

# Some Geometric and Combinatorial Properties of Binary Matrices Related to Discrete Tomography

by

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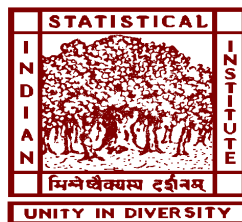
under the supervision of

Prof. Bhargab B. Bhattacharya

Prof. Sandip Das

*submitted in fulfillment of the requirements  
of the degree of M.Tech in Computer Science*

*to the*



Indian Statistical Institute, Kolkata  
July 2013

*Dedicated to*  
*My Parents*



# Certificate

This is to certify that the thesis titled **Some Geometric and Combinatorial Properties of Binary Matrices Related to Discrete Tomography** submitted by **Hirak Sarkar**, in partial fulfillment for the award of the degree of **Master of Technology** is a bonafide record of work carried out by him under our supervision and guidance. The thesis has fulfilled all the requirements as per the regulations of this institute and, in my opinion, has reached the standard needed for submission.

**Prof. Bhargab B. Bhattacharya**

**Prof. Sandip Das**

Indian Statistical Institute

Dated : July, 2013.



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# Abstract

Digital tomography deals with the problem of reconstructing an image from its projections. The image may or may not be reconstructible uniquely. The effective reconstruction also depends on the kind of projections taken. We consider the simplest two-dimensional case in which we have a 2D matrix and the projections are orthogonal. The matrices which are not uniquely reconstructible are known as ambiguous. In this thesis we concentrate on decomposing such an ambiguous matrix into a minimum number of matrices such that each of them are unambiguous. We claim that the *XOR* sum of these component matrices would return the original matrix. As the component matrices can be stored just by storing the row-sum and column-sum (the horizontal and vertical projections), we can store any ambiguous matrix by storing the projections of the components. The space management highly depends on the minimum number of components which we define as *XOR*-dimension. We study the trend of change in *XOR*-dimension first for  $n \times n$  matrices and then for general  $m \times n$  matrices.





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# 1

## Introduction

### 1.1 Motivation

The birth of information age is marked with the invention of computing machines and gradually it became a subject of utmost significance. One of the most important purposes of these machines is to store, represent and process data efficiently. In a digital computer data is stored in binary format. The fact signifies that one step forward in better representation of binary data consequently brings a giant leap in the management of computerized data. Our line of work is motivated from this simple fact apart from the joy of studying the mathematical symmetry and beauty inherent in binary matrices. We scavenge several ideas from different fields

of discrete tomography [3] and graph theory to give a new perspective to an age old problem. Any image, color or gray scale, is stored in the form of a binary matrix. Discrete tomography which has immense applications in medical imaging, deals with reconstruction of a binary matrix from its projections [3]. Such reconstruction of a binary matrix is easy if the matrix satisfies certain properties. In graph theory these matrices, which are known as *Ferrer's digraphs* [1] are well studied. We amalgamate both the ideas to answer some of the unsolved questions about binary matrices in connection to the *reconstruction-from-projection* problem.

### 1.1.1 Digital Tomography

Computerized tomography (CT) based medical-imaging systems generates three-dimensional (3D) images of internal body structures using complex X-ray and computer-aided tomographic imaging techniques.

The X-ray images used to generate the tomographic snapshots are generated first by exposing the patient to a fan-shaped X-ray beam and then detecting the attenuated ray by a thin semicircular, digital X-ray detector. The patient is placed between the source and detector, and the detector is configured with its geometric center located at the X-ray source. Each image is an X-ray projection of a very thin transverse slice of the body. To collect the multitude of X-ray projections necessary to generate a tomographic CT image, both the X-ray source and detector are revolved around a patient within a supporting gantry. As the source and detector are rotated, images are collected and stored. As in a traditional X-ray, the signal levels in the image slice represent the relative radio density of the patient along a line from the X-ray source to the corresponding pixel location [3].

## 1.2 Scope of the Thesis

The matrices that motivated us arise from a problem in discrete tomography as stated earlier. We look at some theoretical problems where we deal with the simple two-dimensional case. Instead of viewing it from all directions, we look at the only horizontal and vertical projections. So, the problem is mapped to that of reconstructing a matrix from its horizontal and vertical projections. Most interestingly, the matrices which can be *uniquely* reconstructed from their horizontal and vertical projections also correspond to ortho-convex polygons, alternatively known as  $h - v$  polygons [3]. In this paper we concentrate on the decomposition of matrices into minimum number of component matrices that are ortho-convex.

In this thesis we first ask the following question:

Let  $M$  be an  $m \times n$  binary matrix such that  $M$  is not uniquely reconstructible from its projections. Can  $M$  be decomposed into  $k$  number of  $m \times n$  matrices  $M_1, M_2, M_3, \dots, M_k$  such that each  $M_i, 1 \leq i \leq k$  is uniquely reconstructible from its projections and  $M = M_1 < op > M_2 < op > \dots < op > M_k$ , where  $< op >$  is some bit-wise logical operation such that AND, OR, XOR?

In context to the above-mentioned problem, we have studied various operations that can generate the original matrix. A binary matrix can also be thought of as the incidence matrix of a directed graph. Then one could represent the underlying graph in terms of intersection of other graphs. Ferrer's [1] digraphs are used as components, intersection of which give the original matrix back. Note that because of the unique reconstruction property [5], it is enough to store the projections of the matrices instead of the matrix itself. In two dimensions, we can have a simplified view of the problem as shown in Figure 1.1.

In this work we surveyed the work done on intersection and studied the properties for the *XOR* operation. Unfortunately the *XOR* operation between two graphs was not well considered earlier in this context and to the best of our knowledge,

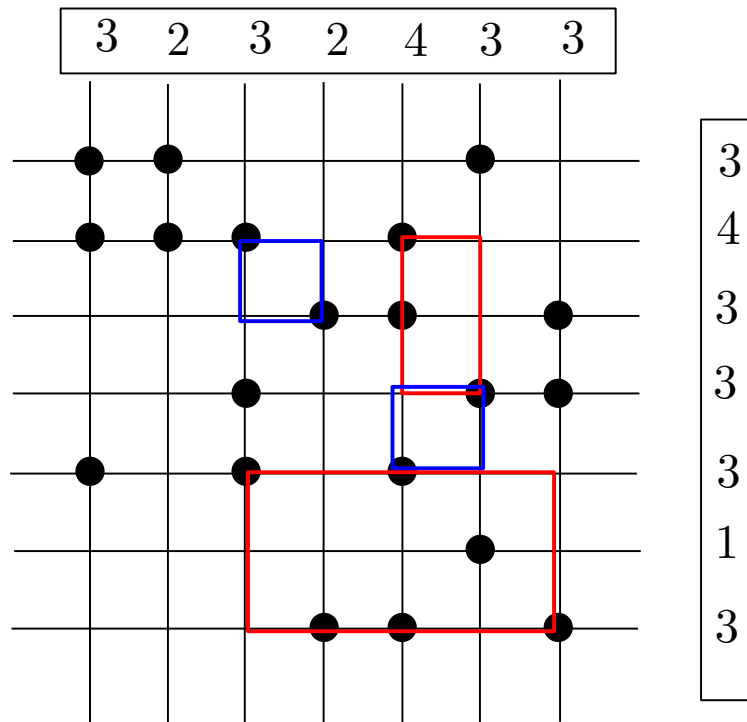


FIGURE 1.1: The dots denote the particle on the grid where the X-ray can get absorbed

their properties were not studied before. Our main goal is to find the components for a given matrix such that each of the components would be a *Ferrer's digraph*, and successive *XOR* operation between them would give back the original matrix. To summarize we have following questions that we are trying to answer in this thesis:

- Is it possible to store an  $n \times n$  matrix in components?
- Is it possible to decompose any matrix in finite components, such that XOR among them would return the matrix?
- If the answer to the above question is yes, then what is the smallest number of components?
- Is it possible to express it as a function of input matrix size?
- Is it possible to extend it to an  $m \times n$  matrix?

