# On Rational Subgroups of Exceptional Groups

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# On Rational Subgroups of Exceptional Groups

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*To My Husband*

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# Contents

# Introduction

The main theme of this thesis is the study of exceptional algebraic groups via their subgroups. This theme has been widely explored by various authors (Martin Leibeck, Gary Seitz, Adam Thomas, Donna Testerman to mention a few), mainly for **split** groups ([26], [27], [28], [60] ). When the field of definition $k$ of the concerned algebraic groups is not algebraically closed, the classification of $k$-subgroups is largely an open problem. In the thesis, we mainly handle the cases of simple groups of type $F_4$ and $G_2$ defined over an arbitrary field. These may not be split over $k$. We first determine the possible simple $k$-subgroups of a fixed simple $k$-algebraic group of type $G_2$ or $F_4$ and then, find conditions for a simple $k$-algebraic group to embed in a given group of type $G_2$ or $F_4$.

One knows that a group of type $G_2$ over a field $k$ arises as the group of automorphisms of an octonion algebra over $k$ and similarly, groups of type $F_4$ over $k$ arise from Albert algebras. We exploit the structure of these algebras to derive our results. On the way we also obtain some results on these algebras, which may be of independent interest. For example, we derive a group theoretic characterization of first Tits construction Albert algebras (Theorem 10.2.3). We also prove a group theoretic characterization of Albert algebras $A$ with $f_5(A) = 0$ (Theorem 9.1.2). Other than these results, we prove some results on generation of the groups discussed above by their simple $k$-subgroups and $k$-tori, determining the number of such subgroups required in each case. The results in this thesis have been partly published in ([10]) and partly under submission ([9]).

We now sketch below an outline of the work done in this thesis, introducing some notation on the way, which will be necessary in the Main results section. Let $K$ be an algebraically closed field. The classification of semisimple algebraic groups over $K$ is well understood.

**Theorem 0.0.1** *(Chevalley Classification Theorem) Two semisimple linear algebraic groups are isomorphic if and only if they have isomorphic root data. For each root* *datum there exists a semisimple algebraic group which realizes it.*

The simple algebraic groups have irreducible root systems or equivalently, have connected Dynkin diagrams. Irreducible root systems fall into nine types, called the **Cartan-Killing types**, labelled as $A_n, B_n, C_n, D_n, E_6, E_7, F_4, G_2$. The first four types exisit for each natural number $n$, while the remaining five types are just one in each case. Simple groups with root system or Dynkin diagram of types $A_n, B_n, C_n, D_n$ are called **classical groups** and the simple groups with root systems of type $E_6, E_7, E_8, F_4, G_2$ are called **exceptional groups**. Let $G$ be a simple algebraic group over a field $k$. By the type of $G$ we mean the Cartan-Killing type of the root system of the group $G \otimes \overline{k}$, obtained by extending scalars to an algebraic closure $\overline{k}$ of $k$.
Let $G$ be a simple linear algebraic group over $K$. Then corresponding to any subdiagram of the Dynkin diagram of $G$, there exists a subgroup of $G$ which realizes it, i.e. has the subdiagram as its Dynkin diagram. But this fails to hold for a non-algebraically closed field. For example, over a non-algebraically closed field $k$ a connected simple algebraic group $G$ may not have any subgroup of type $A_1$, though the Dynkin diagram of $G$ always has $A_1$ as a subdiagram (see Remark 10.2.2). Hence over a non-algebraically closed field $k$, it is important to know what are all simple $k$-subgroups of $G$. In the thesis we answer this for groups of type $A_2$, $G_2$ and $F_4$. We prove that when $G$ is a $k$-group of type $F_4$ (resp. $G_2$) arising from an Albert (resp. octonion) division algebra then the possible type of a simple $k$-subgroup of $G$ is $A_2$ or $D_4$ (resp. $A_1$ or $A_2$). The knowledge of these simple $k$-subgroups is a useful tool in studying these groups. This motivates the
**Problem :** *Find conditions under which a given simple $k$-group of type $A_1$ or $A_2$ embeds over $k$ in a simple $k$-group of type $G_2$ or $F_4$.*
In the thesis we study conditions which control the $k$-embeddings of simple algebraic groups of type $A_1$ and $A_2$ in simple groups of type $G_2$ and $F_4$ as well as $k$-embeddings of rank-2 $k$-tori in simple groups of type $A_2$, $G_2$ and $F_4$. This is done via the mod-2 invariants attached to these groups.

Let us briefly recall the mod-2 invariants of these groups. To a given simple algebraic group $H$ of type $A_2$ (resp. $A_1$) defined over $k$, one attaches an invariant $f_3(H) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$ (resp. $f_2(H) \in H^2(k, \mathbb{Z}/2\mathbb{Z})$), which is the Arason invariant of a 3-fold (resp. 2-fold) Pfister form over $k$, namely the norm form of an octonion (resp. quaternion) algebra (see Remark 7.1.2 of the thesis and [19], Thm. 30.21). For a simply connected, simple algebraic group $H$ of type $A_2$ defined over $k$, there exists a unique (up to isomorphism) degree 3 central simple algebra with center a quadratic étale algebra $K$ and with an involution $\sigma$ of the second kind, such that $H \cong \mathbf{SU}(B, \sigma)$. We call the involution $\sigma$ as **distinguished** if $f_3(H) = 0$.

Let $G$ be a group of type $F_4$ defined over $k$. Then there exists an Albert algebra $A$ over $k$ such that $G = \mathbf{Aut}(A) = Aut(A \otimes_k \overline{k})$, the full group of automorphisms of $A$. Given an octonion algebra $C$ over $k$, it is determined by its **norm form** $n_C$, which is a 3-fold Pfister form over $k$. The groups of type $G_2$ defined over $k$ are precisely of the form $\mathbf{Aut}(C)$ for a suitable octonion algebra $C$ over $k$. These are classified by the Arason invariant $e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$. To any Albert algebra $A$, one attaches a certain reduced Albert algebra $\mathcal{H}_3(C, \Gamma)$, for an octonion algebra $C$ over $k$ and $\Gamma = Diag(\gamma_1, \gamma_2, \gamma_3) \in GL_3(k)$, called the **reduced model** of $A$, such that for any reducing field extension $L/k$ of $A$, we have $A \otimes_k L \cong \mathcal{H}_3(C \otimes_k L, \Gamma)$ ([37]). The reduced model of an Albert algebra is unique up to isomorphism and defines two mod-2 invariants for $G = \mathbf{Aut}(A)$:

$$\begin{aligned} f_3(G) = f_3(A) &:= e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z}), \\ f_5(G) = f_5(A) &:= e_3(n_C).e_2(\langle 1, \gamma_1^{-1}\gamma_2\rangle \otimes \langle 1, \gamma_2^{-1}\gamma_3\rangle) \\ &= e_5(n_C \otimes << -\gamma_1^{-1}\gamma_2, -\gamma_2^{-1}\gamma_3 >>) \in H^5(k, \mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

Let $G$ be a $k$-group of type $A_2, G_2$ or $F_4$. Then the invariant $f_3(G)$ as defined above is a 3-fold Pfister form which is the norm form of a unique octonion algebra $C$ over $k$. We call $C$ the octonion algebra of $G$ and denote it by $Oct(G)$. Let $G$ be a simple, simply connected $k$-group of type $A_2$. We will refer to $G$ as arising from a division algebra if either $G \cong \mathbf{SU}(D, \sigma)$ for some degree 3 central division algebra $D$ over a quadratic field extension $F$ of $k$, with an involution $\sigma$ of the second kind or $G \cong \mathbf{SL_1}(D)$ for some degree 3 central division algebra $D$ over $k$. Let $G$ be a $k$-group of type $F_4$. We will refer to $G$ as arising from a division algebra if $G \cong \mathbf{Aut}(A)$, where $A$ is an Albert division algebra over $k$. Let $G$ be a $k$-group of type $G_2$. We will refer to $G$ as arising from a division algebra if $G \cong \mathbf{Aut}(C)$, where $C$ is an octonion division algebra over $k$.

In the thesis we derive a necessary (resp. necessary and sufficient) condition for a $k$-group $H$ of type $A_1$ or $A_2$ to embed in a $k$-group of type $F_4$ (resp. $G_2$) over $k$, in terms of certain factorization of $f_5(G)$ (resp. $f_3(G)$) with a factor the mod-2 invariant of $H$.

Owing to these results, importance of groups of type $A_1$, $A_2$ becomes evident in studying exceptional groups. The theme of irreducible subgroups and $A_1$-type subgroups of algebraic groups has been thoroughly investigated by several authors over algebraically closed fields and finite fields, see for example ([24], [55], [56], [59], [22], [23], [62], [25]). Since the Galois cohomological invariants of any group of type $G_2$ and $F_4$ over finite fields or algebraically closed fields are all trivial, our results are valid also over such fields.

The next topic of interest in the thesis is the generation of simple $k$-groups of type $G_2$ and $F_4$ by their $k$-subgroups over an arbitrary field $k$. As an easy consequence of simplicity we prove the following:

Let $G$ be a simple algebraic group over a perfect (infinite) field $k$ and $X$ be a fixed type. Suppose $G$ contains a $k$-subgroup of type-$X$. Then $G$ is generated by all $k$-subgroups of type-$X$. Moreover if $G(k)$ is simple then $G(k)$ is generated by the groups of $k$-points of type-$X$ subgroups. Hence, over a prefect (infinite) field $k$, a simple group of type $F_4$ or $G_2$ is generated by all $k$-subgroups of type $A_2$ and similarly $A_1$.

This motivates the following

**Question:** *What is the number of $k$-subgroups of a given type required to generate $G$ over $k$?*

We answer this for simple $k$-groups of type $A_2, G_2$ and $F_4$ (see the table below), in fact we exhibit explicit subgroups of each type generating the group in question. We shall see that the behavior of the $D_4$ type subgroups for groups of type $F_4$ is somewhat analogous to the behavior of the $A_2$ type subgroups for groups of type $G_2$, as far as generation of these groups is our concern.

We also calculate the number of rank-2 $k$-tori required (in fact exhibit such tori explicitly) for the generation of groups of type $A_2$, $G_2$ and $F_4$ arising from division algebras and subgroups of type $D_4$ of $\mathbf{Aut}(A)$, for $A$ an Albert division algebra, over perfect fields (see the table below).

These results motivate the following

**Problem :** *Find conditions so that a rank-2 $k$-torus embeds in a $k$-group of type $A_2$, $G_2$ or $F_4$.*

We give a solution of this for some special rank-2 tori which we refer to as **unitary tori**. We describe these below:

Let $L, K$ be étale algebras over $k$ of dimensions $3, 2$ resp. and $T = \mathbf{SU}(L \otimes K, 1 \otimes {}^-)$, where ${}^-$ denotes the non-trivial involution on $K$. Then $T$ is a torus defined over $k$, referred to in the thesis as the $K$-**unitary torus** associated with the pair $(L, K)$. For this torus, we let $q_T := <1, -\alpha\delta> = N_{k(\sqrt{\alpha\delta})/k}$, where $Disc(L) = k(\sqrt{\delta})$ and $K = k(\sqrt{\alpha})$.

Such tori are important as they occur as maximal tori in simple, simply connected groups of type $A_2$ and $G_2$ (cf. [52]. [6], [4]). We derive conditions under which such tori embed in groups of type $A_2, G_2$ or $F_4$ defined over $k$. We shall see that these embeddings are controlled by the mod-2 invariants of these groups. The behavior of the invariant $f_3$ for groups of type $A_2$ and $G_2$ is somewhat analogous to the behavior of the invariant $f_5$ for groups of type $F_4$, as far as embeddings of unitary tori in such groups is our concern. Towards the end of the thesis, we calculate $H^1(k,T)$ for a unitary rank-2 torus $T$ and see some applications to algebraic groups and étale Tits processes. Let $L, K$ be étale algebras over $k$ of dimensions $3, 2$ resp. and let $(E,\tau) = (L\otimes K, 1\otimes {}^-)$, where $x \mapsto \overline{x}$ is the non-trivial $k$-automorphism of $K$. We define étale Tits process algebras $J_1$ and $J_2$ arising from the pair $(L, K)$ to be $L$-isomorphic, if there exists a $k$-isomorphism $J_1 \to J_2$ which restricts to an automorphism of the subalgebra $L$ of $J_1$ and $J_2$ (see §5.3).

We establish a relation between $H^1(k, \mathbf{SU}(E,\tau))$ and the set of $L$-isomorphism classes of étale Tits process algebras arising from $(L, K)$. We study the effect of the presence of a unitary $k$-torus $T$ in groups of type $A_2, G_2$ and $F_4$ when $H^1(k,T) = 0$.

By a result of Steinberg (see Theorem 4.4.5) it follows that a $k$-group $G$ of type $G_2$ contains a maximal $k$-torus $T \subset G$ such that $H^1(k,T) = 0$ if and only if the associated mod-2 invariant $f_3(G)$ vanishes, i.e, $G$ splits. We give a simpler proof of this result using explicit cohomology computation of $T$. Similarly, let $G$ be a simply connected, simple $k$-group of type $A_2$: if $G$ has a maximal $k$-torus $T$ with $H^1(k,T) = 0$, then $f_3(G) = Oct(G)$ splits. The converse holds in the case when the group arises from a matrix algebra. This gives an algebraic characterization of quasi-split groups of type $A_2$ and $G_2$.

**Remark:** After submission of our paper ([9]), we discovered the paper ([4]) by C. Beli, P. Gille and T.-Y, Lee, posted recently on the math arXiv. The authors of this paper have studied maximal tori in groups of type $G_2$ in terms of the associated octonion algebra $C$. Some of our results on groups of type $G_2$ in ([9]) partially match with results in this paper (see [4], Proposition 4.3.1., Corollary 4.4.2., Remarks 5.2.5. (b), Proposition 5.2.6 (i)), however the scope of our paper and methods of proofs are very different.

## 0.1 Main results

In this section we state the main results proved in the thesis, these are contained in chapters 10, 11, 12 and 13. Let $k$ be a field of characteristic different from 2 and 3.

### Results on Factorization

We begin with the main results proved in chapter 9. We derive a necessary (resp. necessary and sufficient) condition for a $k$-group $H$ of type $A_1$ or $A_2$ to embed in a $k$-group $G$ of type $F_4$ (resp. $G_2$) over $k$ in terms of certain factorization of $f_5(G)$ (resp. $f_3(G)$) with a factor the mod-2 invariant of $H$.

**Theorem.** (Theorem 9.1.2) *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Then $f_5(A) = 0$ if and only if there exists a $k$-embedding $\mathbf{SU}(B, \sigma) \hookrightarrow G$ for some degree 3 central simple algebra $B$ with center a quadratic étale $k$-algebra $K$ and with a distinguished involution $\sigma$.*

More generally, we have

**Theorem.** (Theorem 9.1.1) *Let $K$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $K$ with an involution $\sigma$ of the second kind. Let $A$ be an Albert algebra over $k$. Let $G = \mathbf{Aut}(A)$ be the algebraic group of type $F_4$ associated to $A$. Suppose $\mathbf{SU}(B, \sigma) \hookrightarrow G$ over $k$. Then $f_5(A) = f_3(B, \sigma) \otimes \tau$ for some 2-fold Pfister form $\tau$ over $k$.*

**Theorem.** (Theorem 9.1.9) *Let $Q$ be a quaternion algebra over $k$ and $A$ be an Albert algebra over $k$. Let $G = \mathbf{Aut}(A)$ be the algebraic group of type $F_4$ associated to $A$. Suppose $\mathbf{SL}(1, Q) \hookrightarrow G$ over $k$. Then $f_5(A) = f_2(n_Q) \otimes \tau$ for some three fold Pfister form $\tau$ over $k$.*

It turns out that $G_2$ enjoyes stronger results in comparison to the case of $F_4$, which needs a lot more care.

**Theorem.** (Theorem 9.2.3) *Let $C$ be an octonion algebra over $k$. Let $B$ be a degree 3 central simple algebra over $K$, a quadratic étale extension of $k$, with an involution $\sigma$ of the second kind. Then there exists a $k$-embedding $\mathbf{SU}(B, \sigma) \hookrightarrow \mathbf{Aut}(C)$ if and only $f_3(B, \sigma) = n_C$ and $B \cong M_3(K)$.*

**Theorem** (Theorem 9.2.1) *Let $C$ be an octonion algebra over $k$ and $Q$ be a quaternion algebra over $k$. Then the following are equivalent.*

**(a)** *$Q$ embeds in $C$ as a $k$-subalgebra.*

**(b)** *$n_C = n_Q \otimes \tau$, where $\tau$ is a 1- fold Pfister form over $k$.*

**(c)** *$\mathbf{SL}_1(Q) \hookrightarrow \mathbf{Aut}(C)$ over $k$.*

The proofs of the above results make up chapter 9 of the thesis.

## Results on Embeddings of rank-2 tori

Let $L, K$ be étale algebras of $k$-dimensions $n, 2$ resp. Let $E = L \otimes K$ and $\tau$ be the involution $1 \otimes^{-}$ on $E$. Let $\mathbf{SU}(E, \tau)$ be the $K$-unitary torus associated to the ordered pair $(L, K)$. It turns out that embeddings of unitary $k$-tori in groups of type $A_2$, $G_2$ and $F_4$ are intricately linked to the mod-2 invariants of these groups. We investigate this in chapter 10 of the thesis. We state below the main results in this regard. The proofs of these results form chapter 10 of the thesis.

**Theorem.** (Theorem 10.3.3) *Let $G$ be a simple, simply connected $k$-group of type $G_2$ or $A_2$. Let $L, K$ be étale algebras of dimension $3, 2$ resp. and $T$ be the $K$-unitary torus associated with the pair $(L, K)$.*

**(a)** *Suppose there exists a $k$-embedding $T \hookrightarrow G$. Then $K \subseteq Oct(G)$.*

**(b)** *If $G$ is a simple, simply connected $k$-group of type $F_4$ or $A_2$ arising from a division algebra and $T \hookrightarrow G$ over $k$, then $L$ must be a field extension.*

Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Let $L, K$ be étale algebras of dimension $3, 2$ resp. and $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose there exists a $k$-embedding $T \hookrightarrow G$, then $K$ need not embed in $Oct(G)$ (i.e, $< 1, -\alpha >$ need not divide $f_3(G)$). However,

**Theorem.** (Theorem 10.3.10) *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $L$ be a cubic étale $k$-algebra. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose there exists a $k$-embedding $T \hookrightarrow G$. Then $f_5(A) =< 1, -\alpha > \otimes \gamma$ for some 4-fold Pfister form $\gamma$ over $k$.*

**Theorem.** (Theorem 10.3.6) *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $L$ be a cubic étale $k$-algebra with discriminant $\delta$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose $T \hookrightarrow G$ over $k$. Then $f_5(A) = q_T \otimes \gamma$ for some 4-fold Pfister form $\gamma$ over $k$.*

On exactly similar lines we derive a necessary condition for a rank-2 unitary torus to embed in a connected simple algebraic group of type $A_2$ or $G_2$:

**Theorem.** (Theorem 10.3.8) *Let $G$ be a simple, simply connected $k$-group of type $A_2$ or $G_2$. Let $C := Oct(G)$ and $n_C$ denote the norm form of $C$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $L$ be a cubic étale $k$-algebra with discriminant $\delta$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose there exists a $k$-embedding $T \hookrightarrow G$. Then $n_C = q_T \otimes \gamma$ for some two fold Pfister form $\gamma$ over $k$.*

**Theorem.** (Theorem 10.3.11) *Let $G$ be a simple, simply connected algebraic group defined over $k$. Let $L$ be a cubic étale $k$-algebra with discriminant $K_0$. Suppose there exists an $k$-embedding $\mathbf{L}^{(1)} \hookrightarrow G$. We then have:*

**(a)** *if $G$ is of type $G_2$ or $A_2$ then $Oct(G)$ splits.*

**(b)** *if $G$ is of type $F_4$ then $f_5(G) = 0$ and $K_0 \subset Oct(G)$.*

Apart from these results we study embeddings of distinguished $k$-tori in simply connected, simple algebraic groups of type $A_2$, $G_2$ and $F_4$, defined over a field $k$, in terms of the mod-2 Galois cohomological invariants attached with these groups (see Theorems 10.1.6, 10.1.5, 10.1.4). The next theorem gives criterion for groups of type $A_2$ and $F_4$ to arise from central division algebras.

**Theorem.** (Theorem 10.2.1) *Let $G$ be a simple, simply connected group of type $A_2$ or $F_4$ defined over $k$, arising from a division algebra over $k$. Then,*

**(1)** *$G(k)$ contains no non-trivial involution over $k$.*

**(2)** *There does not exists any rank-1 torus $T$ over $k$ such that $T \hookrightarrow G$ over $k$.*

**(3)** *$G$ is $k$-anisotropic.*

*Moreover, these conditions hold over any field extension of $k$ of degree coprime to 3.*

**Theorem.** (Theorem 10.2.3) *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Then the following are equivalent.*

**(a)** *$f_3(A) = 0$ (i.e, $Oct(G)$ is split).*

**(b)** *There exists a cubic étale $k$-algebra $L$ of trivial discriminant such that $\mathbf{L}^{(1)} \hookrightarrow G$ over $k$.*

**(c)** $\mathbf{SL}_1(D) \hookrightarrow G$ *over $k$, for a degree 3 central simple algebra $D$ over $k$.*

**(d)** *$A$ is a first Tits construction Albert algebra.*

## Results on generation

Let $k$ be a perfect (infinite) field and $G$ be an algebraic group of type $F_4$ (resp. $G_2$) defined over $k$, arising from an Albert (resp. octonion) division algebra. We show as an easy consequence of simplicity that $G$ is generated by all $k$-subgroups of type $A_2$ and similarly by all $k$-subgroups of type $A_1$. More precicely,

**Lemma.** (Lemma 11.2.5) *Let $G$ be a simple algebraic group over a perfect (infinte) field $k$ and $X$ be a fixed type. Suppose $G$ contains a $k$-subgroup of type-$X$. Then $G$ is generated by all $k$-subgroups of type-$X$. Moreover if $G(k)$ is simple then $G(k)$ is generated by the groups of $k$-points of type-$X$ subgroups.*

As a corollary to the above we have the following,

**Theorem.** (Theorem 11.2.6, 11.4.4) *Let $G$ be an simple algebraic group of type $G_2$ or $F_4$ defined over $k$. Then $G$ is generated by subgroups of type $A_2$, defined over $k$. Similarly, $G$ is generated by subgroups of type $A_1$, defined over $k$.*

In chapter 11 we answer the following question: What is the number of $k$-subgroups of type $A_2$ and similarly $A_1$ required to generate $G$ as above? We prove that if $k$ is a perfect (infinite) field and $G$ is an algebraic group of type $F_4$ defined over $k$, arising from an Albert division algebra, then $G$ is generated by two $k$-subgroups of type $D_4$ and three $k$-subgroups of type $A_2$. Similarly, if $G$ is an algebraic group of type $G_2$ defined over $k$, arising from an octonion division algebra, then $G$ is generated by two $k$-subgroups of type $A_2$ and three $k$-subgroups of type $A_1$. Let $A$ be a finite dimensional $k$-algebra and $S \subset A$ be a $k$-subalgebra. Then $\mathbf{Aut}(A)$ is an algebraic group defined over $k$. We shall denote by $\mathbf{Aut}(A/S)$ the (algebraic) $k$-subgroup of all automorphisms of $A$ fixing $S$ pointwise.

The precise results are as follows:

**Theorem.** (Theorem 11.2.2) *Let $A$ be an Albert division algebra over a perfect (infinite) field $k$ and $G = \mathbf{Aut}(A)$. Let $L \subseteq A$ be a cubic subfield and $H = \mathbf{Aut}(A/L)$. Then $H$ is generated by two $k$-subgroups of type $A_2$.*

**Theorem.** (Theorem 11.2.7) *Let $A$ be an Albert division algebra over a perfect (infinite) field $k$ and $G = \mathbf{Aut}(A)$. Let $H_i := \mathbf{Aut}(A/L_i) \subseteq G$, $i = 1, 2$, where $L_1 \neq L_2$ are cubic subfields of $A$. Then $G$ is generated by $H_i$, $i = 1, 2$.*

**Theorem.** (Theorem 11.2.8) *Let $A$ be an Albert division algebra over $k$ and $G = \mathbf{Aut}(A)$. Then $G$ is generated by three $k$-subgroups of type $A_2$.*

Along similar lines we have the following results for groups of type $G_2$;

**Theorem.** (Theorem 11.4.3) *Let $C$ be an octonion division algebra over a perfect (infinite) field $k$ and $G = \mathbf{Aut}(C)$. Let $K \subseteq C$ be a quadratic subfield and $H = \mathbf{Aut}(C/K)$. Then $H$ is generated by two $k$-subgroups of type $A_1$.*

**Theorem.** (Theorem 11.4.5) *Let $C$ be an octonion division algebra over $k$, where $k$ is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by two $k$-subgroups of type $A_2$.*

**Theorem.** (Theorem 11.4.6) *Let $C$ be an octonion division algebra over $k$, where $k$ is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by three $k$-subgroups of type $A_1$.*

We summarize these results in the table below, which gives the number of $k$-subgroups ($k$ a perfect field) generating simple groups of type $G_2$, and $F_4$, arising from division algebras and for $k$-subgroups of type $D_4$ of $\mathbf{Aut}(A)$, where $A$ is an Albert division algebra and $k$-subgroups of type $A_2$ of $\mathbf{Aut}(C)$, where $C$ is an octonion division algebra.

TABLE 1: Number of $k$-subgroups required for generation of groups

| Type of group | Type of $k$-subgroup | Number of $k$-subgroups required for generation |
|---|---|---|
| $F_4$ | $A_2$ | 3 |
| $F_4$ | $D_4$ | 2 |
| $D_4$ | $A_2$ | 2 |
| $G_2$ | $A_1$ | 3 |
| $G_2$ | $A_2$ | 2 |
| $A_2$ | $A_1$ | 2 |

Next we compute the number of rank-2 $k$-tori ($k$ a perfect field) generating simple, simply connected $k$-groups of type $A_2$, $G_2$, and $F_4$, arising from division algebras and for $k$-subgroups of type $D_4$ of $\mathbf{Aut}(A)$, where $A$ is an Albert division algebra. In fact, we explicitly exhibit such $k$-tori in each case (see Theorems 11.1.1, 11.3.1, 11.3.2, 11.5.1). It seems likely that these numbers are minimal in each case. The numbers are mentioned in the table below.

TABLE 2: Number of $k$-tori required for generation of groups

| Type of group | Number of rank-2 $k$-tori required for generation |
|---|---|
| $A_2$ | 2 |
| $G_2$ | 3 |
| $D_4$ | 3 |
| $F_4$ | 4 |

## Results on Cohomology and applications

Let $K$ be a quadratic étale $k$-algebra and $L$ be an étale $k$-algebra of dimension $n = 2r+1$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. In chapter 12, we calculate $H^1(k, T)$ and see some applications to algebraic groups and étale Tits processes. We state below the main results in this regard. The proofs of these results form chapter 12 of the thesis.

**Theorem.** (Theorem 12.1.3, 12.1.5) *Let $K$ be a quadratic étale $k$-algebra and $L$ be an étale $k$-algebra of dimension $n = 2r + 1$. Let $E$ be the $K$-unitary algebra and $T$ the $K$-unitary torus associated with the pair $(L, K)$. Let $K^{(1)}$ (resp. $L^{(1)}$) denote the norm 1 elements of $K$ (resp. $L$). Then,*

$$H^1(k, T) \cong \frac{K^{(1)}}{N_{E/K}(U(E, \tau))} \times \frac{S}{N_{E/L}(E^*)}.$$

*Also,*

$$H^1(k, T) \cong \frac{L^{(1)}}{N_{E/L}(E^{(1)})} \times \frac{M}{N_{E/K}(E^*)},$$

*where*

$$S := \{u \in L^* |\ N_{L/k}(u) \in N_{K/k}(K^*)\},\ M = \{\mu \in K^* |\ \mu\bar{\mu} \in N_{L/k}(L^*)\}$$

.

**Theorem.** (Theorem 12.2.4) *There exists a surjective map from $H^1(k, \mathbf{SU}(E, \tau))$ to the set of $L$-isomorphism classes of étale Tits process algebras arising from $(L, K)$.*

**Theorem.** (Theorem 12.2.5) *Let $L, K$ be a étale $k$-algebras of dimension $3, 2$ resp. and $(E, \tau)$ be the $K$-unitary algebra and $T$ the $K$-unitary torus associated with the pair $(L, K)$. Then $H^1(k, T) = 0$ if and only $J(E, \tau, u, \mu) \cong_L J(E, \tau, 1, 1)$, for all admissible pairs $(u, \mu) \in L^* \times K^*$.*

We study next the effect of the presence of a unitary torus $T$ with $H^1(k, T) = 0$ in groups of type $A_2, G_2$ and $F_4$.

