by equations (5.11), (5.12) and (5.13) we have

$$y^\star = \left(\frac{E}{K}\right)^\star = \frac{\psi(1-\alpha)(\sigma g_m+\rho)}{\alpha g_m\{\alpha-(1-\alpha)(\eta+\psi)\}}; \qquad \ldots\ldots(5D.7)$$

$$z^\star = \left(\frac{G}{E}\right)^\star = (1-\delta)^{-\frac{1}{1-\alpha}}\left[\frac{g_m}{\psi(1-\alpha)}\right]\left[\frac{(\sigma g_m+\rho)}{\alpha\{\alpha-(1-\alpha)(\eta+\psi)\}}\right]^{\frac{\alpha}{1-\alpha}}. \qquad \ldots\ldots(5D.8)$$

Now using equations (5D.7) and (5D.8) and the optimal values of the policy variables, the condition for the trace of the Jacobian matrix to be negative can be written as

$$1 + \frac{\rho}{\sigma}\frac{\alpha^2 g_m\{\alpha-(1-\alpha)(\eta+\psi)\}}{\psi(1-\alpha)(\sigma g_m+\rho)} + \frac{\rho}{\sigma}\{1-\eta(1-\alpha)\}\left\{\frac{\psi(1-\alpha)}{g_m}\right\}(1-\delta)^{\frac{1}{1-\alpha}}\left[\frac{\alpha\{\alpha-(1-\alpha)(\eta+\psi)\}}{(\sigma g_m+\rho)}\right]^{\frac{\alpha}{1-\alpha}}$$

$$< \left[\left\{\frac{\alpha^2}{\sigma} + (1-\alpha)\right\}\frac{\alpha g_m\{\alpha-(1-\alpha)(\eta+\psi)\}}{\psi(1-\alpha)(\sigma g_m+\rho)}\right.$$

$$\left. + \frac{\alpha}{\sigma}\{1-\eta(1-\alpha)\}\left\{\frac{\psi(1-\alpha)}{g_m}\right\}(1-\delta)^{\frac{1}{1-\alpha}}\left[\frac{\alpha\{\alpha-(1-\alpha)(\eta+\psi)\}}{(\sigma g_m+\rho)}\right]^{\frac{\alpha}{1-\alpha}}\right]\frac{(\sigma g_m+\rho)}{\alpha}\left\{\frac{\psi(1-\alpha)(1-\delta)}{g_m}\right\}^{\alpha},$$

or,

$$1 + \left[\frac{\rho\alpha}{\sigma} - \left\{\frac{\alpha^2}{\sigma} + (1-\alpha)\right\}\frac{(\sigma g_m+\rho)}{\alpha}\left\{\frac{\psi(1-\alpha)(1-\delta)}{g_m}\right\}^{\alpha}\right]\frac{\alpha g_m\{\alpha-(1-\alpha)(\eta+\psi)\}}{\psi(1-\alpha)(\sigma g_m+\rho)}$$

$$+\{1-\eta(1-\alpha)\}\left\{\frac{\psi(1-\alpha)}{g_m}\right\}(1-\delta)^{\frac{1}{1-\alpha}}\left[\frac{\alpha\{\alpha-(1-\alpha)(\eta+\psi)\}}{(\sigma g_m+\rho)}\right]^{\frac{\alpha}{1-\alpha}}$$

$$\left[\frac{\rho}{\sigma} - \frac{\alpha}{\sigma}\frac{(\sigma g_m+\rho)}{\alpha}\left\{\frac{\psi(1-\alpha)(1-\delta)}{g_m}\right\}^{\alpha}\right] < 0,$$

or,

$$\left[1 - (1-\alpha)\frac{(\sigma g_m+\rho)}{\alpha}\left\{\frac{\psi(1-\alpha)(1-\delta)}{g_m}\right\}^{\alpha}\frac{\alpha g_m\{\alpha-(1-\alpha)(\eta+\psi)\}}{\psi(1-\alpha)(\sigma g_m+\rho)}\right]$$

$$+\left[\rho - (\sigma g_m+\rho)\left\{\frac{\psi(1-\alpha)(1-\delta)}{g_m}\right\}^{\alpha}\right]\frac{\alpha^2}{\sigma}\frac{g_m\{\alpha-(1-\alpha)(\eta+\psi)\}}{\psi(1-\alpha)(\sigma g_m+\rho)}$$

$$+\left[\rho-(\sigma g_m+\rho)\left\{\frac{\psi(1-\alpha)(1-\delta)}{g_m}\right\}^{\alpha}\right]\frac{\{1-\eta(1-\alpha)\}}{\sigma}\left\{\frac{\psi(1-\alpha)}{g_m}\right\}(1-\delta)^{\frac{1}{1-\alpha}}$$

$$\left[\frac{\alpha\{\alpha-(1-\alpha)(\eta+\psi)\}}{(\sigma g_m+\rho)}\right]^{\frac{\alpha}{1-\alpha}}<0.$$

This will be satisfied if

$$1<g_m^{1-\alpha}\frac{\{\alpha-(1-\alpha)(\eta+\psi)\}}{\psi}\{\psi(1-\alpha)(1-\delta)\}^{\alpha}$$

and if

$$\rho<\frac{(\sigma g_m+\rho)}{g_m^{\alpha}}\{\psi(1-\alpha)(1-\delta)\}^{\alpha}.$$

APPENDIX 5E

DERIVATION OF EQUATION (5.18) IN SECTION 5.5

The relevant Hamiltonian to be maximized by the planner at each point of time is given by

$$\mathcal{H}=e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma}+e^{-\rho t}\lambda_K[K^{\alpha}G^{1-\alpha}-\Pi-C]+e^{-\rho t}\lambda_E[\Omega-\delta K^{\alpha}G^{1-\alpha}]$$
$$+e^{-\rho t}\lambda_G\left[\mu(1-\mu)^{\eta}(\Pi-\Omega)^{1+\eta}E^{\psi}K^{-(\eta+\psi)}\right].$$

The state variables are K, G and E. The control variables are C, Π, Ω and μ. λ_K, λ_G, and λ_E are three co-state variables.

Maximizing L with respect to C, Π, Ω and μ we have

$$C^{-\sigma}=\lambda_K; \qquad \ldots\ldots(5E.1)$$

$$\frac{\lambda_K}{\lambda_G}=(1+\eta)\mu(1-\mu)^{\eta}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{\psi}; \qquad \ldots\ldots(5E.2)$$

$$\frac{\lambda_E}{\lambda_G}=(1+\eta)\mu(1-\mu)^{\eta}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{\psi}; \qquad \ldots\ldots(5E.3)$$

and

$$\mu=\frac{1}{1+\eta}. \qquad \ldots\ldots(5E.4)$$

Using equations (5E.2) and (5E.3) we find that

$$\frac{\lambda_K}{\lambda_E} = 1. \qquad \dots\dots(5E.5)$$

Using equations (5E.3) and (5E.5) we obtain the following.

$$\frac{\lambda_K}{\lambda_G} = \left(\frac{\eta}{1+\eta}\right)^{\eta} \left(\frac{\Pi-\Omega}{K}\right)^{\eta} \left(\frac{E}{K}\right)^{\psi}. \qquad \dots\dots(5E.6)$$

Also, along the optimum path, time behaviour of co-state variables satisfies the followings.

$$(1-\delta)\alpha\left(\frac{E}{K}\right)^{1-\alpha}\left(\frac{G}{E}\right)^{1-\alpha} - \frac{\lambda_G}{\lambda_K}(\psi+\eta)\mu(1-\mu)^{\eta}\left(\frac{\Pi-\Omega}{K}\right)^{1+\eta}\left(\frac{E}{K}\right)^{\psi} = \rho - \frac{\dot{\lambda_K}}{\lambda_K};$$

$\dots\dots(5E.7)$

$$\frac{\lambda_K}{\lambda_G}(1-\alpha)(1-\delta)\left(\frac{E}{K}\right)^{-\alpha}\left(\frac{G}{E}\right)^{-\alpha} = \rho - \frac{\dot{\lambda_G}}{\lambda_G}; \qquad \dots\dots(5E.8)$$

and

$$\frac{\lambda_G}{\lambda_E}\psi\mu(1-\mu)^{\eta}\left(\frac{\Pi-\Omega}{K}\right)^{1+\eta}\left(\frac{E}{K}\right)^{\psi-1} = \rho - \frac{\dot{\lambda_E}}{\lambda_E}. \qquad \dots\dots(5E.9)$$

We assume steady-state equilibrium where all the variables grow at the same rate, g_c. Thus equations (5E.5) and (5E.6) imply that

$$\frac{\dot{\lambda_K}}{\lambda_K} = \frac{\dot{\lambda_G}}{\lambda_G} = \frac{\dot{\lambda_E}}{\lambda_E}. \qquad \dots\dots(5E.10)$$

Using equations (5E.1) and (5E.10) we have

$$-\sigma\frac{\dot{C}}{C} = -\sigma g_c = \frac{\dot{\lambda_K}}{\lambda_K} = \frac{\dot{\lambda_G}}{\lambda_G} = \frac{\dot{\lambda_E}}{\lambda_E}. \qquad \dots\dots(5E.11)$$

Again, using equations (5.2), (5.2.1), (5.4) and (5.3.1) we have

$$\frac{\dot{G}}{G} = g_c = \mu(1-\mu)^{\eta}\left(\frac{\Pi-\Omega}{K}\right)^{1+\eta}\left(\frac{E}{K}\right)^{\psi-1}\left(\frac{G}{E}\right)^{-1};$$

or,

$$\frac{G}{E} = \mu(1-\mu)^{\eta}\left(\frac{\Pi-\Omega}{K}\right)^{1+\eta}\left(\frac{E}{K}\right)^{\psi-1} {g_c}^{-1} \qquad \dots\dots(5E.12)$$

Using equations (5E.6), (5E.7), (5E.11) and (5E.12) we obtain the following equation.

$$(1-\delta)\alpha\{\mu(1-\mu)^{\eta}\}^{1-\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{(1+\eta)(1-\alpha)}\left(\frac{E}{K}\right)^{\psi(1-\alpha)}g_c{}^{\alpha-1}-\frac{(\psi+\eta)}{(1+\eta)}\left(\frac{\Pi-\Omega}{K}\right)=\rho+\sigma g_c. \qquad \dots\dots(5E.13)$$

Again using equations (5E.6), (5E.8), (5E.11) and (5E.12) we obtain the following equation.

$$(1-\delta)(1+\eta)(1-\alpha)\{\mu(1-\mu)^{\eta}\}^{1-\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{\eta-\alpha(1+\eta)}\left(\frac{E}{K}\right)^{\psi(1-\alpha)}g_c{}^{\alpha}=\rho+\sigma g_c. \qquad \dots\dots(5E.14)$$

Using equations (5E.5), (5E.6), (5E.9), (5E.11) and (5E.12) we obtain the following equation.

$$\frac{\psi}{1+\eta}\left(\frac{\Pi-\Omega}{K}\right)\left(\frac{E}{K}\right)^{-1}=\rho+\sigma g_c;$$

or,

$$\frac{E}{K}=\frac{\psi}{1+\eta}\left(\frac{\Pi-\Omega}{K}\right)(\rho+\sigma g_c)^{-1}. \qquad \dots\dots(5E.15)$$

Using equations (5E.14) and (5E.15) we obtain the following.

$$(1-\delta)(1+\eta)^{1-\psi(1-\alpha)}(1-\alpha)\{\mu(1-\mu)^{\eta}\}^{1-\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{(\eta+\psi)(1-\alpha)-\alpha}\psi^{\psi(1-\alpha)}g_c{}^{\alpha}$$

$$=(\rho+\sigma g_c)^{1+\psi(1-\alpha)}.$$

or,

$$\left(\frac{\Pi-\Omega}{K}\right)^{\alpha-(\eta+\psi)(1-\alpha)}$$

$$=\psi^{\psi(1-\alpha)}(1-\delta)(1+\eta)^{1-\psi(1-\alpha)}(1-\alpha)\{\mu(1-\mu)^{\eta}\}^{1-\alpha}(\rho+\sigma g_c)^{\psi(1-\alpha)-1} \qquad \dots\dots(5E.16)$$

Again, using equations (5E.13) and (5E.16) we obtain the following equation.

$$(1-\delta)\alpha\{\mu(1-\mu)^{\eta}\}^{1-\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{1-\alpha+(\psi+\eta)(1-\alpha)}\left(\frac{\psi}{1+\eta}\right)^{\psi(1-\alpha)}g_c{}^{\alpha-1}-\frac{(\psi+\eta)}{(1+\eta)}\left(\frac{\Pi-\Omega}{K}\right)$$

$$=(\rho+\sigma g_c)^{1+\psi(1-\alpha)}. \qquad \dots\dots(5E.17)$$

Finally we use equations (5E.4), (5E.16) and (5E.17) to obtain the following equation that solves for the planned economy growth rate.

$$g_c{}^{-(\eta+\psi)(1-\alpha)}(\rho+\sigma g_c)^{1+\alpha-\eta(1-\alpha)}\{\alpha(\rho+\sigma g_c)-(\eta+\psi)(1-\alpha)g_c\}^{-\alpha+(\eta+\psi)(1-\alpha)}$$

$$=(1-\delta)\left\{\frac{\eta^{\eta}}{(1+\eta)^{1+\eta}}\right\}^{1-\alpha}\{(1-\alpha)(1+\eta)\}^{(1-\alpha)(1+\eta)}\{\psi(1-\alpha)\}^{\psi(1-\alpha)}. \quad \ldots\ldots(5E.18)$$

This is same as equation (5.18) in section 5.5.

CHAPTER 6

6. INFORMAL SECTOR WITH ENVIRONMENTAL POLLUTION AND PUBLIC EXPENDITURE

6.1 INTRODUCTION

In all the previous chapters we have developed one sector aggregate models. However, one-sector framework is not the appropriate one to analyze dynamics of the economy and the optimality of fiscal policy when a substantial part of economic activities remains untaxed. The aggregate of various untaxed economic sectors is known as informal sector or the shadow economy in the literature. The present chapter extends the single-sector basic model developed in section 2.2 in chapter 2 to a two-sector model consisting of a formal sector and an informal sector where both the sectors now generate environmental pollution but government can tax only the formal sector output to finance pollution abatement activity and public expenditure.

In this chapter, we develop a two-sector endogenous growth model consisting of both formal sector and informal sector and analyze the role of public infrastructural expenditure and environmental pollution. The representative household allocates capital between the formal sector and the informal sector. Pollution is generated by both the sectors. Though both sectors pollute the environment, emission-output coefficients in the formal and informal sectors are different. Public infrastructure is a non-rival public good and hence, informal sector cannot be excluded from using it. Government imposes proportional income tax on the formal sector only and finances the abatement expenditure as well as public infrastructural expenditure from this tax revenue. The informal sector avails the benefits of infrastructure without paying for it as government cannot tax this sector. Congestion of public capital or productive role of public health capital from earlier chapters are not considered in this chapter.

We derive following results from this model. First, we prove the existence of the unique steady-state equilibrium growth path in the market economy when the formal and the informal sectors exist side by side. Secondly, the long-run growth rate maximizing income tax rate is dependent upon the emission-output coefficient of the formal sector only; and this result is independent of whether two sectors have identical production technologies or not. The emission-output coefficient of the informal sector does not affect this income tax rate. Thirdly, if there is identical production technologies in the two sectors, the growth rate maximizing abatement expenditure rate and the growth rate maximizing ratio of productive public expenditure to formal sector's output depend not only on the emission-output coefficient of the formal sector but also on that of the informal sector. Fourthly, with identical production technologies in these two sectors, the decentralized steady-state growth equilibrium appears to be saddle-point stable. This result is different from that of Loayza (1996) model that does not show any transitional dynamic property. Lastly, the growth rate maximizing relative size of the informal sector in the steady-state growth equilibrium of the competitive economy exceeds its socially efficient size when this sector pollutes the environment.

Following sections are organized as follows. Section 6.2 describes the basic competitive equilibrium model. Section 6.3 shows the existence of unique steady-state growth equilibrium and analyzes the properties of the long-run growth rate maximizing fiscal policies. Stability property of the decentralized steady-state equilibrium is analyzed in section 6.4. Section 6.5 deals with the properties of steady-state equilibrium in the planned economy.

6.2 THE MODEL

There are two production sectors - formal and informal. Both sectors use private capital, public infrastructure and environmental quality as inputs. The benevolent government imposes a proportional tax on the income of the representative household earned only from the formal sector. The representative household does not pay any tax on her income earned from the informal sector. No penalty[72] is imposed by the government even if tax evasion is detected. The instantaneous utility of the individual is derived from consumption of two goods. The individual also allocates private physical capital between the formal sector and the informal sector while maximizing utility.

We assume a small open economy. We consider trade in final goods only but do not consider international capital mobility. Private physical capital is perfectly immobile between countries. Thus the interest rate is determined internally by the mobility of private capital between the informal sector and the formal sector. The relative price of the formal good in terms of the informal good is exogenous because we assume a small open economy. It is normalized to unity. However, one can explicitly model the determination of the relative price in a closed economy at the cost of additional complications because change in relative price will produce additional effects. A two sector open economy with one traded good sector and one non-traded good sector functions like a closed economy.