At the end of the thesis, we have been able to answer a few questions, others still remain unanswered.

### 1.3 Organization of the Thesis

The document is organized as follows. In the next chapter, we introduce the notion of projections with formal definition. We give a reconstruction algorithm for matrices with well defined properties. We describe the notion of *Ferrer's dimension* which provides the basis of understanding *XOR*-dimension. Starting from Chapter 3, we introduce several new notations and definitions to establish the property of *XOR*-dimension. In Chapter 3 we present a conjecture on the dimension change. Chapter 4 is dedicated towards possible extension of the current study.





# 2

## Problem & Related Works

### 2.1 Introduction

In this section we formally define the problem from light of digital tomography [3]. There are many different kind of objects that are studied in discrete tomography, e.g. lattice sets, binary matrices, digital or label images, measurable sets, dominoes, etc. In general they are represented as function with domain  $X$  and with a given discrete range. Let us denote the class of functions to be studied by  $\mathcal{E}$ . Geometrically lines, strips, hyperplanes. The *projection* of a function in  $\mathcal{E}$  onto a subset  $S$  in  $S$  is an weighted sum or an integral denoted by  $[Pf](S) = \sum_{x \in S} w(x, S) (\int_S w(x, s) f(s))$ .

where  $w : X \times S \rightarrow \mathbb{R}$  is a given weight function. For the matrices we consider

the weight function takes a value 0 or 1, as the contents are binary. With this notation there are few questions that it posed. They are

### Uniqueness( $\mathcal{E}, \mathcal{S}$ )

Given a function  $f \in \mathcal{E}$  does there exist a function  $f' \in \mathcal{E}$  different from  $f$  such that  $f$  and  $f'$  have same projection?

### Existence( $\mathcal{E}, \mathcal{S}$ )

Given a real valued function  $g$  defined on  $\mathcal{S}$  does there exist a function  $f \in \mathcal{E}$  such that  $[Pf](S) = g(S) \forall S \in \mathcal{S}$ ?

### Reconstruction( $\mathcal{E}, \mathcal{S}$ )

Given a real valued function  $g$  defined on  $\mathcal{S}$  construct a function  $f \in \mathcal{E}$  such that  $[Pf](S) = g(S) \forall S \in \mathcal{S}$ .

In our case  $X = \mathbb{Z}^2$ , where  $\mathbb{Z}$  denote set of integers.  $\mathcal{E}$  is the set of  $\{0, 1\}$ -valued functions on  $X$ , such that number of elements of  $X$ . The elements of  $\mathcal{S}$  are the horizontal and vertical lattice lines, that are the sets of the form  $\{i, j\} | j = j_0$  or  $\{i, j\} | i = i_0$ . The weight  $w$  is always 1, so the projections are just sums of the function values on a horizontal or a vertical lattice line. A *switching component* or a *conflict* of an  $f \in \mathcal{E}$  is a set of four points in  $X$  of the form  $x_1 = (i_1, j_1), x_2 = (i_2, j_1), x_3 = (i_1, j_2), x_4 = (i_2, j_2)$ , such that  $f(x_1) \neq f(x_2)$ , but  $f(x_1) = f(x_3)$  and  $f(x_2) = f(x_4)$ . A well known proven result is that under the definitions  $f$  is not *unique* if and only if,  $f$  has a *switching component* [3]. Now we will discuss the reconstruction problem and algorithm for a unique reconstruction if possible.

## 2.2 Unique Reconstruction

Let  $A = (a_{ij})$  be a binary matrix of size  $m \times n$ . As described in the previous paragraph here projection effectively means row sum and column sum. Let the *row*

sum vector to be denoted by  $R(A) = R = (r_1, r_2, \dots, r_m)$ ,

$$r_i = \sum_{j=1}^n a_{ij}, \quad \text{where, } i = 1, 2, \dots, m$$

similarly let the column sum vector denoted by  $S(A) = S = (s_1, s_2, \dots, s_n)$ , so

$$s_j = \sum_{i=1}^m a_{ij}, \quad \text{where, } j = 1, 2, \dots, n$$

Here  $R$  and  $S$  are *projections*. Now let us denote the class of binary matrices with row sum  $R$  and column sum  $S$  by  $\mathcal{A}(R, S)$ . The above set of equations can be shown to be a special case of binary system  $CZ = D$  where  $C$  is an  $mn \times mn$  coefficient matrix of nonnegative integers,  $D$  is an  $mn \times 1$  vector of nonnegative integers. The usual way to solve that set of systems is branch and bound which is exponential in nature [5][4].

Let us present the known result more formally,

**Theorem 2.2.1.** *A binary matrix is non unique (or ambiguous) if and only if it contains a switching element (or conflict).*

It follows from Theorem 2.2.1 that, to decide whether a given binary pattern (or matrix)  $M$  is ambiguous, it suffices to check if  $Z$  contains a *switching component* or not. If  $M$  is of dimension  $m \times n$ , then the exhaustive enumeration requires in the worst case  $\binom{n}{2} \binom{m}{2}$ ,  $2 \times 2$  submatrice of  $Z$ .

A more computationally feasible way is the idea of iterative deletion.

*Primitive Row:* Let  $M$  be a matrix. A primitive row of  $M$  is a row such that all of its entries are either 0's or 1's. A primitive column is defined in the same way.

The idea of iterative deletion comes from the following theorem,

**Theorem 2.2.2.** *Given a matrix  $M$ , an submatrix of  $M$  is obtained by deleting all the primitive rows and all the primitive columns of  $M$ . Repeat the deletion process on successive sub-matrices until no deletion can be made.  $M$  is unique or unambiguous if  $M$  can be deleted completely.*

Considering a binary matrix,

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

It can be easily shown that successive deletion of the rows and columns can completely delete the matrix. Clearly Theorem 2.2.2 can lead to a linear time algorithm 1.