**Theorem.** (Theorem 12.3.2) *Let $F = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $F$ with an involution $\sigma$ of the second kind. Let $T$ be a maximal $k$-torus of $\mathbf{SU}(B, \sigma)$. If $H^1(k, T) = 0$ then $\sigma$ is distinguished.*

**Theorem.** (Theorem 12.3.7) *Let $L, K$ be étale algebras over $k$ of dimension $3, 2$ resp. and $E$ be the $K$-unitary algebra and $T$ the $K$-unitary torus associated with the pair $(L, K)$. Let $G$ be a group of type $F_4$ (resp. $G_2$ or a simple, simply connected group of type $A_2$) defined over $k$. Assume that there is a $k$-embedding $T \hookrightarrow G$. If $H^1(\mathbf{U}(E, \tau)) = 0$ then $f_5(A) = 0$ (resp. Oct(G) splits).*

# Chapter 1

# Pfister forms and algebras

In this chapter we review some basic results on quadratic forms and composition algebras. The exposition in this chapter is mostly based on two books, for the theory of quadratic forms we refer to [20] and [53] for the theory of composition algebras. The first section covers definitions and basic results on quadratic forms that are needed later in the thesis. In the second section we introduce composition algebra and the concept of doubling. In the final section we discuss some results on structure of composition algebras. We fix a field $k$ of characteristic $\neq 2$ for this chapter.

## 1.1 Theory of Quadratic forms

An ($n$-ary) **quadratic form** $q$ over a field $k$ is a polynomial $f$ in $n$ variables over $k$ that is homogeneous of degree 2. It has a general form

$$f(X_1, \cdots, X_n) = \sum_{i,j=1}^{n} a_{ij} X_i X_j \in k[X_1, \cdots, X_n] = k[X].$$

We can make the coefficients symmetric and rewrite $f$ as

$$f(X) = \sum_{i,j} \frac{1}{2} a'_{ij} X_i X_j,$$

where $a'_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$. In this way, $f$ determines uniquely a symmetric matrix $(a'_{ij})$, which we call as the matrix associated with the quadratic form $q$ and denote it by $M_q$. We shall denote by $< d_1, \cdots, d_2 >$ the diagonal form $d_1 X_1^2 + \cdots + d_n X_n^2$.

A **quadratic space** over $k$ is a pair $(V, q)$ where $V$ is a vector space over $k$ equipped

with a quadratic form $q$ on $V$. The dimension of the quadratic space is the dimension of the underlying vector space. Let $V$ be a finite dimensional vector space over $k$ and $B : V \times V \to k$ a symmetric bilinear form on $V$. We associate with it a quadratic form $q = q_B : V \to k$, defined as $q(x) := B(x, x)$ for all $x \in V$. Note that $q$ and $B$ determine each other. Hence any vector space admitting a bilinear form has an induced quadratic form and thus is a quadratic space. We say $q_B$ is a **regular** (or non-degenerate) quadratic form, if for $x \in V$, $B(x, y) = 0$ for all $y \in V$ implies that $x = 0$.
Two quadratic spaces $(V_1, q_1)$ and $(V_2, q_2)$ are said to be **isometric** if there exists a linear isomorphism $T : V_1 \to V_2$ such that for any $v \in V_1$, $q_1(v) = q_2(Tv)$.

**Definition 1.1.1** *Let $(V_1, q_1)$ and $(V_2, q_2)$ be quadratic spaces. Let $V = V_1 \oplus V_2$ and $q_B : V \to k$ be defined as,*

$$q(x_1, x_2) = q_1(x_1) + q_2(x_2),$$

*for $(x_1, x_2) \in V$. Then $(V, q)$ is called the **orthogonal sum** of $(V_1, q_1)$ and $(V_2, q_2)$ and we write $(V, q) = (V_1, q_1) \perp (V_2, q_2)$.*

**Definition 1.1.2** *Let $(V, q)$ be a quadratic space and $v$ be a non-zero vector in $(V, q)$. We call $v \in V$ isotropic if $q(v) = 0$ and call $v$ anisotropic otherwise. The quadratic space $(V, q)$ is said to be **isotropic** if it contains a non-zero isotropic vector and said to be **anisotropic** otherwise. $(V, q)$ is said to be **totally isotropic** if all nonzero vectors in $V$ are isotropic. Let $(V, q)$ be a two dimensional quadratic space. If $V$ is isometric to $< 1, -1 >$ we call $(V, q)$ a **hyperbolic plane**. A quadratic form which is an orthogonal sum of hyperbolic planes is called a **hyperbolic space**.*

**Theorem 1.1.3** *(Witt Decomposition theorem)([20], Theorem 4.1) Any quadratic space $(V, q)$ splits into an orthogonal sum*

$$(V_t, q_t) \perp (V_h, q_h) \perp (V_a, q_a),$$

*where $V_t$ is totally isotropic, $V_h$ is hyperbolic and $V_a$ is anisotropic. Furthermore, the isometry types of $V_t, V_h, V_a$ are uniquely determined.*

**Definition 1.1.4** *The integer $m = (1/2)dimV_h$, uniquely determined in the Witt decomposition above, is called the **Witt index** of the quadratic space $(V, q)$.*

Let $d \in k^*$. We say a quadratic form $q$ **represents** $d$ if there exists $a_1, \cdots, a_n \in k$ such that $q(a_1, \cdots, a_n) = d$. We shall denote by $D_k(q)$ the set of values in $k^*$ represented by $q$. In general $D_k(q)$ is not a subgroup of $k^*$. Let $[D_k(q)]$ denote the subgroup of $k^*$ generated by $D_k(q)$. Let $G_k(q)$ be the group,

$$G_k(q) := \{a \in k^* | \ a.q \cong q\}.$$

The **tensor product** of two quadratic forms is given by,

$$< a_1, \cdots a_n > \otimes < b_1, \cdots, b_n > = < a_1b_1, \cdots a_ib_j, \cdots, a_nb_n > .$$

**Definition 1.1.5** *Let $a_1, \cdots, a_n \in k^*$. An $n$-fold **Pfister form** over $k$, denoted by $<< a_1, a_2, .., a_n >>$, is the $2^n$-dimensional quadratic form $< 1, -a_1 > \otimes < 1, -a_2 > \otimes ... \otimes < 1, -a_n >$.*

Let $K$ be an extension of a field $k$. For a given quadratic space $(V, q)$ over $k$ we construct a quadratic space $(V_K, q_K)$ over $K$ as follows: the underlying vector space $V_K$ is taken to be $K \otimes_k V$, and the $K$-quadratic form $q_K$ is uniquely given by

$$q_K(a \otimes v) = a^2 q(v),$$

for $a \in K, v \in V$. Note that the symmetric matrix of $q$ with respect to a $k$-basis $\{v_1, \cdots, v_n\}$ on $V$ is the same as that of $q_K$ with respect to the $K$-basis $\{1 \otimes v_1, \cdots, 1 \otimes v_n\}$. In particular if $q$ is non-degenerate, so is $q_K$ ([20], Chapter VII).
We list below few useful results about Pfister forms.

**Theorem 1.1.6** *([20], Theorem. 1.7) If a Pfister form $q$ over $k$ is isotropic, then it is hyperbolic.*

**Theorem 1.1.7** *([20], Theorem. 1.8) For any Pfister form $q$ over $k$, $D_k(q) = G_k(q)$.*

**Theorem 1.1.8** *(Knebusch norm principle) ([20], Chap. VII, Thm. 5.1) Let $K/k$ be a finite field extension of degree $n$ and $q$ be a quadratic form over $k$. Let $x \in K^*$. If $x \in D_K(q_K)$ then $N_{K/k}(x)$ is a product of $n$ elements of $D_k(q)$. (In particular $N_{K/k}(x) \in$*

*$[D_k(q)]$). Hence if $q$ is a Pfister form over $F$ and $q_K$ is isotropic, then $N_{K/k}(K^*) \subseteq D_F(q)$.*

**Theorem 1.1.9** *( [20], Chap. IX, Pg. 305, Chap. X, Cor. 4.13) For any quadratic form $\phi$ and any anisotropic quadratic form $\gamma$ over $k$, the following are equivalent,*

**(i)** *$\phi \subseteq \gamma$ ( i.e, $\phi$ is isometric to a subform of the form $\gamma$ over $k$).*
**(ii)** *$D_K(\phi) \subseteq D_K(\gamma)$ for any field $K \supseteq k$. Moreover, if $\phi$ and $\gamma$ are both Pfister forms, then the above conditions are also equivalent to*
**(iii)** *$\gamma = \phi \otimes \tau$ for some Pfister form $\tau$ over $k$ (In this case we will call $\phi$ as a* **factor** *of $\gamma$).*

**Theorem 1.1.10** *Let $\phi$ be a nonzero anisotropic quadratic form and $\psi$ be an irreducible anisotropic quadratic form. Let $k(\phi)$ denotes the function field of $\phi$. Suppose that the form $\phi \otimes k(\psi)$ is hyperbolic. Let $a \in D_k(\phi)$ and $b \in D_k(\psi)$. Then $ab\psi$ is isometric to a subform of $\phi$.*

**Remark 1.1.11** *A regular quadratic form $\phi$ is irreducible if and only if dim $\phi \geq 3$ or dim $\phi = 2$ and $\phi$ is anisotropic.*

**Theorem 1.1.12** *( [20], Chap. VII, Cor. 4.4) Suppose $K/k$ is a finite field extension, and $q$ is a regular quadratic form over $k$. If $q_K$ is hyperbolic over $K$, then $N_{K/k}(K^*) \subseteq G_k(q)$, where $G_k(q)$ is the group of factors of similitudes of $q$. If, in addition, $q$ is a Pfister form, then $N_{K/k}(K^*) \subseteq D_k(q)$, since $G_k(q)$ equals $D_k(q)$ for Pfister forms ([20], Chap. X, Thm. 1.8).*

**Remark 1.1.13** *Ler $q_1, q_2$ be Pfister forms over $k$. We say $q_2$* ***divides*** *$q_1$ over $k$ if there exists a Pfister form $q_3$ over $k$ such that $q_1 = q_2 \otimes q_3$ over $k$. If $q_2$ divides $q_1$ over $k$ then by Theorem 1.1.9, $q_2$ is a subform of $q_1$ over $k$.*

**Arason invariants of Pfister forms:**

Let $k_s$ be a fixed separable closure of $k$. Then $\mathbb{Z}/2\mathbb{Z}$ is a trivial $Gal(k_s/k)$-group. Let $H^n(k, \mathbb{Z}/2\mathbb{Z})$ denote the $n$th Galois cohomology group with mod-2 coefficients (see Chapter 4 for Galois cohomology). For an $n$-fold Pfister form $q = \langle\langle a_1, a_2, \cdots, a_n \rangle\rangle$ the **Arason invariant** $e_n(q)$ is given by,

$$e_n(q) = (a_1) \cup (a_2) \cup \cdots \cup (a_n) \in H^n(k, \mathbb{Z}/2\mathbb{Z}),$$

where, for $a \in k^*$, $(a)$ denotes the class of $a$ in $H^1(k, \mathbb{Z}/2\mathbb{Z})$ (see [1], Pg. 453).

**Definition 1.1.14** *([19], §2.C) Let $V$ be a finite rank free module over a quadratic étale algebra $K$ over $k$ and let $\iota$ denote the non-trivial $k$-automorphism of $K$. A **hermitian form** on $V$ (with respect to $\iota$) is a bi-additive map*

$$h : V \times V \to K$$

*such that*

$$h(\alpha v, \beta w) = \iota(\alpha)h(v, w)\beta \text{ for } v, w \in V \text{ and } \alpha, \beta \in K$$

*and*

$$h(w, v) = \iota(h(v, w)) \text{ for } v, w \in V.$$

Similarly, one can also define hermitian modules over central simple algebras with unitary involutions. We call $h$ to be non-degenerate if the only element $x \in V$ such that $h(x, y) = 0$ for all $y \in V$ is $x = 0$. The pair $(V, h)$ is called a **hermitian space**.

## 1.2 Composition algebras and Doubling

A **composition algebra** $C$ over $k$ is a finite dimensional $k$-algebra with identity element together with a regular quadratic form $N$, called the **norm form** such that

$$N(x)N(y) = N(xy) \text{ for all } x, y \in C.$$

On a composition algebra $C$ there exists an involution $^- : C \to C$ such that $x\overline{x} = \overline{x}x = N(x)$. Note that the norm form $N$ of $C$ is a Pfister form, therefore is either hyperbolic over $k$ or anisotropic over $k$. It follows that any two composition algebras of the same dimension over $k$, which both have isotropic norms, are isomorphic. We call these as **split** composition algebras.

**Theorem 1.2.1** *([53], Cor. 1.2.4) The norm $N$ on a composition algebra is uniquely determined by its algebra structure.*

The following results on on the concept of **doubling** will be needed in the sequel.

**Proposition 1.2.2** *([53], Prop. 1.5.1)(Cayley-Dickson doubling) Let $C$ be a composition algebra and $D$ be a finite dimensional composition subalgebra, $D \neq C$. Choose $a \in D^{\perp}$ with $N(a) = -\gamma \neq 0$. Then $A = D \oplus Da$ is a composition subalgebra of $C$ of dimension twice that of $D$, with multiplication given by;*

$$(u + vt)(x + yt) = (ux + \gamma\overline{y}v) + (yu + v\overline{x})a,$$

*for $u,\ v,\ x,\ y \in D$.*

As a converse to the above proposition we have the following,

**Proposition 1.2.3** *([53], Prop. 1.5.3) Let $D$ be an associative composition algebra over $k$ with norm $N_D$ and $\lambda \in k^*$. Define on $C = D \oplus D$ a product given by*

$$(x, y)(u, v) = (xu + \lambda\overline{v}y, vx + y\overline{u})$$

*for all $x, y, u, v \in D$ and a quadratic form $N$ by*

$$N((x, y)) = N_D(x) - \lambda N_D(y)$$

*for all $x, y \in D$. Then $C$ is a composition subalgebra with $N$ as its norm.*

## 1.3 Structure of composition algebras

The results in the previous section enable us to prove a key result on the structure of a composition algebra. In this section we discuss some basic results on structure and dimension of compositions algebras.

**Theorem 1.3.1** *([53], Theorem 1.6.2) The possible dimensions of a composition algebra over $k$ are $1, 2, 4$ or $8$.*

As a corollary to Proposition 1.2.2 and Theorem 1.3.1 we see that there are no composition algebras of infinite dimension. If not then with such an algebra we could construct a composition subalgebra of dimension 16, which will be a contradiction.

**Theorem 1.3.2** *([53], Theorem 1.8.1) In each dimension* $2, 4$ *or* $8$ *there is, up to isomorphism, exactly one split composition algebra (i.e, composition algebra with isotropic norm). These are the only composition algebras with zero divisors.*

A composition algebra of dimension 2 is isomorphic to either $k \oplus k$ or is a quadratic field extension of $k$. A composition algebra of dimension 4 is called a **quaternion algebra**. The split quaternion algebra over $k$ is isomorphic to the algebra of $2 \times 2$ matrices over $k$ with the determinant as norm. Let $M_2(k)$ denote the $2 \times 2$ matrix algebra. For $x \in M_2(k)$, Let $\overline{x}$ be the adjugate matrix of $x$, i.e,

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then $x \mapsto \overline{x}$ is the involution on $M_2(k)$ with $x\overline{x} = \overline{x}x = det(x)$.

**Definition 1.3.3** *An* ***octonion algebra*** *over* $k$ *is a composition algebra over* $k$ *of dimension* $8$.

Let $C$ be an octonion algebra over $k$ and let $n_C$ denote its norm form. Then $C$ is determined, up to isomorphism by $n_C$, which is a 3-fold Pfister form over $k$. Conversely, any 3-fold Pfister form is the norm form of a unique (up to isomorphism) octonion algebra over $k$. Recall that an octonion algebra $C$ over $k$ is split if and only if the associated norm form $n_C$ is isotropic over $k$. We now describe a model for the split octonion algebra over $k$. Let $C = M_2(k) \oplus M_2(k)$. We define the product on $C$ by

$$(x, y)(u, v) = (xu + \overline{v}y, vx + y\overline{u})$$

and norm form by

$$N((x, y)) = det(x) - det(y).$$

By Proposition 1.2.3 $C$ is an octonion algebra and is split since $N$ is isotropic. Let $C$ be an octonion algebra over $k$ and $K \subseteq C$ be a quadratic étale subalgebra. Then $K^{\perp}$ in $C$ with respect to the norm form on $C$, has a rank-3 hermitian module structure over $K$. We record this below:

**Proposition 1.3.4** *([12], §5) Let* $C$ *be an octonion algebra over* $k$ *and* $K \subseteq C$ *be a quadratic étale subalgebra. Then* $K^{\perp} \subseteq C$ *has a rank-3* $K$*-hermitian module structure* *given as follows:*

*Let* $K = k(\sqrt{\alpha}),\ \alpha \in k^*$. *Define* $h : K^{\perp} \times K^{\perp} \longrightarrow K$ *by*

$$h(x, y) = N(x, y) + \alpha^{-1} N(\alpha x, y),$$

*where* $N(x, y)$ *is the norm bilinear form of* $C$ *and* $K$ *acts on* $K^{\perp}$ *from the left via the multiplication in* $C$.

# Chapter 2

# Involutions on algebras

In this chapter, we will mainly focus on central simple algebras of degree 3 and discuss their involutions of the second kind. These play a central role in the theory of exceptional algebraic groups. Involutions of the second kind on a given central simple algebra of degree 3 are classified, up to conjugation, by a 3-fold Pfister form. The exposition in this chapter is mainly based on [19], [7], [63].

In the first and second section we discuss the theory of central simple algebras and involutions of the second kind on central simple algebras of degree 3. In the third section we introduce the notion of distinguished involutions. In the final section we discuss some basic results on étale algebras.

We fix a field $k$ of characteristic $\neq 2, 3$ for this chapter.

## 2.1 Central simple algebras

A finite dimensional $k$-algebra is called a **central simple algebra** over $k$ if the center $Z(A)$ of $A$ satisfies $Z(A) = k$ and $A$ has no proper two sided ideals. The set of invertible elements of $A$ is denoted by $A^*$. We call a central simple algebra $A$ over $k$ to be **split** over $k$ if $A \cong M_r(k)$ as $k$-algebras, for some $r \in \mathbb{N}$. By $Aut(A)$ we denote the group of all $k$-algebra automorphisms of $A$.

**Example 2.1.1** *$A = M_n(k)$ is a central simple algebra over $k$.*

**Example 2.1.2** *Consider the algebra of Hamilton quaternions $\mathbb{H} \subset M_2(\mathbb{C})$,*

$$\mathbb{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & -\bar{a} \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

*Then $\mathbb{H}$ is a central simple algebra over $\mathbb{R}$. Also $\mathbb{H}$ splits over $\mathbb{C}$.*

**Example 2.1.3** *We can substitute the field $\mathbb{R}$ in Example 2.1.2 by any field $k$ of characteristic $\neq 2$ and $\mathbb{C}$ by a quadratic field extension $K$ of $k$. More generally, let $a, b \in k^*$. Define $(\frac{a,b}{k})$ to be the algebra $k \oplus ks \oplus kt \oplus kst$ with multiplication defined by $st = -ts$, $s^2 = a$, $t^2 = b$. Then $(\frac{a,b}{k})$ is a four dimensional central simple algebra over $k$ which is a quaternion algebra, and all quaternion algebras over $k$ arise this way. In this notation, $\mathbb{H}$ as above is isomorphic to $(\frac{-1,-1}{\mathbb{R}})$. Note that quadratic forms and quaternion algebras are strongly related to each other. Some of the important invariants of quadratic forms are defined in terms of quaternion algebras. Also the theory of central simple algebras with involutions interacts strongly with quaternion algebras.*

**Theorem 2.1.4** *(Wedderburn) Let $A$ be a central simple algebra over $k$. There is a unique central division algebra $D$ and a positive integer $n$ such that $A \cong M_n(D)$.*

Let $A$ be a finite dimensional central simple algebra over $k$. Then there is a field extension $K$ of $k$ such that $A \otimes_k K \cong M_n(K)$, for some $n$, where $A \otimes_k K$ denotes the scaler extension of $A$ to $K$. The field $K$ is called a **splitting field** for $A$. Since the dimension of an algebra does not change under an extension of scalers, it follows that the dimension of every central simple algebra is a square: $dim_k(A) = n^2$ if $A \otimes_k K \cong M_n(K)$ for some extension $K/k$. The integer $n$ is called the **degree** of $A$. Note that since over an algebraically closed field $F$ there are no finite dimensional division algebras, any central simple algebra over an algebraically closed field $F$ is necessarily split, i.e, $A \cong M_r(F)$ for some $r \in \mathbb{N}$.

We now describe the structure of central simple algebras of degree 3. We state Wedderburn's theorem which shows that these algebras are cyclic. We begin with the definition of **cyclic algebras**,

**Definition 2.1.5** *([19], §30.A) Set $C_n = \mathbb{Z}/n\mathbb{Z}$ and $\rho = 1 + n\mathbb{Z} \in C_n$. Given a Galois $C_n$-algebra $L$ over $k$ and an element $a \in k^*$, the* **cyclic algebra** *$(L, a)$ is defined as* *follows:*

$$(L, a) = L \oplus Lz \oplus \cdots \oplus Lz^{n-1}$$

*where $z$ is subject to relations*

$$zl = \rho(l)z, \ z^n = a$$

*for all $l \in L$.*

**Theorem 2.1.6** *(Wedderburn) Every central simple $k$-algebra of degree 3 is cyclic.*

The following theorem classifies all automorphisms of a central simple algebra.

**Theorem 2.1.7** *(Skolem- Noether theorem)([19], Theorem 1.4) Let $A$ be a central simple algebra over $k$ and let $B \subseteq A$ be a simple subalgebra. Every $k$-algebra homomorphism $f : B \to A$ extends to an inner automorphism of $A$, i.e, there exists $a \in A^*$ such that $f(b) = aba^{-1}$ for all $b \in B$. In particular, every automorphism of $A$ is inner.*

As a corollary we have our next theorem:

**Theorem 2.1.8** *Let $A$ be a central simple algebra over $k$. Then $Aut(A) \cong A^*/k^*$.*

Taking $A = M_n(k)$ we immediately deduce that $Aut(M_n(k)) \cong GL_n(k)/k^* = PGL_n(k)$.

### 2.1.1 Reduced norm and reduced trace

Let $A$ be a central simple algebra over $k$ and let $L$ be a splitting field for $A$. Choose an $L$-isomorphism

$$\phi : A \otimes_k L \to M_n(L).$$

For any $x \in A$ $det\ \phi(1 \otimes x)$ belongs to $k$ and is independent of the isomorphism $\phi$ as well as $L$. We will call $det\ \phi(1 \otimes x)$ the **reduced norm** of $x$ and denote it by $N_A(x)$. Similarly, the element $trace\ \phi(1 \otimes x)$ belongs to $k$ and is independent of the isomorphism $\phi$. as well as $L$. We will call $trace\ \phi(1 \otimes x)$ the **reduced trace** of $x$ and denote it by $T_A(x)$.

## 2.2 Involutions of the second kind

We know discuss involutions of the second kind on central simple algebras. We first fix some notations. Let $K = k(\sqrt{\alpha}) = k[X]/(X^2 - \alpha)$ (either $K \cong k \times k$ or $K$ is a field extension) be a quadratic étale algebra. Let $B$ be a finite dimensional $k$-algebra whose center is $K$, and assume that either $B$ is simple (if $K$ is a field) or a direct product of two simple algebras (if $K = k \times k$). An involution of the second kind (also called a **unitary involution**) on $B$ is a $k$-linear map $\sigma : B \to B$ such that for all $x, y \in B$,

**(1)** $\sigma(xy) = \sigma(y)\sigma(x)$,

**(2)** $\sigma^2(x) = x$

**(3)** $\sigma|_K \neq 1$.

For example, Take $K = \mathbb{C}$, $B = M_n(\mathbb{C})$ and $\sigma : B \to B$ be given by the map $X \to \overline{X}^t$. For convenience, we refer to $(B, \sigma)$ as a central simple algebra over $K$ with involution of the second kind, even though the algebra $B$ may not be simple. A homomorphism $f : (B, \sigma) \to (B', \sigma')$ is a $k$-algebra homomorphism $f : B \to B'$ such that $\sigma' \circ f = f \circ \sigma$.

**Proposition 2.2.1** *([19], Proposition 2.14) Let $(B, \sigma)$ as a central simple algebra over $K$ with involution of the second kind. If $K = k \times k$, there is a central simple $k$-algebra $E$ such that*

$$(B, \tau)\sigma(E \times E, \epsilon),$$

*where the involution $\epsilon$ is defined by $\epsilon(x, y) = (y, x)$.*

This involution $\epsilon$ as above is called the **switch involution**. Note that $K \otimes K \cong K \times K$ as $K$-algebras (More generally, let $^-$ denote the non-trivial involution of $K$ then $(K \otimes K, {}^- \otimes 1) \cong (K \times K, \bar{\epsilon})$ where $\bar{\epsilon} : K \times K \to K \times K$ is given by $\bar{\epsilon}(x, y) = (\overline{y}, \overline{x})$. This isomorphism is given by $x \otimes y \mapsto (xy, x\overline{y})$). Hence if the center $K$ of $B$ is a field then $(B \otimes K, \sigma \otimes 1) \cong (B \times B^{op}, \epsilon)$, where $B^{op}$ is the opposite algebra of $B$. We now classify all involutions of the second kind on a given central simple algebra,

**Proposition 2.2.2** *([19], Proposition 2.18) Let $K$ be a quadratic étale extension of $k$ and $(B, \sigma)$ be a central simple algebra over $K$ with involution of the second kind.*

*(1) For every unit $u \in B^*$ such that $\sigma(u) = \lambda u$ with $\lambda \in K^*$, the map $Int(u) \circ \sigma$ is an involution of the second kind on $B$.*

*(2) Conversely, for every involution $\sigma'$ on $B$ which restricts to the non-trivial automorphism of $K/k$, there exists some $u \in B^*$, uniquely determined up to a factor in $k^*$, such that*

$$\sigma' = Int(u) \circ \sigma \text{ and } \sigma(u) = u.$$

## 2.3 Distinguished involutions

We now collect together some results from the theory of unitary involutions on central simple algebras of degree 3 from ([7]) and introduce the notion of a distinguished involution. Let $K$ be a quadratic étale extension of $k$ and let $B$ be a central simple algebra of degree 3 over $K$ with $\sigma$ an involution of the second kind. Let $(B, \sigma)_+$ denote the $k$-subspace of $\sigma$-symmetric elements in $B$. Let $T_B$ denote the reduced trace on $B$ and $Q_\sigma$ denote the restriction of the trace quadratic form $x \mapsto T_B(x^2)$ to $(B, \sigma)_+$. Let $< u >_B$ be the $B$-hermitian form on $B$ (as a right $B$-module) given as

$$< u >_B (x, y) = \sigma(x)uy,$$

for $u \in (B, \sigma)_+ \cap B^*$ and $x, y \in B$. The $B$-hermitian forms $< u_1 >_B$ and $< u_2 >_B$ are **isometric**, written $< u_1 >_B \sim < u_2 >_B$, if there exists $v \in B^*$ such that $\sigma(v)u_2v = u_1$ and are **similar** if there is $\lambda \in k^*$ such that $\lambda\sigma(v)u_2v = u_1$.

**Proposition 2.3.1** *([7], Lemma 1) Let $u_1, u_2 \in (B, \sigma)_+ \cap B^*$ and let $\sigma_i = Int(u_i) \circ \sigma$. Then*

*(1) An isomorphism $(B, \sigma_1) \cong (B, \sigma_2)$ of algebras with involutions induces an isometry $Q_{\sigma_1} \cong Q_{\sigma_2}$.*

*(2) $(B, \sigma_1)$ and $(B, \sigma_2)$ are isomorphic (as $K$-algebras with involution) if and only if the hermitian forms $< u_1 >_B$ and $< u_2 >_B$ are similar.*

Next result provides a decomposition of $Q_\sigma$;

**Proposition 2.3.2** *( [7], §4) Let $K = k(\sqrt{\alpha}) := k[x]/(x^2 - \alpha)$. Then there exist $b, c \in k^*$ such that,*

$$Q_\sigma \cong < 1, 1, 1 > \perp < 2 > . << \alpha >> . < -b, -c, bc > .$$

**Proposition 2.3.3** *( [7], Thm. 15) Let $B$ and $\sigma$ be as above and let $\sigma'$ be another involution of the second kind on $B$ over $K/k$ with, $Q_{\sigma'} \cong < 1, 1, 1 > \perp < 2 > . << \alpha >> . < -b', -c', b'c' >$.*

*Then the following are equivalent,*

**(i)** *The involutions $\sigma$ and $\sigma'$ are isomorphic.*

**(ii)** *The quadratic forms $Q_\sigma$ and $Q_{\sigma'}$ are isometric.*

**(iii)** *The quadratic forms* $<<\alpha>>\otimes<-b,-c,bc>$ *and* $<<\alpha>>\otimes<-b',-c',b'c'>$ *are isometric.*

**(iv)** *The Pfister forms* $<<\alpha,b,c>>$ *and* $<<\alpha,b',c'>>$ *are isometric.*

In view of this, one can assign to an involution $\sigma$ of the second kind on $B$, an invariant in $H^3(k,\mathbb{Z}/2\mathbb{Z})$ denoted by $f_3(B,\sigma)$, which is the Arason invariant of the 3-fold Pfister form $<<\alpha,b,c>>$ associated to $\sigma$ as above i.e, $f_3(B,\sigma)=(a)\cup(b)\cup(c)$.

