SOME APPROXIMATIONS TO THE BINOMIAL DISTRIBUTION FUNCTION

By R. R. Bahadur

Indian Statistical Institute, Calcutta

1. Summary. Let p be given, $0 < p < 1$. Let n and k be positive integers such that $np \leq k \leq n$, and let $B_n(k) = \sum_{r=k}^{n} \binom{n}{r} p^r q^{n-r}$, where $q = 1 - p$. It is shown that

$$B_n(k) = \left[\binom{n}{k} p^k q^{n-k}\right] qF(n + 1, 1; k + 1; p),$$

where F is the hypergeometric function. This representation seems useful for numerical and theoretical investigations of small tail probabilities. The representation yields, in particular, the result that, with $A_n(k) = \left[\binom{n}{k} p^k q^{n-k+1}\right]$ $[(k + 1)/(k + 1 - (n + 1)p)]$, we have $1 \leq A_n(k)/B_n(k) \leq 1 + x^{-2}$, where $x = (k - np)/(npq)^{\frac{1}{2}}$. Next, let $N_n(k)$ denote the normal approximation to $B_n(k)$, and let $C_n(k) = (x + \sqrt{q/np})\sqrt{2\pi} \exp [x^2/2]$. It is shown that

$$(A_n N_n C_n)/B_n \to 1$$

as $n \to \infty$, provided only that k varies with n so that $x \geq 0$ for each n. It follows hence that $A_n/B_n \to 1$ if and only if $x \to \infty$ (i.e. $B_n \to 0$). It also follows that $N_n/B_n \to 1$ if and only if $A_n C_n \to 1$. This last condition reduces to

$$x = o(n^{1/6})$$

for certain values of p, but is weaker for other values; in particular, there are values of p for which N_n/B_n can tend to one without even the requirement that k/n tend to p.

2. Introduction. Let p be given, $0 < p < 1$, and let n and k be positive integers such that

$$np \leq k \leq n. \tag{1}$$

Define

$$B_n(k) = \sum_{r=k}^{n} \binom{n}{r} p^r q^{n-r} \tag{2}$$

where $q = 1 - p$.[1] The following is an apparently new representation of $B_n(k)$:

Received March 28, 1959.

[1] Only upper tail probabilities are discussed in the paper. This involves no loss of generality, since p is arbitrary.

$$B_n(k) = \left[\binom{n}{k} p^k q^{n-k}\right] qF(n+1, 1; k+1; p) \tag{3}$$

where

$$F = 1 + \frac{(n+1)}{(k+1)} p + \frac{(n+1)(n+2)}{(k+1)(k+2)} p^2 + \cdots \text{ ad inf.} \tag{4}$$

To establish (3), consider an unlimited sequence of independent Bernoulli trials each with success probability p. Let S denote the total number of successes in the first n trials, and let T denote the minimum number of trials required in order to obtain a total of $n - k + 1$ failures. Then the event $\{S \geqq k\}$ is identical with the event $\{T \geqq n + 1\}$. Hence $P(S \geqq k) = P(T \geqq n + 1)$, and (3) now follows by referring to the probability distributions of S and T.

Thus (3) expresses a relation between the binomial and negative binomial distributions. Another relation (which, however, is not used in this paper) between these distributions is the following. Let U denote the minimum number of trials required to obtain a total of k successes. Then $\{S \geqq k\}$ is identical with $\{U \leqq n\}$. Hence $P(S \geqq k) = P(U \leqq n)$, and this can be written as

$$B_n(k) = \left[\binom{n}{k} p^k q^{n-k}\right]\left(\frac{k}{n}\right)\left[1 + \frac{(m-1)}{(n-1)}\frac{1}{q} + \frac{(m-1)(m-2)}{(n-1)(n-2)}\frac{1}{q^2} + \cdots + \frac{(m-1)\cdots(2)(1)}{(n-1)\cdots(k+1)(k)} \cdot \frac{1}{q^{n-k}}\right] \tag{5}$$

where $m = n - k + 1$. It may be noted that (3) is valid for each $k = 0, 1, 2, \cdots, n$, while (5) is valid for $k = 1, 2, \cdots, n$.