Let the subscripts F and I stand for the formal sector and the informal sector respectively. Following equations describe the model.

$$Y_F = (\lambda K)^{\alpha} G^{\eta} E^{1-\alpha-\eta} \; with \; 0 < \alpha, \eta < 1; \quad \text{……(6.1)}$$

$$Y_I = \{(1-\lambda)K\}^{\psi} G^{\beta} E^{1-\psi-\beta} \; with \; 0 < \psi, \beta < 1; \quad \text{……(6.2)}$$

$$\dot{K} = (1-\tau)Y_F + Y_I - C_F - C_I; \quad \text{……(6.3)}$$

[72]Introduction of an exogenous penalty rate does not affect the results of this model if the revenue earned from penalty is not used to finance the public expenditure. However, even in this case, results may be different when the penalty rate is made endogenous. Loayza (1996) introduces endogenous penalty rate.

$G = (\tau - T)Y_F;$ ... ... (6.4)

$\dot{E} = (T - \delta_F)Y_F - \delta_I Y_I \ with\ 0 < \delta_F, \delta_I < 1;$ ... ... (6.5)

$u(C_F, C_I) = \frac{\left(C_I^{\theta} C_F^{1-\theta}\right)^{1-\sigma}}{1-\sigma} \ with\ 0 < \theta, \sigma < 1;$ ... ... (6.6)

Equation (6.1) describes the Cobb-Douglas production function in the formal sector which satisfies constant returns to scale in terms of private capital, public capital and environmental quality. Y_F is the level of output produced in the formal sector. K and E are stocks of private capital and environmental quality respectively. G is non-rival flow of public productive input. λ is the fraction of private capital allocated to the formal sector. Elasticities of output with respect to private capital, public capital and environmental quality are denoted by α, η and $(1 - \alpha - \eta)$ respectively.

Equation (6.2) describes the Cobb-Douglas production function in the informal sector. $(1 - \lambda)$ is the fraction of private capital allocated to the informal sector. Elasticities of output of this sector with respect to private capital, public capital and environmental quality are denoted by ψ , β and $(1 - \psi - \beta)$ respectively.

The budget constraint of the representative household is given by equation (6.3). We do not consider depreciation of private capital. Government taxes income of the formal sector only and this proportional income tax rate is denoted by τ. The representative household's income from the informal sector is not taxed. Here C_F and C_I represent the levels of consumption of the formal good and of the informal good respectively; $C_F + C_I$ is total consumption expenditure and $Y_F + Y_I$ is the value of total production of both the sectors[73].

Equation (6.4) describes government's budget constraint. The government finances public infrastructure expenditure and abatement expenditure from the formal sector income tax revenue. T is the abatement expenditure rate defined as the ratio of abatement expenditure to formal sector's output.

[73]We consider a small open economy with the terms-of-trade being assumed to be equal to unity for simplicity of technical analysis.

Equation (6.5) shows how environmental quality changes over time depending upon the magnitudes of emission and abatement activity. Here emission is assumed to be a flow variable and each of the two sectors generates emission as a by-product of its production. Emission level is proportional to the level of production in each of the two sectors; and δ_F and δ_I are the constant emission-output coefficients in the formal sector and in the informal sector respectively. TY_F is the total abatement expenditure made by the government in this model, which is proportional to formal sector output.

Instantaneous utility of the representative consumer being a positive and concave function of the consumption level of each of the two goods is given by equation (6.6). $\{\theta(1-\sigma)-1\}$ and $\{(1-\theta)(1-\sigma)-1\}$ represent the constant elasticities of marginal utility with respect to C_I and C_F respectively. Here we assume $Max\{\theta(1-\sigma),(1-\theta)(1-\sigma)\}<1$ to ensure diminishing marginal utility of consumption of each of these two goods.

6.3 DYNAMIC EQUILIBRIUM

The representative household maximizes $\int_0^{\infty} u(C_I, C_F)\, e^{-\rho t} dt$ with respect to C_I, C_F and λ subject to equations (6.1), (6.2), (6.3) and (6.6). Optimum capital allocation between the two sectors is given by

$$\lambda^{\frac{\alpha(1-\beta)-(1-\eta)}{1-\eta}}(1-\lambda)^{1-\psi} = \frac{\psi}{\alpha}\frac{(\tau-T)^{\frac{\beta-\eta}{1-\eta}}}{(1-\tau)}\left(\frac{E}{K}\right)^{\frac{\alpha(1-\beta)-(1-\eta)}{1-\eta}}. \qquad \text{... ... (6.7)}$$

If we assume $\alpha=\psi$ and $\beta=\eta$, then equation (6.7) is reduced to

$$\left(\frac{1-\lambda}{\lambda}\right)^{1-\alpha} = \frac{1}{(1-\tau)}. \qquad \text{... ... (6.7.1)}$$

The demand rate of growth[74] of consumption is derived as follows.

$$\frac{\dot{C}_I}{C_I} = \frac{\dot{C}_F}{C_F} = \frac{1}{\sigma}\left[\alpha(1-\tau)(\tau-T)^{\frac{\eta}{1-\eta}}\lambda^{\frac{\alpha+\eta-1}{1-\eta}}\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}} - \rho\right]. \qquad \text{... ... (6.8)}$$

[74]Equations (6.7) and (6.8) are derived in Appendix 6A.

Using equations (6.6) and (6A.3) obtained from Appendix 6A, we have

$$u(C_I, C_F) = \left(\frac{\theta}{1-\theta}\right)^{\theta(1-\sigma)} \left(\frac{C_F^{1-\sigma}}{1-\sigma}\right). \qquad \dots\dots (6.6.1)$$

We consider a steady-state growth equilibrium where all macroeconomic variables grow at the same rate, g_m. Hence, we have

$$\frac{\dot{C_F}}{C_F} = \frac{\dot{C_I}}{C_I} = \frac{\dot{Y_F}}{Y_F} = \frac{\dot{Y_I}}{Y_I} = \frac{\dot{K}}{K} = \frac{\dot{E}}{E} = \frac{\dot{G}}{G} = g_m. \qquad \dots\dots (6.9)$$

6.3.1 Existence of the Steady-State Growth Equilibrium

We now turn to show the existence of unique steady-state equilibrium growth rate in the competitive economy; and so we use equations (6.1), (6.2), (6.3), (6.4), (6.5), (6.8) and (6.9) to obtain the following equations.

$$\frac{\dot{C_I}}{C_I} = \frac{\dot{C_F}}{C_F} = \frac{1}{\sigma}\left[\alpha\lambda^{\frac{\alpha+\eta-1}{1-\eta}}(1-\tau)(\tau-T)^{\frac{\eta}{1-\eta}}\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}} - \rho\right] = g_m; \qquad \dots\dots (6.10)$$

$$\frac{\dot{K}}{K} = \left(1 + \frac{\alpha}{\psi}\frac{1-\lambda}{\lambda}\right)\lambda^{\frac{\alpha}{1-\eta}}(1-\tau)(\tau-T)^{\frac{\eta}{1-\eta}}\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}} - \frac{1}{1-\theta}\frac{C_F}{K} = g_m; \qquad \dots\dots (6.11)$$

and

$$\frac{\dot{E}}{E} = \left[T - \delta_F - \delta_I(1-\tau)\frac{\alpha}{\psi}\frac{1-\lambda}{\lambda}\right]\lambda^{\frac{\alpha}{1-\eta}}(\tau-T)^{\frac{\eta}{1-\eta}}\left(\frac{E}{K}\right)^{-\frac{\alpha}{1-\eta}} = g_m. \qquad \dots\dots (6.12)$$

Using equations (6.7), (6.10), (6.11) and (6.12) we obtain the following equation[75] to solve for g_m.

$$g_m(\sigma g_m + \rho)^{\frac{\alpha}{1-\alpha-\eta}} = \{\alpha(1-\tau)\}^{\frac{\alpha}{1-\alpha-\eta}}(\tau-T)^{\frac{\eta}{1-\alpha-\eta}}\Big[(T-\delta_F)$$
$$-\delta_I\psi^{\frac{\psi}{1-\psi}}\{\alpha(1-\tau)\}^{-\frac{\alpha(1-\psi-\beta)}{(1-\psi)(1-\alpha-\eta)}}(\tau-T)^{\frac{\beta(1-\alpha)-\eta(1-\psi)}{(1-\psi)(1-\alpha-\eta)}}(\rho+\sigma g_m)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{(1-\psi)(1-\alpha-\eta)}}\Big]$$

$$\dots\dots (6.13)$$

The L.H.S. of equation (6.13) is an increasing function of g_m. Its R.H.S. is a decreasing function of g_m if $\alpha(1-\beta) - \psi(1-\eta) > 0$, given the income tax rate,

[75]The derivation of equation (6.13) is worked out in Appendix 6B.

τ, and the abatement expenditure rate, T. However, this R.H.S. is independent of g_m when $\alpha(1-\beta) = \psi(1-\eta)$. Thus the existence of the unique g_m is guaranteed if $\alpha(1-\beta) - \psi(1-\eta) \geq 0$ and $0 < \delta_F < T < \tau$. Equation (6B.4) in Appendix 6B shows that $\left(\frac{E}{K}\right)$ is a function of g_m; and then equation (6B.2) shows that $\left(\frac{C_F}{K}\right)$ is a function of g_m. Equation (6.7) then can be used to show that λ is a function of g_m. Also we must have $0 < \lambda < 1$ because R.H.S. of equation (6.7) is always non zero. An equilibrium with $0 < \lambda < 1$ implies simultaneous existence of the formal sector and the informal sector. We can state the following proposition.

Proposition 6.1: There exists unique steady state growth equilibrium in the competitive economy with coexistence of the formal and the informal sector, given the income tax rate and the abatement expenditure rate, if $\alpha(1-\beta) - \psi(1-\eta) \geq 0$ and if $0 < \delta_F < T < \tau$.

We assume perfect inter-sectoral mobility of private capital along with the assumption of diminishing marginal productivity of capital in each of these two sectors. So we can explain the coexistence of both sectors in equilibrium even without assuming endogenous penalty rate on tax evasion in the informal sector. In Loayza (1996), marginal productivity of capital is constant in both these sectors even though capital is perfectly mobile between the sectors. So the assumption of endogenous rate of penalty on informal sector is necessary in that model to ensure the coexistence of these two sectors.

6.3.2 Optimal Fiscal Policy

We assume that government maximizes the steady-state equilibrium growth rate with respect to the fiscal instruments, τ and T, subject to the steady-state equilibrium equation (6.13). Thus, we obtain following expressions of the optimum tax rate and the optimum abatement expenditure rate[76] as shown in equations below.

$$\tau^\star = 1 - \alpha(1 - \delta_F); \qquad \ldots\ldots(6.14)$$

and

$$\{1 - \alpha(1 - \delta_F) - T^\star\}^{\frac{\eta(1-\psi)-\beta(1-\alpha)}{(1-\psi)(1-\alpha-\eta)}}[(T^\star - \delta_F) - (1 - \alpha - \eta)(1 - \delta_F)]$$

$$= \delta_I \psi^{\frac{\psi}{1-\psi}}\left(\frac{\beta}{1-\psi}\right)\{\alpha^2(1 - \delta_F)\}^{-\frac{\alpha(1-\psi-\beta)}{(1-\psi)(1-\alpha-\eta)}}(\rho + \sigma g_m)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{(1-\psi)(1-\alpha-\eta)}}.$$

$$\ldots\ldots(6.15)$$

We derive equations (6.14) and (6.15) without assuming identical production technologies in the two sectors. Optimum income tax rate is found to be independent of the balanced growth rate, g_m , and of technology parameters of the informal sector, because tax is imposed only on income earned from the formal sector and because capital income as well as labour income are taxed at equal rates. On the contrary, optimum abatement expenditure rate is found to depend on the balanced growth rate, and the emission-output coefficients of each of the two sectors.

However, this rate has a complex expression as given by equation (6.15); and so we assume identical production technologies[77,78] in the two sectors at this stage. This implies that $\alpha = \psi$ and $\eta = \beta$. Then equation (6.15) is reduced to the following.

$$T^\star = \delta_F + (1 - \alpha - \eta)(1 - \delta_F) + \delta_I\{\alpha(1 - \delta_F)\}^{-\frac{\alpha}{(1-\alpha)}}\left(\frac{\eta}{1-\alpha}\right) \qquad \ldots\ldots(6.15.1)$$

Using equations (6.14) and (6.15.1) we obtain

$$\tau^\star - T^\star = \eta(1 - \delta_F) - \delta_I\{\alpha(1 - \delta_F)\}^{-\frac{\alpha}{(1-\alpha)}}\left(\frac{\eta}{1-\alpha}\right). \qquad \ldots\ldots(6.16)$$

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[76]The derivation of equations (6.14) and (6.15) from the first order conditions of maximization is worked out in Appendix 6C. Second order conditions of maximization are also satisfied.

[77]Equations (6.13) and (6.15) simultaneously solve for steady-state equilibrium growth rate and optimum value of the abatement expenditure rate. Analytically it is extremely difficult to show the existence of unique value of T lying in the interval (0, 1). The assumption of identical technologies solves this problem. The importance of distinction between formal and informal sector still remains valid because informal sector income is not taxed even with this simplifying assumption.

[78]An identical production technology in the formal sector and the informal sector is assumed by Loayza (1996), Sarte (2000), etc.

$(\tau^{\star} - T^{\star})$ is the ratio of productive public infrastructural expenditure to formal sector's income that maximizes the steady-state equilibrium growth rate in this model. The R.H.S. of equation (6.16) is the competitive share of public input in the unpolluted output of the formal sector less a constant term. This constant term is the fraction of the formal sector's output allocated in order to nullify the effect of pollution generated by the informal sector. Thus, optimum ratio of productive public infrastructural expenditure to taxable income in this model is lower than the competitive output share of public input in the formal sector. This result is different from those obtained in the models of Barro (1990), FMS (1993), Greiner (2005), etc. This is so because the informal sector uses public input without paying any tax and causes pollution. The abatement expenditure is also financed from the tax revenue obtained from the formal sector; and that is why the optimum abatement expenditure rate varies positively with the emission rate of the informal sector.

The model of Loayza (1996) also shows that the ratio of public infrastructural expenditure to income falls short of the competitive output share of the public input. However, none of the two sectors generates emissions in his model. Development of the informal sector there lowers the efficiency of the public input used by the formal sector through congestion effect.

We now state the following proposition.

Proposition 6.2:

(i) When production technologies in the two sectors are identical, the income tax rate and the abatement expenditure rate obtained as solutions to maximization of growth rate in the steady-state equilibrium are given by

$$\tau^{\star} = 1 - \alpha(1 - \delta_F),$$

and

$$T^{\star} = \delta_F + (1 - \alpha - \eta)(1 - \delta_F) + \delta_I\{\alpha(1 - \delta_F)\}^{-\frac{\alpha}{(1-\alpha)}}\left(\frac{\eta}{1-\alpha}\right).$$

(ii) The growth rate maximizing ratio of productive public infrastructural expenditure to taxable income in the steady-state equilibrium is equal to the competitive share of the public input in the unpolluted output of the formal sector less the share of formal sector's income used to negate the polluting effect of the informal sector; and hence this ratio varies inversely with the magnitude of the emission-output coefficient of the formal sector as well as that of the informal sector.

The presence of differential emission-output coefficients in the two production sectors in the economy and their differential role on government's revenue generation make our result different from those found in the existing literature. If $\delta_I = \delta_F = 0$ then we get back the result identical to that of Barro (1990) and FMS (1993) models.

We use equations (6.13), (6.14) and (6.15.1) and obtain

$$g_m(\sigma g_m + \rho)^{\frac{\alpha}{1-\alpha-\eta}} = (1-\alpha)(1-\alpha-\eta)\{\alpha^2(1-\delta_F)\}^{\frac{\alpha}{1-\alpha-\eta}}$$

$$\left[\eta(1-\delta_F) - \delta_I\{\alpha(1-\delta_F)\}^{-\frac{\alpha}{(1-\alpha)}}\left(\frac{\eta}{1-\alpha}\right)\right]^{\frac{\eta}{1-\alpha-\eta}}$$

$$\left[(1-\alpha)(1-\delta_F) - \delta_I\{\alpha(1-\delta_F)\}^{-\frac{\alpha}{(1-\alpha)}}\right]. \qquad \ldots\ldots(6.13.1)$$

Equation (6.13.1) solves for g_m when growth rate maximizing values of fiscal policy variables are chosen. A positive value of g_m is obtained when the right hand side of equation (6.13.1) is positive; and thus the condition for long-run endogenous growth is given by

$$\frac{(1-\delta_F)^{\frac{1}{(1-\alpha)}}}{\delta_I} > \frac{\alpha^{-\frac{\alpha}{(1-\alpha)}}}{(1-\alpha)}.$$

Given the value of α this condition is likely to be satisfied when δ_F and δ_I take very low values. So the economy may not grow at all in the long-run when pollution rates are high in these two sectors.