## 2.3 Motivation from space efficiency

From previous section it is clear that if a binary matrix is unique, then we can reconstruct the matrix from its row sum and column sum in linear time. Now if a matrix is not unique and we can decompose it into matrices which are unique in nature so that the component matrices can give back the matrix. The process is effective in terms of space. The reason is simple, to store an  $n \times n$  matrix we need  $O(n^2)$  space in general. But if it is a unique or unambiguous matrix then storing row-sum and column-sum is fairly suffices. To store row-sum and column-sum we need  $O(n)$  spaces if we consider that the matrix can be stored decompose in  $k$  such

---

**Algorithm 1:** Checking Uniqueness

---

**Input:** A finite set  $A = \{a_1, a_2, \dots, a_n\}$  of integers  
**Output:** The largest element in the set

```

1  $i \leftarrow 0$ 
2 while True do
3   Find a primitive row or column  $D$ 
4   if No such  $D$  exists and  $i < n$  then
5     Report  $M$  to be ambiguous
6     break
7   else
8     Modify  $M$  by deleting  $D$ 
9     if  $M$  is empty then
10      Report  $M$  to be non ambiguous
11      break
12    $i \leftarrow i + 1$ 
13 return max
```

---

matrices then it is easy to see that it would take  $kO(n)$  space. But  $k$  is very important here. If it is also linear to  $n$  then no improvement comes in terms of space. This idea of decomposition of a matrix in unique matrices comes from the theory of intersection graphs.

In this section we will discuss *Ferrer's dimension* which was proposed very early by Cogis et al.[1]. Later different related work added to it by Das et al.[2]. But it needs a fair introduction to understand the matter itself,

### 2.3.1 Ferrer's Digraph

A digraph is a tuple  $G = (V, E)$ , such that  $E \subseteq V^2$ , we assume  $V$  is finite. If  $G$  is reflexive and transitive  $V$  is called *preordering*. A partial ordering is an antisymmetric preordering. A *linear order* is a partial ordering which is complete.

A digraph  $G = (V, E)$  is called **Ferrer's Digraph** when there exists an linear order  $(X, L)$  such that for every  $x, y, z \in V$ , if  $xy \in L$  and  $yz \in E$ , then  $xz \in$

*E.* Dushnik et al. proved that a partial order is intersection of all linear orders containing it and introduced the dimension of the partial order  $G$  as the minimum number of linear orders whose intersection is  $G$ . As linear orders can be equivalently changed into Ferrer's digraphs Cogis et al.[1] extended the dimension to Ferrer's digraphs and called it Ferrer's dimension denoted by  $d_F$ .

### 2.3.2 Ferrer's dimension of a digraph

**Theorem 2.3.1.** *Any digraph is the intersection of the Ferrer's digraph containing it.*

*Proof.* Either  $G = (V, E)$  is the intersection of the family of the family of Ferrer's digraphs  $K_{xy} = (V, V^2 - \{xy\}) \forall xy \notin E$  or it is itself a Ferrer's digraph.  $\square$

**Ferrer's Dimension:** the Ferrer's dimension  $d_F(G)$  of a digraph  $G$  is the smallest number of Ferrer's digraphs whose intersection is  $G$ .

The digraphs of Ferrer's dimension at most 2 were characterized by Cogis. He defined an undirected graph  $H(G)$ , the graph associated to a digraph  $G$  whose vertices correspond to the 0's of its adjacency matrix with two such vertices joined by an edge if and only if the corresponding 0s belong to a couple or conflict. The theorem follows

**Theorem 2.3.2.** *A digraph  $G$  is of Ferrer's dimension at most 2 if and only if  $H(G)$  is bipartite.*

In the general case, if  $d_F(G) = n$ , then there exist Ferrer's digraphs  $F_i, i = 1, 2, \dots, n$ , such that  $G$  can be expressed as  $G = F_1 \cap F_2 \cap \dots \cap F_n$ . Observe that the zeros belonging to any particular  $F_i$  do not form any couple among themselves and consequently form an independent set in  $H(D)$ . Thus  $\chi(H(D)) \leq d_F(D)$  where

$\chi(H(D))$  is the chromatic number of  $H(D)$ .

The theorem that is relevant for the current literature is the following theorem proved by Das et al. [2].

**Theorem 2.3.3.** *The following conditions are equivalent for a digraph  $B$ ,*

*(i)  $B$  is a containment bigraph.*

*(ii)  $B$  is of Ferrer's dimension at most 2.*

*(iii) The rows and columns of biadjacency matrix of  $B$  can be permuted independently, so that in the rearranged matrix no 0 has a 1 both below it and to its right.*

It should be noted that the emphasis from the graph theoretic view is only confined to the intersection due its connection with ordered sets and intervals. The intersection or AND is a very primitive operation when it comes to the family of boolean operations. Unfortunately the graphs having more than Ferrer's dimension 2 are less studied. In this thesis we take the operation to the next level, and have taken XOR operation in consideration and defined *XOR*-dimension, which is more powerful in nature. Our intuition was the matrices even with large Ferrer's dimension can have small XOR-dimension, and the results in the next section evidentially show that there is a fair chance of such decomposition.





# 3

## XOR-dimension & Properties

### 3.1 Introduction

As described in previous chapter the *Ferrer's Dimension* describes the the minimum number of *Ferrer's* matrices, intersection of which generate the original matrix, we can define the same for the XOR operations. Before defining XOR dimension we present the following theorem which helps to understand it better. Let us consider the incidence matrix of a graph again, which is a square matrix.

**Theorem 3.1.1.** *Any digraph  $G = (V, E)$  can be written as XOR of finitely many Ferrer's digraphs, moreover, if a digraph has  $n$  vertices then (i.e adjacency matrix is  $n \times n$ ) it can always be represented as XOR of at most  $n$  Ferrer's digraphs.*

*Proof.* We can write any row of  $n \times n$  matrix as an  $1 \times n$  vector. Now the vector is a combination of binary values. Let  $m_1 \leq n$  be the right most index such that  $m_1$ -th bit is 1. So we can assign an  $1 \times n$  vector with  $m_1$  1's followed by 0's. Let  $m_2 < m_1$  be the second right most index, then there are  $(m_1 - m_2)$  0's. We will define another vector  $(m_1 - 1)$  1's followed by 0's. The XOR of these two vectors will take care of the original vector up to  $m_2$  places. We can repeat the process until we obtain the leftmost 1.

e.g.  $(101010) = (111110) \oplus (111100) \oplus (111000) \oplus (110000) \oplus (100000)$

As the 1's are always together in that method of decomposition therefore the components would be Ferrer's digraphs. Clearly the number of such component vectors can not exceed  $n$ , therefore any  $n \times n$  matrix can be represented as XOR of  $n$  Ferrer's digraphs.  $\square$

## 3.2 XOR dimension

From theorem 3.1.1 we have the intuition of having a dimension for XOR operation. Although the notion is fairly straight forward with XOR operation. Let us now define *XOR dimension* in a formal way.

*XOR dimension:* The *XOR dimension*  $d_X(G)$  of a digraph  $G$  is the smallest number of Ferrer's digraphs, such that successive XOR operation between them would return  $G$ .

From theorem 3.1.1 we have following observation.