Now let

$$z = \frac{(n+1)}{(k+1)} p. \tag{6}$$

Then $0 < z < 1$ by (1). Let us write (4) in the form

$$F = \sum_{s=0}^{\infty} a_s z^s, \tag{7}$$

where

$$a_s = \begin{cases} 1 & \text{for } s = 0, 1 \\ \prod_{i=1}^{s-1} \dfrac{\left(1 + \dfrac{i}{n+1}\right)}{\left(1 + \dfrac{i}{k+1}\right)} & \text{for } s = 2, 3, \cdots. \end{cases} \tag{8}$$

Since $0 < a_s \leqq 1$ for each s, it is clear from (3) and (7) that

$$A_n(k) = \left[\binom{n}{k} p^k q^{n-k}\right]\left[\frac{q}{1-z}\right] \tag{9}$$

is an upper bound for $B_n(k)$. More exact upper bounds, and also lower bounds, are derived from (3) in Section 3.

The case when $n \to \infty$ (and k varies with n so that (1) is satisfied for each n) is considered in Section 4. Since in this case $a_s \to 1$ for each fixed s (cf. (8)) it might seem plausible that $|A_n - B_n| \to 0$. However, $|A_n - B_n|$ tends to zero if and only if B_n tends to zero, and then A_n is a precise estimate of B_n, in the sense that A_n/B_n tends to one. It is also shown in Section 4 that A_n can be modified by multiplying it with a certain factor so that the relative error in the modified estimate always tends to zero. It turns out that the normal approximation to B_n is an explicit divisor of the correction required by A_n in the general case, so that an estimate of the relative error of the normal approximation in the general case is obtained. This last estimate leads, in particular, to necessary and sufficient conditions in order that the relative error of the normal approximation tend to zero as $n \to \infty$.

3. Bounds for $B_n(k)$. The identity (3) suggests the following method for numerical evaluation of $B_n(k)$ to any desired degree of accuracy. Suppose that we compute F only up to the first $j + 1$ terms and thus take

$$B_n^{(j)}(k) = \left[\binom{n}{k} p^k q^{n-k+1}\right]\left[\sum_{s=0}^{j} a_s z^s\right] \tag{10}$$

as an approximation to $B_n(k)$. Since $k \leq n$, it follows easily from (8) that

$$\frac{a_s}{a_t} \leq a_{s-t} \quad \text{for} \quad s > t, \tag{11}$$

the inequality being strict unless $t = 0$. Consequently we have

$$1 - (a_{j+1} z^{j+1}) < \frac{B_n^{(j)}(k)}{B_n(k)} < 1. \tag{12}$$

Since $a_{j+1}z^{j+1} < a_j z^j$, it follows from (12) that the relative error in $B_n^{(j)}(k)$ does not exceed the last term included in the sum on the right side of (10) Moreover, since

$$a_{j+1}z^{j+1} \leq z^{j+1} \tag{13}$$

it is easy to obtain, in advance of undertaking the calculation, an upper bound to the number of terms required to attain a specified relative accuracy. We note also that, since $A_n(k)$ is an overestimate of $B_n(k)$, (12) implies

$$0 < B_n(k) - B_n^{(j)}(k) < (a_{j+1}z^{j+1}) \min\{A_n(k), 1\}. \tag{14}$$

The preceding method of evaluation of $B_n(k)$, although applicable in general, is efficient only when z is appreciably less than one. A parallel method, with similar properties, can be based on (5).

An alternative method of evaluation, which is useful even if z is nearly one, is the following. Let

$$(15)\quad b_r = \frac{(n+r)(k+r-1)}{(k+2r-2)(k+2r-1)}\,p; \qquad c_r = \frac{r(m-r)}{(k+2r-1)(k+2r)}\,p$$

for $r = 1, 2, \cdots$, where $m = n - k + 1$. Then F defined by (4) can be represented (cf., e.g., [1], Chap. XVIII) as a terminating continued fraction thus:

$$(16)\qquad F = \frac{1}{1-}\,\frac{b_1}{1+}\,\frac{c_1}{1-}\,\frac{b_2}{1+}\,\frac{c_2}{1-}\cdots\frac{c_{m-1}}{1-}\,b_m.$$

Let

$$(17)\qquad \begin{aligned} F^{(1)} &= \frac{1}{1-}\,b_1; \qquad F^{(2)} = \frac{1}{1-}\,\frac{b_1}{1+}\,c_1;\\ F^{(3)} &= \frac{1}{1-}\,\frac{b_1}{1+}\,\frac{c_1}{1-}\,b_2; \qquad F^{(4)} = \frac{1}{1-}\,\frac{b_1}{1+}\,\frac{c_1}{1-}\,\frac{b_2}{1+}\,c_2; \quad \text{etc.}\end{aligned}$$

