ON LINEAR OPERATORS LEAVING A CONVEX SET INVARIANT IN BANACH SPACES

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SUMMARY. Here we study the spectral theory of compact operators which leave a convex set invariant in a Banach space. We also obtain a theorem in positive operators by an application of the general minimax theorem of Ky Fan.

1. Introduction

The theory of Positive Operators has acquired much importance as a result of its applications to problems of mechanics and Markov process. The theory in finite dimensions is the well-known theorem of Perron and Frobenius on the spectrum of matrices with non-negative entries. Such matrices leave the positive orthant invariant and Krein and Rutman (1948) extended the theorem to infinite dimensions by studying operators that leave a cone invariant in Banach spaces. Since cones are in particular, convex sets, the following question naturally arises. What can we say about operators that leave a closed convex set with the origin 0 as its extreme point, invariant ? Can we generalize the theorem of Krein to the new case without any restriction ?

By taking the generated cone of the convex set we observe that the conditions of the theorem of Krein are satisfied. If the spectral radius is positive we can only say that the eigen vector corresponding to the spectral radius belongs to the closure of the generated cone. In general no eigen vector for the eigen value need belong to the convex set even when $K-K$ is dense in the whole Banach space E for the convex set K. This follows from the following two examples.

Example 1 : $E = R^2$, $\{K = (x, y) : x \geqslant y^2,\ y \geqslant 0\}$.

$$A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with $AX = (a_{11}x+a_{12}y,\ a_{21}x+a_{22}y)$ for $X = (x, y)'$.

Here we have $AK \subseteq K$ and $K-K = R^2$. The spectral radius is $\frac{1}{2}$ and all the eigen vectors of $\frac{1}{2}$ lie on the y-axis. Thus they belong to the closure of the generated cone and none of them lie in the convex set.

Example 2 : $E = R^4$, $\{K = (x_1, x_2, x_3, x_4)', (x_1+x_3) \geqslant (x_2-x_4)^2 \; x_1, x_2, x_3, x_4 \geqslant 0\}$,

$$A = \begin{bmatrix} \frac{1}{8} & 0 & \frac{1}{8} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad x \to Ax \text{ is defined as above}$$

Here also we find $AK \subseteq K$ with $K-K = R^4$. The spectral radius is $\frac{1}{2}$ and its eigen vectors lie on the subspace $S = \{(x_1, x_2, x_3, x_4)' : x_1 = 0 \; x_3 = 0\}$. Since $S \cap K = 0$ no eigen vector for $\frac{1}{2}$ lies in K. Thus the eigen vectors do not belong to K but they lie in the closure of the generated cone. In fact, we shall show that an eigen vector for the spectral radius lies in the convex set K, whenever the spectral radius is strictly greater than unity.

2. Convex sets and linear functionals

We shall always consider a real Banach space E with its conjugate space E^* of real bounded linear functionals on E.

Unless otherwise stated, K will denote a closed convex set with the origin 0 as an extreme point.

We say $x \geqslant y$ whenever $x-y \in K$. Thus $x \geqslant 0$ for all $x \in K$. Since K is just convex $\geqslant$ is not even a partial order but still we have

(i) $x \leqslant y, \; y \leqslant z \Longrightarrow \frac{x}{2} \leqslant \frac{z}{2}$.

(ii) $x \leqslant y, \; 0 \leqslant \lambda \leqslant 1 \Longrightarrow \lambda x \leqslant \lambda y$

(iii) $x \leqslant y \Longrightarrow -y \leqslant -x$

(iv) $x_1 \leqslant y_1, \; x_2 \leqslant y_2 \Longrightarrow \frac{x_1+x_2}{2} \leqslant \frac{y_1+y_2}{2}$

We say $x < y$ whenever $y-x \in$ interior of K, when it exists.

For K with interior we have

$$(\alpha)\quad x > 0,\ 0 < \lambda \leqslant 1 \Longrightarrow \lambda x > 0$$

$$(\beta)\quad x \leqslant y,\ y < z \Longrightarrow \frac{x}{2} < \frac{z}{2}.$$

$$(\gamma)\quad x_1 \leqslant y_1,\ x_2 < y_2 \Longrightarrow \frac{x_1+x_2}{2} < \frac{y_1+y_2}{2}$$

Properties (α), (β), (γ) are consequences of the fact that every point on the line joining a boundary point and an interior point lies completely in the interior of the convex set except for the one boundary point.