Again, using equations (6.14) and (6.7.1) we obtain the inter-sectoral capital allocation ratio which is given by

$$\frac{1-\lambda}{\lambda} = \{\alpha(1-\delta_F)\}^{-\frac{1}{1-\alpha}}. \qquad \ldots\ldots(6.7.2)$$

Equation (6.7.2) clearly shows that this growth rate maximizing inter-sectoral capital allocation ratio in the market economy is independent of the emission-output coefficient of the informal sector. This is so because the income tax rate is independent of the emission-output coefficient of the informal sector and hence the rate of return on capital in either sector is not disturbed by this coefficient.

6.4 STABILITY PROPERTY

We investigate the stability property of the unique decentralized steady-state equilibrium with given values of policy parameters when production technologies in two sectors are identical[79].

We define $x = \frac{C_F}{K}$ and $y = \frac{E}{K}$; and then using equations (6.1), (6.2), (6.3), (6.4), (6.5), (6.7) and (6.8), we have

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right) \lambda^{\frac{\alpha+\eta-1}{1-\eta}} (1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}} y^{\frac{1-\alpha-\eta}{1-\eta}} - \frac{\rho}{\sigma} + \frac{x}{1-\theta}; \quad \ldots\ldots(6.17)$$

and

$$\frac{\dot{y}}{y} = \left[T - \delta_F - \delta_I(1-\tau)\frac{1-\lambda}{\lambda}\right] \lambda^{\frac{\alpha}{1-\eta}} (\tau - T)^{\frac{\eta}{1-\eta}} y^{-\frac{\alpha}{1-\eta}} - \lambda^{\frac{\alpha+\eta-1}{1-\eta}} (1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}} y^{\frac{1-\alpha-\eta}{1-\eta}} + \frac{x}{1-\theta}. \quad \ldots\ldots(6.18)$$

We then express λ in terms of τ using equation (6.7.1). The determinant of the Jacobian matrix[80] corresponding to differential equations given by (6.17) and (6.18) is given by

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[79]Equation (6.7.1) shows that $\frac{\lambda}{1-\lambda}$ is a linear function of $\frac{E}{K}$ when $\alpha = \psi$ and $\eta = \beta$. So $\frac{\dot{\lambda}}{\lambda}$ is linearly dependent on $\frac{\dot{E}}{E}$ and $\frac{\dot{K}}{K}$ and is independent of λ. Hence we replace the expression of λ from equation (6.7.1) in equations (6.10), (6.11) and (6.12). However, if identical production technology is not assumed then λ can only be expressed as an implicit function of $\frac{E}{K}$ as shown in equation (6.7); and $\frac{\dot{\lambda}}{\lambda}$ depends on λ. Our dynamic system is a 3×3 differential system in that case. Hence, for the sake of technical simplicity, we analyze stability property of the steady-state equilibrium assuming identical production technologies in these two sectors.

[80] The derivation of the determinant is worked out in Appendix 6D.

$$|J| = -\frac{1}{1-\theta}\Bigg[\left(\frac{\alpha}{1-\eta}\right)\left\{T - \delta_F - \delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\right\}\left\{(1-\tau)^{-\frac{1}{1-\alpha}} + 1\right\}^{-\frac{\alpha}{1-\eta}}(\tau - T)^{\frac{\eta}{1-\eta}} y^{-\frac{\alpha}{1-\eta}-1}$$

$$+\frac{\alpha}{\sigma}\left(\frac{1-\alpha-\eta}{1-\eta}\right)\left\{(1-\tau)^{-\frac{1}{1-\alpha}} + 1\right\}^{-\frac{\alpha+\eta-1}{1-\eta}}(1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}} y^{\frac{1-\alpha-\eta}{1-\eta}-1}\Bigg].$$

Here $0 < \delta_F, \delta_I < T < \tau < 1$ and $T > \delta_F + \delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}$ when values of τ and T are chosen maximizing the growth rate in the steady-state equilibrium. So $|J| < 0$ in this case; and hence we can state the following proposition.

Proposition 6.3: If production technologies of the two sectors are identical, the unique steady-state equilibrium is saddle-point stable with a unique saddle path converging to that equilibrium point when fiscal instruments are chosen to maximize the steady-state growth rate.

In Loayza (1996) model, there exists no transitional dynamic property because it behaves like an AK model similar to Barro (1990) model with a flow public expenditure. Our model also assumes flow public expenditure. However, we protect this model from being trapped into the AK model by assuming environmental quality to be an accumulable input; and obtain saddle-point stability property of the long run equilibrium even with a flow public expenditure. FMS (1993) brings back transitional dynamic properties in Barro (1990) model introducing durable public input. Greiner (2005) model exhibits transitional dynamic properties treating environmental pollution as a flow variable but treating public input expenditure as a stock variable.

However, this exercise is not correct when policies are chosen by maximizing the household's intertemporal welfare function subject to the decentralized equilibrium conditions. In this case the policies may be state-dependent and hence may affect dynamic stability.

6.5 THE PROBLEM OF THE SOCIAL PLANNER

The social planner can internalize externalities arising from public infrastructure and environmental quality.

This socially efficient growth rate[81] denoted by g_c and capital allocation between the two sectors in the steady-state equilibrium are derived from the social planner's optimization problem and are given by the following equations.

$$(\rho + \sigma g_c)^{\beta-\eta} = \frac{[\alpha(1-\delta_F)]^{\beta}}{\{\psi(1-\delta_I)\}^{\eta}} \left(\frac{1-\lambda}{\lambda}\right)^{\eta(1-\psi)}; \qquad \ldots\ldots(6.19)$$

and

$$(\rho + \sigma g_c)^{\beta-\eta} = \left[(1-\alpha-\eta) + \frac{\alpha}{\psi}(1-\beta-\psi)\left(\frac{1-\lambda}{\lambda}\right)\right]^{\beta-\eta} (1-\delta_F)^{\frac{\psi-\alpha}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}$$

$$\left[\frac{\alpha}{\psi}\left(\frac{1-\delta_F}{1-\delta_I}\right)\right]^{\frac{\alpha}{\{\alpha(\beta-1)+\psi(1-\eta)\}}} \left[\eta + \beta\frac{\alpha}{\psi}\left(\frac{1-\lambda}{\lambda}\right)\right]^{\frac{\eta\psi-\alpha\beta}{\{\alpha(\beta-1)+\psi(1-\eta)\}}} \left(\frac{1-\lambda}{\lambda}\right)^{\frac{\alpha(1-\psi)}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}.$$

$$\ldots\ldots(6.20)$$

Equations (6.19) and (6.20) solve for g_c and λ; and unique steady-state equilibrium may exist[82].

If two sectors have identical production technologies, i.e., if $\alpha = \psi$ and $\eta = \beta$, then from equation (6.19), we have

$$\frac{1-\lambda}{\lambda} = \left(\frac{1-\delta_I}{1-\delta_F}\right)^{\frac{1}{1-\alpha}}. \qquad \ldots\ldots(6.19.1)$$

Equation (6.19.1) shows socially efficient capital allocation between two sectors. Equations (6.7.1) and (6.19.1) are identical if $\delta_I = 0$; and equation (6.19.1) solves for a higher value of λ when $\delta_I > 0$. This leads to the following proposition.

Proposition 6.4: In the presence (absence) of environmental pollution generated by the informal sector, the relative size of the formal sector in the

[81] Working of the planned economy and derivations of equations (6.19) and (6.20) are described in the Appendix 6E.

[82] The conditions for this existence are also derived in Appendix 6E.

competitive economy in the decentralized steady-state equilibrium falls short of (is equal to) its socially efficient size when the two sectors have identical production technologies.

However, the socially efficient growth rate, g_c, is indeterminate in this case because

$$(\rho + \sigma g_c)^{\beta-\eta} = 1 \text{ for } \eta = \beta.$$

Loayza (1996) neither compares the competitive economy solution to the socially efficient solution nor considers pollution caused by the informal sector. However, informal sector activities generate environmental pollution in reality; and it is well known that fiscal policies are not effectively designed to control informal sector activities and to internalize its negative externalities. This present exercise indicates the importance of appropriate environmental policies to control pollution generated by informal sector and of removing barriers to formalization in order to prevent expansion of informal sector.

APPENDIX 6A

DERIVATION OF EQUATION (6.7) IN SECTION 6.3

The dynamic optimization problem of the representative household is to maximize $\int_0^\infty e^{-\rho t}\frac{\left(C_I^\theta C_F^{1-\theta}\right)^{1-\sigma}}{1-\sigma}dt$ with respect to C_F, C_I and λ subject to equation (6.3) and given $K(0)$. Here C_F, C_I and λ are the control variables satisfying $0 \leq C_F + C_I \leq (1-\tau)Y_F + Y_I$ and $0 \leq \lambda \leq 1$; and K is the state variable.

The Hamiltonian to be maximized at each point of time is given by

$$\mathcal{H} = e^{-\rho t}\frac{\left(C_I^\theta C_F^{1-\theta}\right)^{1-\sigma}}{1-\sigma} + e^{-\rho t}\mu_K[(1-\tau)Y_F + Y_I - C_F - C_I].$$

Here μ_K is the co-state variable representing the shadow price of investment. Maximizing the Hamiltonian with respect to C_F and C_I and assuming an interior solution, we obtain

$$\theta C_I^{\theta(1-\sigma)-1} C_F^{(1-\theta)(1-\sigma)} = \mu_K; \qquad \ldots\ldots(6A.1)$$

and

$$(1-\theta) C_I^{\theta(1-\sigma)} C_F^{(1-\theta)(1-\sigma)-1} = \mu_K. \qquad \ldots\ldots(6.A2)$$

Using equations (6A.1) and (6A.2) we have

$$\frac{\theta}{1-\theta}\frac{C_F}{C_I} = 1,$$

or,

$$C_I = \frac{\theta}{1-\theta} C_F. \qquad \ldots\ldots(6A.3)$$

Now using equations (6A.2) and (6A.3) we obtain the following optimality condition.

$$(1-\theta)\left(\frac{\theta}{1-\theta}\right)^{\theta(1-\sigma)} C_F^{-\sigma} = (1-\theta)\left(\frac{1-\theta}{\theta}\right)^{(1-\theta)(1-\sigma)-1} C_I^{-\sigma} = \mu_K. \qquad \ldots\ldots(6A.4)$$

Also maximizing the Hamiltonian with respect to λ and assuming an interior solution, we obtain

$$(1-\tau)\alpha\frac{Y_F}{\lambda} - \psi\frac{Y_I}{1-\lambda} = 0, \qquad \ldots\ldots(6A.5)$$

Using equations (6.1), (6.2), (6.4) and (6A.5), we obtain

$$\lambda^{\frac{\alpha(1-\beta)-(1-\eta)}{1-\eta}}(1-\lambda)^{1-\psi} = \frac{\psi}{\alpha}\frac{(\tau-T)^{\frac{\beta-\eta}{1-\eta}}}{(1-\tau)}\left(\frac{E}{K}\right)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{1-\eta}}. \qquad \ldots\ldots(6A.6)$$

Also the optimum time path of λ_K satisfies the following.

$$\frac{\dot{\mu_K}}{\mu_K} = \rho - (1-\tau)\alpha\frac{Y_F}{K} - \psi\frac{Y_I}{K}. \qquad \ldots\ldots(6A.7)$$

Using equations (6A.5) and (6A.7) we have

$$\frac{\dot{\mu_K}}{\mu_K} = \rho - \left(1 + \frac{1-\lambda}{\lambda}\right)(1-\tau)\alpha\frac{Y_F}{K}. \qquad \ldots\ldots(6A.8)$$

Now, using equations (6.1), (6.4) and (6A.8), we have

$$\frac{\dot{\mu_K}}{\mu_K} = \rho - \alpha(1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}}\lambda^{\frac{\alpha+\eta-1}{1-\eta}}\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}}. \qquad \dots\dots(6A.9)$$

Using the two optimality conditions (6A.4) and (6A.9), we have

$$\frac{\dot{C_F}}{C_F} = \frac{\dot{C_I}}{C_I} = \frac{1}{\sigma}\left[\alpha(1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}}\lambda^{\frac{\alpha+\eta-1}{1-\eta}}\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}} - \rho\right]. \qquad \dots\dots(6A.10)$$

This equation (6A.10) is same as equation (6.7).

APPENDIX 6B

DERIVATION OF EQUATION (6.13) IN SECTION 6.3.1

Using equations (6.1) to (6.5), (6.7) and (6.8) we have the following equations.

$$\frac{\dot{C_I}}{C_I} = \frac{\dot{C_F}}{C_F} = \frac{1}{\sigma}\left[\alpha\lambda^{\frac{\alpha+\eta-1}{1-\eta}}(1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}}\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}} - \rho\right] = g_m; \qquad \dots\dots(6B.1)$$

$$\frac{\dot{K}}{K} = \left(1 + \frac{\alpha}{\psi}\frac{1-\lambda}{\lambda}\right)\lambda^{\frac{\alpha}{1-\eta}}(1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}}\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}} - \frac{1}{1-\theta}\frac{C_F}{K} = g_m; \qquad \dots\dots(6B.2)$$

and

$$\frac{\dot{E}}{E} = \left[T - \delta_F - \delta_I(1-\tau)\frac{\alpha}{\psi}\frac{1-\lambda}{\lambda}\right]\lambda^{\frac{\alpha}{1-\eta}}(\tau - T)^{\frac{\eta}{1-\eta}}\left(\frac{E}{K}\right)^{-\frac{\alpha}{1-\eta}} = g_m. \qquad \dots\dots(6B.3)$$

From equation (6B.1) we have

$$\left(\frac{E}{K}\right)^{\frac{1-\alpha-\eta}{1-\eta}} = \lambda^{\frac{1-\alpha-\eta}{1-\eta}}(\tau - T)^{-\frac{\eta}{1-\eta}}\frac{(\sigma g_m + \rho)}{\alpha(1-\tau)}. \qquad \dots\dots(6B.4)$$

Using equations (6B.3) and (6B.4) we obtain the following equation.

$$g_m = \left[T - \delta_F - \delta_I(1-\tau)\frac{\alpha}{\psi}\frac{(1-\lambda)}{\lambda}\right]\lambda^{\frac{\alpha}{1-\eta}}(\tau - T)^{\frac{\eta}{1-\eta}}\left[\lambda^{\frac{1-\alpha-\eta}{1-\eta}}(\tau - T)^{\frac{\eta}{1-\eta}}\frac{(\sigma g_m + \rho)}{\alpha(1-\tau)}\right]^{-\frac{\alpha}{1-\alpha-\eta}},$$

or,

$$g_m(\sigma g_m + \rho)^{\frac{\alpha}{1-\alpha-\eta}} = \left[T - \delta_F - \delta_I(1-\tau)\frac{\alpha}{\psi}\frac{(1-\lambda)}{\lambda}\right]\{\alpha(1-\tau)\}^{\frac{\alpha}{1-\alpha-\eta}}(\tau - T)^{\frac{\eta}{1-\alpha-\eta}}.$$

$$\dots\dots(6B.5)$$

Using equations (6A.6) and (6B.4) we obtain the following equation.

$$\left(\frac{1-\lambda}{\lambda}\right)^{1-\psi} = \psi\{\alpha(1-\tau)\}^{\frac{\alpha\beta-(1-\eta)(1-\psi)}{1-\alpha-\eta}}(\tau-T)^{\frac{\beta(1-\alpha)-\eta(1-\psi)}{1-\alpha-\eta}}(\sigma g_m+\rho)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{1-\alpha-\eta}}. \qquad \dots\dots(6B.6)$$

or,

$$\lambda = \frac{1}{\Delta^{\frac{1}{1-\psi}}+1},$$

where

$$\Delta = \psi\{\alpha(1-\tau)\}^{\frac{\alpha\beta-(1-\eta)(1-\psi)}{1-\alpha-\eta}}(\tau-T)^{\frac{\beta(1-\alpha)-\eta(1-\psi)}{1-\alpha-\eta}}(\sigma g_m+\rho)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{1-\alpha-\eta}}. \qquad \dots\dots(6B.7)$$

Now we use equation (6B.6) to substitute the value of $\left(\frac{1-\lambda}{\lambda}\right)$ in equation (6B.5) to obtain the following equation.

$$g_m(\sigma g_m+\rho)^{\frac{\alpha}{1-\alpha-\eta}} = \{\alpha(1-\tau)\}^{\frac{\alpha}{1-\alpha-\eta}}(\tau-T)^{\frac{\eta}{1-\alpha-\eta}}\Big[(T-\delta_F)$$
$$-\delta_I\psi^{\frac{\psi}{1-\psi}}\{\alpha(1-\tau)\}^{-\frac{\alpha(1-\psi-\beta)}{(1-\psi)(1-\alpha-\eta)}}(\tau-T)^{\frac{\beta(1-\alpha)-\eta(1-\psi)}{(1-\psi)(1-\alpha-\eta)}}(\rho+\sigma g_m)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{(1-\psi)(1-\alpha-\eta)}}\Big]. \qquad \dots\dots(6B.8)$$

This equation (6B.8) is same as equation (6.13).