**Observation 3.2.1.** For a digraph  $G$  with  $n$  vertices  $d_X(G) \leq n$ .

### 3.2.1 Decomposition of higher dimension

To have a conflict in a matrix, it must contain an  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  or  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  as sub matrix. We can decompose any matrix into 4 co-ordinates as in figure 3.1, and make diagonal coordinates different, in that way we can prohibit couples whose 4 corners lies in different co-ordinates.

$$\begin{array}{|c|c|} \hline D_{11} & D_{12} \\ \hline D_{21} & D_{22} \\ \hline \end{array} = \begin{array}{|c|c|} \hline D_{11} & 1 \\ \hline 0 & \overline{D_{22}} \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline 0 & \overline{D_{12}} \\ \hline D_{21} & 1 \\ \hline \end{array}$$

FIGURE 3.1: Decoding a permutation Ferrer's element

That decomposition leads to following theorem,

**Theorem 3.2.1.** *If every digraph with  $n$  vertices has dimension  $n$  or less or equal to  $k$ , then any digraph with  $2n$  vertices has a XOR dimension less or equal to  $2k$ .*

*Proof.* Suppose we have  $2n \times 2n$  adjacency matrix  $A$  as

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where each of  $A_1, A_2, A_3, A_4$  are  $n \times n$  adjacency matrices and hence by our assumption can be written as a sum of exactly  $k$  Ferrer's digraphs (we add zero matrices if needed). Let the representations be

$$A_1 = A_{11} \oplus A_{12} \oplus \dots \oplus A_{1k}$$

$$A_2 = A_{21} \oplus A_{22} \oplus \dots \oplus A_{2k}$$

$$A_3 = A_{31} \oplus A_{32} \oplus \dots \oplus A_{3k}$$

$$A_4 = A_{41} \oplus A_{42} \oplus \dots \oplus A_{4k}$$

First suppose  $k$  is even, then we can write  $A$  as

$$\begin{aligned} & \begin{pmatrix} A_{11} & 1 \\ 0 & A_{41} \end{pmatrix} \oplus \begin{pmatrix} A_{12} & 1 \\ 0 & A_{42} \end{pmatrix} \dots \oplus \begin{pmatrix} A_{1k} & 1 \\ 0 & A_{4k} \end{pmatrix} \\ & \oplus \begin{pmatrix} 1 & A_{21} \\ A_{31} & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & A_{22} \\ A_{32} & 0 \end{pmatrix} \dots \oplus \begin{pmatrix} 1 & A_{2k} \\ A_{3k} & 0 \end{pmatrix} \end{aligned}$$

If  $k$  is odd, we can write  $A$  as

$$\begin{aligned} & \begin{pmatrix} \bar{A}_{11} & 1 \\ 0 & A_{41} \end{pmatrix} \oplus \begin{pmatrix} \bar{A}_{12} & 1 \\ 0 & A_{42} \end{pmatrix} \dots \oplus \begin{pmatrix} \bar{A}_{1k} & 1 \\ 0 & A_{4k} \end{pmatrix} \\ & \oplus \begin{pmatrix} 1 & \bar{A}_{21} \\ A_{31} & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & \bar{A}_{22} \\ A_{32} & 0 \end{pmatrix} \dots \oplus \begin{pmatrix} 1 & \bar{A}_{2k} \\ A_{3k} & 0 \end{pmatrix} \end{aligned}$$

where for an adjacency matrix  $A = ((a_{ij}))$ , we define,  $\bar{A} = ((\bar{a}_{ij}))$

Observe that each of these matrices have 4  $n \times n$  sub-blocks which are Ferrer's digraphs. By placing one matrix and zero matrix in the two opposite blocks, we ensure that a couple is not formed from different blocks. Hence, each of these matrices are Ferrer's digraphs and as there are  $2k$  many of them, we get that the  $d_X(A) \leq 2k$ .  $\square$

We can generalize the decomposition method in theorem 3.2.1 for any multiple of  $n$ . The idea goes in a similar way as done for  $l = 2$  above where there are  $ln$  nodes. First observe that as we have proved it for  $l = 2$ , we have a  $ln \times ln$  adjacency

matrix  $A$  as in figure 3.2.

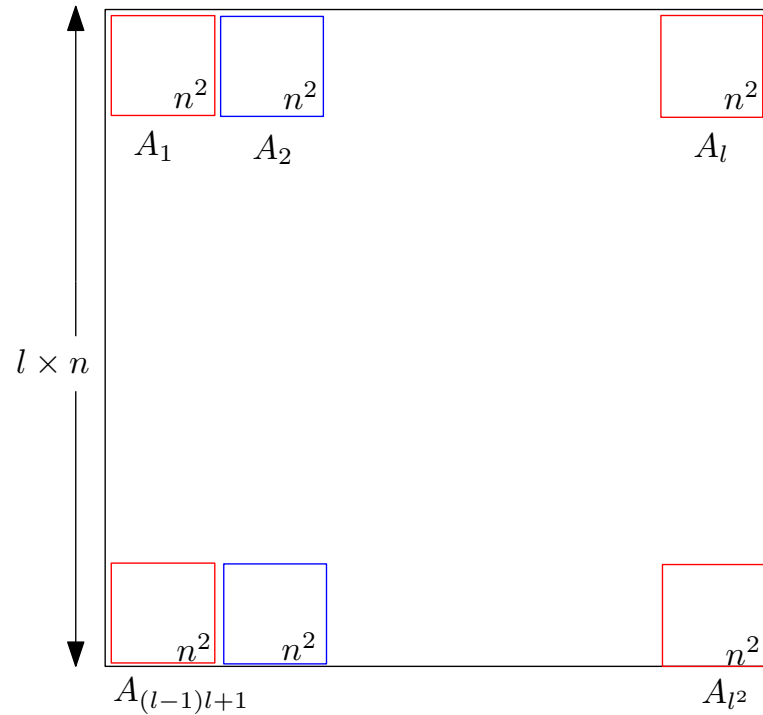


FIGURE 3.2: matrix of  $lk$  dimension

**Theorem 3.2.2.** *If every digraph with  $n$  vertices has XOR dimension less or equal to  $k$ , then any digraph with  $ln$  vertices has XOR dimension at most  $lk$  for  $n, k, l \in \mathbb{N}$ .*

*Proof.* The proof follows the similar track as done for  $l = 2$  in the Theorem 3.2.1. it is sufficient to prove when  $l$  be an even number. Suppose we have a  $ln \times ln$  adjacency matrix  $A$  as

$$\begin{pmatrix} A_1 & \dots & \dots & A_l \\ A_{l+1} & \dots & \dots & A_{2l} \\ \dots & \dots & \dots & \dots \\ A_{(l-1)l+1} & \dots & \dots & A_{l^2} \end{pmatrix}$$

where each of  $A_1, A_2, \dots, A_{l^2}$  are  $n \times n$  adjacency matrices (See figure 3.2) and hence each of the blocks by definition has XOR dimension  $k$  Ferrer's digraphs (we add all ones and all zeros matrices as needed). Let the representations be

$$A_1 = A_{11} \oplus A_{12} \oplus \dots \oplus A_{1k}$$

$$A_2 = A_{21} \oplus A_{22} \oplus \dots \oplus A_{2k}$$

....