Definition 1 : K is finitely reproducing whenever $E = K - K$.

Lemma 1 : *K is finitely reproducing when K has spheres of arbitrary radius.*

Proof: Let $x \epsilon E$ and $|x| = P$ be the norm of x. Take a closed sphere $S(u, r) \subseteq K$ with centre u and radius $r > P$.

Now, $y_1 = u + r\dfrac{x}{|x|}$ and $y_2 = u - r\dfrac{x}{|x|}$ belong to K.

Since $$0 \leqslant \frac{|x|}{2r} < 1, \quad \text{we have } \frac{|x|}{2r}\cdot y_1,\ \ \frac{|x|}{2r}\cdot y_2 \epsilon K.$$

Further $$x = \frac{|x|}{2r}\cdot y_1 - \frac{|x|}{2r}\cdot y_2$$

and this completes the proofs of the lemma.

Remark 1 : In the case of cones, K having non-empty interior is sufficient, but this is not true for convex sets. We consider the following as an example.

Example 3 : $E = R^2 \cdot \{K = (x, y) : (x, y) \geqslant 0,\ y = 1\}$.

K is convex but $K - K \neq E$ even though K is unbounded, with non-empty interior. We state the Lemmas 2, 3, 4, 5 and 6 from Krein and Rutman (1948), without proofs.

Lemma 2 : *If K is a cone with interior then any additive functional, non-negative on K is bounded and hence belongs to E^*.*

Lemma 3 : *Let K be a cone. If $f \epsilon K^*$, (i.e. $f \epsilon E^*$ and $f(y) \geqslant 0$ for all $y \epsilon K$) and u is an interior point of K, then $f(u) \geqslant \rho|f|$, for some $\rho > 0$, such that $S(u, \rho) \subseteq K$.*

Lemma 4 : *Let K be a cone. An element $x_0 \epsilon \bar{K}$, the closure of K if and only if $f(x_0) \geqslant 0$ for every $f \epsilon K^*$.*

Lemma 5 : *If K is a cone with interior then for any $f \epsilon K^* - 0$ and $x \epsilon$ interior of K, $f(x) > 0$.*

Definition 2 : A cone K is normal if and only if for any pair of points $x, y \in K$

$$|x+y| \geqslant \delta \max \{|x|, |y|\}, \delta > 0.$$

or equivalently $|x+y| \geqslant \delta$, when $x, y \in K$ with $|x| = |y| = 1$.

Lemma 6 : *If K is a normal cone with interior then the set $I_u = \{y : -u \leqslant y \leqslant u\}$ is bounded, for every interior point u point $u \in K$.*

Definition 3 : An operator A is strongly positive whenever $A K \subseteq K$ and $A^{n(x)}x > 0$ for every $x \in K$, where $n(x)$ is a positive integer depending on x.

3. Minimax theorem and positive operators in reflexive Banach spaces

In this section we shall use the following theorem of Ky Fan (1953) to prove a theorem of Krein in positive operators.

Proposition (Ky Fan) : If K_1, K_2 are 2 compact convex sets in locally convex linear topological spaces E_1 and E_2 respectively and if $f(x, y)$ is a bilinear functional on $E_1 \times E_2$ possessing continuity in each variable, then

$$\min_{K_2} \max_{K_1} f(x, y) = \max_{K_1} \min_{K_2} f(x, y) = f(x_0, y_0); \; x_0 \in K_1, \; y_0 \in K_2$$

Theorem 1 : *Let K be a closed cone in a reflexive Banach space E. Let K, K^* have non-empty interior. Further let A be a strongly positive bounded linear operator. Then*

(1) *The spectral radius λ_0 is an eigen value of A.*

(2) *There exists an eigen vector z for λ_0 with $z > 0$.*

(3) *The subspace $S_{\lambda_0} = \{y : Az = \lambda_0 z\}$ is one-dimensional.*

(4) *A^* has an eigen vector f for λ_0 which is strictly positive on $K^* - 0$.*

(5) *No other linearly independent eigen vectors of A or A^* lie in K or K^* respectively.*

Proof : Since $E = E^{**}$ we have by Lemma 4, $K = K^{**}$. For any u in the interior of K and $f_1, f_2 \in K^*$

$$f_1(u) + f_2(u) \geqslant 2\rho > 0$$

by Lemma 3, where $|f_1| = |f_2| = 1$ and $S(u, \rho) \subseteq K$.