**Remark 2.3.4** *([19], Remark 19.7) Let* $K$ *be a quadratic étale extension of* $k$ *and let* $(B,\sigma)$ *and* $(B',\sigma')$ *be central simple algebras of degree* 3 *over* $K$ *with an involution of the second kind. Then* $f_3(B,\sigma)\cong f_3(B',\sigma')$ *does not imply that* $(B,\sigma)\cong(B',\sigma')$. *For example, choose* $B\ncong B'$ *and* $K=k\times k$. *In this case* $f_3(B,\sigma)\cong f_3(B',\sigma')$ *are hyperbolic (since both contain the factor of* $<<\alpha>>=<1,-1>$*) but* $(B,\sigma)$ *and* $(B',\sigma')$ *are not isomorphic.*

Following ([7], §4), we have,

**Definition 2.3.5** *A unitary involution* $\sigma$ *on a central simple algebra* $B$ *of degree* 3 *over* $K$ *is called a* ***distinguished involution*** *if* $f_3(B,\sigma)=0$.

One can show that if $\sigma$ is distinguished then either $K=k\times k$ or $<-b,-c,bc>_K\cong<1,-1,-1>_K$ ( [7], Thm. 16). For $B=M_3(K)$, up to automorphisms of $(B,\sigma)$, we have $\sigma=Int(a)\circ\tau$, where $\tau(x_{ij})=(\overline{x_{ij}})^t$ with $a=diag(a_1,a_2,a_3)\in GL_3(k)$. It also follows ([7], Prop. 2) that,

$$Q_\sigma\cong<1,1,1>\perp<2>.<<\alpha>>.<a_1a_2,a_1a_3,a_2a_3>.$$

In this case $f_3(B,\sigma)=<<\alpha,-a_1a_2,-a_2a_3>>$. Hence, if $\sigma$ is distinguished and $K$ is a field, then $<a_1a_2,a_1a_3,a_2a_3>_K\cong<1,-1,-1>_K$. We state below few results which will be essential for our work.

**Proposition 2.3.6** *( [7], Prop. 17) Let* $K$ *be a quadratic étale algebra over* $k$ *and let* $B$ *be a central simple algebra of degree* 3 *over* $K$ *which admits a unitary involution over* $K/k$. *Then* $B$ *admits a distinguished involution over* $K/k$.

**Proposition 2.3.7** *( [7], Cor. 18) The space* $(B,\sigma)_+$ *contains an isomorphic copy of every cubic étale* $k$*-subalgebra* $L$ *of* $B$ *if and only if* $\sigma$ *is distinguished.*

**Proposition 2.3.8** *( [7], Prop. 17) Let $B$ be as above. For every cubic étale $k$-subalgebra $L \subseteq B$, there is a distinguished involution $\sigma$ on $B$ such that $L \subseteq (B, \sigma)_+$.*

## 2.4 Étale algebras

A finite dimensional commutative $k$-algebra $L$ such that $L \cong K_1 \times ... \times K_r$ for some finite separable field extensions $K_i$ of $k$, is called an **étale algebra**. Equivalently, an étale algebra is a finite dimensional commutative $k$-algebra $L$ such that $L \otimes k_{sep} \cong k_{sep} \times ... \times k_{sep}$. Let $k$ be a field of characteristic different from $2, 3$ and $L$ be an étale $k$-algebra of dimension $n$. Let $T : L \times L \to k$ be the bilinear form induced by the trace, $T(x, y) = T_{L/k}(xy)$ for $x, y \in L$, where $T_{L/k}$ denotes the trace map of $L$. Let $d \in k^*$ represents the square class of the determinant of the bilinear form $T$.

**Definition 2.4.1** *([19], Prop. 18.24) Let $L$ be an étale $k$-algebra of dimension $n$. Then the* ***discriminant algebra*** *$\delta(L)$ of $L$ over $k$, is defined to be $k[T]/(t^2 - d)$.*

For the special case when $L$ is a cubic étale $k$-algebra, by ([19], Prop. 18.25) we have,

**Proposition 2.4.2** *Let $L$ be an étale algebra of dimension $3$ over $k$. There is a canonical $k$-isomorphism $L \otimes L \cong L \times L \otimes \delta(L)$ of $k$-algebras.*

In this thesis we will denote $\delta(L)$ by $Disc(L)$ and at times also write $Disc(L) = d$.

# Chapter 3

# Linear Algebraic groups

In this chapter we review some results on linear algebraic groups which will be needed in the thesis. The exposition in this chapter is mostly based on the books [3], [11], [54]. The first section covers definition and examples of algebraic groups. The second section covers results on tori. In the third section we introduce the notion of root systems. The forth section describes the classification of simple algebraic groups. The fifth section gives a brief summary of the Borel-De Siebenthal algorithm. In the final section we study quasi-split groups, especially those of type $G_2$ and $A_2$.

## 3.1 Definition and examples

Fix an algebraically closed field $K$.

**Definition 3.1.1** *An* ***affine algebraic group*** *is an affine variety $G$ defined over $K$ with a group structure such that the product $m : G \times G \to G$ given by $(x, y) \mapsto xy$ and the inversion $i : G \to G$ given by $x \mapsto x^{-1}$ are morphisms of varieties.*

Let the **general linear group**, denoted by $\mathbf{GL}_n$ be the group consisting of all $n \times n$ matrices with non-zero determinant with entries in $K$, together with matrix multiplication as group operation. It can be easily seen that $\mathbf{GL}_n$ is an affine algebraic group. Moreover we have,

**Proposition 3.1.2** *Any affine algebraic group $G$ is a Zariski closed subgroup of $\mathbf{GL}_n$ for some $n$.*

For this reason affine algebraic groups are called **linear algebraic groups**.

For a subfield $k$ of $K$, an algebraic group $G$ is said to be defined over $k$ or a $k$-**group** if, as an algebraic variety, it is defined over $k$. Let $L/k$ be a field extension and $G$ be a $k$-group. Note that the underlying variety of $G$ is also defined over $L$. The $L$-group $G$ thus obtained is denoted by $G \otimes_k L$. Also $G(L)$ denotes the group of $L$-rational points of $G$. We now give some examples of $k$-groups:

**Example 3.1.3** *Consider the group* $\mathbf{GL}_1$ *over* $k$*. It is a* $k$*-group and we denote it by* $\mathbb{G}_m$ *and call it the multiplicative group over* $k$*. When we need to specify the field* $k$ *we also write* $\mathbb{G}_{m,k}$*. We define another* $k$*-group* $\mathbb{G}_a$*, the additive group over* $k$*, for which* $\mathbb{G}_a(L) = L$ *for every extension* $L$ *of* $k$*.*

**Example 3.1.4** *The group of non-singular diagonal matrices,* $\mathbf{D}_n := \{X = (x_{ij}) \in \mathbf{GL}_n : x_{ij} = 0 \text{ if } i \neq j\}$*.*

**Example 3.1.5** *The group of non-singular upper triangular matrices,* $\mathbf{T}_n := \{X = (x_{ij}) \in \mathbf{GL}_n : x_{ij} = 0 \text{ if } i > j\}$*.*

**Example 3.1.6** *The special linear group,* $\mathbf{SL}_n := \{X \in \mathbf{GL}_n : det(X) = 1\}$*.*

**Example 3.1.7** *The orthogonal group,* $\mathbf{O}_n := \{X \in \mathbf{GL}_n : X.X^t = 1\}$*.*

**Example 3.1.8** *The special orthogonal group,* $\mathbf{SO}_n := \mathbf{O}_n \cap \mathbf{SL}_n$*.*

**Example 3.1.9** *Let* $(V, Q)$ *be a quadratic space over* $k$ *and* $Q$ *be a non degenerate quadratic form. Define a* $k$*-group* $\mathbf{SO}(V, Q)$*, the special orthogonal group of* $Q$*, for which* $\mathbf{SO}(V, Q)(L) = \{g \in \mathbf{SL}(V \otimes L)|\ Q(g(v)) = Q(v)\}$*, for all* $v \in V \otimes L$*, where* $\mathbf{SL}(V \otimes L)$ *denotes the group of all bijective linear transformations* $V \otimes L \to V \otimes L$ *of determinant one, for any finite dimensional commutative* $k$*-algebra* $L$*. Note that the group* $\mathbf{SO}_n$ *defined in Example 3.1.8 is the special orthogonal group corresponding to* $q =<< 1, ..., 1 >>$*.*

**Example 3.1.10** ***Unitary groups****: Let* $K$ *be a quadratic étale extension of a field* $k$ *and let* $B$ *be either a central simple* $K$*-algebra or an étale* $K$*-algebra in the sense of ([19], §18.A). Assume that there is an involution* $\sigma$ *on* $B$ *of the second kind over* $K$*, i.e.* $\sigma$ *restricts to* $K$ *as the non-trivial* $k$*-automorphism of* $K$*. Let* $N_B$ *denote the reduced norm* *map of the central simple algebra $B$ or the norm map on the étale algebra $B$. We then define the algebraic groups $\mathbf{U}(B,\sigma)$ and $\mathbf{SU}(B,\sigma)$, by specifying the group of $L$-rational points, for any finite dimensional commutative $k$-algebra $L$, as follows :*

$$\mathbf{U}(B,\sigma)(L) = \{x \in B \otimes L \mid x\sigma(x) = 1\}, \quad \mathbf{SU}(B,\sigma)(L) = \{x \in \mathbf{U}(B,\sigma)(L) \mid N_B(x) = 1\}.$$

*We will denote $\mathbf{U}(B,\sigma)(k)$ by $U(B,\sigma)$ and $\mathbf{SU}(B,\sigma)(k)$ by $SU(B,\sigma)$.*

**Notation:** For a finite dimensional $k$-algebra $A$, the full group of automorphisms $Aut_{\overline{k}}(A \otimes_k \overline{k})$, is an algebraic group defined over $k$. We will denote this algebraic group by $\mathbf{Aut}(A)$ and its group of $k$-rational points will be denoted by $Aut(A)$.

Let $G$ be a connected linear algebraic group. We call $x \in G$ a semisimple (resp. unipotent) if and only if for any isomorphism $\phi$ of $G$ onto a closed subgroup of some $\mathbf{GL}_n$ we have that $\phi(x)$ is semisimple (resp. unipotent). The group $G$ is said to be unipotent if all its elements are unipotent. The maximal closed, connected, solvable, normal subgroup of $G$ is called the **radical** of $G$, denoted by $R(G)$ and the maximal closed, connected, unipotent subgroup of $G$ is called the **unipotent radical** of $G$, denoted by $R_u(G)$. We call a group $G$ **semisimple** if $R(G)$ is trivial. The group $G$ is called **reductive** if $R_u(G)$ is trivial. A torus is a reductive group which is not semisimple.

**Example 3.1.11 The Weil restriction***: Let $G$ be an algebraic group defined over a finite separable field extension $L$ of a field $k$. The* ***Weil's restriction of scalars*** *of $G$ from $L$ to $k$, denoted by $R_{L/k}(G)$ is an algebraic group over $k$ with the property $R_{L/k}(G)(k) = G(L)$. More generally, for any extension $M$ of $k$, $R_{L/k}(M) = G(L \otimes M)$. Also,*

$$Dim_k R_{L/k}(G) = [L : k] Dim_L G.$$

*One can also define $R_{L/k}(\mathbb{G}_m)$ for a finite étale extension $L$ of $k$. For more details see ([64], §3.12).*

## 3.2 Tori

A $k$-**torus** is a $k$-group which is $\overline{k}-$isomorphic to $\mathbb{G}_m^n$ for some $n$. A $k$-torus $T$ is said to be $k$-**split** if it is $k$-isomorphic to some $\mathbb{G}_m^n$. By the **rank** of a torus $T$ we mean its dimension. A **character** of a torus $T$ is an algebraic group homomorphism

$\chi : T \longrightarrow \mathbb{G}_m$. We denote the character group of $T$ by $X(T) = Hom(T, \mathbb{G}_m)$. The multiplication here is defined as (written additively), $(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t)$, for all $\chi_1, \chi_2 \in X(T)$ and $t \in T$. Let $X(T)_k$ denote the subgroup of $X(T)$ consisiting of characters of $T$ defined over $k$. We say $T$ is **$k$-isotropic** if $X(T)_k \neq \{1\}$, **$k$-anisotropic** otherwise. Note the $T$ is $k$-split if and only if $X(T)_k = X(T)$. Let $G$ be an algebraic group defined over $k$. A torus $T \subseteq G$ is called a **maximal torus** of $G$ if it is not properly contained in any other torus in $G$. A maximal $k$-torus need not be $k$-split in general. Let $T$ be a torus in $G$ which is maximal with respect to being split over $k$. The dimension of such a torus $T$ is called the **$k$-rank** of $G$ over $k$. Note that all $k$-tori split over $\overline{k}$. By the **absolute rank** of an algebraic group $G$ defined over $k$ we mean the dimension of a maximal torus in $G$.

**Definition 3.2.1** *Let $G$ be a connected reductive group defined over $k$. We say $G$ is **$k$-anisotropic** if $G$ contains no non-central $k$-split tori and $k$-**isotropic** otherwise. When $G$ is semisimple, $G$ is $k$-isotropic if and only if the $k$-rank of $G$ is positive.*

**Proposition 3.2.2** *([47], Prop. 6.3) Let $G$ be a connected reductive algebraic group defined over a perfect (infinite) field $k$, then $G$ is $k$-anisotropic if and only if $G(k)$ contains no non-trivial unipotents .*

We give some examples of tori below:

**Example 3.2.3** *Let $[L : k] = n$. Let $T = R_{L/k}(\mathbb{G}_m)$ over $L$. Then $T$ is a torus defined over $k$.*

**Example 3.2.4** *Let $L$ be a finite separable field extension of $k$. The norm map $N_{L/k}$ : $L^* \to k^*$ induces a surjective morphism of $k$-tori $R_{L/k}(\mathbb{G}_{m,L}) \xrightarrow{N} \mathbb{G}_{m,k} \to 1$. We denote $Ker(N)$ by $R^{(1)}_{L/k}(\mathbb{G}_{m,L})$ and occasionally also by $\mathbf{L}^{(1)}$ and call it the **norm torus** of $L$. Note that $R_{L/k}(\mathbb{G}_{m,L})$ is $k$-isotropic while $R^{(1)}_{L/k}(\mathbb{G}_{m,L})$ is anisotropic over $k$.*

**Example 3.2.5** *Over the complex numbers, the group $\mathbf{SO}_2$ is conjugate to the diagonal subgroup, since if $a^2 + b^2 = 1$, then*

$$c \begin{pmatrix} a & b \\ -b & a \end{pmatrix} c^{-1} = \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix}, \quad \text{with c} = \frac{1}{\sqrt{2\mathrm{i}}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

*Thus $\mathbf{SO}_2 \subseteq \mathbf{SL}_2$ is a maximal non-split torus, defined over $\mathbb{R}$.*

**Example 3.2.6** $\mathbf{SO}_2 \times \mathbf{SO}_2$ *is a two-dimensional* $\mathbb{R}$*-anisotropic torus.*

**Example 3.2.7** *Let $K$ be a quadratic étale extension of $k$ and $B$ be an étale algebra over $K$ of dimension $n$, then $\mathbf{U}(B,\sigma)$ and $\mathbf{SU}(B,\sigma)$, as defined in the previous section, are tori defined over $k$ (of rank resp. $n$ and $n-1$).*

Next we describe the structure of rank-1 tori over $k$.

**Theorem 3.2.8** *([64], Chap.II, §IV, Example 6) Let $T$ be a rank-1 torus over $k$. Then $T \cong \mathbf{K}^{(1)}$, the norm torus of a quadratic étale extension $K/k$.*

## 3.3 Root systems

Let $E$ be a finite dimensional vector space over $\mathbb{R}$. Define a **reflection**, relative to a non-zero vector $\alpha \in E$, to be a linear transformation which sends $\alpha$ to $-\alpha$ and fixes pointwise a subspace of codimension 1.

**Definition 3.3.1** *([11], Appendix) A* ***root system*** *in the real vector space $E$ is a subset $\Phi$ of $E$ satisfying:*

**(1)** *$\Phi$ is finite, spans $E$, and does not contain 0 (The elements of $\Phi$ are called* ***roots****).*

**(2)** *If $\alpha \in \Phi$, the only multiples of $\alpha$ in $\Phi$ are $+\alpha, -\alpha$.*

**(3)** *If $\alpha \in \Phi$, there exists a reflection $\tau_\alpha$ relative to $\alpha$ which leaves $\Phi$ stable.*

**(4)** *If $\alpha, \beta \in \Phi$, then $\tau_\alpha(\beta) - \beta$ is an integral multiple of $\alpha$.*

If $\Phi'$ is a root system in $E'$, then $\phi'$ is said to be **isomorphic** to $\Phi$ if there exists an isomorphism of vector spaces from $E'$ onto $E$ which maps $\Phi'$ to $\Phi$ and preserves the integers which occur in (**4**). In particular we can talk of an **automorphism** of a root system $\Phi$ in $E$. The reflections $\tau_\alpha$ for $\alpha \in \Phi$, are automorphisms of $\Phi$. The subgroup $W(\Phi)$ of $Aut(\Phi)$ generated by the $\tau_\alpha, \alpha \in \Phi$, is a finite subgroup of $GL(E)$, called the **Weyl group** of $\Phi$. There is an inner product $(\alpha, \beta)$ on $E$ with respect to which $W(\Phi)$ consists of orthogonal transformations.

A subset $\Delta$ of $\Phi$ is called a **base** if $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ is a basis of $E$, relative to which each $\alpha \in \Phi$ has a (unique) expression $\alpha = \Sigma c_i \alpha_i$, where the $c_i$ are integers of like sign. One can prove that a basis $\Delta \subset \Phi$ always exists. The roots belonging to a fixed basis $\Delta$ are called **simple roots**.

To a root system $\Phi$ as above we associate a diagram as follows:

We fix a basis $\Delta$ of $\Phi$. We form a graph with vertex set as $\Delta$ and we join two simple roots $\alpha, \beta$ by $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$ many edges. It is easily seen that the number of edges between two vertices is $0, 1, 2$ or $3$. The diagram thus obtained is called the **Dynkin diagram** of $\Phi$.

By a **subsystem** $\Phi'$ of $\Phi$ we mean a subset $\Phi' \subset \Phi$ which itself forms a root system. A proper subsystem $\Phi_1$ of $\Phi$ is called **maximal** if there is no subsystem $\Phi_2$ satisfying $\Phi_1 \subset \Phi_2 \subset \Phi$.

**Definition 3.3.2** *Let $\Phi$ be a root system. The **height** of the root $\beta = \Sigma_{\alpha\in\Delta} c_\alpha \alpha \in \Phi$ is defined as $ht(\beta) := \Sigma_{\alpha\in\Delta} c_\alpha$. The root $\alpha_0$ which has the largest height among all roots in $\Phi$, is called the highest root of $\Phi$. One can see that there is a unique root with this property.*

A root system is **reducible** if there exist proper subsets $\Phi_1, \Phi_2$ of $\Phi$ such that $\Phi = \Phi_1 \cup \Phi_2$ and each root in $\Phi_1$ is orthogonal to each root in $\Phi_2$. Otherwise, we call $\Phi$ **irreducible**. Note that every root system is decomposed into a disjoint union of irreducible root systems.

Now we shall classify root systems of algebraic groups. Let $G$ be a connected algebraic group over $k$. We call $G$ to be **simple** if $G$ has no proper closed connected normal subgroups. Let $G$ be a connected reductive algebraic group over $k$. Fix a maximal torus $T$ in $G$. We denote the character group of $T$ by $X(T)$. If rank of $G$ is $r$ (i.e, dimension of $T$ is $r$) then, $X \cong \mathbb{Z}^r$. A **cocharacter** of $T$ is an algebraic group homomorphism $\gamma : \mathbb{G}_m \longrightarrow T$. We denote the co-character group of $T$ by $Y(T) = Hom(\mathbb{G}_m, T)$. The multiplication here is defined (written additively) as, $(\gamma_1 + \gamma_2)(t) = \gamma_1(t)\gamma_2(t)$, for all $\gamma_1, \gamma_2 \in Y(T)$ and $t \in \mathbb{G}_m$.

Let $\chi \in X(T)$ and $\gamma \in Y(T)$, then $\chi \circ \gamma$ is a homomorphism of $\mathbb{G}_m$ to itself. Note that any homomorphism $f : \mathbb{G}_m \to \mathbb{G}_m$ is of the form $f(x) = x^n$ for all $x \in \mathbb{G}_m$ and some $n \in \mathbb{Z}$. We denote by $< \chi, \gamma >$ the integer such that $\chi(\gamma(\alpha)) = \alpha^{<\chi,\gamma>}$ for all $\alpha \in \mathbb{G}_m$. Thus we have a non-degenerate bilinear map $\langle , \rangle : X(T) \times Y(T) \to \mathbb{Z}$.

Let $\mathfrak{g}$ denote the Lie algebra of $G$. Then $T$ acts on $\mathfrak{g}$ via the representation $Ad : G \to GL(\mathfrak{g})$ where, for $g \in G$, $Ad(g) = d(Int(g))$, the differential of the inner conjugation automorphism of $G$ is given by $g$. Note that since $T$ is a commuting set of semisimple elements, it acts diagonally on $\mathfrak{g}$ ([54], §7.1) and $\mathfrak{g}$ decomposes as a direct sum of $T$-invariant subspaces,

$$\mathfrak{g} = \oplus_{\chi \in X(T)} \mathfrak{g}_\chi,$$

where $\mathfrak{g}_\chi := \{x \in \mathfrak{g} : Ad(t)(x) = \chi(t)x, \forall t \in T\}$. The subspaces $\mathfrak{g}_\chi \neq 0$ are called the **root spaces** and any non-zero vector in it is called a **root vector**. Those $\chi \in X(T)$ for which $\mathfrak{g}_\chi \neq 0$ are called the **roots** of $G$ with respect to $T$. Let $\Phi(G,T) := \{\chi \in X(T) | \ \mathfrak{g}_\chi \neq 0\}$. Note that $X(T), Y(T), \Phi(G,T)$ are independent of the chosen maximal torus $T$ and we denote these simply by $X, Y, \Phi$ resp. One can show the $\Phi \subseteq X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is a root system. Given a group $G$, $\Phi$ is defined to be the **root system** of $G$.
To each root $\alpha \in \Phi$ we associate a cocharacter $\alpha^* \in Y$ such that $< \alpha, \alpha^* >= 2$. The set $\Phi^* := \{\alpha^* : \ \alpha \in \Phi\}$ the called the set of **coroots** of $G$ with respect to $T$. Let $V := \mathbb{R} \otimes X$. It follows that there exists a subset $\Delta$ of $\Phi$ such that $\Delta$ is a basis of $V$ and any element in $\Phi$ is an integral linear combination of elements of $\Delta$, with all coefficients of the same sign. The elements of $\Delta$ are called **simple** roots.

A homomorphism of algebraic groups $f : G_1 \to G_2$ is called an **isogeny** if $ker(f)$ is finite. If $ker(f)$ is contained in the center of $G_1$, we call $f$ a **central isogeny**.

**Theorem 3.3.3** *([54], §9.6) A connected, split, semisimple $k$-group $G$ is determined, up to a central isogeny, by the isomorphism class of its root system.*

## 3.4 Classification of simple groups

Simple groups correspond to irreducible root systems which eventually can be classified into one of the following Cartan Killing types: $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. The four infinite family of simple groups $A_n, B_n, C_n$ and $D_n$ called the **classical simple groups**. Here the subscript $n$ denotes the rank of the group. In each diagram of type $X_n$ below, there are $n$ vertices, corresponding to the rank of the corresponding group.
**Type $A_n$ $(n \geq 1)$:**
This family of simple $k$-groups corresponds to the special linear group $\mathbf{SL}_{n+1}$. This group is **simply connected** whereas the corresponding **adjoint group** is $\mathbf{PSL}_{n+1}$. Let $K$ be a quadratic étale extension of $k$. Note that the group $\mathbf{SU}(B, \sigma)$ is a simple, simply connected algebraic group of type $A_n$ defined over $k$ when $B$ is a central simple

$k$-algebra of degree $n+1$, with an involution $\sigma$ on $B$ of the second kind over $K$. The Dynkin diagram is given by:

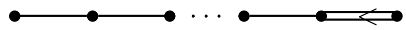


**Type $B_n$ $(n \geq 2)$:**

This family of simple $k$-groups correspond to the special orthogonal groups $\mathbf{SO}_{2n+1}$. This group is adjoint and the corresponding simply connected group is $\mathbf{Spin}_{2n+1}$. The Dynkin diagram is given by:

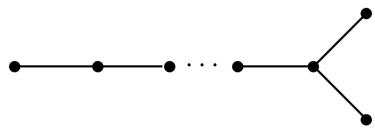
**Type $C_n$ $(n \geq 3)$:**

This family of simple $k$-groups correspond to the symplectic groups $\mathbf{Sp}_{2n}$. This group is simply connected whereas the corresponding adjoint group is $\mathbf{PGSp}_n$. The Dynkin diagram is given by:



**Type $D_n$ $(n \geq 4)$:**

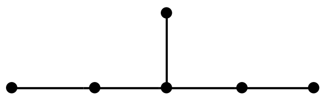
This family of simple $k$-groups correspond to the special orthogonal groups $\mathbf{SO}_{2n}$. This group is neither simply connected nor adjoint. The corresponding simply connected group is given by $\mathbf{Spin}_{2n}$ and the corresponding adjoint group is given by $\mathbf{PGSO}_{2n}$. The Dynkin diagram is given by:



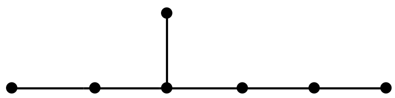
**Exceptional groups:**

In addition to these four families of classical groups there are five **exceptional groups**. These are difficult to handle than the classical groups, since these arise from non-associative algebras. The groups of type $G_2$, $F_4$ arise from octonion algebras and Albert algebra resp. We describe these in Section 5.2. Some groups of type $E_6$, $E_7$, $E_8$ can be described using octonion and Albert algebras, see [53]. We are not going to say much about the rest three exceptional groups $E_6, E_7, E_8$ in this thesis. Following are the Dynkin diagrams of these groups.

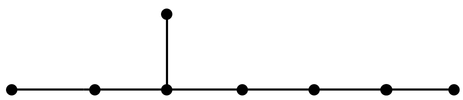
Type $E_6$:



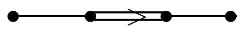
Type $E_7$:



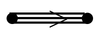
Type $E_8$:



Type $F_4$:



Type $G_2$:



Let $G$ be a semisimple algebraic group over a field $k$. By the **type** of $G$ we mean the Cartan-Killing type of the root system of the group $G \otimes \overline{k}$, obtained by extending the scalars to an algebraic closure $\overline{k}$. Let $G$ be a reductive algebraic group over a field $k$. By the **type** of $G$ we mean the type of its commutator subgroup $[G, G]$.

## 3.5 Borel-De Siebenthal algorithm

For this section we mainly refer to [2],[30]. The **Borel-De Siebenthal** algorithm answers the following question: given a root system, what are all the possible closed subroot systems? (More precisely, the possible maximal closed subroot systems of an irreducible root system). We first define closed subsystems.