$$A_{l^2} = A_{l^2 1} \oplus A_{l^2 2} \oplus \dots \oplus A_{l^2 k}$$

Observe that for Ferrer's digraphs  $D_1, D_2, \dots, D_l$ , the matrix

$$\begin{pmatrix} D_1 & 1 & \dots & 1 \\ 0 & D_2 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & D_l \end{pmatrix}$$

has no couple and hence a Ferrer's digraph. Also observe, as column permutation retains a Ferrer's digraph, rotations of the above matrix i.e.

$$\begin{pmatrix} 1 & D_1 & 1 & \dots & 1 \\ 1 & 0 & D_2 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & D_{l-1} \\ D_l & 0 & 0 & \dots & 0 \end{pmatrix} \dots \dots \begin{pmatrix} 1 & 1 & 1 & \dots & D_1 \\ D_2 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & D_{l-1} & 1 & 0 \\ 0 & \dots & 0 & D_l & 0 \end{pmatrix}$$

are also couple free matrices (By rotation, we mean that put the last block of columns at the beginning of the matrix consecutively).

First suppose  $k$  is even. Then we can write  $A$  as

$$\begin{aligned}
& \left( \begin{array}{cccc} A_{11} & 1 & \dots & 1 \\ 0 & A_{(l+2)1} & \dots & 1 \\ \dots & \dots & & \dots \\ 0 & 0 & \dots & A_{l^2_1} \end{array} \right) \oplus \left( \begin{array}{cccc} A_{12} & 1 & \dots & 1 \\ 0 & A_{(l+2)2} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{l^2_2} \end{array} \right) \dots \oplus \left( \begin{array}{cccc} A_{1k} & 1 & \dots & 1 \\ 0 & A_{(l+2)k} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{l^2_k} \end{array} \right) \\
& \oplus \left( \begin{array}{cccc} 1 & A_{21} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & A_{(l-1)l,1} \\ A_{((l-1)l+1),1} & 0 & \dots & 0 \end{array} \right) \dots \oplus \left( \begin{array}{cccc} 1 & A_{2k} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & A_{(l-1)l,k} \\ A_{((l-1)l+1),k} & 0 & \dots & 0 \end{array} \right) \\
& \dots \oplus \left( \begin{array}{cccc} 1 & \dots & 1 & A_{l1} \\ A_{(l+1)1} & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & A_{(l^2-1)1} & 0 \end{array} \right) \dots \oplus \left( \begin{array}{cccc} 1 & \dots & 1 & A_{lk} \\ A_{(l+1)k} & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & A_{(l^2-1)k} & 0 \end{array} \right)
\end{aligned}$$

because in each block, say in the first block, we have a sum over an even number of ones (and they add up to 0), a few zeros and the  $A_{1j}$ 's which simply add up to  $A_1$ . If  $k$  is odd, then the sum in each block is different depending on the row in which the block is placed. For the odd rows, i.e. the first, third etc. rows, in each block, we have a sum over an even number of ones (since  $l$  is odd, and we sum over  $(l-1)k, (l-3)k \dots$  ones respectively), a few zeros and the corresponding  $A_{ij}$ 's which simply add up to  $A_i$ 's.

For the even rows, we have sum over odd number of ones, few zeros and an odd number of  $A_{ij}$ 's (since  $k$  is odd). Hence, to make the sum  $A_i$ , we replace  $A_{ij}$ 's by  $\bar{A}_{ij}$ 's which are also Ferrer's digraphs. Thus, we can write  $A$  as



$$\begin{aligned}
& \left( \begin{array}{cccc} A_{11} & 1 & \dots & 1 \\ 0 & \bar{A}_{(l+2)1} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{l^2_1} \end{array} \right) \oplus \left( \begin{array}{cccc} A_{12} & 1 & \dots & 1 \\ 0 & \bar{A}_{(l+2)2} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{l^2_2} \end{array} \right) \dots \oplus \left( \begin{array}{cccc} A_{1k} & 1 & \dots & 1 \\ 0 & \bar{A}_{(l+2)k} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{l^2_k} \end{array} \right) \\
& \oplus \left( \begin{array}{cccc} 1 & A_{21} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & \bar{A}_{(l-1)l,1} \\ A_{((l-1)l+1),1} & 0 & \dots & 0 \end{array} \right) \dots \oplus \left( \begin{array}{cccc} 1 & A_{2k} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & \bar{A}_{(l-1)l,k} \\ A_{((l-1)l+1),k} & 0 & \dots & 0 \end{array} \right) \\
& \dots \oplus \left( \begin{array}{cccc} 1 & \dots & 1 & A_{l1} \\ \bar{A}_{(l+1)1} & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & A_{(l^2-1)1} & 0 \end{array} \right) \dots \left( \begin{array}{cccc} 1 & \dots & 1 & A_{lk} \\ \bar{A}_{(l+1)k} & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & A_{(l^2-1)k} & 0 \end{array} \right)
\end{aligned}$$

where for an adjacency matrix  $A = ((a_{ij}))$ ,  $\bar{A}$  is defined as,  $\bar{A} = ((\bar{a}_{ij}))$

By our observation, each of these matrices are Ferrer's digraphs and as there are  $lk$  many of them, we get that the  $d_X(A) \leq lk$ .  $\square$

Now we prove a theorem that guarantees the existence of a digraph with any predefined XOR dimension.

**Theorem 3.2.3.** *For any integer  $k$ , we can have an  $n$  such that there exists an  $n \times n$  matrix with XOR dimension at least  $k$ .*

*Proof.* Let  $n$  and  $k$  be number of vertices and dimension respectively. We know that any Ferrer's digraph can be constructed by independent row and column permutations of matrices where the 1's are clustered in a corner. Now we can construct such a matrix in  $(n+1)^n$  ways (as in each row, we can choose the number of 1's ranging from 0 to  $n$ ). Observe that to construct a Ferrer's digraph, we only need to consider the different column permutations of these  $(n+1)^n$  matrices (since we have already incorporated the row permutations in the count of matrices with 1's clustered in a corner). Suppose in a given matrix with 1's in a corner, the different rows have  $k_1, k_2, \dots, k_l$  1's where  $l \in \mathbb{N}, l \leq n, 0 < k_1 < k_2 < \dots < k_l$ . The  $k_i$ 's take values in  $\{1, 2, \dots, n\}$ . Then the number of different column permutations of this matrix is

$$\frac{n!}{k_1!(k_2 - k_1)! \dots (k_l - k_{l-1})!(n - k_l)!}$$

Hence, the number of Ferrer's digraph with  $n$  vertices is the XOR sum of the quantity, by taking  $k$  of such over the  $(n+1)^n$  matrices. So, an upper bound to the number of Ferrer's digraphs is  $n!(n+1)^n$ . By an use of the AM-GM inequality, we get an upper bound of  $\frac{(n+1)^{2n}}{2^n}$ . (as  $n! < (\frac{n+1}{2})^n$ ).