Therefore $$|f_1 + f_2| \geqslant f_1(u) + f_2(u) \geqslant 2\rho.$$

This shows that K^* is normal. Similarly K is also normal, since K^* has non-empty interior and $K^{**} = K$.

Let us consider the following sets π_1 and π_2.

$$\pi_1 := \{f : f(y_0) = 1\} \cap K^* \quad y_0 \in \text{ interior of } K, |y_0| = 1.$$

$$\pi_2 := \{x : f_0(x) = 1\} \cap K \quad f_0 \in \text{ interior of } K^* |f_0| = 1.$$

By Lemma 3, $f(y_0) \geqslant \rho_1 \|f\|$ and $f_0(x) \geqslant \rho_2 |x|$ when $S(y_0, \rho_1) \subseteq K$ and $S(f_0, \rho_2) \subseteq K^*$. Thus π_1 and π_2 are bounded by $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$ respectively. They are also weakly closed and by the reflexivity of E they are compact in their weak topologies. Moreover they are cross sections of K and K^* (i.e. $f \epsilon K^* - 0 \Longrightarrow \lambda f \epsilon \pi_1$ for some $\lambda > 0$ and $x \epsilon K - 0 \Longrightarrow \mu x \epsilon \pi_2$ for some $\mu > 0$).

Consider the bilinear functional

$$K_\lambda(f, y) = (f, (A-\lambda)y) = f((A-\lambda)\, y).$$

which is weakly continuous in each variable f and y, for all λ, real.

Now by applying the theorem of Ky Fan we have

$$v(\lambda) = \min_{\pi_2} \max_{\pi_1} (f(A-\lambda)y) = \max_{\pi_1} \min_{\pi_2} (f(A-\lambda)\, y) = (f_0 (A-\lambda) y_0)$$

where $f_0 \epsilon \pi_1$ and $y_0 \epsilon \pi_2$.

We observe the following

(1) $v(0) > 0$

(2) $v(\lambda)$ is continuous and non-increasing.

(3) $v(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$.

$v(0) = (f', Ay')$ for some $f' \epsilon \pi_1$ and $y' \epsilon \pi_2$.

But

$(f' Ay') \geqslant (f, Ay')$ for all $f \epsilon \pi_1'$ and in particular

$v(0) = (f' \ Ay') \geqslant (f_0 \, Ay') > 0$ by the strong positivity of A.

Let f_1, y_1, f_2, y_2 be minimax solutions for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. Therefore,

$$v(\lambda_2) \leqslant (f_2 (A-\lambda_2) y_1)$$
$$v(\lambda_1) \geqslant (f_2 (A-\lambda_1) y_1).$$

Thus $$v(\lambda_2) - v(\lambda_1) \leqslant (f_2, (\lambda_1 - \lambda_2) y_1) \leqslant |\lambda_1 - \lambda_2| \, |f_2| \, |y_1|. \quad \ldots (3.1)$$

Changing the suffixes we have

$$v(\lambda_1) - v(\lambda_2) \leqslant |\lambda_1 - \lambda_2| \, |f_1| \, |y_2|. \quad \ldots (3.2)$$

Therefore, we find from (3.1) and (3.2) that

$$|v(\lambda_2) - v(\lambda_1)| \leqslant |\lambda_1 - \lambda_2| \; C(f_1, f_2, y_1, y_2)$$

where C is a positive constant depending on f_1, f_2, y_1 and y_2. Therefore $v(\lambda)$ is continuous.

One can easily show that $v(\lambda)$ is non-increasing. For any λ if f_λ is the maximal solution then

$$v(\lambda) \leqslant (f_\lambda (A-\lambda) y_0) = (f_\lambda \, Ay_0) - \lambda (f \, y_0) \leqslant (f_\lambda \, Ay_0) - \lambda (f \, y_0).$$

If $S(y_0, \rho) \subseteq K$, for an interior point y_0 of K then since π_2 is bounded and $(f \, y_0) > \rho |f|$, we find $v(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow +\infty$.