**Definition 3.5.1** *A subsystem $\Phi' \subset \Phi$ is **closed** if for any $\alpha, \beta \in \Phi'$, $\alpha+\beta \in \Phi$ implies $\alpha + \beta \in \Phi'$*

A proper subsystem $\Phi'$ of a root system $\Phi$ is called a **maximal closed subsystem** if $\Phi$ is closed and if there is no closed subsystem $\Phi''$ satisfying $\Phi' \subset \Phi'' \subset \Phi$.
Let $\Phi$ be an indecomposable root system with base $\Delta$. Let $\alpha_0$ be the highest root of $\Phi$ with respect to $\Delta$. An **extended Dynkin diagram** of $\Phi$ can be obtained from the set $\Delta \cup \{-\alpha_0\}$ in the same way as the ordinary Dynkin diagram of $\Phi$ can be obtained from $\Delta$. We now state the Borel-De Siebenthal theorem:

**Theorem 3.5.2** *(Borel-De Siebenthal) Let $\Phi$ be an indecomposable root system with base $\Delta$ and the highest root $\alpha_0 = \Sigma_{\alpha \in \Delta} n_\alpha \alpha$ with respect to $\Delta$. Then the maximal closed subsystems of $\Phi$, up to conjugation by $W$, the Weyl group of $\Phi$, are those with basis:*

**(1)** $\Delta \setminus \{\alpha\} \cup \{-\alpha_0\}$ *for $\alpha \in \Delta$ with $n_\alpha$ a prime and,*
**(2)** $\Delta \setminus \{\alpha\}$ *for $\alpha \in \Delta$ with $n_\alpha = 1$.*

This Theorem helps us in calculating explicitly all maximal closed subsystems of an indecomposable root system.
We now state the algorithm of Borel-De Siebenthal for the determination of all closed subsystems of a root system.
**(1)** For any proper subset of $\Delta \cup \{\alpha_0\}$, corresponding to a subdiagram of the extended Dynkin diagram, we form the extended Dynkin diagram of each indecomposable part of that subdiagram and repeat this process.
**(2)** At any stage of the process, the set of nodes of the current diagram is a subset $J \subseteq \Phi$ and $\Psi := \mathbb{Z}J \cap \Phi$ is a closed subset of $\Phi$.

**Definition 3.5.3** *A **subsystem subgroup** of a connected reductive group $G$ is a semisimple subgroup normalized by a maximal torus of $G$.*

**Remark 3.5.4** *Let $G$ be a connected reductive group. Note that any maximal rank reductive subgroup of $G$ is a subsystem subgroup since it contains a maximal torus of $G$.*

**Proposition 3.5.5** *([30], Proposition 13.5) Let $G$ be a connected reductive group with root system $\Phi$ and let $H \leq G$ be a subsystem subgroup. Then the root system of $H$ can be naturally regarded as a subsystem of $\Phi$.*

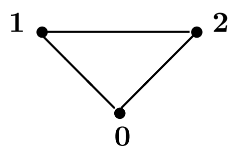
We now discuss the extent to which closed subsystems account for all subsystems subgroups.

**Theorem 3.5.6** *Let $k$ be a field of characteristic different from $2, 3$ and $G$ be a simple algebraic group with root system $\Phi$. The algorithm of Borel-de Siebenthal described above gives all subsystem subgroups of $G$.*

Since by Remark 3.5.4, any maximal rank reductive subgroup of $G$ is a subsystem subgroup, the algorithm of Borel-de Siebenthal gives all maximal rank reductive subgroup of $G$.

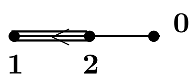
In the extended diagrams below, the node $0$ corresponds to $-\alpha_0$, where $\alpha_0$ is the highest root.

**Example 3.5.7** *We begin with the most simple case, take the root system of type $A_2$. Consider the corresponding extended Dynkin diagram $\tilde{A}_2$.*




*We first remove one node. Observe that on removal of any of the nodes $0, 1, 2$ of $\tilde{A}_2$ we again get the root system of type $A_2$. Lets remove two nodes. Note that removal of any two nodes leads to a Dynkin diagram of type $A_1$. Hence $A_2$-root system contains a closed subsystem of type $A_1$. Since the algorithm of Borel-de Siebenthal gives all maximal rank semisimple subgroups, we see that $A_1 \times A_1, B_2, G_2 \nsubseteq A_2$.*

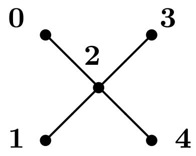
**Example 3.5.8** *Lets start with the root system of type $G_2$. Consider the corresponding extended Dynkin diagram $\tilde{G}_2$.*



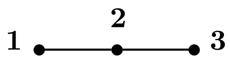

*We first remove one node. Observe that on removal of node $0$ we again get the root system of type $G_2$. On removal of node $2$, we get a the root system of type $A_1 \times A_1$.*

*On removal of node 1 we get a the root system of type $A_2$. Hence $G_2$ contains closed subsystems of type $A_1, A_1 \times A_1$ and $A_2$. Since the algorithm of Borel-de Siebenthal gives all maximal rank semisimple subgroups, we see that $B_2 \nsubseteq G_2$.*

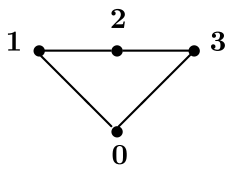
**Example 3.5.9** *We start with the root system of type $D_4$. Consider the corresponding extended Dynkin diagram $\tilde{D}_4$.*




*We first remove one node. Observe that on removal of any of the nodes $0, 1, 3$ or $4$ of $\tilde{D}_4$ we again get the root system of type $D_4$. Lets remove node 2. Removal of node 2 leads to a Dynkin diagram of type $A_1 \times A_1 \times A_1 \times A_1$. Hence $D_4$-root system contains a closed subsystem of type $A_1 \times A_1 \times A_1 \times A_1$. Now we remove two nodes at a time. On removal of nodes 2, 3 we get a Dynkin diagram of type $A_1 \times A_1 \times A_1$. On removal of nodes 3, 4. We get a Dynkin diagram of type $A_3$.*




*Hence $D_4$-root system contains a closed subsystem of type $A_3 \cong D_3$. Again consider the corresponding extended Dynkin diagram $\tilde{A}_3$.*




*By removing any of the nodes $0, 1, 2$ or $3$ we get a Dynkin diagram of type $A_3$. Hence $D_4$-root system contains a closed subsystem of type $A_3$. By removing nodes $2, 0$ we get a Dynkin diagram of type $A_1 \times A_1$. In particular we have inclusions of the subsystem subgroups of $D_4$ as follows:*

$$A_1 \subseteq A_1 \times A_1 \subseteq A_1 \times A_1 \times A_1 \subseteq A_1 \times A_1 \times A_1 \times A_1 \leq D_4,\ A_2 \leq A_3 \leq D_4.$$

*Since the algorithm of Borel-de Siebenthal gives all maximal rank semisimple subgroup of $G$, $A_2 \times A_2 \nsubseteq D_4$.*

## 3.6 Quasi-split groups

A $k$-group $G$ is said to be **quasi-split** over $k$ if there exists a **Borel subgroup** (a maximal connected closed solvable subgroup) of $G$ that is defined over $k$. Clearly, if $G$ splits over $k$ then $G$ is quasi-split over $k$. Another characterization of quasi-siplit groups is given in ( see [61]),

**Theorem 3.6.1** *A semisimple group $G$ over $k$ is quasi-split if and only if the centralizer of a maximal $k$-split torus $S$ of $G$ is a torus.*

**Proposition 3.6.2** *([19], Proposition 27.8) Let $G$ be an algebraic group of type $B_n, C_n, E_7, E_8, F_4$ or $G_2$. If $G$ is quasi-split over $k$ then $G$ splits over $k$.*

### 3.6.1 Quasi-split groups of type $A_n$

Let $G = \mathbf{SU}(V, h)$ be a special unitary group, where dimension of $V$ is $2n+1$ (or $2n$). Then $G$ is quasi-split over $k$ if and only if its $k$-rank is $n$ ([61],Table of Tits indices). In particular, for a simple, simply connected group $G$ of type $A_2$ defined over $k$, $G$ is quasi-split (non-split) over $k$ if and only its $k$-rank is 1. Let $G$ be a simple, simply connected group of type $A_2$ defined over $k$. In the thesis we will give another characterization of $G$ being quasi-split.

We now give another characterization of being quasi-split for the $A_2$-type group.

**Theorem 3.6.3** *Let $G = \mathbf{SU}(B, \sigma)$ be a simple simply connected group of type $A_2$ defined over $k$, where $B$ is a degree 3 central simple algebra with center a quadratic étale $k$-algebra $K$ and with an involution $\sigma$ of the second kind. Then $G$ is quasi-split over $k$ if and only if $B = M_3(K)$ and $\sigma$ is distinguished.*

Proof. Note that $G$ is quasi-split (non-split) over $k$ if and only its $k$-rank is 1 ([61], Table on Tits indices). Assume that $G$ is quasi-split (non-split) over $k$. Hence $k$-rank of $G$ is 1 and therefore $G$ is isotropic over $k$. If follows from Theorem 10.2.1 and Theorem 10.1.1 that $B = M_3(K)$ and $\sigma$ is distinguished. Conversely, assume that $B = M_3(K)$ and $\sigma$ is distinguished. Since $\sigma$ is distinguished, $\sigma \cong \sigma_h$, where $\sigma_h$ is the involution on $M_3(K)$ given by $X \to h\overline{X}^t h^{-1}$, for $h = diag(1, -1, -1) \in GL(3, k)$. From this it follows that $SU(M_3(K), \sigma_h) = SU(K^3, h)$ is $k$-isotropic. Therefore $k$-rank of $G$ is $\geq 1$

and hence $G$ is quasi-split over $k$. $\square$

**Example of a quasi-split non-split group:**

Consider the group $G = \mathbf{SU}(2,1) = \{X \in M_3(\mathbb{C}) |\ det(X) = 1, XJ\overline{X^t} = J\}$ , where $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ defined over $\mathbb{R}$. Note that $G$ has a rank-1 $\mathbb{R}$-split torus $T \cong \mathbb{G}_m$ over $\mathbb{R}$, an isomorphism is given by $x \mapsto Diag(x, 1, x^{-1}) \subseteq \mathbf{SU}(2,1), x \in \mathbb{G}_m$. Hence $G$ is quasi-split over $\mathbb{R}$. But $G$ has no split maximal (rank-2) torus.

# Chapter 4

# Galois Cohomology

In this chapter we introduce Galois cohomology and discuss some results on Galois cohomology of algebraic groups. There are several excellent references for this, [18], [46], [19] to mention a few.

In the first section we introduce the cohomology sets $H^i$ and list few examples. In the second section we describe some cohomological sequences. In the third section we introduce the concept of twisting. The final section mainly deals with Galois cohomology of linear algebraic groups

## 4.1 Cohomology sets

In this section we revise some general constructions of non-abelian cohomology. Let $G$ be a group acting on a set $A$. We call $A$ to be a $G$-set. If $A$ is a group on which $G$ acts such that $g(ab) = g(a)g(b)$ for all $g \in G, a, b \in A$, then we call $A$ to be a $G$-group.

**Definition 4.1.1** *Let $A$ be a $G$-group. The cohomology set $H^0(G, A)$ is defined as follows:*

$$H^0(G, A) = A^G := \{a \in A : \ g(a) = a \ \forall g \in G\},$$

Note that $H^0(G, A)$ forms a subgroup of $A$.

**Definition 4.1.2** *Let $A$ be a $G$-group. Define,*

$$Z^1(G, A) := \{\phi : G \to A : \phi(g_1 g_2) = \phi(g_1)g_1(\phi(g_2))\}.$$

*An element of* $Z^1(G,A)$ *is called a* 1*-cocycle. Define an equivalence on the set of* 1*-cocycles as,* $\phi_1 \sim \phi_2$ *if there exists* $a \in A$ *such that* $\phi_1(g) = a^{-1}\phi_2(g)g(a)$ *for all* $g \in G$*. Define* $H^1(G,A)$ *to be the set of equivalence classes of* 1*-cocycles.*

Note that when $A$ is abelian, the set $Z^1(G,A)$ is an abelian group with the product $(\phi_1\phi_2)(g) := \phi_1(g)\phi_2(g)$ for all $g \in G$. This operation is compatible with the equivalence relation on 1-cocycles, hence it induces an abelian group structure on $H^1(G,A)$.
We will now define general cohomology sets $H^i$. We refer to [18], §1.2 for details.

**Definition 4.1.3** *A topological group* $G$ *is said to be* **profinite** *if it is a projective limit of finite groups, the latter carrying discrete topology.*

Let $G$ be a profinite group and $A$ be any $G$-group with discrete topology. In such a case we shall always assume that the action of $G$ on $A$ is continuous. We shall modify the definition of $H^i(G,A)$ in this case by requiring the cocycles to be continuous. If $U \subset V$ are open normal subgroups of $G$ then the inclusion $\iota : A^V \hookrightarrow A^U$ and the natural projection $\pi : G/U \to G/V$ are compatible (i.e, $\iota(\pi(gU)(a)) = gU(\iota(a))$ for all $a \in A^V$ and $gU \in G/U$), hence we get the induced map $\rho_U^V : H^i(G/V, A^V) \to H^i(G/U, A^U)$ called **inflation**. The sets $H^i(G/U, A^U)$ together with the inflation map form an inductive system and

$$H^i(G,A) := \varinjlim H^i(G/U, A^U),$$

## 4.2 Cohomology sequences

Let $A, B, C$ be $G$-groups. If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of groups, then we have an exact sequence of pointed sets

$$0 \longrightarrow H^0(G,A) \longrightarrow H^0(G,B) \longrightarrow H^0(G,C) \longrightarrow H^1(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C).$$

Since the given sequence is exact we can view $C \cong B/A$, where $B/A := \{bA|\ b \in B\}$ is a homogeneous space for $B$. The inclusion map $i : A \hookrightarrow B$ induces a map from $H^0(G,A) \longrightarrow H^0(G,B)$. Similarly the projection map $\pi : B \to B/A$, induces a map from $H^0(G,B) = B^G \longrightarrow H^0(G,C) = (B/A)^G$. We now describe the connecting map from $H^0(G,C) \longrightarrow H^1(G,A)$ explicitly. Given an element of $H^0(G,B/A) = (B/A)^G$, choose a representative $b$ of it in $B$. Define $a_s = b^{-1}s(b)$ for all $s \in G$. Since $s(bA) = sbA = bA$, $a_s = b^{-1}s(b) \in A$. It is easy to check that $(a_s) \in H^1(G,A)$.

**Few Examples:**

**(1)** If $G$ acts trivially on $A$, then $Z^1(G,A) = Hom(G,A)$ is the set of all homomorphisms from $G$ to $A$. Two cocycles $\phi_1$ and $\phi_2$ in $Z^1(G,A)$ are equivalent if and only if $\phi_2 = Int(a) \circ \phi_1$ for some $a \in A$. Let $\sim$ denote this equivalence relation. Then $H^1(G,A) \cong Hom(G,A)/\sim$.

**(2)** ([19], Theorem 29.2) (Hilbert Theorem 90) For any separable and associative $k$-algebra $A$,

$$H^1(k, \mathbf{GL}_1(A)) = 0.$$

In particular, $H^1(k, \mathbf{GL}_n) = 0$ and $H^1(k, \mathbb{G}_m) = 0$.

**(3)** $H^1(k, \mathbf{SL}_n) = 0$. More generally, ([19], Corollary 29.4) Let $A$ be a central simple $k$-algebra.

$$H^1(k, \mathbf{SL}_1(A)) \cong k^*/N_A(A^*).$$

**(5)** ([54], Ex. 13.2.8) If $T$ is a $k$-split torus then $H^1(k,T) = 0$.

## 4.3 Twisting

Let $A$ be a $G$-group and $E$ be a $G$-set on which $A$ acts. For $x \in A, a \in E$ let $x.a$ denote the action of $A$ on $E$. We call this action to be $G$-compatible if, $g(x.a) = g(x)g(a)$ for all $g \in G, x \in A, a \in E$. For two $G$-sets $E, F$ and a map of sets $f : E \to F$, define ${}^s f : E \to F$ for $s \in G$ as follows, ${}^s f(e) = s(f(s^{-1}e))$ for all $e \in E$. Note that $({}^s f)(s(e)) = s(f(e))$. Let $Aut(E)$ denote the group of all set bijections from $E$ onto itself. In the particular case, when $E = F$, the above action makes $Aut(E)$ into a $G$-group. Define a map $\phi : A \to Aut(E)$ as $\phi(x)(e) := x.e$. It is easy to check that $\phi$ is a $G$-homomorphism. Hence it induces a mapping of cohomologies

$$H^i(G,A) \to H^i(G, Aut(E)).$$

Let $A$ be a $G$ -group and $E$ be a set on which $A$-acts $G$-compatibly. Let $(a_s) \in H^1(G,A)$ be a 1-cocycle. Let $F$ be a copy of $E$ with a bijection $f : E \to F$, namely the identity map. Define an operation of $G$ on $F$ as follows: $s(f(x)) = f(a_s.s(x))$ for $x \in E$ and $s \in G$. With this operation $F$ is a $G$-set and $F$ together with the mapping $f : E \to F$ is said to be obtained from $E$ by twisting with the cocycle $(a_s)$ and is denoted by ${}_aE$. If we replace $(a_s)$ by an equivalent cocycle, $f$ is changed by an automorphism of $E$. If $E$ has, in addition, some algebraic structures and $a_s$ preserves these, then the twisted set ${}_aE$ will carry them. In particular, let $A$ be a $G$-group and $(a_s) \in H^1(G, Aut(A))$. Define a new $G$-action on $A$ by,

$$s * a = a_s.s(a) \text{ for all s} \in G.$$

With this new $G$-action $A$ is called the **twist** of $A$ by $(a_s)$ and is denoted by ${}_aA$.

**Example 4.3.1** *Let $k$ be a field and $B = k \times k$ considered as a $k$-algebra. Let $\mathbf{Aut}(B)$ denote the group of $\overline{k}$-algebra automorphisms of $B \otimes \overline{k} \cong \overline{k} \times \overline{k}$. Note that these $\overline{k}$-algebra automorphisms correspond to permuting the components, hence $Aut(B \otimes \overline{k}) \cong S_2$, where $S_2$ denotes the symmetric group on two symbols. Note that $Gal(\overline{k}/k)$ acts trivially on $S_2$, hence a 1-coclycle $(a_s) \in H^1(Gal(\overline{k}/k), S_2)$ is a group homomorphism from $Gal(\overline{k}/k)$ into $S_2$. Let $(a_s)$ be the non-trivial cocycle in $H^1(Gal(\overline{k}/k), \mathbf{Aut}(B))$ (i.e, the non-trivial homomorphism). Then $B$ twisted by $(a_s)$ is a quadratic field extension of $k$ (with notations as above $A = \mathbf{Aut}(B)$ and $E = \overline{k} \times \overline{k}$).*

**Example 4.3.2** *Let $(a_s) \in H^1(Gal(\overline{k}/k), \mathbf{Aut}(M_{n+1}))$ be a 1-cocycle. We twist $M_{n+1}$ by the 1-cocycle $(a_s)$. The twisted algebra ${}_aM_{n+1}$ is a central simple $k$-algebra $A$ and we have an isomorphism $g : M_{n+1} \otimes \overline{k} \to A \otimes \overline{k}$ such that $s(g(x)) = g(a_s s(x))$ for all $x \in M_{n+1}$.*

**Example 4.3.3** *Consider the algebra $M_{n+1} \oplus M_{n+1}$. It has an involution $I$ of the second kind given by $(X, Y) \to (Y^t, X^t)$ where $t$ denotes the transpose. Consider the group $Aut(M_{n+1} \oplus M_{n+1}, I)$ of those automorphisms of $M_{n+1} \oplus M_{n+1}$ commuting with the involution $I$. Let $(a_s) \in H^1(Gal(\overline{k}/k), \mathbf{Aut}(M_{n+1} \oplus M_{n+1}, I)$ be a 1-cocycle. We now twist $M_{n+1} \oplus M_{n+1}$ by $(a_s)$. We get a twisted algebra $A$ and an isomorphism $g : M_{n+1}(\overline{k}) \oplus M_{n+1}(\overline{k}) \to A \otimes \overline{k}$. Note that $A$ will carry an involution $J$ of the second kind. Also the center $k \oplus k$ of $M_{n+1} \oplus M_{n+1}$ will get twisted into the center of $A$. By Example 4.3.1 above we see that the center $k \oplus k$ of $M_{n+1} \oplus M_{n+1}$ gets twisted into* *a quadratic extension $K$ of $k$. Hence $A$ is a simple $k$-algebra with involution $J$ of the second kind with center a quadratic extension $K$ of $k$.*

## 4.4 Galois Cohomology of Algebraic groups

In this section we will use some definitions and results from ([19], §29). Fix a separable closure $k_{sep}$ of $k$ and let $\Gamma = Gal(k_{sep}/k)$. Let $G$ be an algebraic group defined over $k$ and let $\rho : G \longrightarrow \mathbf{GL}(W)$ be a representation of $G$ in a finite dimensional $k$-vector space W. Fix an element $w \in W$ and identify $W$ with a $k$-subspace of $W_{sep} = W \otimes k_{sep}$. An element $w' \in W_{sep}$ is called a twisted $\rho$-forms of $w$ if $w' = \rho_{sep}(g)(w)$ for some $g \in G(k_{sep})$, where $\rho_{sep} = \rho \otimes k_{sep}$. Let $\overline{A}(\rho, w)$ denote the groupoid whose objects are the twisted $\rho$-form of $w$ and whose morphisms $w' \to w''$ are the elements $g \in G(k_{sep})$ such that $\rho_{sep}(g)(w') = w''$. Let $A(\rho, w)$ denote the groupoid whose objects are the twisted $\rho$-forms of $w$ which lie in $W$ and morphisms $w' \to w''$ are the elements $g \in G(k)$ such that $\rho(g)(w') = w''$. Let $X$ denote the $\Gamma$-set of objects of $\overline{A}(\rho, w)$. Then $X^\Gamma$ the set of $\Gamma$-fixed points in $X$, is the set of objects of $A(\rho, w)$. Also, the set of orbits of $G(k)$ in $X^\Gamma$ is the set of isomorphism classes $\mathrm{Isom}(A(\rho, w))$ of objects of $A(\rho, w)$. Let $w' \in A(\rho, w)$. By $[w'] \in \mathrm{Isom}(A(\rho, w))$ we will denote the isomorphism class of $w'$. Let $\mathbf{Aut}_G(w)$ denote the stabilizer of $w$ in $G$.

**Proposition 4.4.1** *([19], Prop. 29.1) If $H^1(k, G) = 0$, there is a natural bijection of pointed sets*

$$Isom(A(\rho, w)) \leftrightarrow H^1(k, \mathbf{Aut}_G(w))$$

*which maps the isomorphism class of $w$ to the base point of $H^1(k, \mathbf{Aut}_G(w))$.*

The bijection between these sets is as follows: for $w' \in A(\rho, w)$, choose $g \in G(k_{sep})$ such that $\rho_{sep}(g)(w) = w'$. Define a 1-cocycle class $[\alpha_\sigma] \in H^1(k, \mathbf{Aut}_G(w))$ by $\alpha_\sigma = g^{-1}\sigma(g)$. Conversely let $[\alpha] \in H^1(k, \mathbf{Aut}_G(w))$. Since $H^1(k, G) = 0$, $\alpha = g^{-1}\sigma(g)$ for some $g \in G(k_{sep})$. The corresponding object in $A(\rho, w)$ is $\rho_{sep}(g)(w)$.
The next lemma determines the cohomology sets with coefficients in $R_{L/k}(G)$, where $L/k$ is a finite separable field extension of $k$ and $G$ is an algebraic group defined over $L$.

**Lemma 4.4.2** *(**Shapiro's Lemma**)([19], Lemma 29.6) Let $L/k$ be a finite separable field extension and let $G$ be an algebraic group defined over $L$. Then there is a natural* *bijection of pointed sets*

$$H^1(k, R_{L/k}(G)) \to H^1(L, G).$$

### 4.4.1 Forms

Let $L/k$ be an Galois extension. Let $A$ be an algebraic group defined over $k$ and $A_L$ denote the set of $L$-rational points of $A$. Note that the Galois group $Gal(L/k)$ acts on $A_L$. If $L = k_s$, we denote $H^i(Gal(L/k), A_L)$ simply by $H^i(k, A)$. An $L/k$ form of $A$ is an algebraic group defined over $k$ which is isomorphic to $A$ as an algebraic group over $L$. Let $\Phi(k, A)$ denote the set of $k$-isomorphism classes of $k_s/k$-forms of $A$.

**Theorem 4.4.3** *([50], §III.1.3) There exists a bijection from $H^1(k, \mathbf{Aut}(A))$ onto $\Phi(k, A)$, the set of $k$-isomorphism classes of $k_s$-forms of $A$.*

Let $A$, $B$ be algebraic varieties defined over $\overline{k}$. Let $L$ be a field extension of $k$ and $(Aut(A))_L$ denote the group of all automorphisms of $A$ defined over $L$. An algebraic variety $B$ defined over $k$ and isomorphic to $A$ over $L$ is called a $L/k$-**form** of $A$. Suppose $A$ is quasi projective (i.e. isomorphic to a locally closed subvariety of some projective space), then there exists an isomorphism between $H^1(L/K, Aut(A)_L)$ and the set of $k$-isomorphism classes of $L/k$-forms of $A$. Now we see this explicitly in the case of some algebraic groups ([18], §1.6).

We describe first the classification of $k_s/k$ forms of groups of type $A_n$ (i.e, $k_s/k$ forms of $\mathbf{SL}_{n+1}$). These are classified into two subtypes,

(a) *Subtype* 1- These correspond bijectively to central simple $k$-algebras $A$ of dimension $(n+1)^2$, that is, $A \otimes k_s$ isomorphic to $M_{n+1}(k_s)$. The twisted forms of $\mathbf{SL}_{n+1}$ belonging to the subtype 1 are given by $H = \{x \in A| \ N_A(x) = 1\}$.

(b) *Subtype* 2- These correspond bijectively to simple $k$-algebras $A$ of dimension $(n+1)^2$ over the center of $A$ which is a quadratic field extension of $k$, with an involution $\sigma$ of the second kind, namely, the twisted forms of $\mathbf{SL}_{n+1}$ belonging to the subtype 2 are given by $H = \{x \in A| \ X\sigma(X) = 1, N_A(x) = 1\}$.

The homomorphism $\psi : A(k_s) \to \mathbf{Aut}(A)$ defined as $x \to Int(x)$ induces a map from $\theta : H^1(k, A) \to \Phi(k, A)$. An $L/k$-form $G'$ of $G$, is said to be an **inner form** if $G' \in Image(\theta)$. A form that is not inner is called an **outer form**. We shall need

**Proposition 4.4.4** *([54], Proposition 16.4.9) Let $G$ be an connected, reductive group defined over $k$. Then $G$ is an inner $k$-form of a quasi-split $k$-group.*

We now recall the following well known result:

**Theorem 4.4.5** *(Steinberg)([46], Theorem 6.23) Let $G_0$ be a semisimple group defined and quasi-split over a perfect field $k$. Let $\xi \in \mathcal{Z}^1(k, G_0)$ and let $G = {}_\xi G_0$ be the corresponding twisted group. Then for any maximal torus $T$ of $G$ defined over $k$ there is a cocycle $\mu \in \mathcal{Z}^1(k, T)$ such that $G_0 = {}_\mu G$.*

# Chapter 5

# Jordan algebras

The aim of this chapter is to give a quick introduction to the theory of Jordan algebras, especially Jordan algebras of degree 3. The set of symmetric elements in an associative algebra with involution admits the structure of a Jordan algebra. For a detailed exposition on Jordan algebras we refer to ([17]), ([53]) and ([33]), the Fields Institute notes on Albert algebras by H. P. Petersson for an excellent recent survey.
In the first section we define Jordan algebra and give a few examples. In the second section we study Albert algebras and Tits constructions. In the final section we study structure of étale Tit's constructions.

Let $k$ be a field of characteristic different from 2 and 3.

## 5.1 Basic definitions

**Definition 5.1.1** *A commutative algebra over a field $k$ of characteristic $\neq 2$ in which the Jordan identity $x^2(xy) = x(x^2y)$ holds is called a* ***Jordan algebra****.*

**Examples:**

**(1)** Let $A$ be an associative algebra. We define a new product on $A$ as following, $a * b := \frac{1}{2}(ab + ba)$. With this new product $A$ becomes a Jordan algebra and is denoted by $A^+$.

**(2)** Let $B$ be an associative algebra with an involution $\tau$. Let $(B, \tau)_+$ denote the set of $\tau$-symmetric elements of $B$. The set $(B, \tau)_+$ is a Jordan subalgebra of $B^+$.

**(3)** As a special case of example **(2)**, the set of hermitian real, complex or quaternion matrices with multiplication, $X * Y = \frac{XY+YX}{2}$ forms a Jordan algebra.