Now, for a digraph with  $n$  vertices to be of XOR dimension less or equal to  $k$ , we must have

$$2^{n^2} \leq \frac{(n+1)^{2kn}}{2^{kn}}$$

(as there are  $2^{n^2}$  possible matrices and  $(n+1)^{2kn}$  many possible sums)

i.e.,

$$2^{n^2+kn} \leq (n+1)^{2kn}$$

i.e.,

$$2^{n+k} \leq (n+1)^{2k}$$

Now, the RHS of the inequality grows polynomially and the LHS grows exponentially. Hence there exists  $n_k$  such that for all  $n \geq n_k$ ,

$2^{n+k} > (n+1)^{2k}$  and hence for  $n \geq n_k$ , the upper bound is strictly greater than  $k$ . □

NOTE: For  $k = 2$ , we get  $n_2 = 13$ . Hence for an  $n \geq 13$ , we can always get a digraph with  $n$  vertices such that its XOR dimension is at least 3.

Also note that,

$$2^{n_k+k} > (n_k+1)^{2k}$$

i.e.,

$$(n_k+k) \log 2 > 2k \log (n_k+1)$$

i.e.,

$$k < \frac{n_k \log 2}{2 \log (n_k+1) - \log 2}$$

which is  $O\left(\frac{n_k}{\log n_k}\right)$

So, For an  $n \in \mathbb{N}$ , there will always be digraphs with  $n$  vertices such that it has XOR dimension  $k$  which is of the order  $O\left(\frac{n}{\log n}\right)$ . Hence, for digraphs with  $n$  vertices, the upper bound to  $d_X(G)$  is less than or equal to  $n$  and is greater than or equal to

$O\left(\frac{n}{\log n}\right)$ .

### 3.3 XOR-dimension of any $m \times n$ matrix

Until that point we have been discussing matrices which are square in nature. But while doing experiments we also get an idea that we could also handle any  $m \times n$  matrix, since the matrices are binary therefore the number of rows they can produce is bounded by  $2^n$ . We have the following observation from that simple fact. Later we have conjectured another bound, which is much less.

**Observation 3.3.1.** *The XOR dimension of  $n \times n$  matrix is bounded by the dimension of  $2^n \times n$  matrix where rows are non-repeating.*

As in a  $2^n \times n$  matrix exhaust all possible combination of rows and conflicts so the above observation is trivially true.

Generating Ferrer's digraph components for a general  $m \times n$  matrix is computationally a hard job. Therefore we devised several new encoding techniques to write the program efficiently and to minimize repetitions.

#### 3.3.1 Ferrer's Element

Ferrer's digraphs have the adjacency matrix which is conflict free. We generalize the notion of conflict-freeness and recoded a matrix in from a permutation of column numbers. We call a particular permutation a *Ferrer's Element*. For example we can think of any matrix having 4 columns. A permutation of numbers (0, 1, 2, 3) would represent a Ferrer's element. So for  $n$  columns a permutations of numbers  $(a_1, a_2, \dots, a_n)$  where  $i \in \{1, \dots, n\}$  can be interpreted as a matrix whose Ferrer's dimension is 1. The decoding is as follows, if we have columns 0 to  $n$ ,  $a_1$  denotes that  $a_1$ -th bit is made 1 at the first row,  $a_2$  denotes that  $a_2$  th bit is 1 in the second row along with the  $a_1$ -th bit from the first row and so on. In this manner in the  $i$ -th

row all ones are copied from the  $(i - 1)$ -th row and  $a_i$ -th bit is made 1. We have the following observation,

**Observation 3.3.2.** *Every Ferrer's element is conflict free.*

As all the the 1's are carried down from the previous rows so there is no conflict, hence the observation is trivially true.

We will denote Ferrer's elements as  $e_i$  where  $i \in \{1, \dots, n!\}$ .

The decoding is explained in the following Figure 3.3

	4	3	2	1	0
	0	1	0	0	0
	0	1	1	0	0
3 2 1 0 4	0	1	1	1	0
	0	1	1	1	1
	1	1	1	1	1

FIGURE 3.3: Decoding a permutation Ferrer's element

### 3.3.2 Cartesian XOR

While using intersection between the graph and the AND-operation was bitwise. The *Cartesian XOR* is just an extension of normal bitwise XOR operation. Cartesian XOR between two  $m \times n$  matrices (Ferrer's elements) are defined as follows, we XOR each row of the first Ferrer's element with every other row of the second Ferrer's element. As a result we will get  $m^2$  rows not necessarily unique. Our next observation states that in this process we are able to enumerated all possible row combinations.

Table 3.1: Ferrer's Element pairs

Ferrer's element $e_1$	Ferrer's element $e_2$	Missing Rows
(0,1,2,3)	(0,2,1,3)	9-11-13
(0,1,2,3)	(0,2,3,1)	9-11
(0,1,2,3)	(0,3,1,2)	5-13
(0,1,3,2)	(0,2,1,3)	9-13

**Observation 3.3.3.** *Cartesian XOR of two Ferrer's element exhaust all possible row combinations that can be generated by all possible matrices with repetition that can be made from a Ferrer's element.*

We are generating all possible combination, and repetition does not introduce any new row, therefore the observation follows.

**Example:** For 4 columns (i.e.  $n = 4$ ) the rows that are produced by two Ferrer's element (0, 1, 2, 3) and (0, 1, 3, 2) would generate every row from 0 to  $(2^4 - 1) = 15$  rows except 5,9 and 13. Below in the table 3.1 of few Ferrer's element of order 4 with the rows that they can not generate.

It is to be noted that we have fixed the number of bits while working with a particular Ferrer's element. Let us denote the number of bits  $c$  as it is also number of columns too. We have many observations with different values of  $c$ .

### 3.3.3 Observations for $c = 4$ and $c = 5$

During the experiments on  $n \times n$  matrices, it is found from exhaustive search that every square matrix up to  $n = 5$  has XOR-dimension 2. The matrix we first encountered to break that monotonicity was of size  $6 \times 4$ , below is the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

As we have seen in Example 3.3.2 that Ferrer's element XOR-d together they are leaving few rows. In case of  $c = 4$  there are 24 Ferrer's elements. The *best pair* of Ferrer's elements are those which leave minimum number of rows. Below is the picture of such best pairs for  $c = 4$ .