We use equations (6B.2), (6B.4) and (6B.7) to obtain the following values of $\frac{E}{K}$ and $\frac{C_F}{K}$ at the steady-state equilibrium.

$$\frac{E}{K} = \left[\left(\Delta^{\frac{1}{1-\psi}}+1\right)^{-\frac{1-\alpha-\eta}{1-\eta}}(\tau-T)^{-\frac{\eta}{1-\eta}}\frac{(\sigma g_m+\rho)}{\alpha(1-\tau)}\right]^{\frac{1-\eta}{1-\alpha-\eta}}. \qquad \dots\dots(6B.9)$$

and

$$\frac{C_F}{K} = (1-\theta)\left[\frac{\left(1+\frac{\alpha}{\psi}\Delta^{\frac{1}{1-\psi}}\right)}{\left(\Delta^{\frac{1}{1-\psi}}+1\right)}\frac{(\sigma g_m+\rho)}{\alpha} - g_m\right]. \qquad \dots\dots(6B.10)$$

From equations (6A.3) and (6B.10) we obtain

$$\frac{C_I}{K} = \theta\left[\frac{\left(1+\frac{\alpha}{\psi}\Delta^{\frac{1}{1-\psi}}\right)}{\left(\Delta^{\frac{1}{1-\psi}}+1\right)}\frac{(\sigma g_m+\rho)}{\alpha} - g_m\right]. \qquad \dots\dots(6B.11)$$

APPENDIX 6C

DERIVATION OF EQUATIONS (6.14), (6.15) AND (6.15.1) IN SECTION 6.3.2 AND SECOND ORDER CONDITIONS

We denote the L.H.S., the first term $(T-\delta_F)\{\alpha(1-\tau)\}^{\frac{\alpha}{1-\alpha-\eta}}(\tau-T)^{\frac{\eta}{1-\alpha-\eta}}$ and the second term $\delta_I\psi^{\frac{\psi}{1-\psi}}\{\alpha(1-\tau)\}^{\frac{\alpha\beta}{(1-\psi)(1-\alpha-\eta)}}(\tau-T)^{\frac{\beta(1-\alpha)}{(1-\psi)(1-\alpha-\eta)}}(\rho+\sigma g_m)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{(1-\psi)(1-\alpha-\eta)}}$ in the R.H.S. of equation (6.13) by L.H.S.$_{(6.13)}$, R.H.S.$^{1}_{(6.13)}$ and R.H.S.$^{2}_{(6.13)}$ respectively. Maximizing the R.H.S. of equation (6.13) with respect to τ, we obtain the following first order condition.

$$\text{R.H.S.}^1_{(6.13)}\left[\frac{\eta}{1-\alpha-\eta}(\tau-T)^{-1}-\frac{\alpha}{1-\alpha-\eta}(1-\tau)^{-1}\right]$$

$$=\text{R.H.S.}^2_{(6.13)}\left[\frac{\beta(1-\alpha)}{(1-\psi)(1-\alpha-\eta)}(\tau-T)^{-1}-\frac{\alpha\beta}{(1-\psi)(1-\alpha-\eta)}(1-\tau)^{-1}\right]. \quad \ldots\ldots(6C.1)$$

Maximizing the R.H.S. of equation (6.13) with respect to T, we obtain the following first order condition.

$$\text{R.H.S.}^1_{(6.13)}\left[(T-\delta_F)^{-1}-\frac{\eta}{1-\alpha-\eta}(\tau-T)^{-1}\right]=-\text{R.H.S.}^2_{(6.13)}\left[\frac{\beta(1-\alpha)}{(1-\psi)(1-\alpha-\eta)}(\tau-T)^{-1}\right].$$

$$\ldots\ldots(6C.2)$$

Using equations (6C.1) and (6C.2) we get the following equation.

$$\eta(1-\alpha)(1-\tau)(T-\delta_F)-\alpha(1-\alpha)(\tau-T)(T-\delta_F)$$

$$=[\alpha(\tau-T)-(1-\alpha)(1-\tau)][(1-\alpha-\eta)(\tau-T)-\eta(T-\delta_F)],$$

or,

$$\tau=\tau^{\star}=1-\alpha(1-\delta_F). \quad \ldots\ldots(6C.3)$$

This is same as equation (6.14) in section 6.3.2.

Using equations (6C.2) and (6C.3) we obtain

$$\{\alpha^2(1-\delta_F)\}^{\frac{\alpha}{1-\alpha-\eta}}(T-\delta_F)\{1-\alpha(1-\delta_F)-T\}^{\frac{\eta}{1-\alpha-\eta}}$$

$$[\eta(T-\delta_F)-(1-\alpha-\eta)\{1-\alpha(1-\delta_F)-T\}]=\delta_I\psi^{\frac{\psi}{1-\psi}}\frac{\beta(1-\alpha)}{(1-\psi)}$$

$$\{\alpha^2(1-\delta_F)\}^{\frac{\alpha\beta}{(1-\psi)(1-\alpha-\eta)}}(T-\delta_F)\{1-\alpha(1-\delta_F)-T\}^{\frac{\beta(1-\alpha)}{(1-\psi)(1-\alpha-\eta)}}(\rho+\sigma g_m)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{(1-\psi)(1-\alpha-\eta)}},$$

or,

$$\{1-\alpha(1-\delta_F)-T\}^{\frac{\eta(1-\psi)-\beta(1-\alpha)}{(1-\psi)(1-\alpha-\eta)}}[T-\delta_F-(1-\alpha-\eta)(1-\delta_F)]$$

$$=\delta_I\psi^{\frac{\psi}{1-\psi}}\left(\frac{\beta}{1-\psi}\right)\{\alpha^2(1-\delta_F)\}^{-\frac{\alpha(1-\psi-\beta)}{(1-\psi)(1-\alpha-\eta)}}(\rho+\sigma g_m)^{\frac{\alpha(1-\beta)-\psi(1-\eta)}{(1-\psi)(1-\alpha-\eta)}}. \quad \ldots\ldots(6C.4)$$

This is same as equation (6.15) in section 6.3.2. Now if we assume identical production technologies in the two sectors then $\alpha=\psi$ and $\eta=\beta$. Using this assumption in (6C.4) we obtain

$$[T-\delta_F-(1-\alpha-\eta)(1-\delta_F)]=\delta_I\alpha^{-\frac{\alpha}{1-\alpha}}\left(\frac{\eta}{1-\alpha}\right)(1-\delta_F)^{-\frac{\alpha}{(1-\alpha)}},$$

or,

$$T=T^{\star}=\delta_F+(1-\alpha-\eta)(1-\delta_F)+\delta_I(1-\delta_F)^{-\frac{\alpha}{(1-\alpha)}}\alpha^{-\frac{\alpha}{1-\alpha}}\left(\frac{\eta}{1-\alpha}\right). \quad \ldots\ldots(6C.5)$$

This equation (6C.5) is same as equation (6.15.1).

Using equations (6A.3), (6C.3) and (6C.5) in equations (6B.9), (6B.10) and (6B.11) and assuming $\alpha=\psi$ and $\eta=\beta$ we obtain the following.

$$\frac{E}{K}=\left[\left\{\left(\alpha\overline{1-\delta_F}\right)^{-\frac{1}{1-\alpha}}+1\right\}^{-\frac{1-\alpha-\eta}{1-\eta}}\left\{\eta(1-\delta_F)-\delta_I\left(\alpha\overline{1-\delta_F}\right)^{-\frac{\alpha}{(1-\alpha)}}\left(\frac{\eta}{1-\alpha}\right)\right\}^{-\frac{\eta}{1-\eta}}\right.$$

$$\left.\frac{(\sigma g_m+\rho)}{\alpha^2(1-\delta_F)}\right]^{\frac{1-\eta}{1-\alpha-\eta}}; \quad \ldots\ldots(6C.6)$$

$$\frac{C_F}{K}=(1-\theta)\left[\frac{(\sigma g_m+\rho)}{\alpha}-g_m\right]; \quad \ldots\ldots(6C.7)$$

and

$$\frac{C_I}{K}=\theta\left[\frac{(\sigma g_m+\rho)}{\alpha}-g_m\right]. \quad \ldots\ldots(6C.8)$$

We assume $\alpha=\psi$ and $\eta=\beta$ in order to derive the second order conditions for growth rate maximization. Twice differentiating equation (6.13) with respect to τ and T respectively we obtain following two equations.

$$\left[-g_m^{-2}-\frac{\alpha\sigma^2}{1-\alpha-\eta}(\sigma g_m+\rho)^{-2}\right]\frac{\partial g_m}{\partial\tau}+\left[g_m^{-1}+\frac{\alpha\sigma}{1-\alpha-\eta}(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial\tau^2}$$

$$= \Big[-\frac{\alpha}{1-\alpha-\eta}(1-\tau)^{-2} - \frac{\eta}{1-\alpha-\eta}(\tau-T)^{-2} - \delta_I \frac{\alpha}{1-\alpha}\Big(1+\frac{\alpha}{1-\alpha}\Big)(1-\tau)^{-\frac{\alpha}{1-\alpha}-2}$$

$$\Big\{T-\delta_F-\delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\Big\}^{-1} - \Big\{\delta_I \frac{\alpha}{1-\alpha}(1-\tau)^{-\frac{\alpha}{1-\alpha}-1}\Big\}^2 \Big\{T-\delta_F-\delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\Big\}^{-2}\Big];$$

$$\ldots\ldots(6C.9)$$

and

$$\Big[-g_m^{-2} - \frac{\alpha\sigma^2}{1-\alpha-\eta}(\sigma g_m+\rho)^{-2}\Big]\frac{\partial g_m}{\partial T} + \Big[g_m^{-1} + \frac{\alpha\sigma}{1-\alpha-\eta}(\sigma g_m+\rho)^{-1}\Big]\frac{\partial^2 g_m}{\partial T^2}$$

$$= \Big[-\frac{\eta}{1-\alpha-\eta}(\tau-T)^{-2} - \Big\{T-\delta_F-\delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\Big\}^{-2}\Big]. \qquad \ldots\ldots(6C.10)$$

At the balanced growth rate maximizing equilibrium, $\frac{\partial g_m}{\partial \tau} = \frac{\partial g_m}{\partial T} = 0$. Thus equations (6C.9) and (6C.10) take the following form.

$$\frac{\partial^2 g_m}{\partial \tau^2} = \frac{\begin{matrix}\Big[-\frac{\alpha}{1-\alpha-\eta}(1-\tau)^{-2} - \frac{\eta}{1-\alpha-\eta}(\tau-T)^{-2} \\ -\delta_I\frac{\alpha}{1-\alpha}\Big(1+\frac{\alpha}{1-\alpha}\Big)(1-\tau)^{-\frac{\alpha}{1-\alpha}-2}\Big\{T-\delta_F-\delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\Big\}^{-1} \\ -\Big\{\delta_I\frac{\alpha}{1-\alpha}(1-\tau)^{-\frac{\alpha}{1-\alpha}-1}\Big\}^2\Big\{T-\delta_F-\delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\Big\}^{-2}\Big]\end{matrix}}{\Big[g_m^{-1}+\frac{\alpha\sigma}{1-\alpha-\eta}(\sigma g_m+\rho)^{-1}\Big]};$$

and

$$\frac{\partial^2 g_m}{\partial T^2} = \frac{\Big[-\frac{\eta}{1-\alpha-\eta}(\tau-T)^{-2} - \Big\{T-\delta_F-\delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\Big\}^{-2}\Big]}{\Big[g_m^{-1}+\frac{\alpha\sigma}{1-\alpha-\eta}(\sigma g_m+\rho)^{-1}\Big]}.$$

The denominator of the R.H.S. of each of the two above mentioned equations is positive and the corresponding numerator is negative. Thus the second order derivatives are negative in sign.

APPENDIX 6D

DERIVATION OF THE DETERMINANT IN SECTION 6.4

We define the following variables.

$$M = \lambda^{\frac{\alpha+\eta-1}{1-\eta}}(1-\tau)(\tau-T)^{\frac{\eta}{1-\eta}}y^{\frac{1-\alpha-\eta}{1-\eta}}; \quad \ldots\ldots(6\text{D}.1)$$

and

$$N = \left[T - \delta_{\text{F}} - \delta_I(1-\tau)\frac{1-\lambda}{\lambda}\right]\lambda^{\frac{\alpha}{1-\eta}}(\tau-T)^{\frac{\eta}{1-\eta}}y^{-\frac{\alpha}{1-\eta}}. \quad \ldots\ldots(6\text{D}.2)$$

Using $\alpha = \psi$ and $\eta = \beta$ in equation (6A.6) we obtain

$$\lambda = \left\{1 - (1-\tau)^{-\frac{1}{1-\alpha}}\right\}.$$

Thus, if we assume identical technology λ becomes a function of the income tax rate only.

Thus using equations (6D.1) and (6D.2) we modify equations (6.17) and (6.18) in section 6.4 as follows.

$$\frac{\dot{x}}{x} = \left(\frac{\alpha}{\sigma} - 1\right)M + \frac{x}{1-\theta} - \frac{\rho}{\sigma}; \quad \ldots\ldots(6D.3)$$

and

$$\frac{\dot{y}}{y} = N - M + \frac{x}{1-\theta}. \quad \ldots\ldots(6D.4)$$

We obtain following partial derivatives from equations (6D.3) and (6D.4).

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial x} = \frac{1}{1-\theta};$$

$$\frac{\partial\left(\frac{\dot{x}}{x}\right)}{\partial y} = \left(\frac{1-\alpha-\eta}{1-\eta}\right)\left(\frac{\alpha}{\sigma} - 1\right)\frac{M}{y};$$

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial x} = \frac{1}{1-\theta};$$

and

$$\frac{\partial\left(\frac{\dot{y}}{y}\right)}{\partial y} = -\left(\frac{\alpha}{1-\eta}\right)\frac{N}{y} - \left(\frac{1-\alpha-\eta}{1-\eta}\right)\frac{M}{y}.$$

So the determinant of the Jacobian matrix can be written as follows.

$$|J| = \frac{1}{1-\theta}\left[-\frac{\alpha}{(1-\eta)}\frac{N}{y} - \left(\frac{1-\alpha-\eta}{1-\eta}\right)\frac{M}{y} - \left(\frac{1-\alpha-\eta}{1-\eta}\right)\left(\frac{\alpha}{\sigma} - 1\right)\frac{M}{y}\right],$$

or,

$$|J| = \frac{1}{1-\theta}\left[-\frac{\alpha}{(1-\eta)}\frac{N}{y} - \left(\frac{1-\alpha-\eta}{1-\eta}\right)\frac{\alpha}{\sigma}\frac{M}{y}\right],$$

or,

$$|J| = -\frac{1}{1-\theta}\left[\frac{\alpha}{(1-\eta)}\left\{T - \delta_F - \delta_I(1-\tau)\frac{1-\lambda}{\lambda}\right\}\lambda^{\frac{\alpha}{1-\eta}}(\tau - T)^{\frac{\eta}{1-\eta}}y^{-\frac{\alpha}{1-\eta}-1}\right.$$

$$\left. + \left(\frac{1-\alpha-\eta}{1-\eta}\right)\frac{\alpha}{\sigma}\lambda^{\frac{\alpha+\eta-1}{1-\eta}}(1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}}y^{\frac{1-\alpha-\eta}{1-\eta}-1}\right],$$

or,

$$|J| = -\frac{1}{1-\theta}\left[\left(\frac{\alpha}{1-\eta}\right)\left\{T - \delta_F - \delta_I(1-\tau)^{-\frac{\alpha}{1-\alpha}}\right\}\left\{(1-\tau)^{-\frac{1}{1-\alpha}} + 1\right\}^{-\frac{\alpha}{1-\eta}}(\tau - T)^{\frac{\eta}{1-\eta}}y^{-\frac{\alpha}{1-\eta}-1}\right.$$

$$\left. + \frac{\alpha}{\sigma}\left(\frac{1-\alpha-\eta}{1-\eta}\right)\left\{(1-\tau)^{-\frac{1}{1-\alpha}} + 1\right\}^{-\frac{\alpha+\eta-1}{1-\eta}}(1-\tau)(\tau - T)^{\frac{\eta}{1-\eta}}y^{\frac{1-\alpha-\eta}{1-\eta}-1}\right].$$

APPENDIX 6E

DERIVATION OF EQUATIONS (6.19) AND (6.20) IN SECTION 6.5

The social planner's problem is to maximize $\int_0^\infty e^{-\rho t}\frac{\left(C_I^\theta C_F^{1-\theta}\right)^{1-\sigma}}{1-\sigma}dt$. Here, equations (6.1), (6.2) and (6.6) remain unchanged. Equations (6.3), (6.4) and (6.5) are modified as follows.

$$\dot{K} = Y_F + Y_I - \Pi - C_F - C_I; \qquad \text{... ... (6E.1)}$$

$$G = \Pi - \Omega; \qquad \text{... ... (6E.2)}$$

and

$$\dot{E} = \Omega - \delta_F Y_F - \delta_I Y_I. \qquad \text{... ... (6E.3)}$$

Here Π denotes social planner's combined lump sum expenditure on public input and abatement activities; and the abatement expenditure is denoted by Ω.