Thus by the continuity of $v(\lambda)$ we find

$$v(\lambda_0) = 0 \text{ for some } \lambda_0 > 0. \quad \ldots \quad (3.3)$$

Let z, ψ be optimal for $\lambda = \lambda_0$. This gives

$$(\psi\,(A-\lambda_0)y) \geqslant 0 \quad \text{for all,} \quad y \in K \quad (\text{since } \pi_2 \text{ generates } K)$$

$$(f\,(A-\lambda_0)z) \leqslant 0 \quad \text{for all,} \quad f \in K^* \quad (\text{since } \pi_1 \text{ generates } K^*).$$

Therefore by Lemma 4 we have

$$(A-\lambda_0)^*\psi \in K^* \text{ and } (\lambda_0-A)z \in K, \quad \ldots \quad (3.4)$$

and $\quad (\lambda_0-A)z \in K$ implies $Az \leqslant \lambda_0 z$ and that $0 < A^{n(t)}z \leqslant \lambda^{n(t)}z. \quad \ldots \quad (3.5)$

Thus $z > 0$.

If $(A-\lambda_0)^*\psi \neq 0$ then by Lemma 4 and by steps (3.4) and (3.5) it is seen that $(\psi\,(A-\lambda_0)z) = ((A-\lambda_0)^*\psi\; z) > 0$. This contradicts z to be a minimax solution.

Thus

$$A^*\psi = \lambda_0\psi. \quad \ldots \quad (3.6)$$

If $(\lambda_0-A)z \neq 0$ then, since $(A^{*n}\psi)\,(y) = \lambda_0^n\,\psi(y)$ implies $\psi(y) > 0$ for $y \in K-0$ by the strong positivity of A, we have by step (3.4) $(\psi, (\lambda_0-A)z) > 0$ which contradicts the optimality of z as a solution. Therefore

$$Az = \lambda_0 z. \quad \ldots \quad (3.7)$$

Now, if $\tilde{A}y = \lambda_0 y$ for any $y \in \tilde{E}$ linearly independent of z in the complex extension $\tilde{E}$ of E where $\tilde{A}$ is the extension of A to $\tilde{E}$, we can take $y \in E$ without loss of generality by the positivity of λ_0. We can also choose a real α, with $y+\alpha z$ lying on the boundary of K. This shows that an eigen vector belongs to the boundary of K which contradicts the strong positivity of A. Thus S_{λ_0} is 1-dimensional. Similarly one can show that $S^*_{\lambda_0} = \{f : A^*f = \lambda_0 f\}$ is also 1-dimensional.

Since K is normal $I_z = \{y : -z \leqslant y \leqslant z\}$ is bounded.

Further

$$-A^n z \leqslant A^n y \leqslant A^n z,$$

that is,

$$-z \leqslant \frac{A^n y}{\lambda_0 n} \leqslant z \quad \text{and} \quad \frac{|A^n y|}{\lambda_0 n} \leqslant c$$

by the boundedness of I_z. Thus

$$\left|\sum_0^\infty \frac{A^n y}{\lambda^{n+1}}\right| \leqslant \sum_0^\infty \frac{|A^n y|}{|\lambda^{n+1}|} \leqslant c \sum_0^\infty \frac{\lambda_0^n}{|\lambda|^{n+1}} < \infty \text{ for } |\lambda| > \lambda_0.$$

Moreover, $R(\lambda, A)y = -\Sigma \dfrac{A^n y}{\lambda^{n+1}}$ exists for $y \in I_z$.

Since $-z \leqslant \epsilon \dfrac{y}{y} \leqslant z$ for any y and some $\epsilon > 0$. $R(\lambda, A)y = (A-\lambda)^{-1}y$ exists for all y and all λ with $|\lambda| > \lambda_0$. Since λ_0 is also an eigen value, we conclude that λ_0 is the spectral radius of A. Since $A\;\; u = \alpha u$, $u \in K-0$ implies $\alpha(\psi, u) = (\psi, Au) = (A^*\psi, u) = \lambda_0(\psi, u)$. $(\psi, u) > 0$, we have $\lambda_0 = \alpha$ and $u \in S\lambda_0$. This completes the proof of the theorem.