We have $0 < b_r < 1$ for each r, and $0 < c_r < 1$ for $r = 1, 2, \cdots, m-1$, by (1) and (15). It follows hence from (16) that

$$(18)\qquad \begin{aligned} F^{(3)} \leqq F^{(4)} \leqq F^{(7)} \leqq F^{(8)} \leqq \cdots \leqq F&\\ \leqq \cdots \leqq F^{(6)} \leqq F^{(5)} \leqq F^{(2)} \leqq F^{(1)}.&\end{aligned}$$

The equality signs are included here only for the sake of literal accuracy; in fact, $F^{(r)} = F$ for $r \geqq 2(n-k)+1$, but all other inequalities in (18) are strict.

Now let $A_n^{(r)}(k) = \left[\binom{n}{k}p^k q^{n-k+1}\right]\cdot F^{(r)}$ for $r = 1, 2, \cdots$, where $F^{(r)}$ is given by (15) and (17). We then obtain from (3) and (18) sequences of upper and lower bounds to $B_n(k)$, the general form of these bounds being $A^{(4s-1)} \leqq A^{(4s)} \leqq A^{(4s+3)} \leqq B \leqq A^{(4s+1)} \leqq A^{(4s-2)} \leqq A^{(4s-3)}$ for $s = 1, 2, \cdots$. It should be noted that $A_n^{(1)}(k) = A_n(k)$, where $A_n(k)$ is defined by (6) and (9).

Another method of using continued fractions to obtain bounds on B, which is based on (2) itself rather than (3), is given in Uspensky ([2], pp. 52–56). This method, which is attributed in [2] to Markov, does not appear to be generally known, and might therefore be described here. Let

$$(19)\quad \beta_r = \frac{(m-r)(k-1+r)p}{(k-2+2r)(k-1+2r)q}, \qquad \gamma_r = \frac{r(n+r)p}{(k-1+2r)(k+2r)q}$$

for $r = 1, 2, \cdots$, and let

$$(20)\qquad \begin{aligned} G^{(1)} &= \frac{1}{1-}\,\beta_1; \qquad G^{(2)} = \frac{1}{1-}\,\frac{\beta_1}{1+}\,\gamma_1;\\ G^{(3)} &= \frac{1}{1-}\,\frac{\beta_1}{1+}\,\frac{\gamma_1}{1-}\,\beta_2; \qquad G^{(4)} = \frac{1}{1-}\,\frac{\beta_1}{1+}\,\frac{\gamma_1}{1-}\,\frac{\beta_2}{1+}\,\gamma_2; \quad \text{etc.}\end{aligned}$$

Define $M^{(r)}_{(n)}(k) = \left[\binom{n}{k}p^k q^{n-k}\right]\cdot G^{(r)}$ for $r = 1, 2, \cdots$, where $G^{(r)}$ is given by (19) and (20). Suppose $k > np + 1$. It can then be shown [2] that, as with the A's, we have $M^{(4s-1)} \leqq M^{(4s)} \leqq M^{(4s+3)} \leqq B \leqq M^{(4s+1)} \leqq M^{(4s-2)} \leqq M^{(4s-3)}$ for $s = 1, 2, \cdots$. Here $B = M^{(r)}$ for $r \geqq 2(n-k)$, but all other inequalities are strict. The writer conjectures that we always have

$$(21) \qquad M^{(2r-1)} = A^{(2r-1)}, \qquad |B - M^{(2r)}| \leqq |B - A^{(2r)}|$$

for $r = 1, 2, \cdots$, the inequality being strict for $r \leqq (n-k)$. If so, Markov's method of computation is superior, by one step, to the one described in the preceding paragraph.

The following Theorem 1 shows that, if n and k are large and z is appreciably less than one (i.e. if $B_n(k)$ is very small), then $A_n(k)$ is a good estimate of $B_n(k)$. The theorem is an expression of the fact that, under the conditions stated, even the first few continued fraction approximations to B are very close to B.