The relevant Hamiltonian to be maximized by the social planner at each point of time is given by

$$\mathcal{H} = e^{-\rho t}\frac{\left(C_I^{\theta} C_F^{1-\theta}\right)^{1-\sigma}}{1-\sigma} + e^{-\rho t}\mu_K[Y_I + Y_F - \Pi - C_F - C_I] + e^{-\rho t}\mu_E[\Omega - \delta_F Y_F - \delta_I Y_I].$$

The state variables are K and E. The control variables are C_F, C_I, λ, Π and Ω; and μ_K and μ_E are two co-state variables.

Maximizing $\mathcal{H}$ with respect to C_I and C_F, we obtain equations identical to (6A.1) and (6A.2). Now maximizing $\mathcal{H}$ with respect to λ, Π and Ω we have

$$\left(\frac{\mu_K}{\mu_E} - \delta_F\right)\eta\left(\frac{Y_F}{\Pi-\Omega}\right) + \left(\frac{\mu_K}{\mu_E} - \delta_I\right)\beta\left(\frac{Y_I}{\Pi-\Omega}\right) = \frac{\mu_K}{\mu_E}; \qquad \dots\dots(6E.4)$$

$$\left(\frac{\mu_K}{\mu_E} - \delta_F\right)\eta\left(\frac{Y_F}{\Pi-\Omega}\right) + \left(\frac{\mu_K}{\mu_E} - \delta_I\right)\beta\left(\frac{Y_I}{\Pi-\Omega}\right) = 1; \qquad \dots\dots(6E.5)$$

and

$$\left(\frac{\mu_K}{\mu_E} - \delta_F\right)\alpha\frac{Y_F}{\lambda} = \left(\frac{\mu_K}{\mu_E} - \delta_I\right)\psi\frac{Y_I}{1-\lambda}. \qquad \dots\dots(6E.6)$$

Using equations (6E.4) and (6E.5) we find that

$$\frac{\mu_K}{\mu_E} = 1. \qquad \dots\dots(6E.7)$$

Using equations (6E.6) and (6E.7) we obtain the following.

$$(1-\delta_I)Y_I = \frac{(1-\lambda)}{\lambda}\frac{\alpha}{\psi}(1-\delta_F)Y_F. \qquad \dots\dots(6E.8)$$

Also, along the optimum path, time behaviour of the co-state variables satisfies the followings.

$$\left(1 - \delta_F\frac{\mu_E}{\mu_K}\right)\alpha\frac{Y_F}{K} + \left(1 - \delta_I\frac{\mu_E}{\mu_K}\right)\psi\frac{Y_I}{K} = \rho - \frac{\dot{\mu_K}}{\mu_K}; \qquad \dots\dots(6E.9)$$

and

$$\left(\frac{\mu_K}{\mu_E} - \delta_F\right)(1-\alpha-\eta)\frac{Y_F}{E} + \left(\frac{\mu_K}{\mu_E} - \delta_I\right)(1-\beta-\psi)\frac{Y_I}{E} = \rho - \frac{\dot{\mu_E}}{\mu_E}. \qquad \dots\dots(6E.10)$$

Using equations (6E.5) and (6E.8) we obtain the following equation.

$$\frac{E}{K} = \left[(1-\delta_F)\left\{\eta + \beta\frac{\alpha}{\psi}\left(\frac{1-\lambda}{\lambda}\right)\right\}\lambda^{\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{\eta-1}\right]^{-\frac{1}{1-\alpha-\eta}}. \qquad \dots\dots(6E.11)$$

From equation (6E.8) we obtain

$$\left(\frac{\Pi-\Omega}{K}\right)^{\beta-\eta} = \frac{\alpha}{\psi}\left(\frac{1-\delta_F}{1-\delta_I}\right)\frac{(1-\lambda)^{1-\psi}}{\lambda^{1-\alpha}}\left(\frac{E}{K}\right)^{\beta+\psi-\alpha-\eta}. \qquad \dots\dots(6E.12)$$

From equations (6A.1) and (6A.2), we have

$$-\sigma\frac{\dot{C}_I}{C_I} = -\sigma\frac{\dot{C}_F}{C_F} = \frac{\dot{\mu_K}}{\mu_K}. \qquad \dots\dots(6E.13)$$

Using equations (6E.7), (6E.8), (6E.9) and (6E.13) we obtain the following.

$$\frac{\dot{C}_F}{C_F} = \frac{\dot{C}_I}{C_I} = \frac{1}{\sigma}\left[\alpha(1-\delta_F)\lambda^{\alpha-1}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{1-\alpha-\eta} - \rho\right]. \qquad \dots\dots(6E.14)$$

Again, using equations (6E.7), (6E.8), (6E.10) and (6E.13) we obtain the following.

$$\frac{\dot{C}_F}{C_F} = \frac{\dot{C}_I}{C_I} = \frac{1}{\sigma}\left[\left\{(1-\alpha-\eta) + \left(\frac{1-\lambda}{\lambda}\right)\frac{\alpha}{\psi}(1-\beta-\psi)\right\}(1-\delta_F)\lambda^{\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{-\alpha-\eta} - \rho\right]$$

$$\dots\dots(6E.15)$$

In the steady state growth equilibrium,

$$\frac{\dot{C}_F}{C_F} = \frac{\dot{C}_I}{C_I} = \frac{1}{\sigma}\left[\alpha(1-\delta_F)\lambda^{\alpha-1}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{1-\alpha-\eta} - \rho\right] = g_c; \qquad \dots\dots(6E.16)$$

$$\frac{\dot{C}_F}{C_F} = \frac{\dot{C}_I}{C_I} = \frac{1}{\sigma}\left[\left\{(1-\alpha-\eta) + \left(\frac{1-\lambda}{\lambda}\right)\frac{\alpha}{\psi}(1-\beta-\psi)\right\}(1-\delta_F)\lambda^{\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{-\alpha-\eta} - \rho\right]$$

$$= g_c; \quad \dots\dots(6E.17)$$

$$\frac{\dot{K}}{K} = \left[1 + \frac{\alpha}{\psi}\left(\frac{1-\delta_F}{1-\delta_I}\right)\left(\frac{1-\lambda}{\lambda}\right)\right]\lambda^{\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{1-\alpha-\eta} - \frac{\Pi}{K} - \frac{C_F}{K} - \frac{C_I}{K} = g_c; \qquad \dots\dots(6E.18)$$

and

$$\frac{\dot{E}}{E} = \frac{\Omega}{E} - \left[\delta_F + \delta_I\frac{\alpha}{\psi}\left(\frac{1-\delta_F}{1-\delta_I}\right)\left(\frac{1-\lambda}{\lambda}\right)\right]\lambda^{\alpha}\left(\frac{\Pi-\Omega}{K}\right)^{\eta}\left(\frac{E}{K}\right)^{-\alpha-\eta} = g_c. \qquad \dots\dots(6E.19)$$

Assuming $\beta(1-\alpha) = \eta(1-\psi)$ and using equations (6E.11), (6E.12) and (6E.16) we obtain the following equation

$$(\rho + \sigma g_c)^{\beta-\eta} = \frac{[\alpha(1-\delta_F)]^{\beta}}{\{\psi(1-\delta_I)\}^{\eta}}\left(\frac{1-\lambda}{\lambda}\right)^{\eta(1-\psi)}. \qquad \dots\dots(6E.20)$$

Again, using equations (6E.11), (6E.12) and (6E.15) we arrive at the following equation.

$$(\rho + \sigma g_c)^{\beta-\eta} = \left[(1-\alpha-\eta) + \frac{\alpha}{\psi}(1-\beta-\psi)\left(\frac{1-\lambda}{\lambda}\right)\right]^{\beta-\eta}(1-\delta_F)^{\overline{\{\alpha(\beta-1)+\psi(1-\eta)\}}^{\,\psi-\alpha}}$$

$$\left[\frac{\alpha}{\psi}\left(\frac{1-\delta_F}{1-\delta_I}\right)\right]^{\frac{\alpha}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\left[\eta+\beta\frac{\alpha}{\psi}\left(\frac{1-\lambda}{\lambda}\right)\right]^{\frac{\eta\psi-\alpha\beta}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\left(\frac{1-\lambda}{\lambda}\right)^{\frac{\alpha(1-\psi)}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}.$$

$$\dots\dots(6E.21)$$

Equations (6E.20) and (6E.21) are same as equations (6.22) and (6.23) in section 6.5.

We derive the Jacobian determinant of equations (6E.20) and (6E.21) where g_c and $\left(\frac{1-\lambda}{\lambda}\right)$ are the two variables and arrive at the following.

$$|J|=\sigma\left[(1-\delta_F)^{\frac{\psi-\alpha}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\left[\frac{\alpha}{\psi}\left(\frac{1-\delta_F}{1-\delta_I}\right)\right]^{\frac{\alpha}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\right]^{\frac{1}{\beta-\eta}}\left[\eta+\beta\frac{\alpha}{\psi}\left(\frac{1-\lambda}{\lambda}\right)\right]^{\frac{\eta\psi-\alpha\beta}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}$$

$$\left(\frac{1-\lambda}{\lambda}\right)^{\frac{\alpha(1-\psi)}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\left[\frac{\alpha}{\psi}(1-\beta-\psi)+\frac{\left\{(1-\alpha-\eta)+\frac{\alpha}{\psi}(1-\beta-\psi)\left(\frac{1-\lambda}{\lambda}\right)\right\}}{\{\alpha(\beta-1)+\psi(1-\eta)\}(\beta-\eta)}\left\{\frac{(\eta\psi-\alpha\beta)\beta\frac{\alpha}{\psi}}{\eta+\beta\frac{\alpha}{\psi}\left(\frac{1-\lambda}{\lambda}\right)}+\frac{\alpha(1-\psi)}{\left(\frac{1-\lambda}{\lambda}\right)}\right\}\right]$$

$$-\sigma\frac{[\alpha(1-\delta_F)]^{\beta}}{\{\psi(1-\delta_I)\}^{\eta}}\frac{\eta(1-\psi)}{(\beta-\eta)}\left(\frac{1-\lambda}{\lambda}\right)^{\frac{\eta(1-\psi)}{(\beta-\eta)}-1}.$$

The Jacobian determinant should be non-zero for the existence of a unique solution of g_c and $\left(\frac{1-\lambda}{\lambda}\right)$; and this is true if

$$\left[(1-\delta_F)^{\frac{\psi-\alpha}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\left\{\frac{\alpha}{\psi}\left(\frac{1-\delta_F}{1-\delta_I}\right)\right\}^{\frac{\alpha}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\right]^{\frac{1}{\beta-\eta}}\left[\eta+\beta\frac{\alpha}{\psi}\left(\frac{1-\lambda}{\lambda}\right)\right]^{\frac{\eta\psi-\alpha\beta}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}$$

$$\left(\frac{1-\lambda}{\lambda}\right)^{\frac{\alpha(1-\psi)}{\{\alpha(\beta-1)+\psi(1-\eta)\}}}\left[\frac{\alpha}{\psi}(1-\beta-\psi)+\frac{\left\{(1-\alpha-\eta)+\frac{\alpha}{\psi}(1-\beta-\psi)\left(\frac{1-\lambda}{\lambda}\right)\right\}}{\{\alpha(\beta-1)+\psi(1-\eta)\}(\beta-\eta)}\left\{\frac{(\eta\psi-\alpha\beta)\beta\frac{\alpha}{\psi}}{\eta+\beta\frac{\alpha}{\psi}\left(\frac{1-\lambda}{\lambda}\right)}+\frac{\alpha(1-\psi)}{\left(\frac{1-\lambda}{\lambda}\right)}\right\}\right]$$

$$\neq\left[\frac{[\alpha(1-\delta_F)]^{\beta}}{\{\psi(1-\delta_I)\}^{\eta}}\right]^{\frac{1}{(\beta-\eta)}}\frac{\eta(1-\psi)}{(\beta-\eta)}\left(\frac{1-\lambda}{\lambda}\right)^{\frac{\eta(1-\psi)}{(\beta-\eta)}-1}.$$

CHAPTER 7

7. HUMAN CAPITAL ACCUMULATION AND ENDOGENOUS POLLUTION RATE

7.1 INTRODUCTION

In this chapter we develop two extensions to the basic model introducing human capital accumulation in both cases. The first extension done in section 7.2 focuses on human capital accumulation jointly financed by private expenditure and public expenditure; and human capital here is used as a productive input like that in Lucas (1988), Rebelo (1991), etc. The second extension developed in section 7.3 emphasizes the role of human capital in the improvement of the stock of environmental quality when pollution rate is endogenous.

Models of Lucas (1988), Rebelo (1991), Glomm and Ravikumar (2001), CL (2006), Agenor (2011), etc deal with the role of human capital on economic growth and these contributions are surveyed in chapter 1. However, none of these models deals with the problem of environmental pollution and its interconnections with human capital accumulation. Only HK (2005) considers human capital as a pollution free productive input in a dynamic model developed to explain the features of environmental Kuznet's curve where pollution is a by-product of the final output and where pollution can be reduced only by sacrificing the use of physical capital from production. HK (2005), however, neither analyzes the problem of allocation of tax revenue to human capital accumulation nor analyzes the role of productive public expenditure on economic growth.

The objective of the present chapter is to fill the gap, i.e., to develop two models of endogenous economic growth with congestible public good which would enable us to analyze the properties of optimal fiscal policy in the presence of human capital accumulation and environmental pollution when each affects the other.

7.2 PUBLIC EXPENDITURE ON HUMAN CAPITAL AND ENVIRONMENTAL POLLUTION

This section develops the model where human capital accumulation is financed by private expenditure as well as by public expenditure and where human capital is used as a productive input. However, it does not affect environmental quality. This section is organized as follows. Subsection 7.2.1 describes the model. Subsection 7.2.2 analyzes its dynamic equilibrium properties. Subsection 7.2.2.1 shows the possibility of existence of unique steady-state equilibrium growth path in the market economy; and subsection 7.2.2.2 analyzes the properties of optimal fiscal policy along the steady-state equilibrium growth path.

7.2.1 THE MODEL

The human capital is used as a productive input to produce the final good. Human capital accumulation is financed by private expenditure and public expenditure. Government spends a part of its tax revenue on public education while the representative consumer spends a part of her disposable income on education. The rest of government's tax revenue is spent on public infrastructure investment and on abatement activity. Productive public expenditure is considered to be a flow variable. There is neither any role of health expenditure nor any informal sector in this model.

Following equations describe the model.

$$Y = K^{\alpha} H^{\beta} \hat{G}^{1-\alpha-\beta} with\ 0 < \alpha, \beta < 1; \qquad \dots \dots (7.2.1)$$

$$\hat{G} = G(\bar{K})^{-\theta} E^{\theta} with\ \theta > 0; \qquad \dots \dots (7.2.2)$$

$$\dot{H} = \{v_H Y\}^{\eta} (e_H)^{1-\eta}\ with\ 0 < T < \tau < 1\ and\ 0 < \eta < 1; \qquad \dots \dots (7.2.3)$$

$$\dot{K} = (1-\tau)Y - C - e_H; \qquad \dots \dots (7.2.4)$$

$$\dot{E} = TY - \delta Y\ ; \qquad \dots \dots (4.3)$$

$$G = v_G Y; \qquad \dots \dots (7.2.5)$$

$$v_G + v_H + T = \tau; \qquad \dots \dots (7.2.6)$$

and

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma}\ with\ \sigma > 0. \qquad \dots \dots (2.2.6)$$

Equation (7.2.1) describes the Cobb-Douglas production function of the final good. Y is the level of output produced. K is the stock of physical capital and H is the stock of human capital. Elasticities of output with respect to physical capital and human capital are denoted by α and β respectively. $\hat{G}$ is the effective benefit derived from public input with its output elasticity being $1 - \alpha - \beta$.

Equation (7.2.2) describes the nature of the combined effect of congestion and environment on the effectiveness of public productive input.

Equation (7.2.3) describes the law of motion of human capital. The stock of human capital is augmented by expenditure made by the government on public educational amenities and by private educational spending. $v_H Y$ is the part of tax revenue spent on human capital investment. e_H is expenditure that the representative consumer makes on education to augment human capital. η and $(1-\eta)$ are the appropriate elasticity parameters. CL (2006) and Agenor (2011) also consider public expenditure as an argument in the human capital accumulation function.

Equation (7.2.4) describes the budget constraint of the household who allocates its post tax disposable income between consumption, C, education expenditure, e_H, and savings (investment); and there is no depreciation of physical capital.

Equation (4.3) is taken from chapter 4 and is interpreted similarly.

Equation (7.2.5) shows how public productive input is financed. $v_G Y$ is the part of total tax revenue that is spent on public input. Government budget constraint is described by equation (7.2.6). The left hand side of this equation stands for the sum of income shares on productive public input, public education and abatement while its right hand side represents the income tax rate.

Equation (2.2.6) is borrowed from section 2.2 of chapter 2.

7.2.2 DYNAMIC EQUILIBRIUM

The representative household maximizes $\int_0^\infty u(C)\, e^{-\rho t} dt$ with respect to C subject to equations (7.2.1), (7.2.3), (7.2.4) and (2.2.6). The demand rate of growth of consumption[83] is derived from this maximizing problem as follows.