## 5.2 Albert algebras

In this subsection, we recall briefly some basic results on Albert algebras needed in the thesis. Let $C$ be an octonion algebra over $k$. Let $M_3(C)$ denote the space of $3 \times 3$ matrices with entries in $C$. Let $\Gamma = diag(\gamma_1, \gamma_2, \gamma_3) \in GL_3(k)$ be a diagonal matrix and let

$$\mathcal{H}_3(C,\Gamma) = \{X \in M_3(C) | \Gamma^{-1}\overline{X}^t\Gamma = X\},$$

where, for $X = (x_{ij}), \overline{X} = (\overline{x_{ij}})$, $x \mapsto \overline{x}$ denotes the involution on $C$ and $X^t$ is the transpose of $X$. With the multiplication $X \circ Y = \frac{1}{2}(XY + YX)$, $\mathcal{H}_3(C,\Gamma)$ is an Jordan algebra over $k$. These are called **reduced** Albert algebras. It follows that $\mathcal{H}_3(C,\Gamma)$ consists of all $3 \times 3$ matrices

$$X = \begin{pmatrix} \alpha_1 & c & \gamma_1^{-1}\gamma_3\overline{b} \\ \gamma_2^{-1}\gamma_1\overline{c} & \alpha_2 & a \\ b & \gamma_3^{-1}\gamma_2\overline{a} & \alpha_3 \end{pmatrix},$$

where $\alpha_i \in k,\ a, b, c \in C$. It is known that the octonion algebra $C$ is determined, up to isomorphism, by the algebra $\mathcal{H}_3(C,\Gamma)$ and is called the **coordinate octonion algebra** of $\mathcal{H}_3(C,\Gamma)$.

For $X = (x_{ij}) \in \mathcal{H}_3(C,\Gamma)$, we define its **trace** by $T(X) = \sum_{i=1}^3 x_{ii} \in k$. A $k$-algebra $A$ is called an **Albert algebra** if $A \otimes_k \overline{k} \cong \mathcal{H}_3(C, I)$ for the (split) octonion algebra $C$ over $\overline{k}$ and $I \in GL_3(\overline{k})$ the identity matrix, $\overline{k}$ denotes an algebraic closure of $k$. The **split** Albert algebra over $k$ is an Albert algebra $A$ isomorphic over $k$ to $\mathcal{H}_3(C, I)$, where $C$ is the split octonion algebra over $k$ and $I \in GL_3(k)$ is the identity matrix. An Albert algebra is either reduced or a division algebra ( [17], Chap. IX, §1, Pg. 359). Tits has given two rational constructions of Albert algebras, which are exhaustive, i.e, all Albert algebras arise from these constructions. We briefly describe these for the convenience of the reader.

**Tits's first construction**

Let $A$ be a central simple algebra of degree 3 over a field $k$ and let $\mu \in k^* := k - \{0\}$. For $a, b \in A$ define,

$$a.b = \frac{1}{2}(ab + ba),\ a \times b = a.b - \frac{1}{2}t(a)b - \frac{1}{2}t(b)a + \frac{1}{2}(t(a)t(b) - t(a.b)),$$

here $t = T_A$ is the reduced trace on $A$. Further, for $x \in A$, $\overline{x} = \frac{1}{2}(t(x) - x)$. To this data, one attaches an Albert algebra $J(A, \mu)$ as follows: $J(A, \mu) = A_0 \oplus A_1 \oplus A_2$, where $A_i = A$ for $i = 1, 2, 3$, with multiplication,

$$(a_0,\ a_1,\ a_2)(b_0,\ b_1,\ b_2) = (a_0.b_0 + \overline{a_1 b_2} + \overline{b_1 a_2},\ \overline{a_0} b_1 + \overline{b_0} a_1 + \mu^{-1} a_2 \times b_2,\ a_2 \overline{b_0} + b_2 \overline{a_0} + \mu a_1 \times b_1).$$

With this multiplication, $J(A, \mu)$ is an Albert algebra over $k$ and is referred to as a **first Tits construction** Albert algebra. The Albert algebra $J(A, \mu)$ has $(1, 0, 0)$ as identity and

$$N(x, y, z) = N_A(x) + \mu N_A(y) + \mu^{-1} N_A(z) - T_A(xyz),\ x,\ y,\ z \in A,$$

as cubic norm. Moreover, $J(A, \mu)$ is a division algebra if and only if $A$ is a division algebra and $\mu$ is not a reduced norm from $A$. Note that $A_+ = A_0$ is a Jordan subalgebra of $J(A, \mu)$. A first construction Albert algebra is either split or division ( [17], Chap. IX, Thm. 20).

**Tits's second construction**

Let $K$ be a quadratic field extension of $k$ and $B$ be a central simple algebra of degree 3 over $K$ and let $\sigma$ be an involution of the second kind on $B$. Let $x \mapsto \overline{x}$ be the non-trivial Galois automorphism of $K/k$. Let $(B, \sigma)_+$ denote the $k$-subspace of $B$ of $\sigma$-symmetric elements in $B$. Fix a unit $u$ in $(B, \sigma)_+$ such that $N(u) = \mu\overline{\mu}$ for some $\mu \in K^*$. Let $J(B, \sigma, u, \mu) = (B, \sigma)_+ \oplus B$. We define a multiplication on $J(B, \sigma, u, \mu)$ as follows,

$$(a_0,\ a)(b_0,\ b) = (a_0.b_0 + \overline{au\sigma(b)} + \overline{bu\sigma(a)},\ \overline{a_0} b + \overline{b_0} a + \overline{\mu}(\sigma(a) \times \sigma(b))u^{-1}),$$

where the notation is same as above. With this multiplication, $J(B, \sigma, u, \mu)$ is an Albert algebra over $k$ and is referred to as a **second Tits construction** Albert algebra. The Albert algebra $J(B, \sigma, u, \mu)$ has $(1, 0)$ as identity and

$$N(a, x) = N_B(a) + \mu N_B(x) + \overline{\mu} N_B(\sigma(a)) - T_B(axu\sigma(a)),\ a \in (B, \sigma)_+,\ x \in B,$$

as cubic norm. Moreover $J(B, \sigma, u, \mu)$ is a division algebra if and only if $B$ is a division algebra and $\mu$ is not a reduced norm from $B$. Note that $(B, \sigma)_+$ is a Jordan subalgebra of $J(B, \sigma, u, \mu)$. The second construction becomes a first construction over $K$, more precisely, $J(B, \sigma, u, \mu) \otimes K \cong J(B, \mu)$ as $K$-algebras ( [29], [17], Chap. IX, Exercise 5, Pg. 422).

We call a field extension $F/k$ a **reducing field** of an Albert algebra $A$ over $k$ if the extended algebra $A \otimes_k F$ over $F$ is reduced.

Let $B$ be a degree 3 central simple algebra over a quadratic étale extension $K$ of $k$ with an involution $\sigma$ of the second kind. Let $A = J(B, \sigma, u, \mu)$ be a second Tits construction Albert algebra and $G = \mathbf{Aut}(A)$. Then we have a $k$-embedding of the special unitary group $\mathbf{SU}(B, \sigma)$ in $G$,

$\mathbf{SU}(B, \sigma) \hookrightarrow G$ via $p \to \phi_p$, where $\phi_p : (x, y) \to (px\sigma(p), py)$, for all $(x, y) \in A$.

**Remark 5.2.1** *An Albert algebra $A$ over $k$ is a pure first (resp. second) construction if it cannot be expressed as a second (resp. first) construction. It is known that $A$ is a first construction if and only if $f_3(A) = 0$. Hence if $f_3(A) \neq 0$ then $A$ must be a pure second construction ([19], Chap. IX, Prop. 40.5). Albert algebras of both pure types as well as mixed types exist (see [42], [39]).*

**Theorem 5.2.2** *([43], §7) For an Albert algebra $A$ over $k$, there exists, up to a $k$-isomorphism, a unique reduced Albert algebra $\mathcal{H}_3(C, \Gamma)$ over $k$, such that for any reducing field $L/k$ of $A$, $A \otimes_k L \cong \mathcal{H}_3(C \otimes_k L, \Gamma)$.*

The reduced Albert algebra in the theorem is called the **reduced model** of $A$. The coordinate octonion algebra of a reduced model of an Albert algebra $A$ is called the **octonion algebra** of $A$ and denoted by $Oct(A)$. We note that when $A$ is the reduced Albert algebra $\mathcal{H}_3(C, \Gamma)$ over $k$, then $Oct(A) = C$. In particular, when $A$ is split, $Oct(A)$ is split as well.

Let $A$ be an absolutely simple Jordan algebra of degree 3 and dimension 9 over $k$. By the structure theory, there exists a central simple associative algebra $(B, *)$ of degree 3 over $k$ with an involution $*$ of the second kind, unique up to isomorphism, satisfying $A \cong (B, *)_+$. We define the octonion algebra of $A$, written as $Oct(A)$, to be the coordinate octonion algebra of the reduced Albert algebra $J(B, *, 1, 1)$ (see [37], 1.11).

Let $K/k$ be a quadratic extension. Let $*_\Gamma$ denote the involution on $M_3(K)$ given by $*_\Gamma(X) = \Gamma^{-1}\overline{X}^t\Gamma$, where $\Gamma = diag(\gamma_1, \gamma_2, \gamma_3) \in GL_3(k)$ with $\gamma_1\gamma_2\gamma_3 = 1$ and $^-$ denotes the entrywise action of the automorphism $^-$ of $K$. Let $V \in GL_3(K)$ with $*_\Gamma(V) = V$. Suppose further that $det\ V = \mu\overline{\mu}$ for some $\mu \in K^*$. Then one has the second Tits construction $J(M_3(K), *_\Gamma, V, \mu)$ with the underlying vector space $(M_3(K), *_\Gamma)_+ \oplus M_3(K)$.

The matrix $U = V\Gamma^{-1}$ is hermitian, i.e, $\overline{U}^t = U$. Further, $det\ U = det\ V = \mu\overline{\mu}$. Let $h$ denote the hermitian form on $K^3$ given by $h(x, y) = x\overline{U}\overline{y}^t$. Then the discriminant of $h$, denoted $disc\ h$, is trivial. Let $\psi : (\wedge^3 K^3, \wedge^3 h) \simeq (K, < 1 >)$ be the trivialization of $disc\ h$ given by $e_1 \wedge e_2 \wedge e_3 \mapsto \overline{\mu}$, $e_i$ being the standard basis vectors of $K^3$. We then have the octonion algebra $C = C(K^3, h, \psi) = K \oplus K^3$, with the multiplication given by

$$(a, v)(a', v') = (aa' - h(v, v'),\ av' + \overline{a'}v + \theta(v, v')),$$

where $\theta$ is defined by the identity

$$h(v'', \theta(v, v')) = \psi(v'' \wedge v \wedge v'),$$

for all $v, v', v'' \in K^3$. Also, the norm $n_C$ is given by $n_C(a, v) = n_{K/k}(a) + h(v)$, where $h(v) = h(v, v)$. The involution on $C = K \oplus K^3$ is given by $\overline{(\alpha, v)} = (\overline{\alpha}, -v)$. For more details see ([58]). Then one has the reduced Albert algebra $\mathcal{H}_3(C, \Gamma)$.

Note that $\mathcal{H}_3(C, \Gamma)$ contains $\mathcal{H}_3(K, \Gamma) = (M_3(K), *_\Gamma)_+$ as a Jordan subalgebra. For more details see ( [34], §1).

**Theorem 5.2.3** *( [34], §1, Thm. 1.1 ) With notations as above, there exists an isomorphism of Jordan algebras between $J(M_3(K), *_\Gamma, V, \mu)$ and $\mathcal{H}_3(C, \Gamma)$. Further, the norm form $n_C$ of $C = K \oplus K^3$ is given by $n_C = tr_{K/k}(< 1 > \perp h)$, where $h$ is the hermitian form given by the matrix $\overline{V\Gamma^{-1}}$.*

**Proposition 5.2.4** *Let $D$ be a degree 3 central division algebra or an Albert division algebra over $k$. Then $D$ remains a division algebra over field extensions of degree coprime to 3.*

Proof. First, let $D$ be a degree 3 central division algebra over $k$. By ([13], Exercise 9, Section 4.6), it follows that, for a field extension $L$ of $k$ of degree coprime to 3, $D \otimes L$ is a division algebra. When $D$ is an Albert division algebra, the result follows from ([39], Cor., p. 205). □

## 5.3 Étale Tits processes

Let $L$ be a cubic étale algebra and $K$ be a quadratic étale algebra over an arbitrary base field $k$ and $E = L \otimes K$. Let $\tau = 1 \otimes \bar{\ }$, where $\bar{\ }$ denotes the non-trivial involution on $K$. Suppose $(u, \mu) \in L^* \times K^*$ is such that $N_{L/k}(u) = N_{K/k}(\mu)$, we then call the pair $(u, \mu)$ an **admissible pair**. The étale Tits process produces an absolutely simple Jordan algebra $J = J(E, \tau, u, \mu)$ of degree 3 and dimension 9, with the underlying vector space $L \oplus E$ and with $L = \{(l, 0)|\ l \in L\}$ as a subalgebra. The Jordan algebra $J = J(E, \tau, u, \mu)$ has a cubic norm given by,

$$N((a, b)) = N_{L/k}(a) + \mu N_{E/k}(b) + \overline{\mu N_{E/k}(b)} - t_{L/k}(abu\tau(b)),$$

for $a \in L, b \in E$. Let $(B, \sigma)$ be a central simple algebra over $K$ with an involution $\sigma$ of the second kind and suppose $(B, \sigma)_+$ contains a cyclic étale algebra $L$ over $k$. Then there exists $z \in B^*$ such that $B = L \otimes K \oplus (L \otimes K)z \oplus (L \otimes K)z^2$ with $z^3 = \mu \in K^*$ and $N_{K/k}(\mu) = 1$. Also the involution $\sigma$ is given by $\sigma(z) = uz^{-1}$ with $u \in L$ such that $N_B(u) = 1$. In this case $(B, \sigma)_+ \cong J(L \otimes K, \tau, u, \mu)$ (see [19], Pg. 527 for details). The next theorem gives us the center of $B$ in the above situation.

**Theorem 5.3.1** *([44], Theorem 1, cf. [36], Theorem 1.4) Let $L, K$ be étale $k$-algebras of dimension $3, 2$ respectively, and suppose $(u, \mu)$ is an admissible pair. If $(B, \sigma)$ is a central simple algebra of degree $3$ over $k$ with involution of the second kind such that $J(L \otimes K, \tau, u, \mu)$ becomes isomorphic over $k$ to $(B, \sigma)_+$, then the center of $B$, as a quadratic étale $k$-algebra, corresponds to the element $\delta(L) + \delta(K) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ where $\delta(L)$ (resp. $\delta(K)$) denotes the discriminant algebra of $L$ (resp. $K$) over $k$.*

We saw that starting with a cubic étale $k$-algebra $L$ and a quadratic étale $k$-algebra $K$, as well as an admissible pair $(u, \mu)$, the étale Tits process produces an absolutely simple Jordan algebra $J(L \otimes K, \tau, u, \mu)$ of degree 3 and dimension 9, which turns out to be the Jordan algebra of symmetric elements of a central simple algebra of degree 3, with involution of the second kind. The theorem below provides a converse to this result.

**Theorem 5.3.2** *(Extension Theorem)([36], Theorem 1.6) Let $L$ be a cubic étale $k$-algebra, $(B, \sigma)$ a central simple algebra of degree $3$ with an involution of the second kind over $k$ and suppose $\iota$ is an isomorphic embedding from $L$ to $J = (B, \sigma)_+$, the Jordan* *algebra over $k$ of $\sigma$-symmetric elements in $B$. Writing $K$ for the center of $B$ and $E$ for the quadratic étale $k$-algebra corresponding to the element*

$$\delta(K) + \delta(L) \in H^1(k, \mathbb{Z}/2\mathbb{Z}),$$

*there are invertible elements $u \in L, \mu \in E$ satisfying $N_{L/k}(u) = N_{E/K}(\mu)$ such that $\iota$ extends to an isomorphism from the étale Tits process $J(L \otimes E, \tau, u, \mu)$ onto $J$.*

We define étale Tits processes $J_1$ and $J_2$ arising from étale algebras $L$ and $K$ of dimensions $3, 2$ resp., to be $L$-isomorphic, denoted by $J_1 \cong_L J_2$, if there exists a $k$-isomorphism $J_1 \to J_2$, which restricts to an automorphism of the subalgebra $L$ of $J_1$ and $J_2$. By ([38], Prop. 3.7) we have the following

**Theorem 5.3.3** *([38], Prop. 3.7) Let $L, K$ and $E$ be as above. Let $(u, \mu) \in L^* \times K^*$ be an admissible pair. For any $w \in E$, $(wu\tau(w), \mu N_{E/K}(w)) \in L^* \times K^*$ is again an admissible pair and*

$$J(E, \tau, u, \mu) \cong_L J(E, \tau, wu\tau(w), \mu N_{E/K}(w)),$$

*via $(a, b) \mapsto (a, bw)$.*

**Remark 5.3.4** *Note that $J(E, \tau, 1, 1)$ has zero divisors. Choose $x \in E$ such that $\tau(x) = -x$. Then $(0, x)$ is a zero divisor in $J(E, \tau, 1, 1)$, since it is a norm zero element. More generally, as an easy consequence of Theorem 5.3.3, one can see that if $\mu \in N_{E/K}(E^*)$, then $J(E, \tau, u, \mu)$ has zero divisors.*

**Theorem 5.3.5** *For any étale Tits construction $J(E, \tau, u, \mu)$, there exists an $L$-isomorphic Tits process $J(E, \tau, u', \mu')$ with $N_{L/k}(u') = 1 = \mu'\tau(\mu')$.*

Proof. Take $w = \mu^{-1}u$ and apply Theorem 5.3.3. □

In §5.2, 5.3, we described a cubic norm structure on Albert algebras and étale Tits processes. A Jordan algebra $J$ admits a generic minimal polynomial and generic norm and trace are defined (see [19], §37).

**Definition 5.3.6** *Let $J$, $J'$ be Jordan algebras with identity. Then a bijective linear map $\eta$ of $J$ into $J'$ is called a* ***norm isometry*** *of $J$ onto $J'$ if $N'(\eta(a)) = N(a)$ for all $a \in J$ where $N$ and $N'$ are the generic norms in $J$ and $J'$ respectively.*

The next theorem gives a criterion for isomorphism of finite dimensional Jordan algebras.

**Theorem 5.3.7** *([17], Chapter. VI, Thm. 7) Let $k$ be a field of characteristic different from $2$ and $3$. Let $J$, $J'$ be finite dimensional Jordan algebras over $k$ with identity. If $\eta$ is a norm isometry of $J$ onto $J'$ preserving identities, then $\eta$ is an isomorphism of $k$-algebras.*

# Chapter 6

# Groups $G_2$ and $F_4$

The aim of this chapter is to describe groups of type $G_2$ and $F_4$ defined over a field $k$ and some of their $k$-subgroups. These groups are obtained as automorphism groups of octonion algebras and Albert algebras respectively. For a detailed exposition we refer to [53], [54]. We fix a field $k$ of characteristic different from 2 and 3.
In the first section we describe the basic structural properties of these algebras. In the second section we study structure of certain $k$-subgroups of the groups of type $G_2$ and $F_4$. In the last section we study some $k$-embeddings of $k$-groups of type $A_1, A_2$ and $D_4$ in $F_4$.

## 6.1 Structural properties

Octonion algebras over a field $k$ describe the $k$-groups of type $G_2$. We state below two important theorems in this regard:

**Theorem 6.1.1** *([54], §17.4) Let $G$ be a simple group of type $G_2$ over $k$. Then there exists an octonion algebra $C$, unique up to isomorphism, defined over $k$, such that $G$ is $k$-isomorphic to the group $\mathbf{Aut}(C)$.*

**Theorem 6.1.2** *([53], §2.3, Thm. 2.3.5, [54], Prop. 17.4.2, 17.4.5) Let $C$ be an octonion algebra over a field $k$ and let $G = \mathbf{Aut}(C)$ be the associated algebraic group of automorphisms of $C$. Then $G$ is a connected simple algebraic group of type $G_2$ defined over $k$ and $G$ is either $k$-anisotropic or $k$-split. Moreover, $G$ is $k$-split if and only if $C$ is split, if and only if $n_C$ is $k$-isotropic (i.e. hyperbolic).*

Albert algebras over the field $k$ describe the $k$-groups of type $F_4$.

**Theorem 6.1.3** *([54], §17.6) Let $G$ be a simple algebraic group of type $F_4$ over $k$. Then there exists an Albert algebra $A$, unique up to isomorphism, defined over $k$ such that $G$ is $k$-isomorphic to the group $\mathbf{Aut}(A)$.*

**Theorem 6.1.4** *( [53], Thm. 7.2.1, [39], Pg. 205) Let $A$ be an Albert algebra over a field $k$. Let $G = \mathbf{Aut}(A)$ be the associated algebraic group of automorphisms of $A$. Then $G$ is a connected simple algebraic group defined over $k$ of type $F_4$.*

**Remark 6.1.5** *Let $G$ be a split simple group of type $G_2$. Then $G$ is simply connected, adjoint and is isomorphic to $\mathbf{Aut}(C)$ where $C$ is a split octonion algebra (see [19], Theorem 25.14). Since $G$ is simply connected and adjoint, by ([19], Theorem 25.16) $G = \mathbf{Aut}(G)$. Hence by Theorem 4.4.3, $H^1(k, G)$ is in bijection with the set of $k$-isomorphism classes of $k_s$-forms of $G$. Since any group of type $G_2$ is split over $k_s$, $H^1(k, G)$ is in bijection with the set of $k$-isomorphism classes of simple $k$-groups of type $G_2$. Also $H^1(k, G)$ is in bijection with the set of $k$-isomorphism classes of $k_s$-forms of $C$. Since any octonion algebra is split over $k_s$, $H^1(k, G)$ is also in bijection with the set of $k$-isomorphism classes of octonion algebras over $k$. By a similar argument we can show that for a simple split group $G$ of type $F_4$, $H^1(k, G)$ is in bijection with the set of $k$-isomorphism classes of simple $k$-groups of type $F_4$. Also $H^1(k, G)$ is in bijection with the set of $k$-isomorphism classes of $k_s$-forms of $A$, where $A$ is the unique split Albert algebra such that $G = \mathbf{Aut}(A)$. Hence, just as above, $H^1(k, G)$, for $G$ the $k$-split group of type $F_4$, classifies the $k$-isomorphism classes of Albert algebras over $k$.*

## 6.2 Subgroups of $G_2, F_4$

We begin with few known results which describe some $k$-subgroups of groups of type $G_2$ and $F_4$ defined over $k$. We recall that the groups of type $G_2$ over a field $k$ are precisely the groups of automorphisms of octonion algebras defined over $k$ ([54], §17.4). Similarly groups of type $F_4$ over $k$ occur as groups of automorphisms of Albert algebras defined over $k$ (see ([54], §17.6). Let $A$ be a finite dimensional $k$-algebra and $S \subset A$ be a $k$-subalgebra. Then $\mathbf{Aut}(A)$ is an algebraic group defined over $k$. In the thesis , we shall denote by $\mathbf{Aut}(A/S)$ the (algebraic) $k$-subgroup of all automorphisms of $A$ fixing $S$ pointwise and $\mathbf{Aut}(A, S)$ will denote the $k$-subgroup of $\mathbf{Aut}(A)$ mapping $S$ to $S$.

**Proposition 6.2.1** *(Jacobson)([19], Remark 39.13) Let $A$ be an Albert algebra defined over $k$ and let $G = \mathbf{Aut}(A)$ denote the algebraic group of type $F_4$ associated with $A$. Let $L \subset A$ be a cubic étale subalgebra. Then the subgroup $\mathbf{Aut}(A/L)$ of $G$ is a simply connected, simple group of type $D_4$ defined over $k$.*

Proof. To see this we base change to $\bar{k}$ and apply a theorem of Jacobson ([17], Chap. IX, Pg. 378, Exercise 2) which asserts that for a reduced Albert algebra $A = \mathcal{H}_3(C, \Gamma)$, the subgroup $\mathbf{Aut}(A/(k.e_1 + k.e_2 + k.e_3))$ is isomorphic to $Spin(C, n_C)$ over $k$, here $e_i$, $1 \leq i \leq 3$, are the diagonal (primitive) idempotents of $A$. The result now follows from the fact that the diagonal subalgebra of $\mathcal{H}_3(C_{\bar{k}}, \Gamma)$ is conjugate to $L_{\bar{k}}$ by an automorphism of $A_{\bar{k}}$ ([17], Chap. IX, Exercise 3, Pg. 389). $\square$

**Proposition 6.2.2** *([57], Thm. 4.2) Let $k$ be a field with $char(k) \neq 2, 3$ and $A$ an Albert algebra over $k$. Suppose $\phi \in \mathbf{Aut}(A)(k)$ is semisimple. Then $\phi$ fixes a cubic étale subalgebra $L \subset A$ pointwise.*

**Lemma 6.2.3** *Let $C$ be an octonion algebra over $k$. Let $G = \mathbf{Aut}(C)$ and let $\phi \in G(k)$. Then $\phi$ fixes a non-zero element of $C$ with trace $0$.*

Proof. Since $\phi$ is an automorphism of $C$, $\phi(1) = 1$. Therefore $\phi(C_0) = C_0$, where $C_0$ denotes the subspace of trace zero elements in $C$. Hence $\phi \in SO(C_0, n|_{C_0})(k)$. Since $(C_0, n|_{C_0})$ is a regular quadratic space of odd dimension, by a classical theorem of Cartan and Dieudonné, $\phi$ fixes a non-zero element of $C_0$ ([16], §6.6, Chap. VI). $\square$

**Theorem 6.2.4** *([15], Thm. 9, [19], §39, Chap. IX) Let $A$ be an Albert algebra over $k$ and let $S$ be a $9$-dimensional cubic separable Jordan subalgebra of $A$. The subgroup $\mathbf{Aut}(A/S)$ (resp. $\mathbf{Aut}(A, S)$) is a simply connected, simple algebraic group of type $A_2$ (resp. $A_2 \times A_2$) defined over $k$.*

**Theorem 6.2.5** *([12], Thm. 3, Thm. 4, Thm. 5) Let $C$ be an octonion algebra over $k$ and let $K$ be a quadratic étale (resp. quaternion) subalgebra of $C$. Then the subgroup $\mathbf{Aut}(C/K)$ is a simply connected, simple group of type $A_2$ (resp. $A_1$) defined over $k$.*

## 6.3 Embeddings in $F_4$

In this section we describe some $k$-embeddings of $k$-groups of type $A_1, A_2$ and $D_4$ in $F_4$.

**(1) $k$-Embeddings of $A_2$ and $G_2$:**

Let $B$ be a degree 3 central simple algebra over a quadratic étale extension $K$ of $k$ with an involution $\sigma$ of the second kind. Let $A = J(B, \sigma, u, \mu)$ be a second Tits construction Albert algebra. Let $G = \mathbf{Aut}(A)$. Then $G$ is a group of type $F_4$ over $k$. Let $\sigma_u = Int(u) \circ \sigma$. We have the following $k$-embedding of the special unitary groups $\mathbf{SU}(B, \sigma), \mathbf{SU}(B, \sigma_u)$ in $G$,

$$\mathbf{SU}(B, \sigma) \hookrightarrow G \text{ via } p \mapsto \phi_p, \text{ where } \phi_p : (x, y) \mapsto (px\sigma(p), py), \text{ for all } (x, y) \in A.$$

$$\mathbf{SU}(B, \sigma_u) \hookrightarrow G \text{ via } q \mapsto \psi_q, \text{ where } \psi_q : (x, y) \mapsto (x, yq^{-1}), \text{ for all } (x, y) \in A.$$

Let $A = \mathcal{H}_3(C, \Gamma)$ be a reduced Albert algebra over $k$. Let $G = \mathbf{Aut}(A)$. Note that $H = \mathbf{Aut}(C) \hookrightarrow G$ (via $\phi \mapsto \widetilde{\phi}$, where $\widetilde{\phi}(X) = (\phi(x_{ij}))$, $X = (x_{ij}) \in A$). Fix $K \subseteq C$, a quadratic étale algebra. Then $K^{\perp} \subseteq C$ has a $K$-hermitian form $h$ defined on it (see Proposition 1.3.4) and

$$\mathbf{Aut}(C/K) \subseteq \mathbf{Aut}(C) \hookrightarrow \mathbf{Aut}(A),$$

over $k$. Note that $\mathbf{Aut}(C/K) \cong \mathbf{SU}(K^{\perp}, h) \cong \mathbf{SU}(M_3(K), *_h)$ ([52], Proposition 3.1) is a $k$-group of type $A_2$. Let $D$ be a central simple algebra over $k$. Let $A = J(D, \mu)$ be a first Tits construction Albert algebra and $G = \mathbf{Aut}(A)$. The $k$-group $\mathbf{SL}_1(D)$ has type $A_2$ and we have the following $k$-embedding of $\mathbf{SL}_1(D)$ in $G$,

$$\mathbf{SL}_1(D) \hookrightarrow G \text{ via } p \mapsto \phi_p, \text{ where } \phi_p : (x, y, z) \mapsto (x, yp, p^{-1}z), \text{ for all } (x, y, z) \in A.$$

**(2) $k$-Embeddings of $A_1$**

Let $A = \mathcal{H}_3(C, \Gamma)$. Fix a quaternion subalgebra $Q \subseteq C$. Then

$$\mathbf{Aut}(C/Q) \hookrightarrow \mathbf{Aut}(C) \hookrightarrow \mathbf{Aut}(A),$$

over $k$. Note that $\mathbf{Aut}(C/Q) \cong \mathbf{SL}(1, Q)$ is a group of type $A_1$.