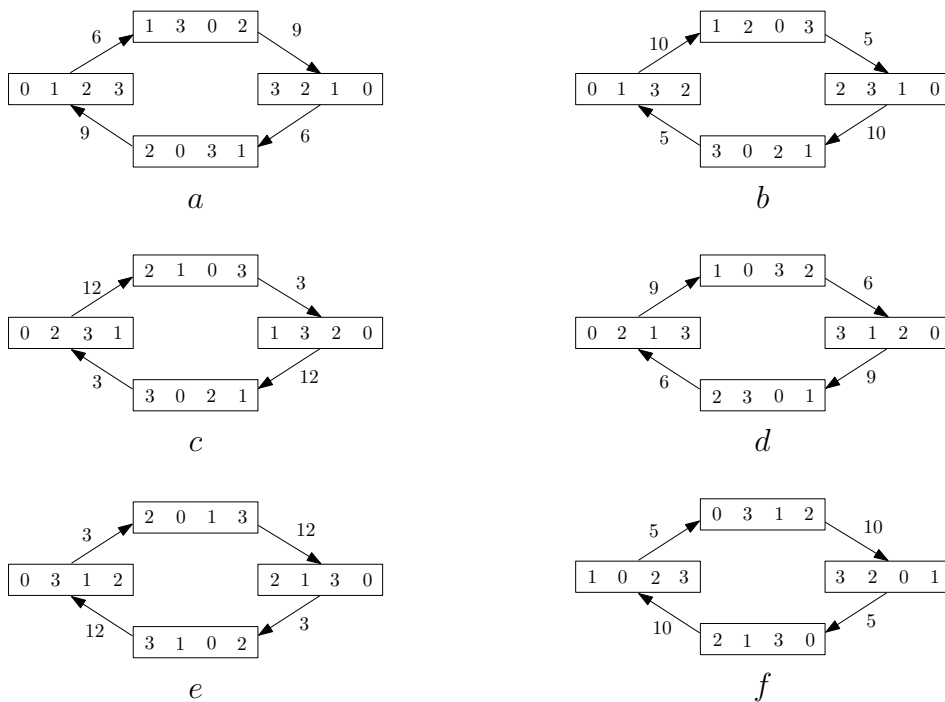


FIGURE 3.4: Best pairs for  $c = 4$

The rows that a pair misses is written on the corresponding arrow. It is to be noted that if a pair misses a row then the reverse of the Ferrer's elements misses

its  $\text{mod}(2^4 - 1)$  complement. To construct the  $6 \times 4$  matrix we have taken these missing rows all together so that no pair can generate them, and the barrier of XOR dim 2 is broken. Another thing is to be noted that the matrix is in general has the form of  $\binom{n}{\frac{n}{2} \times n}$  where each row has equal number of 1's and 0's. We will prove some interesting property with that intuition.

Just as we have seen that the best pairs for  $c = 4$ , (Figure 3.3.3) forms a 2-regular graph, similarly in case of  $c = 5$  the best pairs form a 20 node 6 regular graph,

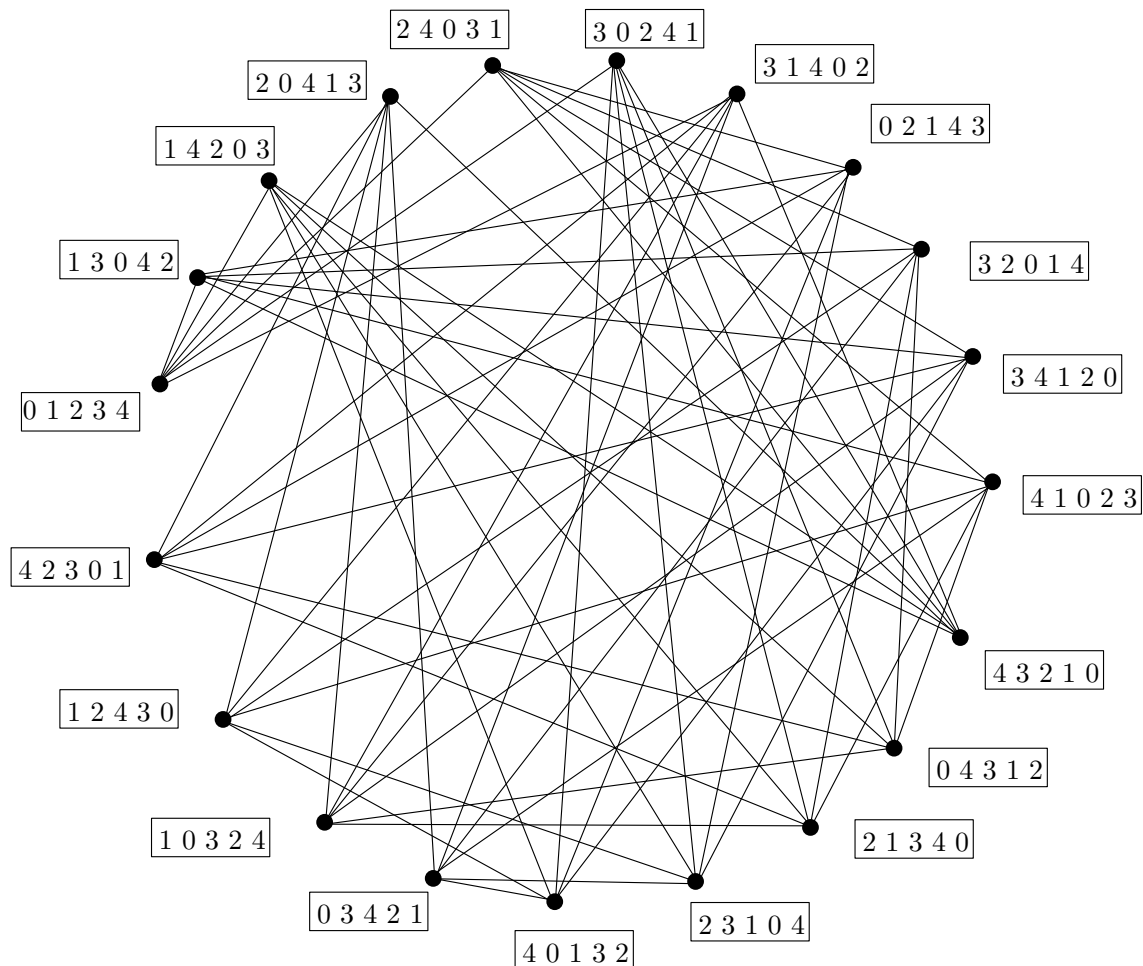


FIGURE 3.5: 6 regular graph for  $c = 5$



There are 6 such graphs, As there are 120 Ferrer's elements.

### 3.3.4 Detecting Dimension

Now we will give a general construction technique for making a matrix with higher XOR-dimension. Here we will assume that the graph between different Ferrer's elements, let us call it  $G_P$  is already given and for fixed number of columns the graph is also fixed. Now if we move from a pair to triplet or a  $k$  subset then  $G_P$  would become a hypergraph. Let us farther assume that the matrix of the higher dimension contain  $m$  rows and the rows are  $r_1, r_2, \dots, r_m$ . We will label those subsets that misses  $r_1$ , let the subgraph be  $G_P^{r_1}$ , then we take  $G_P \setminus G_P^{r_1}$ , and do the same for rows successively until the graph becomes empty, if the graph is still not empty after  $m$  iterations then the taken matrix has dimension less or equal to  $k$  as it can be generated by  $k$  components. In this way we can construct and check the XOR-dimension of a matrix, but the task is computationally hard as the process of generating the hypergraph is combinatorially exhaustive. Algorithm 2 is a iterative deletion algorithm to find out the dimension