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)v_G^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]. \qquad \text{... ... (7.2.7)}$$

The steady-state growth equilibrium is defined as

$$\frac{\dot{C}}{C} = \frac{\dot{e_H}}{e_H} = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{H}}{H} = \frac{\dot{E}}{E} = g_m. \qquad \text{... ... (7.2.8)}$$

7.2.2.1 Existence of Steady-State Growth Equilibrium

We use equations (7.2.1) to (7.2.5), (4.3), (7.2.7) and (7.2.8) to obtain the following equations.

$$\frac{1}{\sigma}\left[\alpha(1-\tau)v_G^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right] = g_m\,; \qquad \text{... ... (7.2.9)}$$

[83] The derivation is worked out in Appendix 7.2A.

$$v_H{}^{\eta} v_G{}^{\eta\left(\frac{1-\alpha-\beta}{\alpha+\beta}\right)} \left(\frac{E}{K}\right)^{\eta\left\{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}\right\}} \left(\frac{H}{K}\right)^{\frac{\eta\beta}{\alpha+\beta}-1} \left(\frac{e_H}{K}\right)^{1-\eta} = g_m; \quad \text{……(7.2.10)}$$

$$(1-\tau) v_G{}^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \frac{C}{K} - \frac{e_H}{K} = g_m; \quad \text{……(7.2.11)}$$

and

$$(T-\delta) v_G{}^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}-1} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} = g_m. \quad \text{……(7.2.12)}$$

We solve for g_m using equations (7.2.9), (7.2.10) and (7.2.12) and obtain the following equation.

$$g_m{}^{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}} (\sigma g_m + \rho)^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}} = (\eta\beta)^{\frac{\beta-\eta\beta}{\theta(1-\alpha-\beta)}} \alpha^{\frac{\alpha-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}} v_G{}^{\frac{1}{\theta}} v_H{}^{\frac{\eta\beta}{\theta(1-\alpha-\beta)}} (1-\tau)^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}} (T-\delta). \quad \text{……(7.2.13)}$$

The existence of unique steady-state equilibrium growth rate[84] can be shown using equation (7.2.13) given τ, T, v_G and v_H, under the condition that physical capital is socially productive, i.e., $\alpha - \theta(1-\alpha-\beta) > 0$. Then equations (7.2.9) to (7.2.12) can be used to show that equilibrium values of $\frac{H}{K}$, $\frac{E}{K}$, $\frac{e_H}{K}$ and $\frac{C}{K}$ are also uniquely determined in this case.

We can state the following proposition.

Proposition 7.2.1: There exists unique positive steady-state equilibrium growth rate in the market economy only if the fiscal instruments satisfy $0 < v_G, v_H, T < \tau < 1$ and $\alpha - \theta(1-\alpha-\beta) > 0$.

7.2.2.2 Optimal Taxation

We call taxes and expenditures to be optimal when they ensure growth rate maximization at the steady-state growth equilibrium in a decentralized

[84] The complete derivation is shown in Appendix 7.2B.

economy[85]. The government maximizes the steady-state equilibrium growth rate given by equation (7.2.13) with respect to fiscal instruments, τ, T, v_G and v_H. We obtain following expressions of these four fiscal instruments when they are chosen optimally[86].

$$\tau^{\star} = 1 - (1-\delta)\{\alpha - \theta(1-\alpha-\beta) + \beta(1-\eta)\}\,; \qquad \text{……(7.2.14)}$$

$$T^{\star} = \delta + (1-\delta)\theta(1-\alpha-\beta); \qquad \text{……(7.2.15)}$$

$$v_G{}^{\star} = (1-\delta)(1-\alpha-\beta); \qquad \text{……(7.2.16)}$$

and

$$v_H{}^{\star} = (1-\delta)\eta\beta. \qquad \text{……(7.2.17)}$$

Using equations (7.2.14) and (7.2.15), we have

$$\tau^{\star} - T^{\star} = (1-\delta)\{1-\alpha-\beta(1-\eta)\}. \qquad \text{……(7.2.18)}$$

These are interpreted as follows. To ensure that the growth rate is non-negative deterioration of environmental quality due to pollution is neutralized by allocating δ fraction of the total output to abatement expenditure. The optimum net abatement expenditure rate is then $(T^{\star} - \delta)$; and $(1-\delta)\theta(1-\alpha-\beta)$ is the competitive unpolluted output share of environmental quality. So equation (7.2.15) implies that the optimum net abatement expenditure rate is equal to the competitive share of environmental input in the unpolluted output. Equation (7.2.6) shows that $(\tau^{\star} - T^{\star}) = v_G{}^{\star} + v_H{}^{\star}$ is the optimum ratio of net aggregate public expenditure to national income where net aggregate public expenditure consists of expenditure on productive public input and educational expenditure. Similarly, $(1-\delta)(1-\alpha-\beta+\eta\beta)$ is the sum of the competitive unpolluted output shares of productive public input and public expenditure on human capital. So the ratio of net optimum aggregate expenditure to national income is equal to the aggregate of the competitive shares of each of these public inputs in the unpolluted output. If $\eta = 0$ and $\delta > 0$, then equation (7.2.18) is identical to equation (2.2.15). If $\eta = \delta = 0$, we get back the result

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[85] This tax system may not implement the Ramsey optimum solution in the decentralized economy.

[86] The detailed derivation is worked out in Appendix 7.2 C.

obtained in models of Barro (1990), FMS (1993), etc. If $\eta > 0$, we find, from equation (7.2.18), that $(\tau^\star - T^\star)$ varies positively with η. Here η represents the elasticity of human capital accumulation with respect to public expenditure on education.

We now examine whether growth rate maximizing solution also achieves maximum social welfare in the steady-state equilibrium. Using equations (7.2.9), (7.2.11) and (7.2B.7) and assuming that the economy is on the steady-state equilibrium growth path, it can be shown that

$$C = \frac{1}{\alpha}[\rho - (\alpha + \eta\beta - \sigma)g_m]K(0)e^{g_m t}. \qquad \text{… … (7.2.19)}$$

We do not assume human capital to affect utility of the household. Therefore, the social welfare function is identical to that in the basic model in section 2.2 of chapter 2. It can be shown[87] that

$$W = \alpha^{\sigma-1}\frac{K(0)^{1-\sigma}}{1-\sigma}\left[\frac{\rho-(\alpha+\eta\beta-\sigma)g_m}{\rho-(1-\sigma)g_m}\right][\rho - (\alpha + \eta\beta - \sigma)g_m]^{-\sigma}. \qquad \text{… … (7.2.20)}$$

W, in this extension too, varies positively with g_m. If $\eta = 0$, then equation (7.2.20) is identical to equation (2.2.16) obtained in chapter 2.

Thus the level of social welfare in the steady-state growth equilibrium is maximized whenever the steady-state equilibrium growth rate is maximized[88]. We now can state the following proposition.

Proposition 7.2.2: (i) The optimum income tax rate, the optimum abatement expenditure rate and the optimum rates of public expenditure on productive public input and human capital, in the steady-state growth equilibrium, are given by

$$\tau^\star = 1 - (1-\delta)\{\alpha - \theta(1-\alpha-\beta) + \beta(1-\eta)\};$$

$$T^\star = \delta + (1-\delta)\theta(1-\alpha-\beta);$$

$$v_G{}^\star = (1-\delta)(1-\alpha-\beta);$$

[87] The detailed derivation is worked out in Appendix 7.2D.

[88] As with analysis of the basic model in chapter 2, here too, we abstain from analyzing social welfare maximization along the transitional path.

and

$$v_H{}^{\star} = (1-\delta)\eta\beta.$$

(ii) The optimum ratio of combined net public expenditure on productive public input and on human capital to national income in the steady-state equilibrium is equal to the combined competitive unpolluted output share of public input and public expenditure financed human capital; and hence this optimum ratio varies inversely with the magnitude of the pollution-output coefficient and directly with the coefficient representing elasticity coefficient of human capital accumulation with respect to public expenditure on education.

The optimal income tax rate derived here is less than that obtained in section 2.2 of chapter 2. The productivity of public intermediate input is less in this model than that in section 2.2 of chapter 2 because $(1-\alpha-\beta) < (1-\alpha)$. Human capital accumulation technology requires both public input and private input. Thus investment in human capital is not financed by government fund alone. The optimal abatement expenditure rate and the optimal share of tax expenditure on public intermediate input are less than their corresponding values in section 2.2 of chapter 2 due to lower productivity of public input in this model.

7.3 HUMAN CAPITAL ACCUMULATION AND ENDOGENOUS POLLUTION RATE

This section develops an alternative extension of the basic model which emphasizes the role of human capital on the improvement of environmental quality when pollution rate is endogenously determined. Human capital is assumed to be a productive input in the final good production in this model like that in the model developed in section 7.2. However, accumulation of human capital is now wholly funded by private spending. Here also environmental quality is considered to be a stock variable which deteriorates over time with pollution caused from the production of the final good and is improved by the abatement activities of the government. However, the pollution rate per unit of output is assumed to be endogenous in this model; and this rate varies positively with the stock of human capital and inversely with the stock of physical capital[89].

The following subsections are organized as follows. Subsection 7.3.1 describes the basic model. Subsection 7.3.2 analyzes its dynamic equilibrium properties. Subsection 7.3.2.1 shows the possibility of the existence of unique steady-state equilibrium growth path in the market economy; and subsection 7.3.2.2 analyzes properties of optimal fiscal policy along the steady-state equilibrium path.

7.3.1 THE MODEL

Following equations describe the model. Equations (7.2.1), (7.2.2) and (7.2.4) of section 7.2.1 remain unchanged and so does equation (2.2.6) taken from chapter 2. The rest of the equations describing the model in this section are shown below.

$$G = (\tau - T)Y \; with \; 0 < T < \tau < 1; \qquad \dots\dots(7.3.1)$$

$$\dot{H} = e_H \; ; \qquad \dots\dots(7.3.2)$$

and

$$\dot{E} = \left(T - \delta_e K^{\psi} H^{-\psi}\right) Y \; ; \qquad \dots\dots(7.3.3)$$

[89] Generation of pollution is traced to usage of capital by various authors. The literature includes works of Bertinelli, Strobl and Zou (2008), Itaya (2008), Benarroch and Weder (2006), Hartman and Kwon (2005), Cassou and Hamilton (2004), Hart (2004), Oueslati (2002), Byrne (1997), Elbasha and Roe (1996), Smulders and Gradus (1996), Mohtadi (1996), Bovenberg and Smulders (1995), etc. Only Hartman and Kwon (2005) consider physical capital that is used to reduce pollution while Grimaud and Tournemaine (2007) consider knowledge as an input in pollution abatement. Gürlük (2009) empirically justifies the hypothesis that there is a significant non-linear relationship between modified human development index and biological oxygen demand in the Mediterranean countries implying that increased investments in education and health infrastructure produce valuable human capital that is less pollution intensive than physical capital as productive input. However, none of these models treats pollution rate to be endogenously determined by physical capital nor by human capital; pollution is considered to be an exogenous by-product of either production or physical capital.

Equation (7.3.1) shows how productive public input is financed. A part of the total tax revenue is spent on public input. τ is the proportional income tax rate and T is the abatement expenditure rate. $(\tau - T)$ is the expenditure rate on public input. If $v_H = 0$, from equations (7.2.5) and (7.2.6), we obtain equation (7.3.1).

Equation (7.3.2) describes how the stock of human capital grows over time. It is a linear function of private expenditure on education only. This equation (7.3.2) is a special case of equation (7.2.3) when $\eta = 0$.

Equation (7.3.3) shows how environmental quality changes over time. We assume the rate of emission to be endogenous and to depend on stocks of private physical capital and human capital. Emission rate increases with increase in physical capital stock and is reduced by an increase in human capital stock. An educated individual chooses less polluting technologies or less-polluting inputs in production. On the other hand, heavier physical capital usage has been found to be pollution intensive[90]. δ_e is an exogenous parameter.

7.3.2 DYNAMIC EQUILIBRIUM

The representative household maximizes $\int_0^\infty u(C)\, e^{-\rho t} dt$ with respect to C subject to equations (7.2.1), (7.2.4) and (7.3.2). The demand rate of growth of consumption[91] is derived from this maximizing problem as follows.

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]. \qquad \ldots\ldots (7.3.4)$$

We consider a steady-state growth equilibrium where all macroeconomic variables grow at the same rate, g_m. Hence, we have

[90] We assume no R&D sector which uses both physical capital and human capital as inputs. Hence we assume away non-polluting usage of private physical capital when it is used as input only in final good production. For example, Aloi and Tournemaine (2011), Grimaud and Tournemaine (2007), Ricci (2004), etc., treat R&D activity as clean and non pollution-generating in their models.

[91] The derivation is shown in Appendix 7.3A.

$$\frac{\dot{C}}{C} = \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\dot{H}}{H} = \frac{\dot{E}}{E} = g_m. \qquad \ldots\ldots (7.3.5)$$

7.3.2.1 Existence Of Steady-State Growth Equilibrium

We now turn to show the existence of unique steady state equilibrium growth rate in the market economy; and so we use equations (7.2.1), (7.2.2), (7.2.4), (7.3.1), (7.3.2), (7.3.3), (7.3.4) and (7.3.5) to obtain the following equations.

$$\frac{1}{\sigma}\left[\alpha(1-\tau)(\tau-T)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right] = g_m; \qquad \ldots\ldots (7.3.6)$$

$$\frac{e_H}{H} = \left(\frac{e_H}{K}\right)\left(\frac{H}{K}\right)^{-1} = g_m; \qquad \ldots\ldots (7.3.7)$$

$$(1-\tau)(\tau-T)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \frac{C}{K} - \frac{e_H}{K} = g_m; \qquad \ldots\ldots (7.3.8)$$

and

$$\left[T - \delta_e\left(\frac{H}{K}\right)^{-\psi}\right](\tau-T)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}-1}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} = g_m. \qquad \ldots\ldots (7.3.9)$$

Using equations (7.3.6), (7.3.7) and (7.3.9) we obtain the following equation[92] to solve for g_m.

$$g_m(\sigma g_m + \rho)^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}} = \left[T - \delta_e\left(\frac{\beta}{\alpha}\right)^{-\psi}\right]\left(\frac{\beta}{\alpha}\right)^{\beta}(\tau-T)^{\frac{1}{\theta}}\{\alpha(1-\tau)\}^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}}.$$

$$\ldots\ldots (7.3.10)$$

The existence of unique steady-state equilibrium growth rate can be shown given income tax rate, τ, and abatement expenditure rate, T, with $0 < T < \tau < 1$ and $T > \delta_e\left(\frac{\beta}{\alpha}\right)^{-\psi}$. Then equations (7.3.6), (7.3.7), (7.3.8) and (7.3.9) show that equilibrium values of $\frac{H}{K}$, $\frac{E}{K}$, $\frac{e_H}{K}$ and $\frac{C}{K}$ are also unique in this case.

We can state the following proposition.

[92] Derivation of equation (7.3.10) is worked out in Appendix 7.3B.

Proposition 7.3.1: There exists unique positive steady-state equilibrium growth rate in the market economy only if the fiscal instruments satisfy $0 < T < \tau < 1$ and $T > \delta_e \left(\frac{\beta}{\alpha}\right)^{-\psi}$.

7.3.2.2 Optimal Fiscal Policy

Maximizing the steady-state equilibrium growth rate with respect to the fiscal instrument rates, τ and T, we obtain following expressions of optimum tax rate and abatement expenditure rate.

$$\tau^{\star} = 1 - \left\{1 - \delta_e \left(\frac{\alpha}{\beta}\right)^{\psi}\right\}\{\alpha + \beta - \theta(1 - \alpha - \beta)\}; \qquad \dots\dots (7.3.11)$$

and

$$T^{\star} = \delta_e \left(\frac{\alpha}{\beta}\right)^{\psi} + \left\{1 - \delta_e \left(\frac{\alpha}{\beta}\right)^{\psi}\right\}\theta(1 - \alpha - \beta); \qquad \dots\dots (7.3.12)$$

Using equations (7.3.11) and (7.3.12), we have

$$(\tau^{\star} - T^{\star}) = \left\{1 - \delta_e \left(\frac{\alpha}{\beta}\right)^{\psi}\right\}(1 - \alpha - \beta); \qquad \dots\dots (7.3.13)$$

The growth rate maximizing tax rate is equal to the sum of competitive unpolluted output shares of productive public input and abatement expenditure. The abatement expenditure rate is equal to the sum of competitive unpolluted output share of environmental quality and a fraction $\delta_e \left(\frac{\alpha}{\beta}\right)^{\psi}$, which is the effective pollution rate adjusted for the cumulative effect of human capital and physical capital on total pollution. The fraction $\delta_e \left(\frac{\alpha}{\beta}\right)^{\psi}$ signifies the non-productive use of abatement expenditure; because it is used to negate the effect of pollution on environmental quality. If $\psi = 0$, then $\delta_e \left(\frac{\alpha}{\beta}\right)^{\psi} = \delta_e$. So the effective pollution rate is constant. Models developed in chapters 2 to 6 assume $\psi = 0$. The growth rate maximizing tax rate varies positively with the pollution parameter δ_e. Also, the share of unpolluted output now varies inversely not only with the pollution parameter, δ_e, but also with the parameter ψ. This means that the value of ψ is higher whenever human capital, per unit of physical capital, is more effective in reducing pollution. Hence, this implies that the fraction of unpolluted output available to allocate to productive uses, i.e., to public productive input and to augment environmental quality, is higher whenever the value of ψ is higher.