Let

$$(22) \qquad \alpha_n(k) = \left(\frac{1}{k+1} - \frac{1}{n+1}\right)\left(\frac{k+1}{k+2}\right)z,$$

where z is given by (6), and let

$$(23) \qquad x_n(k) = \frac{k - np}{\sqrt{npq}} \qquad (0 \leqq x_n(k) \leqq \sqrt{nq/p}).$$

THEOREM 1: *Given integers n and k satisfying* (1), *let B be defined by* (2), *A by* (6) *and* (9), *and α and x by* (22), (23). *Then*

$$(24) \quad 1 + \alpha[z/(1-z)](1+\alpha)^{-1} \leqq A/B \leqq 1 + \alpha[z/(1-z)](1+\alpha-z)^{-1}.$$

In particular,

$$(25) \qquad 1 \leqq A/B < 1 + x^{-2}.$$

PROOF: Since $b_1 = z$ by (6) and (15), we have $F^{(1)} = 1/(1-z)$ from (17). Hence $A/B = F^{(1)}/F$ by (3) and (9). Consequently,

$$(26) \qquad \frac{F^{(1)} - F^{(2)}}{F^{(2)}} \leqq \frac{A}{B} - 1 \leqq \frac{F^{(1)} - F^{(3)}}{F^{(3)}}$$

by (18). A straightforward calculation shows that the lower bound in (26) equals $\alpha[z/(1-z)](1+\alpha)^{-1}$, and that the upper bound equals

$$\alpha[z/(1-z)](1 + \alpha - z + \delta)^{-1},$$

where $\delta = b_1 - b_2$. Since $\delta > 0$, (24) therefore follows from (26). It is easily seen that $\alpha[z/(1-z)](1+\alpha-z)^{-1} \leqq \alpha z/(1-z)^2 < x^{-2}$, so that (25) follows from (24). This completes the proof.

In concluding this section reference may be made to certain bounds given by

Hodges and Lehmann [3] for any probability of the form $\sum_{r=a}^{a+b} \binom{n}{r} p^r q^{n-r}$. The reader may verify that the bounds for B obtainable by taking $a = k$ and $b = n - k$ in [3], p. 331, (3.1), say L and U, always satisfy $L \leq A^{(2)} \leq B \leq U \leq A^{(1)}$.

4. Asymptotic estimates. The normal approximation. In this section we consider a given sequence of positive integers $k_1, k_2, \cdots$ such that

$$np \leq k_n \leq n \qquad (n = 1, 2, \cdots), \tag{27}$$

and we study the behaviour of $B_n(k_n)$ as $n \to \infty$. Since the sequence $\{k_n\}$ remains fixed throughout the discussion, we abbreviate $B_n(k_n)$ to B_n, and $A_n(k_n)$ to A_n. Similarly, $x_n(k_n)$ defined by putting $k = k_n$ in (23) is abbreviated to x_n

Let N_n denote the usual normal approximation to B_n, i.e.

$$N_n = \int_{x_n}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.^2 \tag{28}$$

Let

$$C_n = (x_n + \sqrt{q/np})\sqrt{2\pi} \exp[\tfrac{1}{2}x_n^2]. \tag{29}$$

THEOREM 2: $(A_n N_n C_n)/B_n = 1 + \epsilon_n$, *where* $\epsilon_n \to 0$ *as* $n \to \infty$.

This result is valid without any restriction on the sequence $\{k_n\}$ other than (27). As may be seen from the proof of Theorem 2, ϵ_n is at most of the order $1/\sqrt{n}$ if $\{x_n\}$ is a bounded sequence, and at most of the order $1/x_n^2$ if $x_n \to \infty$ If, however, the sequence $\{x_n\}$ has finite limit points of arbitrarily large magnitude, the order of ϵ_n is indeterminate from the present proof.

To prove Theorem 2, we note first that

$$A_n C_n = \left[\binom{n}{k_n} p^{k_n} q^{n-k_n}\right]\left(1 + x_n\sqrt{q/np} + \frac{1}{np}\right)\sqrt{2\pi npq} \exp[\tfrac{1}{2}x_n^2], \tag{30}$$

by a straightforward computation using (6), (9), (23) and (29). Suppose now that $\{x_n\}$ is a bounded sequence. In this case,

$$n - k_n \to \infty \tag{31}$$

as $n \to \infty$. Since k_n certainly tends to infinity by (27), Stirling's formula can be applied to the binomial coefficient on the right side of (30). This application shows that

$$A_n C_n = 1 + O\left(\frac{1}{\sqrt{n}}\right). \tag{32}$$

Since, by the De Moivre-Laplace limit theorem, we have

$$B_n - N_n = O\left(\frac{1}{\sqrt{n}}\right) \tag{33}$$

---

[2] The case when the normal approximation includes a continuity correction is discussed at the end of this section

in any case, and since N_n is bounded away from zero in the present case, it follows from (32) and (33) that $(A_nN_nC_n)/B_n = 1 + O(1/\sqrt{n})$.