**(3) $k$-Embedding of $D_4$:**

Let $A = \mathcal{H}_3(C, \Gamma)$ be a reduced Albert algebra over $k$. Let $H \subseteq \mathbf{Aut}(A)$ be the algebraic subgroup of $\mathbf{Aut}(\mathrm{A})$ consisting of all automorphisms of $A$ which fix the three diagonal idempotents in $A$. Then $H$ is a $k$-subgroup of type $D_4$ and $H \cong \mathbf{Spin}(n_C)$ ([19], Remark 39.13). As a consequence of this, we have:

Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Let $L \subset A$ be a cubic étale subalgebra. Let

$$G_L := \{\phi \in G | \ \phi(l) = l \text{ for all } l \in L\}.$$

Then $G_L$ is a $k$-subgroup of type $D_4$.

**Remark 6.3.1** *Note that the $k$-embeddings described in $(\mathbf{1}), (\mathbf{2})$ of groups of type $A_1$ and $A_2$, in groups of type $F_4$ arising from reduced Albert algebras, factor through the subgroup $\mathbf{Aut}(C)$ of type $G_2$.*

# Chapter 7

# Mod-$2$ invariants of groups

This chapter is a basic yet most essential part of the thesis. Here we describe the mod-2 Galois cohomological invariants for simple, simply connected groups of type $A_1$, $A_2$, $G_2$ and $F_4$. The exposition in this chapter is mostly based on [54], [46], [61]. We fix a field $k$ of characteristic different from 2 and 3.

In the first section we describe the mod-2 invariants for simple, simply connected $k$-groups of type $A_1$ and $A_2$. In the second section we describe the mod-2 invariants for the $k$-groups of type $G_2$ and $F_4$ as well as octonion algebras for simple, simply connected $k$-groups of type $A_2$, $G_2$ and $F_4$.

## 7.1 Mod-$2$ invariants of $A_1$ and $A_2$

Let $G$ be a simple, simply connected $k$-group of type $A_1$, then there is a quaternion algebra $Q$ over $k$, unique up to isomorphism, such that $G \cong \mathbf{SL}(1, Q)$ over $k$. One knows that quaternion algebras are determined by their norm forms and hence by the corresponding Arason invariants. Therefore we get an invariant for $G = \mathbf{SL}(1, Q)$, namely $f_2(G) := e_2(n_Q) \in H^2(k, \mathbb{Z}/2\mathbb{Z})$. Let $G$ be a simple, simply connected $k$-group of type $A_2$, then there is a central simple algebra $B$ of degree 3 over a quadratic étale extension $K$ of $k$ with an involution $\sigma$ of the second kind such that $G \cong \mathbf{SU}(B, \sigma)$ over $k$.

We now prove a lemma which helps us to define an invariant for a simple simply connected $k$-group of type $A_2$.

**Lemma 7.1.1** *([10], Lemma 2.1) Let $B_i$, $i = 1, 2$, be central simple algebras of degree* $3$*, with unitary involutions $\sigma_i$, $i = 1, 2$, such that $\mathbf{SU}(B_1, \sigma_1) \cong \mathbf{SU}(B_2, \sigma_2)$ as* *algebraic groups. Then $(B_1,\sigma_1) \cong (B_2,\sigma_2)$ as algebras with involutions. In particular, $f_3(B_1,\sigma_1) = f_3(B_2,\sigma_2)$.*

Since $\mathbf{SU}(B_1,\sigma_1) \cong \mathbf{SU}(B_1,\sigma_2)$, we have

$$Lie(\mathbf{SU}(B_1,\sigma_1)) \cong Lie(\mathbf{SU}(B_2,\sigma_2)).$$

By ([19], Pg. 346) we have

$$Lie(\mathbf{SU}(B_i,\sigma_i)) = Skew(B_i,\sigma_i)^\circ := \{x \in B_i | \sigma_i(x) = -x,\ T_{B_i}(x) = 0\},\ i = 1, 2.$$

Now applying ([14], Chap. X, Thm. 11), we get $(B_1,\sigma_1) \cong (B_2,\sigma_2)$ (see also [19], Prop. 2.25, Pg. 29). $\square$

Hence the Arason invariant $f_3(B,\sigma) \in H^3(k,\mathbb{Z}/2\mathbb{Z})$ of the norm of the octonion algebra of $(B,\sigma)_+$ is an invariant of $G$ as well. We summarize this in the remark below,

**Remark 7.1.2** *When $G$ is simple simply connected $k$-group of type $A_1$, there is a quaternion algebra $Q$ over $k$, unique up to isomorphism, such that $G \cong \mathbf{SL}_1(Q)$ over $k$ ([46], Chap. II, Prop. 2.17). Hence we get an invariant for $G = \mathbf{SL}_1(Q)$, namely $f_2(G) := e_2(n_Q) \in H^2(k,\mathbb{Z}/2\mathbb{Z})$. If $G$ is simple simply connected $k$-group of type $A_2$, then there is a central simple algebra $B$ of degree 3 over a quadratic étale extension $K$ of $k$ with an involution $\sigma$ of the second kind such that $G \cong \mathbf{SU}(B,\sigma)$ over $k$ ([46], Chap. II, Prop. 2.18). By Lemma 7.1.1, the Arason invariant $f_3(B,\sigma) \in H^3(k,\mathbb{Z}/2\mathbb{Z})$ of the norm of the octonion algebra of $(B,\sigma)_+$ is an invariant of $G$ as well. For a connected reductive group $G$ of type $A_2$ defined over $k$, we define $f_3(G)$ to be the invariant thus obtained for the simply connected cover of $[G,G]$.*

## 7.2 Octonion algebras of $A_2$, $G_2$ and $F_4$

In this section we describe the mod-2 invariants for the groups of type $G_2$ and $F_4$ and octonion algebras of the groups of type $A_2$, $G_2$ and $F_4$.

Let $G$ be a simple, simply connected $k$-group of type $A_2$ and $G \cong \mathbf{SU}(B,\sigma)$ over $k$(see Remark 7.1.2). As in Remark 7.1.2, the Arason invariant $f_3(B,\sigma)$ of the norm of the octonion algebra of $(B,\sigma)_+$ is an invariant of $G$. Define $Oct(G) := C$, where $C$ is the octonion algebra determined by the 3-fold Pfister form $f_3(B,\sigma)$. Note that $\sigma$ is distinguished if and only $f_3(B,\sigma) = 0$ if and only if $Oct(G)$ splits.

Let $G$ be a group of type $G_2$ defined over $k$. Then $G \cong \mathbf{Aut}(C)$ for a unique octonion algebra $C$ over $k$ ([54], §17.4). Recall that $C$ is determined by its **norm form** $n_C$, which is a 3-fold Pfister form over $k$. Hence the groups $G$ over $k$ of type $G_2$ are classified by the Arason invariant $f_3(G) := e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$, where $G \cong \mathbf{Aut}(C)$ as above. Define $Oct(G) := C$. Observe that $f_3(G) = 0$ if and only if $G$ splits, if and only if $Oct(G)$ splits.

Let $G$ be a group of type $F_4$ defined over $k$. Then there exists an Albert algebra $A$ over $k$ such that $G \cong \mathbf{Aut}(A)$, the full group of automorphisms of $A$ ([54], §17.6). Let $A$ be an Albert algebra over $k$ and let $A_{red} = \mathcal{H}_3(C,\Gamma)$ be the reduced model for $A$, where $C$ is an octonion algebra over $k$ (see §5.2) and $\Gamma = diag(\gamma_1,\gamma_2,\gamma_3) \in GL_3(k)$. This defines two mod-2 invariants for $G = \mathbf{Aut}(A)$:

$$f_3(G) = f_3(A) := e_3(n_C) \in H^3(k, \mathbb{Z}/2\mathbb{Z}),$$

$$f_5(G) = f_5(A) = e_5(n_C \otimes \langle 1, \gamma_1^{-1}\gamma_2\rangle \otimes \langle 1, \gamma_2^{-1}\gamma_3\rangle) \in H^5(k, \mathbb{Z}/2\mathbb{Z}).$$

Define $Oct(G) := Oct(A) = C$. Observe that $f_3(A) = 0$ if and only if $Oct(G)$ splits if and only if $A$ ia a first Tits construction.

**Proposition 7.2.1** *([39], Pg. 205) Let $G$ be a connected, simple algebraic group of type $F_4$ defined over $k$. Then $G$ is $k$-isotropic if and only if $A$ is reduced and $f_5(A) = 0$.*

**Remark on notation :** In the thesis we need to deal with both invariants $f_3$ and $f_5$ of Albert algebras and the $f_3$-invariant of degree 3 central simple algebras with unitary involutions and the corresponding Pfister forms simultaneously. To avoid an unpleasant surplus of notation, we shall use the same notation for both Pfister quadratic form in question as well as its mod-2 invariant as defined above, whenever no confusion is likely to arise. In the thesis, unadorned tensor products will be understood to be over base fields.

# Chapter 8

# Maximal Tori

The main aim of this chapter is to study the structure of maximal tori in groups of type $G_2$ and simple, simply connected groups of type $A_2$. We refer to the maximal tori in simple, simply connected groups of type $A_n$ or $G_2$ as **unitary** tori. In this chapter we define the terminology of a unitary and distinguished torus which we use extensively in the thesis.

Fix a field $k$ of characteristic different from 2, 3.

## 8.1 Maximal tori of special unitary groups

Let $K$ a quadratic field extension of $k$ with the non-trivial $k$-automorphism $^-$. Let $V$ be a $K$-vector space of dimension $n$. Let $h$ be a non-degenerate hermitian form on $V$. By ([52], Theorem 5.1 and Corollary 5.2), we have the following well known explicit description of maximal tori in a special unitary group of a non-degenerate hermitian space,

**Theorem 8.1.1** **(a)** *Let $k$ be a field and $K$ a quadratic field extension of $k$. Let $V$ be a $K$-vector space of dimension $n$ with a non-degenerate hermitian form $h$. Let $T \subseteq \mathbf{U}(V,h)$ be a maximal $k$-torus. Then there exists an étale algebra $E_T$ of dimension $n$ over $K$, with an involution $\sigma_h$ restricting to the non-trivial $k$-automorphism of $K$, such that $T = \mathbf{U}(E_T, \sigma_h)$.*

**(b)** *Let $T \subset \mathbf{SU}(V,h)$ be a maximal $k$-torus. Then there exists an étale algebra $E_T$ over $K$ of dimension $n$, such that $T = \mathbf{SU}(E_T, \sigma_h)$.*

Note that in Theorem 8.1.1, $E_T = Z_{End_K(V)}(T)$, the centralizer algebra of $T$ in $End_K(V)$, and $\sigma_h$ is the involution on $End_K(V)$ adjoint to $h$, where $T$ is a maximal torus in $\mathbf{U}(V,h)$ or $\mathbf{SU}(V,h)$.

**Remark 8.1.2** *Let $(B,\sigma)$ be a central simple algebra of degree $n$ over $K$, with $\sigma$ an involution of the second kind. Let $\mathbf{SU}(B,\sigma)$ be the associated algebraic group. Let $T \subset \mathbf{SU}(B,\sigma)$ be a maximal $k$-torus. By the above it follows that there is an étale $K$-subalgebra $E_T \subset B$, of dimension $n$ over $K$ (the centralizer of $T$ in $B$), stable under the involution $\sigma$, such that $T \cong \mathbf{SU}(E_T,\sigma)$ over $k$.*

By ([52], Remark after Lemma 5.1) we have,

**Lemma 8.1.3** *Let $K$ be a quadratic étale extension of $k$. Let $E$ be an étale algebra of dimension $2n$ over $k$ containing $K$, equipped with an involution $\sigma$, restricting to the non-trivial $k$-automorphism of $K$. Let $L = E^\sigma = \{x \in E | \sigma(x) = x\}$. Then $E = L \otimes_k K$ and $(E,\sigma) = (L \otimes_k K, 1 \otimes \bar{\ })$, where $x \mapsto \overline{x}$ is the non-trivial $k$-automorphism of $K$.*

In view of the above lemma, $dim(E^\sigma) = n$ over $k$. Let $k$ be a field and $L, K$ be étale $k$-algebras of $k$-dimension $n, 2$ resp. and $E = L \otimes K$. Then $E$ is an étale algebra of dimension $2n$ over $k$. Let $\bar{\ }$ denote the non-trivial $k$-automorphism of $K$ and $\tau$ the involution $1 \otimes \bar{\ }$ on $E$. We will refer to $(E,\tau)$ as the $K$-**unitary algebra** associated with the ordered pair $(L,K)$.

**Lemma 8.1.4** *Let $L, K$ be étale algebras of $k$-dimensions $n, 2$ resp. Let $(E,\tau)$ be the $K$-unitary algebra associated with $(L,K)$. Then $E^\tau = \{x \in E\ |\tau(x) = x\} = L$.*

Proof. Let $K$ be a quadratic field extension. By Lemma 8.1.3, $dim(E^\tau) = n = dim(L)$. Since $L \subseteq E^\tau$, and the dimensions are equal, we have $L = E^\tau$. Let $K = k \times k$. Then $(E,\tau) = (L \times L, \epsilon)$, where $\epsilon$ is the switch involution on $L \times L$. Clearly $E^\tau = L$. $\square$

## 8.2 Unitary and Distinguished tori

Let $L, K$ be étale algebras of $k$-dimensions $n, 2$ resp. and $(E,\tau)$ be the $K$-unitary algebra associated with the pair $(L,K)$. We call the torus $\mathbf{SU}(E,\tau)$ as the $K$-**unitary torus** associated to the ordered pair $(L,K)$. With such a $K$-unitary torus $T$, we associate

the quadratic form $q_T :=< 1, -\alpha\delta >$, where $Disc(L) = k(\sqrt{\delta})$ and $K = k(\sqrt{\alpha})$. Such tori are important as they occur as maximal tori in simple, simply connected groups of type $A_n$ and $G_2$. In the thesis, we will call a $k$-torus $T$ as a **distinguished torus** if there exists étale $k$-algebras $L, K$ of $k$-dimensions $3, 2$ resp. such that $disc(L) = K$ and $T = \mathbf{SU}(E, \tau) \cong \mathbf{SU}(L \otimes K, 1 \otimes^-)$, where $(E, \tau) \cong (L \otimes K, 1 \otimes^-)$ is the $K$-unitary algebra associated to the pair $(L, K)$. Also observe that when $T$ is a distinguished $k$-torus, the associated quadratic form $q_T$ splits over $k$. Note also that $T$ has rank-2.
We record below an evident, yet useful result:

**Lemma 8.2.1** *Let $K$ be a quadratic étale algebra over $k$ and $B$ be a degree $3$ central simple algebra over $K$ with an involution $\sigma$ of the second kind. Let $L \subseteq (B, \sigma)_+$ be a cubic étale subalgebra. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Then there exists a $k$-embedding $T \hookrightarrow \mathbf{SU}(B, \sigma)$.*

## 8.3 Maximal tori in $G_2$

In this section we describe the structure of maximal tori in groups of type $G_2$. The structure of such tori is well studied (cf. for example, [51]), we supply a proof for convenience of the reader.

**Proposition 8.3.1** *([9], Prop. 2.11 ) Let $G$ be a group of type $G_2$ over $k$ and $T$ be a maximal $k$-torus of $G$. Then there exists étale algebras $L, K$ of $k$-dimensions $3, 2$ resp. such that $T$ is isomorphic to the $K$-unitary torus associated to the pair $(L, K)$.*

Proof. Let $G$ be as in the hypothesis. Then there exists an octonion algebra $C$ over $k$ such that $G = \mathbf{Aut}(C)$. Let $T \subset G$ be a maximal $k$-torus in $G$.
**Claim:** There exists a quadratic étale subalgebra $K$ of $C$ such that $K = C^T$, the fixed points of the octonion algebra $C$ under the action of $T$.

To see this, we may assume that the dimension $[C^t : k] = 4$ for all $t \in T(k)$ (Note that since $t$ is semisimple $C^t$ is a composition subalgebra of $C$ ([65])). If not, then there exists $t \in T(k)$ such that $C^t = K$ is a quadratic étale subalgebra of $C$ ([65], cf. also [21], paragraph before Theorem 4 and [32]). Now $T$ stabilizes, and hence, by a connectedness argument, fixes $K$ pointwise. Hence $K \subseteq C^T \subseteq C^t = K$. Let $t \in T(k)$ be such that $C^t = Q$ for some quaternion subalgebra $Q$ of $C$. Since $T$ centralizes $t$, we see that $T \subseteq \mathbf{Aut}(C, Q)$. We write, by doubling process, $C = Q \oplus Qb$ for some $b \in Q^{\perp}$. Let

$c \in Q^*, p \in \mathbf{SL}_1(Q)$, define $\phi_{c,p} : C \to C$ as $\phi_{c,p}(x + yb) = cxc^{-1} + (pcyc^{-1})b, \forall x, y \in Q$. Then by ([53], §2.1),

$$\mathbf{Aut}(C, Q) = \{\phi_{c,p} | c \in Q^*, p \in \mathbf{SL}_1(Q)\}.$$

By an easy computation it follows that for $c, c' \in Q^*$, $p, p' \in SL_1(Q)$ if $\phi_{c,p}\phi_{c',p'} = \phi_{c',p'}\phi_{c,p}$, then there exists $a \in k^*$ such that $cc' = ac'c$.

**Claim:** There exists $\phi_{c,p} \in T(k)$ such that $c \notin k^*$.

If not, then for all $\phi_{c,p} \in T(k)$ we have $c \in k^*$. Let $x \in Q$ be arbitary and $y = 0$. Then, for any $\phi_{c,p} \in T$, $\phi_{c,p}(x) = x$. Thus $Q \subseteq C^T$ and hence $T \subseteq \mathbf{Aut}(C/Q)$, where $\mathbf{Aut}(C/Q)$ denotes subgroup of $\mathbf{Aut}(C)$ consisting of automorphisms of $C$ which fix $Q$ pointwise. This is a contradiction, since $T$ is a rank-2 torus and $\mathbf{Aut}(C/Q)$ is a simple group of type $A_1$ (see Theorem 6.2.5).

Thus there exists $\phi_{c,p} \in T(k)$ such that $c \notin k^*$. Since $\phi_{c,p} \in T$ is semisimple and $c \notin k^*$, $c$ generates a quadratic étale subalgebra, $K := k(c)$ of $Q$. Let $\phi_{c',p'} \in T$. Since $\phi_{c',p'}$ commutes with $\phi_{c,p}$, we have $cc' = ac'c$ for some $a \in k^*$. Any element $\gamma \in K$ is a polynomial expression in $c$ with coefficients from $k$, say,

$$\gamma = a_0 + a_1 c + ... + a_m c^m \text{ for } a_i \in k,\ m \in \mathbb{N}.$$

Now $\phi_{c',p'}(\gamma) = c'\gamma c'^{-1} = a_0 + aa_1 c + \cdots + aa^m a_m c^m \in K$. Hence $\phi_{c',p'}(\gamma) \in K$ for all $\gamma \in K$. Since $\phi_{c',p'}$ was chosen arbitrarily in $T$, we see that $T$ stabilizes, and hence, fixes $K$ pointwise. Hence $K \subseteq C^T$. Therefore $T \subseteq \mathbf{SU}(K^\perp, h)$, where $h$ is the non-degenerate hermitian form on $K^\perp \subseteq C$ over $K$, induced by the norm bilinear form $n_C$ (see [12], §5, cf. Prop. 1.3.4). Note that $\mathbf{SU}(K^\perp, h) = \mathbf{SU}(M_3(K), *_h)$, where $*_h$ is the involution on $M_3(K)$ given by $*_h(X) = h^{-1}\overline{X}^t h$ ([52], Proposition 3.1). By Theorem 8.1.1, any maximal torus of $\mathbf{SU}(M_3(K), *_h)$ is of the form $\mathbf{SU}(E, *_h)$ for some six dimensional $K$-unitary algebra $E$ over $k$. Hence $T = \mathbf{SU}(E, *_h)$ for some $E$ as above. $\square$

# Chapter 9

# Factorization results

This chapter reports the work done in [10]. We fix a field $k$ of characteristic different form 2, 3. In this chapter, we shall discuss $k$-embeddings of $k$-groups of type $A_1$ and $A_2$ in algebraic groups of type $F_4$ and $G_2$ defined over $k$ and derive a factorization of the mod-2 invariant of the groups of type $F_4$ and $G_2$ in terms of the mod-2 invariant of the embedded groups of type $A_1$ and $A_2$.

## 9.1 Embedding of $A_1$, $A_2$ in $F_4$

We begin with a factorization result for the mod-2 invariant $f_5(G)$ associated to a $k$-group $G$ of type $F_4$, given a $k$-embedding of a $k$-group of type $A_2$ in $G$.

**Theorem 9.1.1** *Let $K$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $K$ with an involution $\sigma$ of the second kind. Let $A$ be an Albert algebra over $k$. Let $\mathbf{Aut}(A)$ be the algebraic group of type $F_4$ associated to $A$. Suppose $\mathbf{SU}(B,\sigma) \hookrightarrow \mathbf{Aut}(A)$ over $k$. Then $f_5(A) = f_3(B,\sigma) \otimes \tau$ for some two fold Pfister form $\tau$ over $k$.*

We will first prove a special case of this theorem. This is a group theoretic characterization of Albert algebras with zero $f_5$ invariant. It follows from Remark 5.2.1 that Albert algebras with $f_5 \neq 0$ are pure second Tits constructions. Hence this gives examples of pure second construction Albert algebras. We note that the result below also gives an alternative proof of the fact that a second Tits construction Albert algebra $A = J(B,\sigma,u,\mu)$, with $\sigma$ distinguished, satisfies $f_5(A) = 0$ ([19], Chap. IX, Prop. 40.7).

**Theorem 9.1.2** *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Then $f_5(A) = 0$ if and only if there exists a $k$-embedding $\mathbf{SU}(B, \sigma) \hookrightarrow G$ for some degree 3 central simple algebra $B$ with center a quadratic étale $k$-algebra $K$ and with a distinguished involution $\sigma$.*

Proof. Suppose there exists a $k$-embedding $\mathbf{SU}(B, \sigma) \hookrightarrow G$ for some degree 3 central simple algebra $B$ with center a quadratic étale $k$-algebra $K$ and with a distinguished involution $\sigma$. We divide the proof in two cases.

**Case (i)** $K = k \times k$.

In this case $(B, \sigma) \cong (D \times D^{op}, \epsilon)$ for some central simple $k$-algebra $D$, where $\epsilon$ denotes the switch involution $(x, y) \mapsto (y, x)$ and $D^{op}$ is the opposite algebra of $D$. Now, $\mathbf{SU}(D \times D^{op}, \epsilon) \cong \mathbf{SL}(1, D)$ over $k$. Hence, $\mathbf{SU}(B, \sigma) \cong \mathbf{SL}(1, D) \hookrightarrow G$. If $D$ is a division algebra, choose a cubic separable extension $L$ over $k$, $L \subseteq D$. Now,

$$D \otimes_k L \cong M_3(L) \text{ and } \mathbf{SL}(1, D \otimes_k L) \cong \mathbf{SL}_3 \hookrightarrow G_L.$$

Hence $G_L$ has L-rank $\geqq 2$. We conclude that $G$ splits over $L$ ([61], Pg. 60). Hence $A$ and thereby $Oct(A)$ splits over $L$ ([54], Chap. 17, §17.6.4, [8], Pg. 164). Since $[L : K] = 3$, by Springer's theorem $Oct(A)$ must split over $k$. In the case when $D$ is split, $\mathbf{SL}(1, D) \cong \mathbf{SL}_3 \hookrightarrow G$ over $k$. Hence $G$ is split over $k$. In both of the above cases $f_3(G) = 0$, hence, $f_5(G) = f_5(A) = 0$.

**Case (ii)** $K$ is a field.

If $B$ is a division algebra, choose a cubic separable field extension $L$ over $k$ such that $L \subset (B, \sigma)_+$. In fact any $a \in (B, \sigma)_+ \setminus k$ generates a cubic separable subfield $L = k(a) \subseteq (B, \sigma)_+$ (To see this, observe that $B$ is a degree 3 division algebra over $K$ and $a$ is symmetric. Hence $a$ satisfies an irreducible cubic polynomial over $k$, this polynomial is separable as $char(k) \neq 3$). Notice that $[K : k] = 2$ and $[L : k] = 3$, so $L \cap K = k$. Hence, $L \otimes K \cong LK$. Since $B \otimes_k L$ is split over $LK$, we have the embedding

$$\mathbf{SU}(B \otimes_k L, \sigma \otimes_k 1) \cong \mathbf{SU}(M_3(LK), \sigma_h) \hookrightarrow G_L,$$

where $\sigma_h$ is the involution on $M_3(LK)$ given by $X \mapsto h\bar{X}^t h^{-1}$, for some $h = diag(h_1, h_2, h_3) \in GL_3(k)$. We may assume that $det(h) = 1$ (modulo squares). We note that $\sigma_h$, being isomorphic to $\sigma \otimes 1$, is distinguished. Therefore we have $< h_1, h_2, h_3 >_{LK} \cong < 1, -1, -1 >_{LK}$

as hermitian forms (see discussion after Prop. 2.3.3). Note that,

$$SU(M_3(LK),\sigma_h) = SU((LK)^3,h) \text{ and } SO(L^3,h) \subseteq SU((LK)^3,h).$$

Since $< h_1,h_2,h_3 >_{LK} \cong < 1,-1,-1 >_{LK}$ as hermitian forms, we have,

$$SU((LK)^3,h) \cong SU((LK)^3, < 1,-1,-1 >_{LK}).$$

Now $SO(L^3, < 1,-1,-1 >_L)$ is $L$-isotropic (since, for a non-degenerate form $Q$, $SO(Q)$ is $L$-isotropic iff $Q$ has a $L$-zero). Hence $SU((LK)^3,h)$ is $L$-isotropic. Therefore, $L$-rank of $G_L \geqq 1$. Hence $G_L$ is L-isotropic. Thus $f_5(G_L) = 0$ ([39], Pg. 205). By Springer's theorem, $f_5(G) = 0 = f_5(A)$. When $B = M_3(K)$, the proof follows along same lines without base changing.

We give another proof of the above theorem when $K$ is a field:
Suppose there exists an embedding $\mathbf{SU}(B,\sigma) \hookrightarrow G$, with $\sigma$ distinguished and $K$ a field extension. Recall that a simple, simply connected $k$-group $H = \mathbf{SU}(B,\sigma)$, where $B$ is a degree 3 central simple algebra with center a quadratic étale $k$-algebra $K$ and with an involution $\sigma$ of the second kind, is quasi-split over $k$ if and only if $B = M_3(K)$ and $H$ is distinguished (see Theorem 3.6.3).