### 3.3.5 Properties of $\binom{n}{\frac{n}{2}} \times n$ matrices

From the observations of 3.3.3, it would be intuitively realizable that  $\binom{n}{\frac{n}{2}} \times n$  matrices are special, as they have a terminal dimension. After a simple observation we would prove that for a given type of matrices  $\binom{n}{\frac{n}{2}} \times n$  has maximum XOR-dimension. First we have the following simple observation,

**Observation 3.3.4.** *XOR-dimension is non-decreasing with increasing size of matrices.*

The observation comes from the fact, that row column permutation does not change the XOR dimension. If a matrix  $M \times N$  matrix contain a submatrix of

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**Algorithm 2:** Checking XOR-dimension of a matrix

---

**Input:** The hypergraph  $G_P$  labeled with missing rows by a subset,  
 $Matrix M = \{r_1, r_2, \dots, r_m\}$ , dimension  $k$

**Output:**  $M$  has a dimension  $k$  or not

```

1  $i \leftarrow 0$ 
2 repeat
3   Choose subgraph  $G_P^{r_i}$  such that the subsets missed row  $r_i$ 
4    $G_P = G_P \setminus G_P^{r_i}$ 
5   if  $G_P$  is empty then
6     Report that matrix  $M$  has XOR dimension greater than  $k$ 
7     STOP
8    $i \leftarrow i + 1$ 
9 until  $G_P$  is not empty and  $i \leq m$ 
10 if  $i > m$  and  $G_P$  is not empty then
11   Report that matrix  $M$  has dimension less or equal to  $k$ 
12   STOP

```

---

size  $m \times n$ , which has XOR-dimension  $k$ , then XOR dimension of  $M \times N$  matrix is greater or equal to  $k$ . If it is less than  $k$ , then we take the components and could omit the extra rows and columns and find less than  $k$  components for  $m \times n$ , which is a contradiction.

Now we consider matrices with same number ones in each row, we would prove that such matrices for a fixed  $c$ , the matrix with highest dimension would contain  $\binom{c}{\frac{c}{2}}$  rows.

**Theorem 3.3.1.** *For a fixed number of columns  $c$ , the matrix of size  $\binom{c}{\frac{c}{2}} \times c$  has higher dimension among the matrices of column size  $c$  and have same number of ones at each row.*

*Proof.* We would prove the theorem in inductive way, Let us fix a  $c$ . Now the matrix with no ones at each row is a zero matrix, which is conflict free by nature. Let us call a matrix  $M_s$  if it contains  $s$  1's in every row. The XOR-dimension of  $M_1$  is 2 as depicted below

$$\begin{aligned}
& \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \\
= & \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 0 \end{pmatrix}
\end{aligned}$$

It is to be noted that when we have the components for  $M_1$  then we can easily calculate the components for  $M_{c-1}$  by complementing one component and they have the same dimension. For  $M_2$  we know the matrix must contain a submatrix  $6 \times 4$  as in section 3.3.3, so  $M_2$  must have dimension more than or equal to 3. Let us denote the function as  $d_X$  for XOR-dimension. We know that  $d_X(M_i) = d_X(M_{c-i})$ . As  $d_X(M_0) = d_X(M_c)$ ,  $d_X(M_1) = d_X(M_{c-1})$  the function is symmetric with respect to  $x = \frac{c}{2}$  line. It has the same value for  $s = i$ ,  $s = c - i$ , so the  $d_X$  must attain maximum for some  $i = \frac{c}{2}$ , for which there complements are same. For  $c = 2k$  that matrix is  $M_{(\frac{c}{2})}$ . For odd values of  $c$  we can take floor or ceiling.  $\square$

We will end this section with a conjecture that we found to be true for  $c = 10$  experimentally. If we fix  $c$  and call any matrix with  $r$  rows as  $\mathcal{M}_r$  then.

**A conjecture:** For a given  $c$ ,  $d_X(\mathcal{M}_{\frac{c}{2}}) \geq d_X(\mathcal{M}_i)$  where  $i \leq 2^c, i \in \mathbb{N}$ .

# 4

## Conclusion and Future Research

We began with several questions that are asked from the point of view of decomposition of binary matrices. The major question was to determine if a matrix can be decomposed into several matrices that do not contain conflicts, and we wanted the number of components (*XOR-dimension*) to be minimized. We have been able to answer that question for some matrices. We have proved that the XOR-dimension is finite for a given matrix. We have also given a decomposition technique, and shown that the upper bound on the XOR-dimension of the original matrix can be computed from those of its component matrices. Next we have proved that there always exists a matrix with any arbitrary XOR-dimension  $k$ . To construct a general  $m \times n$  matrix with a given XOR-dimension, we have used an encoding technique

Table 4.1: Ferrer's Element pairs

Number of Columns	XOR-dimension
2	2
3	2
4	3
5	3
6	3
7	3
8	4
9	4
10	4
12	5
13	5
14	5
15	6
16	6
17	6
18	6
19	6

(*Ferrer's elements*), where we fix the number of columns. The combination of Ferrer's elements would exhaust the combination of matrices that could be generated by any permutation of rows. We have described an enumeration graph (for a set of Ferrer's elements) that can be constructed by the possible combinations of Ferrer's elements. An iterative deletion algorithm on such a graph that determines the XOR-dimension of a matrix is also presented.

As the construction of the graph is computationally intensive, an efficient enumeration could speed up the algorithm. However it is yet to be resolved whether or not such an algorithm exists. Experimental results on enumeration of matrices are given in Table 4.1.

The result is known for a matrix with 11 columns. However it should lie between 4 and 5. For higher dimension matrices we could not derive a closed-form formula for it neither do we know if such formula exists or not. So the questions that remain

are following :

- What is the dimension of an  $n \times n$  square matrix in general?
- Is the XOR-dimension a function of the number of columns?
- Can we determine the minimal components in polynomial time?

We were motivated to study the problem as the challenge of reconstructing a matrix from its projection has a direct correspondence to discrete tomography. The idea of decomposition of such matrices arises from its relation to intersection graphs. However the XOR operation seems more promising than simple intersection or AND operation. We have analyzed the problem with simple square matrices as they represent graphs, and then we have generalized it for rectangular matrices. The idea of this decomposition seems powerful, as we have several options for choosing the bit-wise logical operation. Altogether, there are lot of opportunities to expand the boundaries.



# Bibliography

- [1] Olivier Cogis. On the ferrers dimension of a digraph. *Discrete Mathematics*, 38(1):47 – 52, 1982.
- [2] S. Das, M. Sen, A. B. Roy, and D. B. West. Interval digraphs: An analogue of interval graphs. *Journal of Graph Theory*, 13(2):189–202, 1989.
- [3] Gabor T. Herman and Attila Kuba. *Advances in Discrete Tomography and Its Applications (Applied and Numerical Harmonic Analysis)*. Birkhauser, 2007.
- [4] Attila Kuba. Reconstruction of unique binary matrices with prescribed elements. *Acta Cybern.*, 12(1):57–70, 1995.
- [5] Y. R. Wang. Characterization of binary patterns and their projections. *IEEE Trans. Comput.*, 24(10):1032–1035, oct 1975.