Suppose next that $x_n \to \infty$ as $n \to \infty$. In this case it follows from (28) and (29) by a property of the normal distribution ([4], p. 166) that

$$N_n C_n = 1 + O\left(\frac{1}{x_n^2}\right). \tag{34}$$

It is plain from (25) and (34) that $(A_nN_nC_n)/B_n = 1 + O(1/x_n^2)$.

To treat the general case write $r_n = (A_nN_nC_n)/B_n$ for $n = 1, 2, \cdots$, and let l be a limit point of the sequence $\{r_n\}$, $0 \leqq l \leqq \infty$. Then there exists a strictly increasing sequence of positive integers, say $i_1, i_2, \cdots$, such that $r_n \to l$ as $n \to \infty$ through the sequence i. The sequence i surely contains a subsequence, say $j_1, j_2, \cdots$, such that x_n tends to a finite or infinite limit as $n \to \infty$ through the sequence j. Hence $l = 1$ by the preceding two paragraphs. Thus 1 is the only limit point of the sequence $\{r_n\}$. This completes the proof.

Corollary 1: *As $n \to \infty$, the following four statements are mutually equivalent:*

$$| A_n - B_n | \to 0, \tag{35}$$

$$A_n/B_n \to 1, \tag{36}$$

$$B_n \to 0, \tag{37}$$

$$x_n \to \infty. \tag{38}$$

Proof: The equivalence of (37) and (38) is immediate from (28) and (33). It is evident from (25) that (38) implies (36). Since (36) always implies (35), it will now suffice to show that (35) implies (38). Suppose to the contrary that (35) holds, but that the sequence $\{x_n\}$ has a finite limit point, say a, $0 \leqq a < \infty$. Let $i_1 < i_2 < \cdots$ be a sequence of integers such that $x_n \to a$ as $n \to \infty$ through the sequence i. With n restricted to this sequence, B_n is bounded away from zero, by (28) and (33); hence A_n is also bounded away from zero by (35). It follows from Theorem 2 and (35) that $A_n(1 - N_nC_n) \to 0$. Hence $N_nC_n \to 1$ as $n \to \infty$ through i. This is a contradiction, since $x \to a$ implies ([4], p. 166) that $NC \to b$, where $b = 0$ if $a = 0$ and $0 < b < 1$ if $0 < a < \infty$. This completes the proof.

Next, define

$$f(y) = \left(1 + \frac{y}{p}\right)^{p+y} \left(1 - \frac{y}{q}\right)^{q-y} e^{-y^2/(2pq)}, \tag{39}$$

and

$$g(y) = \sqrt{\frac{1 - (y/q)}{1 + (y/p)}} \tag{40}$$

for $-p < y < q$. Write

$$y_n = \frac{k_n}{n} - p = \frac{x_n}{\sqrt{n}} \sqrt{pq} \qquad (0 \leqq y_n \leqq q). \tag{41}$$

COROLLARY 2: *If* (31) *is satisfied then*

$$N_n/B_n = [f(y_n)]^n g(y_n)(1 + \epsilon_n), \tag{42}$$

where $\epsilon_n \to 0$ *as* $n \to \infty$.

This corollary to Theorem 2 follows from (30), (39), (40) and (41) by an application of Stirling's formula. We omit the detailed calculation. The corollary is a generalization, in the present very special case, of estimates of the type introduced by Cramér and developed by Feller and Petrov. Petrov has recently given the best versions of such estimates [5][3]. The generalization consists in replacing the condition

$$y_n \to 0 \tag{43}$$

of Petrov's theorems with the much weaker condition (31). However, the order of the ϵ_n in (42) remains indeterminate.

Corollary 2 is useful in certain applications [3], [6] where B_n tends to zero very rapidly. Another application, with which the remainder of this section is concerned, is to the study of exact conditions under which

$$\lim_{n\to\infty} \{R_n\} = 1 \tag{44}$$

or at least

$$0 < \liminf_{n\to\infty} \{R_n\} \leq \limsup_{n\to\infty} \{R_n\} < \infty, \tag{45}$$

where

$$R_n = N_n/B_n. \tag{46}$$

Define

$$\varphi(t) = \log(1/t) + \frac{(t-1)}{2t} \tag{47}$$

for $0 < t < 1$. It is easily seen that, as t increases from 0 to 1, φ increases steadily from $-\infty$ to a positive maximum at $t = \frac{1}{2}$ and then decreases to zero. Let ρ denote the root of the equation $\varphi(t) = 0$, $0 < \rho < \frac{1}{2}$. ($\rho \doteq .2847$).