We now examine whether steady-state equilibrium growth rate maximizing solution is identical to social welfare maximizing solution in the steady-state equilibrium.

Using equations (7.3.6), (7.3.8) and (7.3B.7) and assuming that the economy is on the steady-state equilibrium growth path, we have

$$C = \frac{1}{\alpha}[\rho - (\alpha + \beta - \sigma)g_m]K(0)e^{g_m t}. \qquad \ldots\ldots(7.3.14)$$

Here too, the social welfare function is identical to that in the basic model in section 2.2. It can be shown[93] that

$$W = \alpha^{\sigma-1}\frac{K(0)^{1-\sigma}}{1-\sigma}\left[\frac{\rho-(\alpha+\beta-\sigma)g_m}{\rho-(1-\sigma)g_m}\right][\rho - (\alpha + \beta - \sigma)g_m]^{-\sigma}. \qquad \ldots\ldots(7.3.15)$$

Hence, W, in this extended model also varies positively with g_m.

Thus the level of social welfare in the steady-state growth equilibrium is maximized whenever the steady-state equilibrium growth rate is maximized[94]. We can now state the following proposition.

Proposition 7.3.2: (i) The optimum income tax rate, the optimum abatement expenditure rate and the optimum rate of productive public expenditure in the steady-state growth equilibrium are given by

$$\tau^* = 1 - \left\{1 - \delta_e\left(\frac{\alpha}{\beta}\right)^{\psi}\right\}\{\alpha + \beta - \theta(1 - \alpha - \beta)\},$$

$$T^* = \delta_e\left(\frac{\alpha}{\beta}\right)^{\psi} + \left\{1 - \delta_e\left(\frac{\alpha}{\beta}\right)^{\psi}\right\}\theta(1 - \alpha - \beta);$$

[93] The derivation is worked out in Appendix 7.3D.

[94] We abstain from analyzing social welfare maximization along the transitional path.

and

$$\tau^\star - T^\star = \left\{1 - \delta_e \left(\frac{\alpha}{\beta}\right)^{\psi}\right\}(1 - \alpha - \beta).$$

(ii) The optimum ratio of net public expenditure on productive public input to national income in the steady-state equilibrium is equal to the competitive unpolluted output share of public input; and hence this optimum ratio varies inversely with the magnitude of the parameter representing the pollution elasticity with respect to the ratio of human capital to physical capital.

The optimal income tax rate derived in this model is less than that obtained in section 2.2 of chapter 2 if $\delta = \delta_e \left(\frac{\alpha}{\beta}\right)^{\psi}$. Here also productivity of public intermediate good is less and accumulation of human capital is fully financed by private households. Under the same parametric restriction on pollution rate, the optimal abatement expenditure rate and the optimal share of expenditure on public intermediate good are also less than their corresponding values in section 2.2 of chapter 2.

APPENDIX 7.2A

DERIVATION OF EQUATION (7.2.7) IN SECTION 7.2.2:

The dynamic optimization problem of the representative household is to maximize $\int_0^\infty e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} dt$ with respect to C subject to equation (7.2.4) and given $K(0)$. Here C is the control variable satisfying $0 \leq C \leq (1-\tau)Y$; and K is the state variable.

The Hamiltonian to be maximized at each point of time is given by

$$\mathcal{H} = e^{-\rho t} \frac{C^{1-\sigma}}{1-\sigma} + e^{-\rho t}\lambda_K[(1-\tau)Y - C - e_H] + e^{-\rho t}\lambda_H[(v_H Y)^{\eta}(e_H)^{1-\eta}].$$

Here λ_K and λ_H are the co-state variables representing the shadow price of physical capital investment and the shadow price of human capital. Maximizing the Hamiltonian with respect to C and e_H and assuming an interior solution, we obtain

$$C^{-\sigma} = \lambda_K; \qquad \text{... ... } (7.2A.1)$$

and

$$\frac{\lambda_K}{\lambda_H} = \eta (v_H Y)^{\eta} (e_H)^{-\eta}. \qquad \text{... ... } (7.2A.2)$$

Using equations (7.2.1), (7.2.2), (7.2.5) and (7.2A.2) we have

$$\frac{\lambda_K}{\lambda_H} = \eta {v_H}^{\eta} {v_G}^{\eta\left(\frac{1-\alpha-\beta}{\alpha+\beta}\right)} \left(\frac{E}{K}\right)^{\eta\left\{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}\right\}} \left(\frac{H}{K}\right)^{\frac{\eta\beta}{\alpha+\beta}} \left(\frac{e_H}{K}\right)^{-\eta}. \qquad \text{... ... } (7.2A.3)$$

Also the optimum time paths of λ_K and λ_H satisfy the followings.

$$\frac{\dot{\lambda}_K}{\lambda_K} = \rho - \alpha(1-\tau) K^{\alpha-1} H^{\beta} \hat{G}^{1-\alpha-\beta}; \qquad \text{... ... } (7.2A.4)$$

and

$$\frac{\dot{\lambda}_H}{\lambda_H} = \rho - \frac{\lambda_K}{\lambda_H} \beta(1-\tau) K^{\alpha} H^{\beta-1} \hat{G}^{1-\alpha-\beta}. \qquad \text{... ... } (7.2A.5)$$

Using equations (7.2.1), (7.2.2), (7.2.5) and (7.2A.4) we have

$$\frac{\dot{\lambda}_K}{\lambda_K} = \rho - \alpha(1-\tau) {v_G}^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}}. \qquad \text{... ... } (7.2A.6)$$

Similarly, using equations (7.2.1), (7.2.2), (7.2.5), (7.2A.3) and (7.2A.5) we have

$$\frac{\dot{\lambda}_H}{\lambda_H} = \rho - \eta\beta(1-\tau) {v_H}^{\eta} {v_G}^{\left(\frac{1-\alpha-\beta}{\alpha+\beta}\right)(1+\eta)} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)(1+\eta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\eta\beta-\alpha}{\alpha+\beta}} \left(\frac{e_H}{K}\right)^{-\eta}.$$

$$\text{... ... } (7.2A.7)$$

Using the two optimality conditions (7.2A.1) and (7.2A.6), we have

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau) {v_G}^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]. \qquad \text{... ... } (7.2A.8)$$

This is same as equation (7.2.7).

APPENDIX 7.2B

DERIVATION OF EQUATION (7.2.13) IN SECTION 7.2.2.1

Using equations (7.2.1) to (7.2.5), (4.3), (7.2.7) and (7.2.8) we have the following equations.

$$g_m = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau){v_G}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]; \qquad \text{... ... }(7.2B.1)$$

$$g_m = \frac{\dot{H}}{H} = {v_H}^{\eta}{v_G}^{\eta\left(\frac{1-\alpha-\beta}{\alpha+\beta}\right)}\left(\frac{E}{K}\right)^{\eta\left\{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}\right\}}\left(\frac{H}{K}\right)^{\frac{\eta\beta}{\alpha+\beta}-1}\left(\frac{e_H}{K}\right)^{1-\eta}; \qquad \text{... ... }(7.2B.2)$$

$$g_m = \frac{\dot{K}}{K} = (1-\tau){v_G}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \frac{C}{K} - \frac{e_H}{K}; \qquad \text{... ... }(7.2B.3)$$

and

$$g_m = \frac{\dot{E}}{E} = (T-\delta){v_G}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}}. \qquad \text{... ... }(7.2B.4)$$

From equation (7.2B.1) we have,

$$\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} = \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}{v_G}^{-\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{H}{K}\right)^{-\frac{\beta}{\alpha+\beta}}. \qquad \text{... ... }(7.2B.5)$$

At the steady-state equilibrium the ratios $\frac{E}{K}$, $\frac{H}{K}$ and $\frac{e_H}{K}$ are constants; and so are the policy variables. Hence $\frac{\lambda_K}{\lambda_H}$ in equation (7.2A.3) is also constant.

Thus

$$\frac{\dot{\lambda}_H}{\lambda_H} = \frac{\dot{\lambda}_K}{\lambda_K}. \qquad \text{... ... }(7.2B.6)$$

Using equations (7.2B.2), (7.2B.5), (7.2A.7) and (7.2B.6) we derive the following equation.

$$\frac{e_H}{K} = \frac{\eta\beta}{\alpha}g_m. \qquad \text{... ... }(7.2B.7)$$

Using equations (7.2B.2), (7.2B.5) and (7.2B.7) we get

$$\frac{H}{K} = \left(\frac{\eta\beta}{\alpha}\right)^{1-\eta} v_H{}^{\eta} \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}^{\eta} g_m{}^{-\eta}. \qquad \ldots\ldots(7.2B.8)$$

Thus, using equations (7.2B.5) and (7.2B.8) we get

$$\frac{E}{K} = \left(\frac{\eta\beta}{\alpha}\right)^{-\frac{(1-\eta)\beta}{\theta(1-\alpha-\beta)}} v_H{}^{-\frac{\eta\beta}{\theta(1-\alpha-\beta)}} v_G{}^{-\frac{1}{\theta}} \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}^{\frac{\alpha+\beta-\eta\beta}{\theta(1-\alpha-\beta)}} g_m{}^{\frac{\eta\beta}{\theta(1-\alpha-\beta)}}. \qquad \ldots\ldots(7.2B.9)$$

Now, using equations (7.2B.4), (7.2B.8) and (7.2B.9) we get

$$g_m = (T-\delta) v_G{}^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left[\left(\frac{\eta\beta}{\alpha}\right)^{-\frac{(1-\eta)\beta}{\theta(1-\alpha-\beta)}} v_H{}^{-\frac{\eta\beta}{\theta(1-\alpha-\beta)}} v_G{}^{-\frac{1}{\theta}} \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}^{\frac{\alpha+\beta-\eta\beta}{\theta(1-\alpha-\beta)}} g_m{}^{\frac{\eta\beta}{\theta(1-\alpha-\beta)}}\right]^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left[\left(\frac{\eta\beta}{\alpha}\right)^{1-\eta} v_H{}^{\eta} \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}^{\eta} g_m{}^{-\eta}\right]^{\frac{\beta}{\alpha+\beta}},$$

or,

$$g_m{}^{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}} (\sigma g_m + \rho)^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}} = (\eta\beta)^{\frac{\beta-\eta\beta}{\theta(1-\alpha-\beta)}} \alpha^{\frac{\alpha-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}} v_G{}^{\frac{1}{\theta}} v_H{}^{\frac{\eta\beta}{\theta(1-\alpha-\beta)}} (1-\tau)^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}} (T-\delta).$$

$$\ldots\ldots(7.2B.10)$$

This is same as equation (7.2.13).

APPENDIX 7.2C

DERIVATION OF EQUATIONS (7.2.14) TO (7.2.17) AND THE SECOND ORDER CONDITIONS IN SECTION 7.2.2.2:

We denote the L.H.S. and the R.H.S. of equation (7.2.13) by L.H.S.$_{(7.2.13)}$ and R.H.S.$_{(7.2.13)}$ respectively. Maximizing the R.H.S. of equation (7.2.13) with respect to τ and using equation (7.2.6), we obtain the following first order condition.

$$\text{R.H.S.}_{(7.2.13)} \left[\frac{1}{\theta} v_G{}^{-1} - \frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)} (1-\tau)^{-1}\right] = 0;$$

or,

$$\frac{1}{\theta}v_G{}^{-1} = \frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}(1-\tau)^{-1}. \qquad \ldots\ldots(7.2C.1)$$

Maximizing the R.H.S. of equation (7.2.13) with respect to T and using equation (7.2.6), we obtain the following first order condition.

$$\text{R.H.S.}_{(7.2.13)}\left[-\frac{1}{\theta}v_G{}^{-1} + (T-\delta)^{-1}\right] = 0;$$

or,

$$\frac{1}{\theta}v_G{}^{-1} = (T-\delta)^{-1}. \qquad \ldots\ldots(7.2C.2)$$

Maximizing the R.H.S. of equation (7.2.13) with respect to v_H and using equation (7.2.6), we obtain the following first order condition.

$$\text{R.H.S.}_{(7.2.13)}\left[-\frac{1}{\theta}v_G{}^{-1} + \frac{\eta\beta}{\theta(1-\alpha-\beta)}v_H{}^{-1}\right] = 0;$$

or,

$$\frac{1}{\theta}v_G{}^{-1} = \frac{\eta\beta}{\theta(1-\alpha-\beta)}v_H{}^{-1}. \qquad \ldots\ldots(7.2C.3)$$

Using equations (7.2C.1), (7.2C.2), (7.2C.3) and (7.2.6) we obtain the following expressions.

$$\tau^\star = 1-(1-\delta)\{\alpha-\theta(1-\alpha-\beta)+\beta(1-\eta)\}; \qquad \ldots\ldots(7.2C.4)$$

$$T^\star = \delta+(1-\delta)\theta(1-\alpha-\beta); \qquad \ldots\ldots(7.2C.5)$$

$$v_G{}^\star = (1-\delta)(1-\alpha-\beta); \qquad \ldots\ldots(7.2C.6)$$

and

$$v_H{}^\star = (1-\delta)\eta\beta; \qquad \ldots\ldots(7.2C.7)$$

These are same as equations (7.2.14), (7.2.15), (7.2.16) and (7.2.17) in the section 7.2.2.2.

To check the second order conditions for optimality we twice differentiate equation (7.2.13), with respect to τ, T and v_H respectively. At the equilibrium point $\text{L.H.S.}_{(7.2.13)} = \text{R.H.S.}_{(7.2.13)}$. We arrive at the following three second order conditions.

$$-\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}g_m{}^{-2} + \left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma^2(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]^2\left(\frac{\partial g_m}{\partial \tau}\right)^2$$

$$+\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial \tau^2}$$

$$=\left[\frac{1}{\theta}{v_G}^{-1}-\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}(1-\tau)^{-1}\right]^2$$

$$-\left[\left(\frac{1}{\theta}\right){v_G}^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}(1-\tau)^{-2}\right];$$

$$-\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma^2(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]^2\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial T^2}$$

$$=\left[-\frac{1}{\theta}{v_G}^{-1}+(T-\delta)^{-1}\right]^2-\left[\frac{1}{\theta}{v_G}^{-2}+(T-\delta)^{-2}\right];$$

and

$$-\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma^2(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial v_H}\right)^2$$

$$+\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]^2\left(\frac{\partial g_m}{\partial v_H}\right)^2$$

$$+\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial {v_H}^2}$$

$$=\left[-\frac{1}{\theta}{v_G}^{-1}+\left\{\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{v_H}^{-1}\right]^2-\left[\frac{1}{\theta}{v_G}^{-2}+\left\{\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{v_H}^{-2}\right].$$

Now we evaluate the above two second order conditions at $\tau=\tau^\star$, $T=T^\star$ and $v_H={v_H}^\star$ where $\frac{\partial g_m}{\partial\tau}=\frac{\partial g_m}{\partial T}=\frac{\partial g_m}{\partial v_H}=0$ at the optimum, using equations (7.2C.1), (7.2C.2) and (7.2C.3). Hence we obtain the followings.

$$\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial \tau^2}$$

$$=-\left[\left(\frac{1}{\theta}\right){v_G}^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}(1-\tau)^{-2}\right];$$

$$\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}g_m{}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial T^2}$$

$$=-\left[\frac{1}{\theta}v_G{}^{-2}+(T-\delta)^{-2}\right];$$

and

$$\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}g_m{}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial v_H{}^2}$$

$$=-\left[\frac{1}{\theta}v_G{}^{-2}+\left\{\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}v_H{}^{-2}\right].$$

$\left[\left\{1+\frac{\eta\beta}{\theta(1-\alpha-\beta)}\right\}g_m{}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)-\eta\beta}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]>0$ and the R.H.S. of each of these three equations is negative. Hence the sign of both the second order derivatives is negative.