Remark 2 : Even if E is not reflexive and K^* has no interior still we can by the same proof show that all the assertions of the theorem except for the first one (i.e. λ_0 is the spectral radius) are true, provided the cone K has a weakly compact cross-section or even a compact cross-section in some locally convex topology in which the operator A is also continuous when restricted to the cross-section. The closed bounded cross-section in K^* is always weakly compact, as closed bounded set in E^* are weakly compact.

Thus the general minimax theorem, yields also Theorem 1 of Schaefer (1960) in a restricted sense.

4. Operators leaving a convex set invariant

First of all we shall prove a theorem in finite dimensions.

Theorem 2 : *Let 0 be an extreme point of a closed convex set in* R^n *(real Euclidean n-space). Let A be a linear transformation with* $AK \subseteq K$. *Further, let* $A^*f - f$ *be non-negative on K for some* $f \epsilon R^n$ *and* $T = \{x : f(x) = c, c > 0\} \cap K$ *be bounded for some c.*

Then

(1) *A has an eigen value* $\lambda_0 \geqslant 1$ *and*

(2) *A has an eigen vector* $y_0 \epsilon K$ *for* λ_0.

Proof: T being closed and bounded T is compact. Moreover T is convex. Since $A^*f - f$ is non-negative on K, then

$$(A^*f - f)(x) = f(Ax) - f(x) \geqslant 0 \text{ for all } x \epsilon K.$$

Thus $(f, Ax) \geqslant c$ for all $x \epsilon T$.

If $K_1 = \{x : f(x) \leqslant c\} \cap K$, then $0 \epsilon K$ evidently, and $Ax \notin K_1 - T$. Thus for any $x \epsilon T$, the line joining Ax and 0 cuts T at a unique point, say $\mu_x \cdot Ax$. Consider the map $\phi : x \rightarrow \mu_x \cdot Ax$ of T into itself. Since $Ax \notin K_1 - T$, $\mu_x \leqslant 1$. It is easily seen that ϕ is also continuous. Therefore by Browor's fixed point theorem $\phi(y_0) = y_0$ for some $y_0 \epsilon T$. i.e. $\mu_{y_0} \cdot Ay_0 = y_0$ which shows that

$$Ay_0 = \lambda_0 y_0, \ \lambda_0 = \frac{1}{\mu y_0} \geqslant 1, \ y_0 \epsilon T.$$

This completes the proof of the theorem.

As stated in the beginning of Section 2, we will take K to be a closed convex set with 0 as an extreme point in the real Banach space E. Further we assume $\overline{K - K} = E$. By Lemma 1, we know that it is true if K contains spheres of arbitrary radius.

We shall give the first infinite dimensional generalization of Krein's theorem to convex sets as follows.

Theorem 3 : *Let A be a compact operator with* $AK \subseteq K$. *If the spectral radius* λ_0 *of A is strictly greater than unity then,*

$$Ay_0 = \lambda_0 y_0 \text{ for some } y_0 \epsilon K.$$

We shall prove the theorem under different cases.

Case 1 : The spectral radius $\lambda_0 \geqslant 2$ and it belongs to the spectrum $\sigma(A)$.

Proof : By the well known theory of compact operators in Banach spaces, we have

$$(A-\lambda)^{-1} = R(\lambda_0 A) = \sum_{R=-n}^{\infty} (\lambda-\lambda_0)^R \Gamma_R$$

where the Laurant's series as an analytic operator function has only finite poles. Further Γ_R, $R = -n, \ldots$ are bounded operators. Since $\overline{K-K} = E$, there exists a $u \epsilon K$ with $\Gamma_{-n} u \neq \theta$.

Now $$R(\lambda, A)u = -\sum_{0}^{\infty} \frac{A^n u}{\lambda^{n+1}} \qquad \text{for } \lambda > \lambda_0 \geqslant 2.$$

since $$AK \subseteq K,\ A^n u\ \epsilon K \qquad \text{for all } n.$$

For any finite N, $\sum_{0}^{N} \frac{A^n u}{\lambda^{n+1}} + \left(1 - \sum_{0} \frac{1}{\lambda^{n+1}}\right) \theta$ is a convex linear combination of elements of K. This follows from the fact that

$$\sum_{0}^{N} \frac{1}{\lambda^{n+1}} \leqslant \sum_{0}^{N} \frac{1}{2^{n+1}} < 1.$$

Thus $$\sum_{0}^{N} \frac{A^n u}{\lambda^{n+1}}\ \epsilon K \text{ for all } N.$$

Since K is closed, $\lim_{N \to \infty} \sum_{0}^{N} \frac{A^n u}{\lambda^{n+1}}\ \epsilon K$, where limit is taken in the strong topology, i.e. $-R(\lambda, A)u \epsilon K$.