COROLLARY 3: *If* $p > \frac{1}{2}$ *or* $p \leq \rho$, *then* (44) *holds if and only if*

$$ny_n^2 = o(1). \tag{48}$$

If $p = \frac{1}{2}$, (44) *holds if and only if*

$$ny_n^4 = o(1). \tag{49}$$

If $\rho < p < \frac{1}{2}$, (48) *is sufficient for* (44), *but may not be necessary*; *if, however,* $\lim_{n\to\infty} y_n$ *exists, then* (48) *is necessary for* (44).

---

[3] Petrov's work was pointed out to the writer by Mr. Ranga Rao of the Indian Statistical Institute. The writer wishes to thank Mr. Ranga Rao for valuable suggestions and discussions during the preparation of this paper.

In view of (41), conditions (48) and (49) are restrictions on the rate at which x_n becomes large, if it does so at all.[4] It is therefore rather surprising that (44) can hold in the case $\rho < p < \frac{1}{2}$ even when $\{x_n\}$ contains a subsequence which tends to infinity very rapidly. The details of this exceptional case are given in the course of the following proof.

To prove Corollary 3, suppose first that we have $n - k_n = m$ for all n, where m is a fixed non-negative integer. In this case it follows from Theorem 2 by (23), (30), (46) that

$$\log R_n = n\,\varphi(p) - \log\binom{n}{m} - \tfrac{1}{2}\log n + O(1), \tag{50}$$

where φ is given by (47). The right side of (50) $\to +\infty$ or $-\infty$ according as $p > \rho$ or $p \leqq \rho$, so that (45) does not hold.

It now follows that (31) is necessary for (45). For, if (31) is not satisfied, there exists an m such that $n - k_n = m$ for infinitely many n, and $\log R_n$ is therefore unbounded, by the preceding paragraph. It will be shown presently that in fact

$$\limsup_{n\to\infty} \{y_n\} < q \tag{51}$$

is necessary for (45).

Let

$$h(y) = \log f(y) \tag{52}$$

for $0 \leqq y < q$, where f is given by (39). We shall require the following easily verified properties of h regarded as a function of y. (i) In the neighborhood of $y = 0$, h is of the order y^3 if $p \neq \frac{1}{2}$ and of the order y^4 if $p = \frac{1}{2}$. (ii) $h(y) \to \varphi(p)$ as $y \to q$, where φ is given by (47). (iii) If $p \geqq \frac{1}{2}$, h is positive and steadily increasing in the interval $(0, q)$. (iv) If $p \leqq \rho$, where ρ is the number defined in the paragraph containing (47), then h is negative in the interval $(0, q)$. (v) If $\rho < p < \frac{1}{2}$ then the equation $h = 0$ has a root, a say, in the interval $(0, q)$; h is negative in $(0, a)$ and positive and increasing in (a, q); the derivative of h is positive at $y = a$.

Let us write

$$w_n = n\,h(y_n) + \log g(y_n). \tag{53}$$

It then follows from (41), (42), (46), (52) and (53) that

$$\log R_n = w_n + o(1) \tag{54}$$

provided only that (31) is satisfied.

We can now show that (51) is necessary for (45). Since (45) is already known to imply (31), it follows from (54) that it will suffice to show that $w_n = O(1)$ and (31) imply (51). First consider the case when $p \leqq \rho$. In this case $h(y_n) \leqq$

[4] It is well known that (48) always implies (44): [4], pp. 178–181 and [5].

0 and hence $w_n \leqq \log g(y_n)$ for every n, by (53). It now follows by referring to (40) that (51) must hold, for otherwise $\liminf \{w_n\} = -\infty$. Now consider the case when $\rho < p < 1$, and suppose that $\{y_n\}$ contains a subsequence tending to q. Since in the present case $h(y)$ tends to a positive limit as $y \to q$, it follows from (40) and (53), using the hypothesis $w_n = O(1)$, that there exist positive constants c_1 and c_2 such that $\log (q - y_n) < c_1 - c_2 n$ for infinitely many n. Hence $\liminf \{n(q - y_n)\} = 0$. This contradicts (31), since $n(q - y_n) = (n - k_n)$ by (41).