Coming back to the proof of Theorem 9.1.2, if $B = M_3(K)$, then $H = \mathbf{SU}(B,\sigma)$ is quasi-split. Hence $k$-rank of $H$ is atleast 1 and therefore $H$ is isotropic over $k$. Thus $G$ is isotropic over $k$ and $f_5(G) = 0$. Now suppose $B$ is a division algebra. Choose a cubic separable field extension $L$ over $k$ such that $L \subset (B,\sigma)_+$. Since $B \otimes_k L$ is split over $LK$, we have the embedding

$$\mathbf{SU}(B \otimes_k L, \sigma \otimes_k 1) \cong \mathbf{SU}(M_3(LK),\sigma_h) \hookrightarrow G \otimes L,$$

where $\sigma_h$ is the involution on $M_3(LK)$ given by $X \mapsto h\bar{X}^t h^{-1}$, for some
$h = diag(h_1,h_2,h_3) \in GL_3(k)$. We note that $\sigma_h$, being isomorphic to $\sigma \otimes 1$, is distinguished. Hence by the previous case, $f_5(G \otimes L) = 0$. By Springer's theorem, $f_5(G) = 0 = f_5(A)$.
Conversely, if $f_5(A) = 0$ then $A \cong J(B,\sigma,u,\mu)$ for a central simple algebra $B$ over a quadratic étale extension $K$ of $k$, with a distinguished involution $\sigma$ ([19], Prop.40.7). Then by §6.3, $\mathbf{SU}(B,\sigma) \hookrightarrow G$. □

We now prove Theorem 9.1.1. PROOF. Let $\mathbf{SU}(B,\sigma) \hookrightarrow \mathbf{Aut}(A)$ be as in the hypothesis. We first need to settle the following

**Claim**: $D_L(f_3(B,\sigma)\otimes L) \subseteq D_L(f_5(A)\otimes L)$ for all field extensions $L$ of $k$.
Recall that for a quadratic form $q$ over $L$, $D_L(q)$ denotes the set of non-zero values of $q$ in $L$. Let $C'$ be the unique octonion algebra over $k$ such that $n_{C'} = f_3(B,\sigma)$ (Cayley-Dickson doubling, [53], §1.5).
Note that,

$$n_{C'}\otimes L = n_{C'\otimes L} \text{ and } f_5(A)\otimes L = f_5(A\otimes L).$$

Let $\alpha \in D_L(n_{C'\otimes L})$. We may assume that both $n_{C'\otimes L}$ and $f_5(A\otimes L)$ are anisotropic (otherwise $n_{C'\otimes L}$ is hyperbolic over $L$ and by the above theorem $f_5(A\otimes L)=0$) . We therefore assume that $C'\otimes L$ is a division algebra.
Let $x\in C'\otimes L$ be such that $n_{C'\otimes L}(x)=\alpha$. We may assume that $K' := L(x) \subseteq C'\otimes L$ is a quadratic field extension of $L$ (otherwise $x\in L$ and $\alpha$ is a square in $L$, which is represented by $f_5(A\otimes L)$). Base changing to $L$ we have, $\mathbf{SU}(B\otimes_k L,\sigma)\hookrightarrow \mathbf{Aut}(A\otimes_k L)$. Further base changing to $K'$ we get,

$$\mathbf{SU}(B\otimes_k L\otimes_L K',\sigma) \hookrightarrow \mathbf{Aut}((A\otimes_k L)\otimes_L K'). \qquad (*)$$

Since $K'\subseteq C'\otimes L$ is a quadratic subfield, $n_{C'}\otimes_k L\otimes_L K'$ is split ([53], Thm. 1.8.1). Taking $L\otimes_L K'$ ($\cong K'$) as the base field and $K\otimes_k L\otimes_L K'$ as the quadratic étale algebra over the base field and applying Theorem 9.1.2 to the embedding (*), we get $f_5((A\otimes_k L)\otimes_L K') = 0$. Now, since $K'$ over $L$ is a finite field extension and $f_5(A\otimes_k L)\otimes_L K'$ is split, we have by Theorem 1.1.12,

$$\alpha \in N_{K'/L}(K'^*) \subseteq D_L(f_5(A\otimes_k L)).$$

Hence $D_L(f_3(B,\sigma)\otimes L)\subseteq D_L(f_5(A)\otimes L)$ for all extensions $L$ of $k$. Hence by Theorem 1.1.9, $n_{C'}$ is isometric to a subform of $f_5(A)$ and we have, $f_5(A) = n_{C'}\otimes\tau$, for some 2-fold Pfister form $\tau$ over $k$. $\square$

**Remark 9.1.3** *The converse of the above theorem fails to hold. For example, let $A$ be an Albert division algebra. Now,* $\mathbf{SU}(M_3(K),*_\gamma)$ *does not embed in* $\mathbf{Aut}(A)$ *for any* *central simple algebra $(M_3(K), *_\gamma)$, where $K$ is a quadratic field extension of $k$. To see this we observe that $A$ continues to be a division algebra over $K$ (Proposition 5.2.4), and over $K$,*

$$\mathbf{SU}(M_3(K), *_\gamma) \cong \mathbf{SL}_3 \hookrightarrow \mathbf{Aut}(A \otimes K).$$

*Therefore $K$-rank of $\mathbf{Aut}(A \otimes K)$ is at least 2, hence $A \otimes K$ is split ([61], Pg. 60), a contradiction. In particular, if $f_5(A) = 0$, then we have,*

$$f_5(A) = 0 = f_3(M_3(K), *_\gamma) \otimes < 1, -1, 1, -1 > .$$

**Corollary 9.1.4** *Let $A$ be an Albert algebra over $k$, $G = \mathbf{Aut}(A)$ be the associated algebraic group of type $F_4$. Let $H$ be a connected reductive group of type $A_2$ defined over $k$. Suppose there is a $k$-embedding $H \hookrightarrow G$. Then $f_5(A) = f_3(H) \otimes \tau$ for some 2-fold Pfister form $\tau$ over $k$.*

Proof. Let $\widetilde{[H,H]}$ be the simply connected cover of $[H,H]$ and $\psi : \widetilde{[H,H]} \to H$ be the covering map. Let $\phi : \widetilde{[H,H]} \to G$ be the composite $\widetilde{[H,H]} \xrightarrow{\psi} [H,H] \hookrightarrow H \hookrightarrow G$. Then $ker(\phi)^\circ = 1$, owing to the simplicity of $\widetilde{[H,H]}$. Hence, over an extension $L$ of $k$, $\phi$ maps any non-trivial torus in $\widetilde{[H,H]}_L$ to a non-trivial torus in $G \otimes L$. Now, $\widetilde{[H,H]}$ is a simply connected simple $k$-group of type $A_2$. Hence, by Remark 7.1.2, there exists a central simple algebra $B$ of degree 3 over a quadratic étale extension $K$ of $k$, with an involution $\sigma$ of the second kind, such that $\widetilde{[H,H]} \cong \mathbf{SU}(B, \sigma)$. Also, by definition (Remark 7.1.2) $f_3(H) = f_3(\widetilde{[H,H]})$. The result now follows along exactly same lines as in the proofs of Theorem 9.1.2 and Theorem 9.1.1. □

We give few examples to illustrate the above theorems:

**Examples**: Let $A = J(D, \tau, u, \mu)$ be a second Tits construction Albert algebra. Let the center of $D$ be $K$ and let $< \alpha_1, \alpha_2, \alpha_3 >$ denote the hermitian form corresponding to $K^\perp$ in $C = Oct(A)$ with respect to the norm form of $C$ ([12], §5) and $\mathcal{H}_3(C, \Gamma)$ be the reduced model for $A$, $\Gamma = diag(\gamma_1, \gamma_2, \gamma_3)$. Let $\tau_u = Int(u) \circ \tau$. As seen in §6.3, we have the $k$-embeddings of $\mathbf{SU}(D, \tau)$ and $\mathbf{SU}(D, \tau_u)$ in $\mathbf{Aut}(A)$,

**(a)** Under the $k$-embedding $\mathbf{SU}(D, \tau) \hookrightarrow \mathbf{Aut}(A)$ we have,

$$f_5(A) = f_3(D, \tau) \otimes < 1, \alpha_1, \alpha_2, \alpha_3 > .$$

**(b)** Under the $k$-embedding $\mathbf{SU}(D, \tau_u) \hookrightarrow \mathbf{Aut}(A)$ we have,

$$f_5(A) = f_3(D, \tau_u) \otimes < 1, \gamma_1, \gamma_2, \gamma_3 > .$$

**Lemma 9.1.5** *Let $A$ be an Albert algebra of the form $J(D, \tau, 1, \mu)$. Then*

$$f_5(A) = n_C \oplus n_C \oplus n_C \oplus n_C = n_C \otimes << -1, -1 >>,$$

*where $n_C$ denotes the norm form of the octonion algebra $C$ of $A$.*

Proof. By base changing to a suitable cubic extension, we may assume that

$$A = J(M_3(K), *_\Gamma, 1, \mu),\ \Gamma = diag(\gamma_1, \gamma_2, \gamma_3) \text{ with } \gamma_1\gamma_2\gamma_3 = 1,$$

and $K$ is a quadratic étale extension of $k$. Then $A \cong \mathcal{H}_3(C, \Gamma)$ and $n_C = n_K \otimes < 1, \gamma_1, \gamma_2, \gamma_3 >$, where $\Gamma = Diag(\gamma_1, \gamma_2, \gamma_3)$ (Theorem 5.2.3). Now, $f_5(A) = n_C \otimes < 1, \gamma_1, \gamma_2, \gamma_3 >$ (see §7.2). In this special case, $n_C$ represents $\gamma_i, 1 \leq i \leq 3$ (since $n_K$ represents 1). Therefore we get

$$f_5(A) = n_C \oplus \gamma_1 n_C \oplus \gamma_2 n_C \oplus \gamma_3 n_C = n_C \oplus n_C \oplus n_C \oplus n_C.$$

□

**Lemma 9.1.6** *Let $K$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $K$ with an involution $\sigma$ of the second kind. Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Suppose $\mathbf{SU}(B, \sigma) \hookrightarrow G$ over $k$. Then $K \subseteq Oct(A)$.*

Proof. Base changing to $K$ we get, $\mathbf{SU}(B \otimes_k K, \sigma \otimes_k 1) \hookrightarrow G \otimes K$. Also $(B \otimes_k K, \sigma \otimes_k 1) \cong (D \times D^{op}, \epsilon)$, where $D$ is a degree 3 central simple $K$-algebra and $\epsilon$ denotes the switch involution $(x, y) \mapsto (y, x)$ and $D^{op}$ is the opposite algebra of $D$. Hence, $\mathbf{SU}(B \otimes_k K, \sigma \otimes_k 1) \cong \mathbf{SL}_1(D) \hookrightarrow G \otimes K$. As in the first case of Theorem 9.1.2 we have $f_3(G \otimes K) = 0$. Therefore $Oct(A) \otimes K$ is split and hence $K \subset Oct(A)$ ( follows from [5], Lemma 5). □

**Remark 9.1.7** *Let $K$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $K$ with a distinguished involution $\sigma$. Let $A$ be an Albert algebra over $k$. It is possible to have $\mathbf{SU}(B,\sigma) \hookrightarrow \mathbf{Aut}(A)$ over $k$, yet $f_3(A) \neq 0$. Let $C$ denote the octonion division algebra represented by the 3-fold (anisotropic) Pfister form $< 1, -x > \otimes < 1, -y > \otimes < 1, -z >$ over $k = \mathbb{C}(x, y, z)$. Let $K \subset C$ be a quadratic subfield and let $h = diag(h_1, h_2, h_3)$ denote the hermitian form on $K^{\perp} \subset C$ (see [12], §5). Let $A = J(M_3(K), *_h, 1, \mu)$ where $*_h(X) = h^{-1}\overline{X}^t h$ and $\mu \in K$ satisfies $\mu\overline{\mu} = 1$. Then $C = Oct(A)$ and $f_5(A) = n_C \otimes << -1, -1 >>$ by Lemma 9.1.5. Since $-1$ is a square in $k$, $f_5(A) = 0$. So by ([19], Prop. 40.7), we can write $A \cong J(B, \sigma, u', \mu')$ where $\sigma$ is distinguished. Hence, $\mathbf{SU}(B,\sigma) \hookrightarrow \mathbf{Aut}(A)$ (see §6.3). Moreover $f_3(B,\sigma) = 0$ and $f_3(A) \neq 0$.*

**Proposition 9.1.8** *Let $A$ be an Albert division algebra and $G = \mathbf{Aut}(A)$. Then there exists a bijection between the set of $k$-conjugacy classes of subgroups $\mathbf{Aut}(A/D)$ in $G$, $D$ a 9-dimensional subalgebra, and the set of isomorphism classes of 9-dimensional subalgebras of $A$. The map $[D] \mapsto f_3(D)$, from the set of isomorphism classes of 9-dimensional subalgebras of $A$ to the set of isometry classes of 3-fold Pfister divisors of $f_5(A)$, is injective, when restricted to the subset $\{(B,\sigma)_+ \mid \sigma$ a unitary involution on $B\}$ for a fixed $B$.*

Proof. Let $S_i$, $i = 1, 2$, be 9-dimensional subalgebras of $A$ such that $\mathbf{Aut}(A/S_i)$, $i = 1, 2$, are $k$-conjugate in $\mathbf{Aut}(A)$. Then there exists $\phi \in \mathbf{Aut}(A)(k)$ such that

$$\phi \mathbf{Aut}(A/S_1)\phi^{-1} = \mathbf{Aut}(A/\phi(S_1)) = \mathbf{Aut}(A/S_2).$$

Now, by taking fixed points of these subgroups in $A$, we get $\phi(S_1) = S_2$. Hence $S_1$ is isomorphic to $S_2$. Thus we have a map $[\mathbf{Aut}(A/D)] \mapsto [D]$ from the set of $k$-conjugacy classes of subgroups $\mathbf{Aut}(A/D)$ in $G$ to the set of isomorphism classes of 9-dimensional subalgebras $D$ of $A$. Let now $S_i$, $i = 1, 2$ be 9-dimensional subalgebras such that $\phi : S_1 \to S_2$ is an isomorphism. By the Skolem-Noether theorem for Albert algebras ([34], Theorem 3.1, see also [41], §5), $\phi$ extends to an automorphism $\widetilde{\phi}$ of $A$. It follows that $\widetilde{\phi}\mathbf{Aut}(A/S_1)\widetilde{\phi}^{-1} = \mathbf{Aut}(A/S_2)$. Hence we have a map $[D] \mapsto [\mathbf{Aut}(A/D)]$, which is the required inverse of the above map. By Theorem 9.1.1 we have the map $[D] \mapsto f_3(D)$. That this map is injective on the set $\{(B,\sigma)_+ \subset A\}$ for a fixed $B$, follows from Proposition 2.3.3. $\square$

**Embeddings of $A_1$ in $F_4$:**

Let $G$ be a group of type $F_2$ defined over $k$. Along similar lines of Theorem 9.1.1, we now derive a factorization result for the mod-2 invariant $f_5(G)$ of $G$ in terms of the mod-2 invariant of the embedded $k$-group of type $A_1$.

**Theorem 9.1.9** *Let $Q$ be a quaternion algebra over $k$ and $A$ be an Albert algebra over $k$. Let $G = \mathbf{Aut}(A)$ be the algebraic group of type $F_4$ associated to $A$. Suppose $\mathbf{SL}(1, Q) \hookrightarrow G$ over $k$. Then $f_5(A) = f_2(n_Q) \otimes \tau$ for some three fold Pfister form $\tau$ over $k$.*

**Remark 9.1.10** *We note that the converse of the above theorem does not hold. For example, if $A$ is an Albert division algebra over $k$, then for no quaternion algebra $Q$, $\mathbf{SL}(1, Q)$ can embed in $G = \mathbf{Aut}(A)$, since for any quadratic subfield $K \subset Q$, the $K$-rank of $G \otimes K$ is positive. Hence $A \otimes K$ must be reduced (Prop. 7.2.1), whereas $A$ continues to be a division algebra over $K$ (Proposition 5.2.4), a contradiction. In particular if $f_5(A) = 0$, then for any quaternion algebra $Q$, $f_5(A) = 0 = f_2(n_Q) \otimes \tau$, where $\tau$ is an 8-dimensional hyperbolic form.*

## 9.2 Embeddings of $A_1, A_2$ in $G_2$

Let $G$ be a group of type $G_2$ defined over $k$. In this section we derive factorization of the mod-2 invariant $f_3(G)$ of $G$ in terms of the mod-2 invariant of the embedded $k$-groups of type $A_1$ and $A_2$.

**Theorem 9.2.1** *Let $C$ be an octonion algebra over $k$ and $Q$ be a quaternion algebra over $k$. Then the following are equivalent.*

**(a)** *$Q$ embeds in $C$ as a subalgebra.*
**(b)** *$n_C = n_Q \otimes \tau$, where $\tau$ is a 1- fold Pfister form over $k$.*
**(c)** *$\mathbf{SL}_1(Q) \hookrightarrow \mathbf{Aut}(C)$ over $k$.*

Proof. (**1**) (**a**) $\Longrightarrow$ (**c**) : Since $Q$ embeds in $C$ as a subalgebra, we can write $C$, up to an isomorphism, as $C = Q \oplus Q$ (by Cayley-Dickson doubling). Consider the automorphism of $C$ given by

$$\phi_p : (x, y) \mapsto (x, py),\ \forall\ x, y \in Q,\ p \in \mathbf{SL}_1(Q).$$

Then, $\mathbf{SL}_1(Q) \hookrightarrow \mathbf{Aut}(Q)$ via $p \to \phi_p$ is a $k$-embedding.

**(2)** **(c)** $\Longrightarrow$ **(b)** : We can assume that $n_Q$ and $n_C$ are anisotropic. Let $X$ be the conic attached to $Q$. Then $Q$ (hence $\mathbf{SL}_1(Q)$) splits over the function field $k(X)$. Therefore, base changing to $k(X)$ we get,

$$\mathbf{SL}_1(Q \otimes_k k(X)) \hookrightarrow \mathbf{Aut}(C \otimes_k k(X)).$$

Hence $k(X)$ splits the Pfister form $n_C$. Therefore by Theorem 1.1.10, taking $a, b = 1$, $n_Q$ is isometric to a subform of $n_C$ and hence is a factor of $n_C$.

**(3)** **(b)** $\Longrightarrow$ **(a)** : Let $\tau = < 1, -\alpha >$, for some $\alpha \in k$. Let $C' = Q \oplus Q$ be the $\alpha$-double of $Q$. Now, $n_{C'} = n_Q \otimes < 1, -\alpha >= n_C$. Since octonion algebras with isometric norms are isomorphic ([19], Thm. 33.19), we have $C' \cong C$ over $k$. Hence, $Q$ embeds in $C$. □

**Lemma 9.2.2** *Let $C$ be an octonion algebra over $k$. Let $G \cong \mathbf{Aut}(C)$. Let $K$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $K$ with an distinguished involution $\sigma$ of the second kind. If there exists a $k$-embedding $\mathbf{SU}(B, \sigma) \hookrightarrow G$ then $G$ splits.*

Proof. By Theorem 10.1.1, $H$ becomes isotropic over an odd degree extension of $k$, hence $G$ becomes isotropic over an odd degree extension and splits over it. By Springer's theorem, $G$ splits over $k$. □

**Theorem 9.2.3** *Let $C$ be an octonion algebra over $k$. Let $K$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $K$ with an involution $\sigma$ of the second kind. Then there exists a $k$-embedding $\mathbf{SU}(B, \sigma) \hookrightarrow \mathbf{Aut}(C)$ if and only $f_3(B, \sigma) = n_C$ and $B \cong M_3(K)$.*

PROOF. Suppose there exists a $k$-embedding $\mathbf{SU}(B,\sigma) \hookrightarrow \mathbf{Aut}(C)$. Assume that $B$ is a division algebra. Then $\mathbf{SL}_1(B) \hookrightarrow \mathbf{Aut}(C \otimes_k K)$ over $K$. Note that $C_o := \{x \in C | Tr(x) = 0\}$ is a 7-dimensional faithful representation of $\mathbf{Aut}(C)$ defined over $k$. Hence $\mathbf{SL}_1(B)$ has a 7-dimensional faithful representation defined over $K$. This is a contradiction, since the only irreducible non-trivial representation of $\mathbf{SL}_1(B)$ of dimension $\leq 8$, for $B$ a division algebra, is the 8-dimensional adjoint representation. Hence $B \cong M_3(K)$.

**Claim**: $f_3(B,\sigma) = n_C$.

We can assume that both $n_C$ and $f_3(M_3(K),\sigma)$ are irreducible anisotropic forms. Let $Z$ be the quadric attached to the Pfister form $f_3(M_3(K),\sigma)$. Base Changing to $k(Z)$ ,

$$\mathbf{SU}(M_3(K) \otimes k(Z), \sigma \otimes 1) \hookrightarrow G \otimes k(Z).$$

Now $f_3(M_3(K),\sigma)$ splits over $k(Z)$. Hence, by the Lemma 9.2.2, $k(Z)$ splits $G$ and hence $n_C$ is split by $k(Z)$ and by the Subform theorem (Theorem 1.1.10), $f_3(B,\sigma) = n_C$. $\square$

We have the following result linking rank-1 $k$-tori in a $k$-group $G$ of type $G_2$ and 1-fold Pfister divisors of $f_3(G)$,

**Theorem 9.2.4** *Let $C$ be an octonion division algebra over $k$ and $G = \mathbf{Aut}(C)$ be the corresponding $k$-group of type $G_2$. Then the following sets are in natural bijection with each other*

**(i)** *Isomorphism classes of quadratic étale subalgebras $K$ of $C$.*

**(ii)** *$k$-Conjugacy classes of subgroups $\mathbf{Aut}(C/K)$ in $G$, for quadratic étale subalgebras $K$ of $C$.*

**(iii)** *Isometry classes of 1-fold Pfister divisors of $n_C$, the norm form of $C$.*

**(iv)** *$k$-isomorphism classes of rank-1 $k$-tori in $G$.*

PROOF. The map **(i)** $\rightarrow$ **(ii)** is given by $[K] \mapsto [\mathbf{Aut}(C/K)]$ and is well defined. The map $[\mathbf{Aut}(C/K)] \mapsto [K]$ is a well defined inverse of this map. To see this, let $K_i$, $i = 1,2$, be quadratic étale subalgebras of $C$ and $\phi\mathbf{Aut}(C/K_1)\phi^{-1} = \mathbf{Aut}(C/\phi(K_1)) = \mathbf{Aut}(C/K_2)$ for some $\phi \in \mathbf{Aut}(C)(k)$. We show $\phi(K_1) = K_2$. If not, consider the subalgebra $Q$ generated by $\phi(K_1)$ and $K_2$. Then $Q$ is a quaternion subalgebra. Let $x \neq 0 \in K_2$ be arbitrary and consider $Int(x) : Q \rightarrow Q$. By the Skolem-Noether theorem for composition algebras ([53], Cor. 1.7.3) $Int(x)$ extends to an automorphism $\psi$ of $C$. Observe that $Int(x)$, hence $\psi$, fixes $K_2$ pointwise. Hence $\psi$ fixes $\phi(K_1)$ pointwise. Therefore $x$ centralizes $\phi(K_1)$ in $Q$ and hence $x \in \phi(K_1)$ since $\phi(K_1)$ is maximal commutative in $Q$. This is a contradiction. Therefore we have a bijection $(\mathbf{i}) \leftrightarrow (\mathbf{ii})$. The map $(\mathbf{ii}) \mapsto (\mathbf{i})$ is given by $[H = \mathbf{Aut}(C/K)] \mapsto [C^H]$. Note that $C^H = K$. For the bijection $(\mathbf{i}) \leftrightarrow (\mathbf{iii})$, the construction is as in the *case*$(\mathbf{2})$ of the proof of Theorem 9.2.1. The map $(\mathbf{i}) \to (\mathbf{iv})$ is given by $[K] \mapsto [K^{(1)}]$ where $K^{(1)}$ is the norm torus of $K$. Let $S_i,\ i = 1, 2$, be rank-1 $k$-isomorphic tori in $G$. By ([64], Example 6, Pg. 57 ) there exists quadratic étale extensions $K_i,\ i = 1, 2$, of $k$ such that $S_i \cong K_i^{(1)},\ i = 1, 2$. Observe that $S_i,\ i = 1, 2$, split over $K_i$. Hence $G$, therefore $C$, splits over $K_i,\ i = 1, 2$. By ([5], Lemma 5) $K_i,\ i = 1, 2$, embeds in $C$. Since $S_1$ and $S_2$ are $k$-isomorphic, it follows that $K_1$ and $K_2$ are isomorphic. Therefore we have a bijection $(\mathbf{i}) \leftrightarrow (\mathbf{iv})$. □

**Proposition 9.2.5** *The following sets are in natural bijection with each other*

**(1)** *$k$-Conjugacy classes of involutions in $\mathbf{Aut}(C)(k)$.*

**(2)** *Isomorphism classes of quaternion subalgebras $D$ of $C$.*

**(3)** *$k$-Conjugacy classes of subgroups $\mathbf{Aut}(C/D) \subseteq \mathbf{Aut}(C)$.*

**(4)** *Isometry classes of 2-fold Pfister divisors of $n_C$.*

Proof. The existence of bijection $(\mathbf{1}) \leftrightarrow (\mathbf{2})$ follows from ([52], Prop. 4.1).
The existence of bijection $(\mathbf{2}) \leftrightarrow (\mathbf{3})$ follows along similar lines as in the proof of bijection $(\mathbf{i}) \leftrightarrow (\mathbf{ii})$ of Theorem 9.2.4. The existence of bijection $(\mathbf{2}) \leftrightarrow (\mathbf{4})$ follows from Theorem 9.2.1. □

# Chapter 10

# Embeddings of rank-2 tori

This chapter reports the work done in [9]. In chapter 9, we studied $k$-embeddings of connected, simple algebraic groups of type $A_1$ and $A_2$ in simple groups of type $G_2$ and $F_4$, defined over $k$, in terms of their respective mod-2 Galois cohomological invariants. Fix a field $k$ of characteristic different from 2 and 3. In this chapter we investigate $k$-embeddings of certain rank-2 $k$-tori in $k$-groups of type $G_2, F_4$ and $A_2$ and study the mod-2 invariants of these groups via these embeddings.

In §8.2 we described the notion of a distinguished torus. In the first section of this chapter we study $k$-embeddings of distinguished $k$-tori in simply connected, simple algebraic groups of type $A_2$, $G_2$ and $F_4$, defined over a field $k$, in terms of the mod-2 Galois cohomological invariants attached with these groups. In the second section we list some results on groups of type $A_2$, $G_2$ and $F_4$ which arise from division algebras, which will be useful in the thesis. In the third section we study $k$-embeddings of unitary $k$-tori in simply connected, simple $k$-groups of type $A_2$, $G_2$ and $F_4$ in terms of mod-2 invariants of these groups.

## 10.1 Embeddings of Distinguished tori

Let $G$ of a $k$-group of type $F_4$. We show that $G$ contains a distinguished $k$-torus if and only if $f_5(G) = 0$. A stronger version of this result holds for groups of type $G_2$ and $A_2$. Let $G$ be a simple, simply connected $k$-group of type $G_2$ or $A_2$ defined over $k$. We prove that $Oct(G)$ splits over $k$ if and only if $G$ contains a distinguished (maximal) $k$-torus.

We begin with a group theoretic characterization of distinguished involutions on degree 3 central simple algebras:

**Theorem 10.1.1** *Let $F$ be a quadratic étale $k$-algebra and $B$ be a degree $3$ central simple algebra over $F$ with an involution $\sigma$ of the second kind. Then $\sigma$ is distinguished over $k$ if and only if $\mathbf{SU}(B,\sigma)$ becomes isotropic over an odd degree extension.*

Proof. Assume that $\mathbf{SU}(B,\sigma)$ becomes isotropic over an odd degree extension $L$ of $k$.
**Claim**: The degree 3 $L$-Jordan algebra $(B,\sigma)_+ \otimes L \cong (B\otimes L, \sigma\otimes 1)_+$ contains a non-zero nilpotent.
Let $V = (B,\sigma)_+$. Note that $\mathbf{SU}(B,\sigma)$ acts via automorphisms on $V$ by the map $\phi : \mathbf{SU}(B,\sigma) \to \mathbf{Aut}(V)$ of algebraic groups given by

$$x \mapsto \phi(x)(p) = px\sigma(p) = pxp^{-1},\ x \in V,\ p \in \mathbf{SU}(B,\sigma).$$

Since $\mathbf{SU}(B,\sigma)$ becomes isotropic over $L$, we have an embedding $\mathbb{G}_m \hookrightarrow \mathbf{SU}(B\otimes L, \sigma\otimes 1)$ over $L$. We now decompose $V\otimes L$ under the action of $\mathbb{G}_m$ as

$$V\otimes L = \bigoplus_{\lambda\in\chi(\mathbb{G}_m)} V_\lambda,\ \text{where } \mathrm{V}_\lambda := \{\mathrm{w} \in \mathrm{V}\otimes\mathrm{L} : \mathrm{t(w)} = \lambda(\mathrm{t})\mathrm{w}\ \forall\ \mathrm{t} \in \mathbb{G}_\mathrm{m}\}.$$

Note that $\ker(\phi_L)^\circ = \{1\}$, owing to the simplicity of $\mathbf{SU}(B,\sigma)$, where $\phi_L$ is the base change of $\phi$ to $L$. Hence the above embedded $\mathbb{G}_m$ does not act trivially on $V\otimes L$. Thus there exists $\lambda \in \chi(\mathbb{G}_m),\ \lambda \neq 1$ such that $V_\lambda \neq 0$. Now choose $t \in \mathbb{G}_m$ such that $\lambda(t) \neq 1$, $\lambda(t^2) \neq 1$, $\lambda(t^3) \neq 1$ and $0 \neq w \in V_\lambda$. Let $Q(X) = \frac{1}{2}T_B(X^2)$ and $N$ be the reduced norm on $B$. Since $t$ acts via automorphisms on $(B,\sigma)_+$, it follows, by considering the equations,

$$T_B(t(w)) = T_B(w) = \lambda(t)T_B(w), \qquad Q(t(w)) = Q(w) = \lambda(t)^2 Q(w),$$

$$N(t(w)) = N(w) = \lambda(t)^3 N(w)$$

that $T_B(w) = Q(w) = N(w) = 0$. Hence $w$ is nilpotent. Therefore, $V\otimes L$ contains a non-zero nilpotent element. Hence the claim.
By ([37], Thm. 2.11., Pg. 376), $Oct(B,\sigma)_+ \otimes L$ splits. Hence $\sigma$ is distinguished over $L$ ([7], Thm. 16). By Springer's theorem, it follows that $\sigma$ is distinguished over $k$. The converse follows by an argument as in the $case(ii)$ of the proof of Theorem 9.1.2. $\square$

We now prove a lemma which will be used extensively in the thesis.