Thus $$\lim_{\lambda \to \lambda_0+0} -(\lambda-\lambda_0)^n R(\lambda, A)u = -\Gamma_{-n} u \epsilon K.$$

Let $y_0 = -\Gamma_{-n} u$. Now, we have

$$AR(\lambda, A)u = \lambda R(\lambda, A)u + u$$

and that

$$-\lim_{\lambda \to \lambda_0+0} A(\lambda-\lambda_0)^n R(\lambda, A)u = -\lim_{\lambda \to \lambda_0+0} \lambda(\lambda-\lambda_0)^n R(\lambda, A)u - \lim_{\lambda \to \lambda_0+0} (\lambda-\lambda_0)^n u \qquad \ldots (4.1)$$

In (4.1) the last term on the right hand side is zero.

We have $$Ay_0 = \lambda_0 y_0 \text{ where } y_0 = -\Gamma_{-n}\ u.$$

Case 2 : Among the characteristic numbers of A of maximal modulus there is a root of a positive number.

Let $Au = \lambda_0 u$. Since $\lambda_0 > 1$, by the assumption in this case, we have $\lambda_0^n > 2$ for some n. Since $A^n u = \lambda_0^n u$, λ_0^n is also the spectral radius of A and therefore by Case 1, we have $A^n y_0 = \lambda_0^n y_0$ for some $y_0 \epsilon K$.

Consider $$y_0 = \left(\frac{A^{2n-1}y_0}{|\lambda_0^{2n}|} + \frac{A^{2n-2}y_0}{|\lambda_0^{2n-1}|} + \dots + \frac{A^{n+1}y_0}{|\lambda_0^{n+2}|} + \frac{A^n y_0}{|\lambda_0^{n+1}|}\right)$$

We have $$Ay_0 = \left(\frac{A^{2n}y_0}{|\lambda_0^{2n}|} + \frac{A^{2n-1}y_0}{|\lambda_0^{2n-1}|} + \dots \dots + \frac{A^{n+2}y_0}{|\lambda_0^{n+1}|}\right)$$

$$= |\lambda_0|\left(\frac{A^{2n-1}y_0}{|\lambda_0^{2n}|} + \frac{A^{2n-2}y_0}{|\lambda_0^{2n-1}|} + \dots + \frac{A^{n+1}y_0}{|\lambda_0^{n+2}|} + \frac{\lambda_0^{2n}y_0}{|\lambda_0^{2n+1}|}\right)$$

(Here $\frac{A^{2n}y^*}{\lambda_0^{2n}}$ is written as $|\lambda_0| \frac{|\lambda_0^{2n}y_0|}{|\lambda_0^{2n+1}|}$). Therefore $Ay_0 = \lambda_0 y_0$.

Since $A^n K \subseteq K$ for all n and since $|\lambda_0| > 1$ with $\lambda_0^n > 2$ $y_0 \epsilon K$. This completes the proof of the theorem when Case 2 is true.

Case 3 : None of the eigen values of maximal modulus is a root of a positive number.

Let λ_0 be an eigen value with maximal real part among roots of maximal modulus.

i.e. $$Ay_0 = \lambda_0 y_0 \qquad |\lambda_0| = P\exp(i\phi_0) \qquad 0 < \phi_0 < 2\pi$$

and $\exp(iN\phi_0) \neq 1$ for any integer N. By spectral mapping theorem (Danford and Schwatz (1958))

we have $$\sigma((1-\varepsilon)A + \varepsilon A^2) = \{(1-\varepsilon)\lambda + \varepsilon\lambda^2;\ \lambda\epsilon\sigma(A)\}$$
where $\sigma(A)$ is the spectrum of A.

Suppose $\lambda\epsilon\sigma(A)$ and $|\lambda| = |\lambda_0| = P$, then $\lambda = Pe^{i\phi}$ and $\cos\phi \leqslant \cos\phi_0$ by assumption.