Since (51) evidently implies that (31) holds, and also that $\log g(y_n) = O(1)$, the following general criterion is now plain from (53) and (54): (45) *holds if and only if* (51) *is satisfied and* $n \cdot h(y_n) = O(1)$. By reference to the properties of the function h we see that this criterion reduces to (48) with o replaced by O in case $p > \frac{1}{2}$ or $p \leqq \rho$, and to (49) with the same modification in case $p = \frac{1}{2}$. The reduction of the criterion in the case $\rho < p < \frac{1}{2}$ is also straightforward and is omitted.

It follows easily from the preceding criterion and (53) and (54) that (44) *holds if and only if* (51) *is satisfied and* $w_n = o(1)$. This reduces to (48) if $p > \frac{1}{2}$, or if $p \leqq \rho$, and to (49) if $p = \frac{1}{2}$. In case $\rho < p < \frac{1}{2}$, the present criterion reduces to the following: 1) the sequence $\{y_n\}$ has no limit points other than 0 and $a = a(p)$, where a is the positive root of the equation $h(y) = 0$, $0 < a < q$; 2) if $i_1, i_2, \cdots$ is any increasing sequence of positive integers such that $y_n \to 0$ as $n \to \infty$ through the sequence i, then (48) holds for n restricted to i; and 3) if $j_1, j_2, \cdots$ is any increasing sequence of positive integers such that $y_n \to a$ as $n \to \infty$ through the sequence j, then

$$y_n = a + \frac{b}{n} + o\left(\frac{1}{n}\right) \tag{55}$$

for n restricted to j, where $b = [h'(a)]^{-1} \log [g(a)]^{-1}$.

It is clear that if (48) holds then 1), 2) and 3) are satisfied, 3) being vacuous. We shall now show that in general 3) is not vacuous, i.e. there are values of p and corresponding sequences $\{k_n\}$ of *integers* for which (55) holds for n restricted to some sequence j. For any non-negative number r, let $[r]$ denote the greatest integer contained in r. Then (55) can be written as

$$k_n - \{[b] + [n(p + a)]\} = \theta + \xi_n + \epsilon_n \tag{56}$$

where θ is a constant, $0 \leqq \theta < 1$, $0 \leqq \xi_n = n(p + a) - [n(p + a)] < 1$, and $\epsilon_n \to 0$. Suppose that $p + a$ is irrational. Then, as is well known, each point in $[0, 1]$ is a limit point of the sequence $\{\xi_n\}$. Consequently, there exists a sequence $j_1, j_2, \cdots$ such that $\xi_n \to 1 - \theta$ as $n \to \infty$ through j. If we let

$$k_n = [n(p + a)] + [b] + 1$$

for $n = j_1, j_2, \cdots$ and $k_n = [np + \sqrt{n}]$ (say) for all other values of n, it follows that 1), 2) and 3) are satisfied. Thus 3) is non-vacuous at least when $p + a(p)$ is irrational. It is not difficult to see that $p + a(p)$ is a non-constant and continuous function of p, so that it does in fact assume irrational values as p varies from ρ to $\frac{1}{2}$.

To complete the proof of Corollary 3, consider an arbitrary but fixed p in $(\rho, \frac{1}{2})$, and suppose that $\lim_{n\to\infty} y_n$ exists for the given sequence $\{k_n\}$. Assume, contrary to the last statement in Corollary 3, that (44) holds but (48) does not. It then follows from the necessary and sufficient conditions 1), 2) and 3) that in the present case (55) holds as $n \to \infty$ through the entire sequence 1, 2, 3, $\cdots$. Express (55) in the form (56). Since the left side of (56) is an integer, since $\epsilon_n \to 0$, and since θ and ξ_n are in $[0, 1)$ for each n, it follows that, for all sufficiently large n, $\theta + \xi_n + \epsilon_n = 0$, or 1 or 2. Let L denote the set of all limit points of the sequence $\theta + \xi_n$. We then have $L \subset \{0, 1, 2\}$.

The conclusion of the preceding paragraph implies that $p + a$ cannot be irrational. Suppose therefore that $p + a = u/v$, where u and v are integers such that $0 < u < v$. Assuming that u/v is in its lowest terms, the limit points of the sequence ξ_n are 0, $1/v$, $2/v$, $\cdots$, and $(v - 1)/v$. Hence

$$L = \{\theta + (r/v): r = 0, 1, 2, \cdots, v - 1\}.$$

This implies, in particular, that θ is in L. Hence $\theta = 0$, or 1, or 2 by the preceding paragraph; hence $\theta = 0$, since $0 \leq \theta < 1$ in any case. We now see that $L = \{(r/v): r = 0, 1, \cdots, v - 1\}$. This cannot be a subset of $\{0, 1, 2\}$ unless $v = 1$. However, $v = 1$ implies $0 < u < 1$ and is therefore a contradiction. This completes the proof.