APPENDIX 7.2D

DERIVATION OF EQUATION (7.2.19) IN SECTION 7.2.2.2

Here the social welfare functional is given by $W=\int_0^\infty e^{-\rho t}u(C)\,dt$. From equation (7.2.9), we have

$$\alpha(1-\tau)v_G{}^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}}=\frac{\sigma g_m+\rho}{\alpha}. \qquad \ldots\ldots(7.2D.1)$$

Using equations (7.2.11), (7.2B.7) and (7.2D.1), we have

$$\frac{C}{K}=\frac{\sigma g_m+\rho}{\alpha}-\frac{\eta\beta}{\alpha}g_m-g_m=\frac{1}{\alpha}[\rho-(\alpha+\eta\beta-\sigma)g_m]. \qquad \ldots\ldots(7.2D.2)$$

At the steady state equilibrium, $K=K(0)e^{g_m t}$; where $K(0)$ is the initial value of K. Thus equation (7.2D.2) can be written as

$$C=\frac{1}{\alpha}[\rho-(\alpha+\eta\beta-\sigma)g_m]K(0)e^{g_m t}. \qquad \ldots\ldots(7.2D.3)$$

Using equations (2.2.6) and (7.2D.3) and the social welfare functional we have

$$W = \int_0^\infty e^{-\rho t} \frac{[\rho - (\alpha + \eta\beta - \sigma) g_m]^{1-\sigma} [K(0)]^{1-\sigma} e^{g_m (1-\sigma) t}}{\alpha^{1-\sigma}(1-\sigma)} dt,$$

or,

$$W = \frac{[K(0)]^{1-\sigma}}{\alpha^{1-\sigma}(1-\sigma)} [\rho - (\alpha + \eta\beta - \sigma) g_m]^{1-\sigma} \int_0^\infty e^{[g_m(1-\sigma) - \rho]t} \, dt.$$

For convergence we assume $\rho - g_m(1-\sigma) > 0$. Thus,

$$W = \frac{[K(0)]^{1-\sigma}}{\alpha^{1-\sigma}(1-\sigma)} [\rho - (\alpha + \eta\beta - \sigma) g_m]^{1-\sigma} [\rho - (1-\sigma) g_m]^{-1}, \qquad \ldots\ldots (7.2\text{D}.4)$$

or,

$$W = \alpha^{\sigma - 1} \frac{[K(0)]^{1-\sigma}}{1-\sigma} \left[\frac{\rho - (\alpha + \eta\beta - \sigma) g_m}{\rho - (1-\sigma) g_m}\right] [\rho - (\alpha + \eta\beta - \sigma) g_m]^{-\sigma}. \qquad \ldots\ldots (7.2D.4')$$

This is same as equation (7.2.19).

We can show that the social welfare functional given by (7.2D.4) is an increasing function of g_m.

Case 1a: $\sigma < 1$ and $\alpha + \eta\beta < \sigma$. In this case the social welfare functional given by equation (7.2D.4) is unambiguously increasing in the growth rate, g_m.

Case 1b: $\sigma < 1$ and $\alpha + \eta\beta > \sigma$. In this case, since $\alpha + \eta\beta < 1$, the effect of an increase in g_m on the denominator of the term $\left[\frac{\rho - (\alpha + \eta\beta - \sigma) g_m}{\rho - (1-\sigma) g_m}\right]$ given in (7.2D.4′) dominates that on the numerator. Hence, $\left[\frac{\rho - (\alpha + \eta\beta - \sigma) g_m}{\rho - (1-\sigma) g_m}\right]$ increases as g_m increases and so does $[\rho - (\alpha + \eta\beta - \sigma) g_m]^{-\sigma}$. Thus, the social welfare functional, alternatively given by equation (7.2D.4'), increases with the growth rate.

Case 2: $\sigma > 1$ and $\alpha + \eta\beta < 1 < \sigma$. In this case, $\frac{[\rho - (\alpha + \eta\beta - \sigma) g_m]^{1-\sigma} [\rho - (1-\sigma) g_m]^{-1}}{1-\sigma}$ term in equation (7.2D.4) is an increasing function of g_m as $\sigma > 1$. Therefore, in this case too, the social welfare function increases with the growth rate.

APPENDIX 7.3A

DERIVATION OF EQUATION (7.3.4) IN SECTION 7.3.2

The dynamic optimization problem of the representative household is to maximize $\int_0^\infty e^{-\rho t}\frac{C^{1-\sigma}}{1-\sigma}dt$ with respect to C subject to equation (7.2.4) and given $K(0)$. Here C is the control variable satisfying $0 \le C \le (1-\tau)Y$; and K is the state variable.

The Hamiltonian to be maximized at each point of time is given by

$$\mathcal{H} = e^{-\rho t}\frac{c^{1-\sigma}}{1-\sigma} + e^{-\rho t}\lambda_K[(1-\tau)Y - C - e_H] + e^{-\rho t}\lambda_H[e_H].$$

Here λ_K and λ_H are the co-state variables representing the shadow price of physical capital investment and the shadow price of human capital. Maximizing the Hamiltonian with respect to C and e_H and assuming an interior solution, we obtain

$$C^{-\sigma} = \lambda_K; \qquad \ldots\ldots(7.3A.1)$$

and

$$\frac{\lambda_K}{\lambda_H} = 1. \qquad \ldots\ldots(7.3A.2)$$

Also the optimum time paths of λ_K and λ_H satisfy the followings.

$$\frac{\dot{\lambda}_K}{\lambda_K} = \rho - \alpha(1-\tau)K^{\alpha-1}H^{\beta}\hat{G}^{1-\alpha-\beta}; \qquad \ldots\ldots(7.3A.3)$$

and

$$\frac{\dot{\lambda}_H}{\lambda_H} = \rho - \frac{\lambda_K}{\lambda_H}\beta(1-\tau)K^{\alpha}H^{\beta-1}\hat{G}^{1-\alpha-\beta}. \qquad \ldots\ldots(7.3A.4)$$

Using equations (7.2.1), (7.2.2), (7.3.1) and (7.3A.3) we have

$$\frac{\dot{\lambda}_K}{\lambda_K} = \rho - \alpha(1-\tau)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}}. \qquad \ldots\ldots(7.3A.5)$$

Similarly, using equations (7.2.1), (7.2.2), (7.3.1), (7.3A.2) and (7.3A.4) we have

$$\frac{\dot{\lambda_H}}{\lambda_H} = \rho - \beta(1-\tau)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{-\alpha}{\alpha+\beta}}. \qquad \dots\dots(7.3A.6)$$

Using the two optimality conditions (7.3A.1) and (7.3A.5), we have

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]. \qquad \dots\dots(7.2A.7)$$

which is same as equation (7.3.4).

APPENDIX 7.3B

DERIVATION OF EQUATION (7.3.10) IN SECTION 7.3.2.1

Using equations (7.2.1), (7.2.2), (7.2.4), (7.3.1), (7.3.2), (7.3.3), (7.3.4) and (7.3.5) we have the following equations.

$$g_m = \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[\alpha(1-\tau)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \rho\right]; \qquad \dots\dots(7.3B.1)$$

$$g_m = \frac{\dot{H}}{H} = \left(\frac{e_H}{K}\right)\left(\frac{H}{K}\right)^{-1}; \qquad \dots\dots(7.3B.2)$$

$$g_m = \frac{\dot{K}}{K} = (1-\tau)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}} - \frac{C}{K} - \frac{e_H}{K}; \qquad \dots\dots(7.3B.3)$$

and

$$g_m = \frac{\dot{E}}{E} = (T-\delta)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}}. \qquad \dots\dots(7.3B.4)$$

From equation (7.3B.1) we have,

$$\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} = \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}(\tau - T)^{-\frac{1-\alpha-\beta}{\alpha+\beta}} \left(\frac{H}{K}\right)^{-\frac{\beta}{\alpha+\beta}}. \qquad \dots\dots(7.3B.5)$$

Using equations (7.3A.1), (7.3A.2), (7.3A.6) and (7.3B.1) we get

$$\frac{H}{K} = \frac{\beta}{\alpha}. \qquad \dots\dots(7.3B.6)$$

Using equations (7.3B.2) and (7.3B.6) we derive the following equation.

$\frac{e_H}{K} = \frac{\beta}{\alpha} g_m.$ ...... (7.3B.7)

Thus, using equations (7.3B.5) and (7.3B.6) we get

$$\frac{E}{K} = \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} (\tau - T)^{-\frac{1-\alpha-\beta}{\alpha+\beta}} \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}. \quad \text{...... (7.3B.8)}$$

Now, using equations (7.3B.4), (7.3B.6) and (7.3B.8) we get

$$g_m = (T - \delta)(\tau - T)^{\frac{1-\alpha-\beta}{\alpha+\beta}} \left[\left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} (\tau - T)^{-\frac{1-\alpha-\beta}{\alpha+\beta}} \left\{\frac{\rho+\sigma g_m}{\alpha(1-\tau)}\right\}\right]^{-\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}} \left[\frac{\beta}{\alpha}\right]^{\frac{\beta}{\alpha+\beta}},$$

or,

$$g_m(\sigma g_m + \rho)^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}} = \left(\frac{\beta}{\alpha}\right)^{\beta} \left[T - \delta_e \left(\frac{\beta}{\alpha}\right)^{-\psi}\right] (\tau - T)^{\frac{1}{\theta}} \{\alpha(1-\tau)\}^{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}}$$

...... (7.3B.9)

This is same as equation (7.3.10).

APPENDIX 7.3C

DERIVATION OF EQUATIONS (7.3.11) AND (7.3.12) AND THE SECOND ORDER CONDITIONS IN SECTION 7.3.2.2:

We denote the L.H.S. and the R.H.S. of equation (7.3.10) by L.H.S.$_{(7.3.10)}$ and R.H.S.$_{(7.3.10)}$ respectively. Maximizing the R.H.S. of equation (7.3.10) with respect to τ, we obtain the following first order condition.

$$\text{R.H.S.}_{(7.3.10)} \left[\frac{1}{\theta}(\tau - T)^{-1} - \frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}(1-\tau)^{-1}\right] = 0;$$

or,

$$\frac{1}{\theta}(\tau - T)^{-1} = \frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}(1-\tau)^{-1}. \quad \text{...... (7.3C.1)}$$

Maximizing the R.H.S. of equation (7.3.10) with respect to T, we obtain the following first order condition.

$$\text{R.H.S.}_{(7.3.10)}\left[-\frac{1}{\theta}(\tau-T)^{-1}+\left\{T-\delta_e\left(\frac{\beta}{\alpha}\right)^{-\psi}\right\}^{-1}\right]=0;$$

or,

$$\frac{1}{\theta}(\tau-T)^{-1}=\left\{T-\delta_e\left(\frac{\beta}{\alpha}\right)^{-\psi}\right\}^{-1}. \qquad \ldots\ldots(7.3C.2)$$

Using equations (7.3C.1) and (7.3C.2) we obtain the following expressions.

$$\tau^{\star}=1-\left\{1-\delta_e\left(\frac{\alpha}{\beta}\right)^{\psi}\right\}\{\alpha+\beta-\theta(1-\alpha-\beta)\}; \qquad \ldots\ldots(7.3C.3)$$

and

$$T^{\star}=\delta_e\left(\frac{\alpha}{\beta}\right)^{\psi}+\left\{1-\delta_e\left(\frac{\alpha}{\beta}\right)^{\psi}\right\}\theta(1-\alpha-\beta); \qquad \ldots\ldots(7.3C.4)$$

These are same as equations (7.3.11) and (7.3.12) in section 7.3.2.2.

To check the second order conditions for optimality we twice differentiate equation (7.3.10), with respect to τ and T respectively. At the equilibrium point $\text{L.H.S.}_{(7.3.10)}=\text{R.H.S.}_{(7.3.10)}$. We arrive at the following two second order conditions.

$$-\left[g_m{}^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma^2(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+\left[g_m{}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]^2\left(\frac{\partial g_m}{\partial\tau}\right)^2$$

$$+\left[g_m{}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial\tau^2}$$

$$=\left[\frac{1}{\theta}(\tau-T)^{-1}-\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}(1-\tau)^{-1}\right]^2$$

$$-\left[\left(\frac{1}{\theta}\right)(\tau-T)^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}(1-\tau)^{-2}\right];$$

and

$$-\left[g_m{}^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma^2(\sigma g_m+\rho)^{-2}\right]\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+\left[g_m{}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]^2\left(\frac{\partial g_m}{\partial T}\right)^2$$

$$+\left[{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial T^2}$$

$$=\left[-\frac{1}{\theta}(\tau-T)^{-1}+\left\{T-\delta_e\left(\frac{\beta}{\alpha}\right)^{-\psi}\right\}^{-1}\right]^2-\left[\frac{1}{\theta}(\tau-T)^{-2}+\left\{T-\delta_e\left(\frac{\beta}{\alpha}\right)^{-\psi}\right\}^{-2}\right].$$

Now we evaluate the above two second order conditions using equations (7.3C.1) and (7.3C.2) at $\tau=\tau^\star$ and $T=T^\star$ where $\frac{\partial g_m}{\partial \tau}=\frac{\partial g_m}{\partial T}=0$ at the optimum. Hence we obtain the followings.

$$\left[{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial \tau^2}$$

$$=-\left[\left(\frac{1}{\theta}\right)(\tau-T)^{-2}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}(1-\tau)^{-2}\right];$$

and

$$\left[{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]\frac{\partial^2 g_m}{\partial T^2}$$

$$=-\left[\frac{1}{\theta}(\tau-T)^{-2}+\left\{T-\delta_e\left(\frac{\beta}{\alpha}\right)^{-\psi}\right\}^{-2}\right];$$

$$\left[{g_m}^{-1}+\left\{\frac{\alpha+\beta-\theta(1-\alpha-\beta)}{\theta(1-\alpha-\beta)}\right\}\sigma(\sigma g_m+\rho)^{-1}\right]>0$$

The R.H.S. of each of these two equations is negative. Hence the sign of both the second order derivatives is negative.

APPENDIX 7.3D

DERIVATION OF EQUATION (7.3.15) IN SECTION 7.3.2.2

Here the social welfare functional is given by $W=\int_0^\infty e^{-\rho t}u(C)\,dt$. From equation (7.3.6), we have

$$\alpha(1-\tau)(\tau-T)^{\frac{1-\alpha-\beta}{\alpha+\beta}}\left(\frac{E}{K}\right)^{\frac{\theta(1-\alpha-\beta)}{\alpha+\beta}}\left(\frac{H}{K}\right)^{\frac{\beta}{\alpha+\beta}}=\frac{\sigma g_m+\rho}{\alpha}. \qquad \ldots\ldots(7.3D.1)$$

Using equations (7.3.8), (7.3B.7) and (7.3D.1), we have

$$\frac{C}{K} = \frac{\sigma g_m + \rho}{\alpha} - \frac{\beta}{\alpha} g_m - g_m = \frac{1}{\alpha}[\rho - (\alpha + \beta - \sigma) g_m]. \qquad \dots\dots (7.3D.2)$$

At the steady state equilibrium, $K = K(0)e^{g_m t}$; where $K(0)$ is the initial value of K. Thus equation (7.3D.2) can be written as

$$C = \frac{1}{\alpha}[\rho - (\alpha + \beta - \sigma) g_m] K(0) e^{g_m t}. \qquad \dots\dots (7.3D.3)$$

Using equations (2.2.6) and (7.3D.3) and the social welfare functional we have

$$W = \int_0^\infty e^{-\rho t} \frac{[\rho - (\alpha+\beta-\sigma) g_m]^{1-\sigma} [K(0)]^{1-\sigma} e^{g_m (1-\sigma) t}}{\alpha^{1-\sigma}(1-\sigma)} dt,$$

or,

$$W = \frac{[K(0)]^{1-\sigma}}{\alpha^{1-\sigma}(1-\sigma)} [\rho - (\alpha + \beta - \sigma) g_m]^{1-\sigma} \int_0^\infty e^{[g_m(1-\sigma) - \rho] t} dt.$$

For convergence we assume $\rho - g_m(1-\sigma) > 0$. Thus,

$$W = \frac{[K(0)]^{1-\sigma}}{\alpha^{1-\sigma}(1-\sigma)} [\rho - (\alpha + \beta - \sigma) g_m]^{1-\sigma} [\rho - (1-\sigma) g_m]^{-1}, \qquad \dots\dots (7.3D.4)$$

or,

$$W = \alpha^{\sigma-1} \frac{[K(0)]^{1-\sigma}}{1-\sigma} \left[\frac{\rho - (\alpha+\beta-\sigma) g_m}{\rho - (1-\sigma) g_m}\right] [\rho - (\alpha + \beta - \sigma) g_m]^{-\sigma}. \qquad \dots\dots (7.3D.4')$$

This is same as equation (7.3.15).

We can show that the social welfare functional given by (7.3D.4) is an increasing function of g_m.

Case 1a: $\sigma < 1$ and $\alpha + \beta < \sigma$. In this case the social welfare functional given by equation (7.3D.4) is unambiguously increasing in the growth rate, g_m.

Case 1b: $\sigma < 1$ and $\alpha + \beta > \sigma$. In this case, since $\alpha + \beta < 1$, the effect of an increase in g_m on the denominator of the term $\left[\frac{\rho-(\alpha+\beta-\sigma) g_m}{\rho-(1-\sigma) g_m}\right]$ given in (7.3D.4'), dominates that on the numerator. Hence, $\left[\frac{\rho-(\alpha+\beta-\sigma) g_m}{\rho-(1-\sigma) g_m}\right]$ increases as g_m increases and so does $[\rho - (\alpha + \beta - \sigma) g_m]^{-\sigma}$. Thus, the social welfare functional, alternatively given by equation (7.3D.4'), increases with the growth rate.

Case 2: $\sigma > 1$ and $\alpha + \beta < 1 < \sigma$. In this case, $\frac{[\rho-(\alpha+\beta-\sigma)g_m]^{1-\sigma}[\rho-(1-\sigma)g_m]^{-1}}{1-\sigma}$ term in equation (7.3D.4) is an increasing function of g_m as $\sigma > 1$. Therefore, in this case too, the social welfare function increases with the growth rate.

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