Further $$(1-\varepsilon)\lambda + \varepsilon\lambda^2 = P(1-\varepsilon)e^{i.\phi} + \varepsilon P^2 e^{2i.\phi}$$

and $$|(1-\varepsilon)\lambda + \varepsilon\lambda^2| = P.\sqrt{(1-\varepsilon)^2 + \varepsilon^2 P^2 + 2\varepsilon(1-\varepsilon)P.\cos\phi} \leqslant |(1-\varepsilon)\lambda_0 + \varepsilon\lambda_0^2|$$

by the choice of λ_0 among maximal roots in modulus. Thus $(1-\varepsilon)\lambda_0 + \varepsilon\lambda_0^2$, $(1-\varepsilon)\bar{\lambda}_0 + \varepsilon\bar{\lambda}_0^2$ are eigenvalues of $A_\varepsilon = (1-\varepsilon)A + \varepsilon A^2$. By a proper choice of $\varepsilon > 0$ we can make the argument of $(1-\varepsilon)\lambda_0 + \varepsilon\lambda_0^2$ commensurable with 2π. Further for arbitrarily small $\varepsilon > 0$ we can assume $(1-\varepsilon)\lambda_0 + \varepsilon\lambda_0^2 > 1$ and therefore Case 2 is applicable here since $(1-\varepsilon)\lambda_0 + \varepsilon\lambda_0^2 > 1$. Actually $(1-\varepsilon)A + \varepsilon A^2$ leaves K invariant for a proper choice of ε, satisfying also the commensurability condition. As ε is arbitrary, it follows that $\lambda_0 > 1$. Thus the proof of the theorem is complete.

Theorem 4 : *If A is a compact operator and*

(a) $AK \subseteq K$

(b) $Ay - cy\epsilon K$, $y\epsilon K$, $|y| = 1$, $c > 1$.

(c) *There exists an $f \epsilon E^*$ with $f(y) > 0$ for all $y\epsilon K - 0$.*

Then $Ay_0 = \lambda_0 y_0$ $y_0 \epsilon K$, $\lambda_0 \geqslant c$, where λ_0 is the spectral radius of A.

Proof: $AK \subseteq K$ with $Ay-cy\epsilon K$ we have $A^n y - cA^{n-1}y\epsilon K$. As in cones we cannot conclude that $A^n y - c^n y \epsilon K$.

Let $a_1, a_2, \ldots, a_N$ be N positive numbers with

$$\text{(i)} \quad 0 < a_i < 1, \quad i = 1, 2, \ldots, N$$

$$\text{(ii)} \quad \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = \frac{a_{N-1}}{a_N} = c$$

$$\text{(iii)} \quad \Sigma a_i < 1.$$

Since $A^n y - cA^{n-1}y\epsilon K$ for all n, by the convexity of K

$$\sum_{1}^{N} a_i(A^i y - cA^{i-1}y) + (1-\Sigma a_i)0 \, \epsilon K.$$

That is $$(a_N A^N y - a_1 cy)\epsilon K,$$

which gives $$a_N(A^N y - c^N y)\epsilon K.$$

Since $f(y) > 0$ for all $y\epsilon K-0$,

$$f(a_N(A^N y - c^N y) = a_N f(A^N y - c^N y) \geqslant 0.$$

By the positivity of a_N we have

$$f(A^N y) \geqslant c^N f(y) > 0.$$

Thus $$|f| \;\; |A^N| \geqslant C^N f(y)$$

which shows $$|A^N| \geqslant C^N \frac{f(y)}{|f|}.$$

Since $$0 < \frac{f(y)}{|f|} < 1, \overline{\lim_{N \to \infty}} \;\; \sqrt[N]{A^N} \geqslant c > 1.$$

i.e. the spectral radius of $A > 1$. By an application of Theorem 3, we get the required result and hence the theorem.

Remark 3 : If we take K to be a cone and $AK \subseteq K$ then $pAK \subseteq K$ for any positive constant p. Thus by proper choice of p we can make the spectral radius of pA, strictly greater than unity, whenever it is positive. We can now apply the theorem by treating K as a convex set and therefore we get $pAy_0 = p\lambda_0 \cdot y_0$ where $p \cdot \lambda_0$ is the spectral radius of pA. Thus $Ay_0 = \lambda_0 y_0$ where $y_0 \epsilon K$.

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