It may be of some interest to examine the modifications required in Corollary 3 when the normal approximation includes a correction for continuity, e.g., when N_n is defined by (28) but with $x_n = (k_n - \frac{1}{2} - np)/(npq)^{\frac{1}{2}}$. It turns out that Corollary 3 requires no modification for this particular continuity correction. For certain more general 'corrections', the only modification is that (48) is not necessary for (44) if $\rho < p < \frac{1}{2}$, even if $\lim y_n$ exists.

The conclusions just stated are readily derived as follows. Let $\{c_n\}$ be a bounded sequence, and let

$$(57)\qquad \begin{aligned} x_n^* &= \frac{k_n - c_n - np}{\sqrt{npq}} = x_n - \frac{c_n}{\sqrt{npq}}, \\ N_n^* &= \int_{x_n^*}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}t^2}\, dt, \\ R_n^* &= N_n^*/B_n. \end{aligned}$$

We wish to know whether

$$(58)\qquad \lim_{n\to\infty} \{R_n^*\} = 1.$$

Suppose for the moment that $x_n \to \infty$. Then x_n^* also $\to \infty$, and it follows easily from [4], p. 166 that $N_n/N_n^* = (1 + \epsilon_n) \exp(-c_n y_n/pq)$, where $\epsilon_n \to 0$. This asymptotic formula is surely valid if $\{x_n\}$ is bounded, for then $y_n \to 0$ and

$$N_n/N_n^* \to 1.$$

It follows therefore (cf. the paragraph preceding Corollary 1) that the formula is valid in general, i.e.

$$\log N_n^* = \log N_n + (c_n y_n/pq) + o(1) \tag{59}$$

in general.

Since $\{c_n y_n\}$ is a bounded sequence, it follows from (46), (57), (59), and the proof of Corollary 3, that (58) holds if and only if

$$\limsup_{n\to\infty} \{y_n\} < q, \; n\,h(y_n) + \log g(y_n) + (c_n y_n/pq) = o(1). \tag{60}$$

If $p > \frac{1}{2}$, or if $p \leq \rho$, (60) reduces to (48). (60) reduces to (49) if $p = \frac{1}{2}$. Suppose next that $\rho < p < \frac{1}{2}$, and that c_n is a constant, say $c_n = c$. In this case (60) reduces to conditions 1), 2) and 3) of the paragraph containing (55), but with b replaced by $(-\log g(a) - ca/pq)/h'(a)$. Since the value of b is immaterial to the arguments following (55), we conclude that in the present case (48) suffices for (58) but may not be necessary, unless $\lim y_n$ exists.

It remains therefore to consider the case when $\rho < p < \frac{1}{2}$ but the c_n are not constant. Here (48) is sufficient for (58), but is not necessary, even if $\lim y_n$ exists. Indeed, if $k_n = n(p + a) + b_n$ for each n, where $\{b_n\}$ is a bounded sequence, and we take $c_n = -(pq/a)(b_n h'(a) + \log g(a))$, then (60) and therefore (58) is satisfied.

REFERENCES

[1] H. S. WALL, *Analytic Theory of Continued Fractions*, D. Van Nostrand Co. Inc., New York, 1948.

[2] J. V. USPENSKY, *Introduction to Mathematical Probability*, McGraw Hill Book Co., New York, 1937.

[3] J. L. HODGES, JR., AND E. L. LEHMANN, "The efficiency of some nonparametric competitors of the t test," *Ann. Math. Stat.*, Vol. 27 (1956), p. 324.

[4] W. FELLER, *An Introduction to Probability Theory and Its Applications*, Vol. I (second edition), John Wiley and Sons, New York, 1957.

[5] V. V. PETROV, "Generalization of Cramér's limit theorem," *Uspekhi Mat. Nauk.*, Vol. 9 (1954), pp. 195–202 (in Russian).

[6] R. R. BAHADUR, "Simultaneous comparison of the optimum and sign tests of a normal mean," *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling*, Stanford University Press, to be published in 1960.