**Lemma 10.1.2** **(a)** *Let $G$ be a group of type $F_4$ defined over $k$. If $G$ has $k$-rank $\geqq 1$, then $f_5(G) = 0$. Moreover, if $G$ has $k$-rank $\geqq 2$, then $G$ splits over $k$ and $f_3(G) = 0 = f_5(G)$.*

**(b)** *Let $G$ be a simple, simply connected group of type $G_2$ or $A_2$ defined over $k$. If $k$-rank of $G \geq 1$, then $Oct(G)$ splits.*

Proof. **(a)** Let $G$ be a group of type $F_4$ defined over $k$. If $G$ has k-rank $\geqq 1$ then $G$ is $k$-isotropic and by ([39], Pg. 205), $f_5(G) = 0$. If $G$ has $k$-rank $\geqq 2$ then by ([61], Pg. 60), $G$ splits over $k$. Hence $A$ and thereby $Oct(A)$ splits over $k$ ([54], Chap. 17, §17.6.4, [8], Pg. 164).

**(b)** Let $G$ be a group of type $G_2$ defined over $k$. If $k$-rank of $G \geq 1$, then by ([54], Chap. 17, §17.4.2), $G$ is $k$-split. Hence $Oct(G)$ is split ([54], Chap. 17, §17.4.5, [61], Pg. 60). Let $G$ be a simple, simply connected group of type $A_2$ defined over $k$. If $k$-rank of $G \geq 1$ then $G$ is $k$-isotropic and by Theorem 10.1.1 above, $Oct(G)$ splits. □

**Theorem 10.1.3** *Let $T$ be a distinguished torus defined over $k$. Then $T$ is isotropic over an odd degree extension of $k$.*

Proof. Let $T$ be a distinguished torus over $k$. Then, by definition, there exists étale $k$-algebras $L, K$ of $k$-dimensions $3, 2$ resp. such that $disc(L) = K$ and $T = \mathbf{SU}(E, \tau)$, where $(E, \tau)$ is the $K$-unitary algebra associated to $(L, K)$. By Lemma 8.1.4, $L = E^\tau = \{x \in E \mid \tau(x) = x\}$. We divide the proof into three cases.

**Case (i)** $L = k \times k \times k$.

Since $T$ is distinguished, we have $disc(L) = K = k \times k$. Hence $(E, \tau) \cong (L \times L, \epsilon)$, where $\epsilon : L \times L \to L \times L$ is given by $\epsilon(x, y) = (y, x)$, the switch involution on $L \times L$. Now

$$SU(E, \tau) \cong \{(x, y) \in L \times L | (x, y)\epsilon(x, y) = 1, (N_{L/k}(x), N_{L/k}(y)) = (1, 1)\} \cong L^{(1)} \cong k^* \times k^*,$$

where $L^{(1)}$ denotes the group of norm 1 elements of $L$. It follows that $\mathbf{SU}(E, \tau) \cong \mathbb{G}_m \times \mathbb{G}_m$ over $k$, and hence $T = \mathbf{SU}(E, \tau)$ splits over $k$ in this case.

**Case (ii)** $L = k \times K$, $K$ is a field.

Let $\overline{\epsilon} : K \times K \to K \times K$ be given by $\overline{\epsilon}(x, y) = (\overline{y}, \overline{x})$. Then $(E, \tau) = ((k \times K) \otimes K, 1 \otimes^{-}) \cong (K \times (K \times K), (^{-}, \overline{\epsilon}))$. We have therefore,

$$\begin{aligned} SU(E,\tau) &\cong \{(x,y,z) \in K \times K \times K | (x,y,z)(\overline{x},\overline{z},\overline{y}) = (1,1,1), xyz = 1\}. \\ &\cong \{(\overline{z}z^{-1}, \overline{z}^{-1}, z)|\ z \in K^*\} = \{(z^{-2}N(z), zN(z^{-1}), z)|\ z \in K^*\}. \\ &\cong K^*. \end{aligned}$$

Hence $T = \mathbf{SU}(E, \tau) \cong R_{K/k}(\mathbb{G}_m)$ is isotropic over $k$.

**Case (iii)** $L$ is a field.

Base changing to $L$ we get, $L \otimes L \cong L \times K_0$ as $L$-algebras, where $K_0 = L \otimes \Delta$, and $\Delta = Disc(L)$ (see Prop. 2.4.2). By case **(i)** and **(ii)**, it follows that $\mathbf{SU}(E, \tau) \otimes L$ is isotropic. Hence $T = \mathbf{SU}(E, \tau)$ is isotropic over $L$. □

Note that this property characterizes distinguished tori among unitary ones. We now study the presence of distinguished $k$-tori in groups of type $A_2$, $G_2$ and $F_4$ defined over $k$. We see that existence of such tori has a direct relation with the mod-2 invariants attached to these groups. We obtain as an immediate consequence of the above theorem the following,

**Theorem 10.1.4** *Let $G$ be a group of type $G_2$ over $k$. Then $G$ splits over $k$ (equivalently, $Oct(G)$ splits over $k$) if and only if there exists a maximal $k$-torus in $G$ which is distinguished.*

Proof. Let $T \subseteq G$ be a distinguished maximal $k$-torus. By Theorem 10.1.3, $T$ becomes isotropic over an odd degree extension, say $M$, of $k$. Hence $M$-rank of $G \geq 1$. Thus $Oct(G) \otimes M$ is split (Lemma 10.1.2). By Springer's theorem, $Oct(G)$ splits over $k$ itself and consequently $G$ is $k$-split. Conversely, suppose $G$ splits over $k$. Let $L = k \times k \times k$ and $K = k \times k$ and $T = \mathbf{SU}(L \otimes K, 1 \otimes^{-})$. By case **(i)** of the proof of Theorem 10.1.3, $T \cong \mathbb{G}_m \times \mathbb{G}_m$ and $\mathbb{G}_m \times \mathbb{G}_m \hookrightarrow G$ over $k$ as $G$ is $k$-split. Hence $T$ is the required distinguished $k$-torus. □

A similar result holds for groups of type $A_2$.

**Theorem 10.1.5** *Let $G$ be a simple, simply connected group of type $A_2$ over $k$. Then $Oct(G)$ splits over $k$ if and only if there exists a maximal $k$-torus in $G$ which is distinguished.*

Proof. Let $G$ be as in the hypothesis. Then $G \cong \mathbf{SU}(B,\sigma)$ for some degree 3 central simple algebra $B$ over a quadratic étale extension $F$ of $k$ with an involution $\sigma$ of the second kind. Let $T \hookrightarrow G$ be a maximal $k$-torus which is distinguished. Then, by Theorem 10.1.3, $T$ is isotropic over an odd degree extension $M$ of $k$. Thus $G$ is isotropic over $M$. Hence by Theorem 10.1.1, $\sigma$ is distinguished over $k$. Hence, $f_3(B,\sigma) = 0$ and $Oct(G)$ is split over $k$. Conversely, if $Oct(G)$ is split over $k$, then $f_3(B,\sigma) = 0$ and hence $\sigma$ is distinguished over $k$. By ([7], Theorem 16, pg. 317), $(B,\sigma)_+$ contains a cubic étale $k$- algebra $L$ with $F$ as its discriminant algebra. Let $E = L \otimes F$. Then $E \hookrightarrow B$ and $\sigma$ restricted to $E$ equals $\tau := 1 \otimes \bar{\ }$, where $\bar{\ }$ denotes the non-trivial $k$-automorphism of $F$. Hence $\mathbf{SU}(E,\sigma)$ is a distinguished $k$-torus and, by Lemma 8.2.1, $\mathbf{SU}(E,\tau) \hookrightarrow G \cong \mathbf{SU}(B,\sigma)$ over $k$. □

For groups of type $F_4$ we have the following,

**Theorem 10.1.6** *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Then $f_5(A) = 0$ if and only if $G$ contains a distinguished $k$-torus.*

Proof. Assume that $G$ contains a distinguished $k$-torus $T$. Then by Theorem 10.1.3, $T$ is isotropic over an odd degree extension $M$ of $k$, hence $G$ becomes isotropic over $M$. Therefore $M$-rank of $G \otimes M \geq 1$ and $f_5(A \otimes M) = f_5(A) \otimes M = 0$ (Lemma 10.1.2). By Springer's theorem $f_5(A) = 0$. Conversely, if $f_5(A) = 0$, by ([19], Prop. 40.7), $A \cong J(B,\sigma,u,\mu)$ for a central simple algebra $B$ over a quadratic étale extension $F$ of $k$, with a distinguished involution $\sigma$. Since $\sigma$ is distinguished, by Theorem 10.1.5 there exists a $k$-embedding of a distinguished $k$-torus $T$ in $\mathbf{SU}(B,\sigma)$. Now $\mathbf{SU}(B,\sigma) \hookrightarrow G$ over $k$ (see §6.3). Hence $T \hookrightarrow G$ over $k$ and $T$ is distinguished. □

As a consequence of the above theorem, we have an alternative proof of (Theorem 9.1.2, §9.1).

**Corollary 10.1.7** *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Then $f_5(A) = 0$ if and only if there exists a $k$-embedding $\mathbf{SU}(B,\sigma) \hookrightarrow G$ for some degree 3 central simple algebra $B$ with center a quadratic étale $k$-algebra $F$ and with a distinguished involution $\sigma$.*

Proof. Suppose $\mathbf{SU}(B,\sigma) \hookrightarrow G$ over $k$ for $(B,\sigma)$ as in the hypothesis. Since $\sigma$ is distinguished, by Theorem 10.1.5, there exists a $k$-embedding $T \hookrightarrow \mathbf{SU}(B,\sigma)$ for a distinguished $k$-torus $T$. Hence $T \hookrightarrow \mathbf{SU}(B,\sigma) \hookrightarrow G$ over $k$. Therefore, by Theorem 10.1.6,

$f_5(A) = 0$. The proof of the converse follows exactly along the same lines as in the proof of Theorem 10.1.6. $\square$

## 10.2 Groups arising from division algebras

Let $G$ be a simple, simply connected $k$-group of type $A_2$. We will refer to $G$ as arising from a division algebra if either $G \cong \mathbf{SU}(D, \sigma)$ for some degree 3 central division algebra $D$ over a quadratic field extension $F$ of $k$, with an involution $\sigma$ of the second kind or $G \cong \mathbf{SL_1}(D)$ for some degree 3 central division algebra $D$ over $k$. Let $G$ be a $k$-group of type $F_4$. We will refer to $G$ as arising from a division algebra if $G \cong \mathbf{Aut}(A)$, where $A$ is an Albert division algebra over $k$. Let $G$ be a $k$-group of type $G_2$. We will refer to $G$ as arising from a division algebra if $G \cong \mathbf{Aut}(C)$, where $C$ is an octonion division algebra over $k$.

**Theorem 10.2.1** *Let $G$ be a simple, simply connected group of type $A_2$ or $F_4$ defined over $k$, arising from a division algebra over $k$. Then,*
**(1)** *$G(k)$ contains no non-trivial involution over $k$.*
**(2)** *There does not exists any rank-1 torus $T$ over $k$ such that $T \hookrightarrow G$ over $k$.*
**(3)** *$G$ is $k$-anisotropic.*
*These conditions hold over any field extension of $k$ of degree coprime to 3.*

Proof. First we prove (**1**). Recall that an involution in a group is an element of order atmost 2. Let $G$ be a simple, simply connected group of type $A_2$, arising from a division algebra $D$ over $k$. Let $Z(D)$ denote the center of $D$. Then $[D : Z(D)] = 9$. Let $\theta \in G(k) \subseteq D^*$ be an involution. Then $\theta^2 = 1$ and $N_D(\theta) = 1$. Since $\theta^2 = 1$, $\theta$ generates the field extension $k(\theta)$ of $k$ of degree $\leq 2$ over $Z(D)$. Since the dimension $[D : Z(D)] = 9$, it follows that $\theta \in Z(D)$. Since $\theta^2 = 1$ and $\theta \in Z(D)$, $\theta = 1$ or $-1$ ($Z(D)$ is a field). Since $N_D(\theta) = 1$ we have $\theta = 1$. Hence $G(k)$ does not contain any non-trivial involutions. When $G$ is a group of type $F_4$, the result follows from a theorem of Jacobson ( [17], Chap. IX, Theorem 9). Moreover, let $M$ be any field extension of $k$ of degree coprime to 3. As seen above, if $G(M)$ contains a non-trivial involution, then $G \otimes M$ cannot arise from a division algebra. By Proposition 5.2.4, $G$ cannot arise from a division algebra.

We now prove (**2**). Suppose there exists a rank-1 torus $T$ over $k$ such that $T \hookrightarrow G$ over $k$. Necessarily, $T = K^{(1)}$, the norm torus of a quadratic étale extension $K/k$ ([64], Chap.II, §IV, Example 6). But then $T$ splits over $K$, which in turn implies that $G$ becomes isotropic over $K$. Suppose $G$ is a group of type $F_4$ over $k$. Then $G = \mathbf{Aut}(A)$ for some Albert algebra $A$ over $k$. Since $G$ becomes isotropic over $K$, $A \otimes K$ is reduced (see Prop. 7.2.1). Hence $G$ does not arise from a division algebra over $k$, since no extension of degree coprime to 3 can reduce a Albert division algebra (Proposition 5.2.4). This is a contradiction. Now suppose $G$ is a group of type $A_2$ over $k$. Since $G$ becomes isotropic over $K$, by ([61],Table of Tits indices), $G \otimes K$ does not arise from a division algebra over $K$. By Proposition 5.2.4, $G$ does not arise from a division algebra over $k$, a contradiction. Moreover, let $M$ be any field extension of $k$ of degree coprime to 3. Suppose there exists a rank-1 torus $T$ over $M$ such that $T \hookrightarrow G$ over $M$. Then, as seen above, $G$ does not arise from a division algebra over $M$. Hence by Proposition 5.2.4, $G$ does not arise from a division algebra over $k$. This is a contradiction. The proof of (**3**) follows from ([61], Remark on Page 61, Table of Tits indices). $\square$

**Remark 10.2.2** **(1)** *Let $G$ be a simple, simply connected $k$-group of type $A_2$ arising from a division algebra. Let $H$ be a simple, simple connected $k$-group of type $A_1$. Then $H$ does not embed in $G$ over $k$. By ([46], Chap. II, Prop. 2.17) there is a quaternion algebra $Q$ over $k$, unique up to isomorphism, such that $H \cong \mathbf{SL}_1(Q)$ over $k$. Suppose $H$ embeds in $G$ over $k$. If $Q$ is split, then $H \cong \mathbf{SL}_2$ and $k$-rank of $G \geq 1$. Hence $G$ is $k$-isotropic and by Theorem 10.2.1, $G$ cannot arise from a division algebra. If $Q$ is a quaternion division algebra, we choose a quadratic subfield $K \subseteq Q$. Then $Q$ splits over $K$ and the $K$-rank of $H$ is one. Hence $K$-rank of $G \geq 1$ and $G$ is isotropic over $K$. By Theorem 10.2.1, $G$ cannot arise from a division algebra.*

**(2)** *For a simple group $G$ of type $A_2$, conditions* **(1)**, **(2)**, **(3)** *of Theorem 10.2.1 are all equivalent to the condition that $G$ arises from a division algebra.*

**Theorem 10.2.3** *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Then the following are equivalent.*

**(a)** *$f_3(A) = 0$ (i.e, $Oct(G)$ is split).*

**(b)** *There exists a cubic étale $k$-algebra $L$ of trivial discriminant such that $\mathbf{L}^{(1)} \hookrightarrow G$ over $k$.*

**(c)** *There exists a $k$-embedding $\mathbf{SL}_1(D) \hookrightarrow G$ over $k$, for a degree 3 central simple algebra $D$ over $k$.*

**(d)** *$A$ is a first Tits construction Albert algebra.*

Proof. Let $f_3(A) = 0$. Then, by ([19], Prop. 40.5), $A$ is a first Tits construction and $A \cong J(D, \mu)$, where $D$ is a degree 3 central simple algebra over $k$. If $D$ is split, let $L = k \times k \times k$ and if $D$ is a division algebra, let $L$ a cubic cyclic extension of $k$ such that $L \subseteq D_+$ (This is possible by Weddernburn's Theorem [19], Pg. 303, 19.2). In either case, since $\mathbf{SL}_1(D) \hookrightarrow G$ (see §6.3), $\mathbf{L}^{(1)} \hookrightarrow G$ over $k$. Hence (**a**) $\Rightarrow$ (**b**) and (**a**) $\Rightarrow$ (**c**) follows.

For the proof of (**b**) $\Rightarrow$ (**a**), let $\mathbf{L}^{(1)} \hookrightarrow G$ over $k$, where $L$ is a cubic étale $k$-algebra of trivial discriminant. Clearly $L \cong k \times k \times k$ or $L$ is a cubic cyclic field extension of $k$. If $L \cong k \times k \times k$ then $\mathbf{L}^{(1)} \cong \mathbb{G}_m \times \mathbb{G}_m$. Hence the $k$-rank of $G \geq 2$ and, by Lemma 10.1.2, $f_3(A) = 0$. Let $L$ be a cubic cyclic field extension of $k$. Observe that $\mathbf{L}^{(1)} \otimes L \cong \mathbf{E}^{(1)}$, where $E = L \otimes L$. By Proposition 2.4.2, $L \otimes L \cong L \times L \times L$ and hence $\mathbf{E}^{(1)}$ is an $L$-split torus of rank-2, embedding in $G \otimes L$. Hence the $L$-rank of $G \otimes L \geq 2$ and thus, by Lemma 10.1.2, $f_3(A \otimes L) = 0$. By Springer's theorem, $f_3(A) = 0$. We now prove (**c**) $\Rightarrow$ (**a**). Let $\mathbf{SL}_1(D) \hookrightarrow G$ over $k$, where $D$ is a degree 3 central simple algebra over $k$. If $D$ is a division algebra, choose a cubic separable extension $L$ over $k$, $L \subseteq D$. Now,

$$D \otimes_k L \cong M_3(L) \text{ and } \mathbf{SL}_1(\mathrm{D} \otimes_{\mathrm{k}} \mathrm{L}) \cong \mathbf{SL}_3 \hookrightarrow \mathrm{G} \otimes \mathrm{L}.$$

Hence $G \otimes L$ has L-rank $\geqq 2$. By Lemma 10.1.2, $Oct(A)$ splits over $L$. Since $[L : K] = 3$, by Springer's theorem, $Oct(A)$ must split over $k$ and $f_3(A) = 0$. In the case when $D$ is split, $\mathbf{SL}_1(D) \cong \mathbf{SL}_3 \hookrightarrow G$ over $k$. Hence $G$ is split over $k$ and $f_3(A) = 0$. The implication (**a**) $\Leftrightarrow$ (**d**) holds by ([19], Proposition 40.5). □

## 10.3 Embeddings of Unitary tori

It turns out that embeddings of unitary tori in groups of type $A_2$, $G_2$ and $F_4$ are intricately linked to the mod-2 invariants of these groups. We discuss this below.

**Lemma 10.3.1** *Let $L = k \times K_0$ be a cubic étale algebra over $k$, where $K_0$ is a quadratic étale extension of $k$. Let $K = k \times k$ and $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Then $T \cong R_{K_0/k}(\mathbb{G}_m)$.*

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes {}^{-})$. Note that

$$(L \otimes K, 1 \otimes {}^{-}) \cong (L \times L, \epsilon) \cong ((k \times K_0) \times (k \times K_0), \epsilon),$$

where $\epsilon : L \times L \mapsto L \times L$ is the switch involution. Hence

$$\mathbf{SU}(E, \tau) \cong \mathbf{SU}((k \times K_0) \times (k \times K_0), \epsilon).$$

For $((a, x), (b, y)) \in (k \times K_0) \times (k \times K_0)$ we have,

$$((a, x), (b, y))\epsilon((a, x), (b, y)) = (((a, x), (b, y))((b, y), (a, x)) = ((ab, xy), (ba, yx)), \text{ and}$$

$$N_{E/K}((a, x), (b, y)) = N_{(k \times K_0) \times (k \times K_0)/(k \times k)} = (a.N_{K_0/k}(x), b.N_{K_0/k}(y)).$$

Hence,

$$SU(E, \tau) \cong \{((a, x), (a^{-1}, x^{-1})) \in (k \times K_0) \times (k \times K_0) \ | a.N_{K_0/k}(x) = 1\} \cong K_0^*.$$

From this it follows that $\mathbf{SU}(E, \tau) \cong R_{K_0/k}(\mathbb{G}_m)$. □

**Lemma 10.3.2** *Let $L = k \times k \times k$ and $K$ be a quadratic étale extension of $k$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Then $T \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$.*

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes {}^{-})$. It is immediate that $(E, \tau) \cong (K \times K \times K, ({}^{-}, {}^{-}, {}^{-}))$. Hence,

$$SU(E, \tau) \cong \{(x, y, z) \in K \times K \times K | x\overline{x} = y\overline{y} = z\overline{z} = 1, xyz = 1\} \cong K^{(1)} \times K^{(1)}.$$

It follows that $\mathbf{SU}(E, \tau) \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$. □

**Theorem 10.3.3** **(a)** *Let $G$ be a $k$-group of type $G_2$ or a simply connected, simple group of type $A_2$. Let $L, K$ be étale algebras of dimension $3, 2$ resp. and $T$ be the $K$-unitary* *torus associated with the pair $(L, K)$. Suppose there exists a $k$-embedding $T \hookrightarrow G$. Then $K \subseteq Oct(G)$.*

**(b)** *If $G$ is a $k$-group of type $F_4$ or a simply connected, simple group of type $A_2$ arising from a division algebra and $T \hookrightarrow G$ over $k$, then $L$ must be a field extension.*

Proof. Let $(E, \tau)$ and $T$ be the $K$-unitary algebra and torus resp. associated with the pair $(L, K)$. By definition $T = \mathbf{SU}(E, \tau)$. For the assertion (**a**), we divide the proof into two cases.

**Case 1:** $L = k \times K_0$ for some quadratic étale extension $K_0$ of $k$.

Let $K = k \times k$. By Lemma 10.3.1, $T \cong R_{K_0/k}(\mathbb{G}_m) \hookrightarrow G$. Therefore, the $k$-rank of $G \geq 1$. Thus by Lemma 10.1.2, $Oct(G)$ is split. When $K$ is a field extension, base changing to $K$ and applying the same argument, it follows that $Oct(G) \otimes K$ is split. Hence $K \subseteq Oct(G)$ ([5], Lemma 5).

**Case 2:** L is a field extension.

Base changing to $L$, by Proposition 2.4.2, we have, $L \otimes L \cong L \times K_0$ for $K_0 = L \otimes \Delta$, where $\Delta$ is the discriminant algebra of $L$ over $k$. By case **1**, $K \otimes L \subseteq Oct(G) \otimes L$. Therefore if $K = k \times k$, $Oct(G) \otimes L$ is split and by Springer's theorem, $Oct(G)$ splits and $K \subseteq Oct(G)$. Hence we may assume that $K$ is a field. Then $K \otimes L$ is a cubic field extension of $K$ and

$$(Oct(G) \otimes L) \otimes_L (L \otimes K) \cong Oct(G) \otimes L \otimes K \cong (Oct(G) \otimes K) \otimes_K (K \otimes L)$$

is split, since $K \otimes L \subseteq Oct(G) \otimes L$. Hence $(Oct(G) \otimes K)$ is split over the cubic extension $(K \otimes L)$ of $K$. Therefore by Springer's theorem, $Oct(G) \otimes K$ is split. Hence $K \subseteq Oct(G)$ ([5], Lemma 5).

Now we prove **(b)**. Let $G$ be a $k$-group of type $F_4$ or $A_2$ as in the hypothesis and let $T \hookrightarrow G$ over $k$, where $T$ is the $K$-unitary torus associated to the pair $(L, K)$ as in the hypothesis. Assume that $L$ is not a field. Let $L = k \times K_0$ for some quadratic field extension $K_0$ of $k$. If $K = k \times k$ then, as in the proof of case **1**, $G$ is $k$-isotropic. Therefore, by Theorem 10.2.1, $G$ cannot arise from a division algebra. Let $K$ be a field extension. By an easy calculation we see that $T \otimes K_0 = \mathbf{SU}(E \otimes K_0, \tau) \cong \mathbf{M}^{(1)} \times \mathbf{M}^{(1)}$, where $M = (K \otimes K_0)$. Note that $M^{(1)} \times M^{(1)}$ contains the involution $(-1, 1)$ defined over $K_0$. Hence $G(K_0)$ contains a non-trivial involution. Therefore, by Theorem 10.2.1,

$G$ cannot arise from a division algebra. In the case when $L = k \times k \times k$, by Lemma 10.3.2, $T \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$. Again $K^{(1)} \times K^{(1)}$ contains the involution $(-1, 1)$ defined over $k$. Hence $G(k)$ contains a non-trivial involution. Therefore, by Theorem 10.2.1, $G$ cannot arise from a division algebra. Hence (**b**) follows. □

**Remark 10.3.4** **(1)** *For $k$-groups of type $G_2$,* **(b)** *fails to hold. To see this, let $L = k \times k \times k$ and $K$ be a quadratic field extension of $k$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. By Lemma 10.3.2, $T \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$. Such a torus embeds in a $k$-group of type $G_2$ arising from a division algebra (see [53], §2.1).*

**(2)** *For $k$-groups of type $F_4$,* **(a)** *fails to hold. Let $C$ be an octonion division algebra over $k$. Let $\Gamma = diag(1, -1, -1) \in GL_3(k)$. Consider the reduced Albert algebra $A := \mathcal{H}_3(C, \Gamma)$. Let $G = \mathbf{Aut}(A)$. Then $C = Oct(G)$ (see §5.2). Let $F \subseteq C$ be a quadratic subfield. By ([34], §1, Thm. 1.1), there exists an isomorphism of Jordan algebras $\mathcal{H}_3(C, \Gamma) \cong J(M_3(F), *_\Gamma, V, \mu)$, where $*_\Gamma(X) = \Gamma^{-1}\overline{X}^t\Gamma$, $V \in GL_3(F)$ with $*_\Gamma(V) = V$ and $detV = \mu\overline{\mu}$ for some $\mu \in F^*$. Let $L = k \times F$. Note that $L \subseteq M_3(F)$ as a $k$-subalgebra (via the embedding $(\gamma, x) \to diag(\gamma, x, x)$, $\gamma \in k$, $x \in F$). Since $*_\Gamma$ is a distinguished involution on $M_3(F)$ ([7], Theorem 16), by ([7], Cor. 18), it follows that $L \hookrightarrow (M_3(F), *_\Gamma)_+$ over $k$. Let $T$ be the $F$-unitary torus associated with the pair $(L, F)$. Then $T \hookrightarrow \mathbf{SU}(M_3(F), *_\Gamma) \hookrightarrow G$ over $k$ (see §6.3). By case* **(ii)** *of Theorem 10.1.3, $T \cong R_{F/k}(\mathbb{G}_m)$. Hence $R_{F/k}(\mathbb{G}_m) \hookrightarrow G$ over $k$. Now consider $K = k \times k$. By Lemma 10.3.1, $\mathbf{SU}(L \otimes K, \tau) \cong R_{F/k}(\mathbb{G}_m)$. Hence $\mathbf{SU}(L \otimes K, \tau) \hookrightarrow G$ over $k$ but $K$ does not embed in $C = Oct(G)$, since $C$ is a division algebra.*

However we have the following,

**Theorem 10.3.5** *Let $G$ be a group of type $F_4$ defined over $k$. Let $K$ be a quadratic étale $k$-algebra and $L$ be a cubic étale $k$-algebra with trivial discriminant. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose $T \hookrightarrow G$ over $k$. Then $K \subseteq Oct(G)$.*

Proof. Let $L$ be as in the hypothesis. When $K = k \times k$, we have $(L \otimes K, \tau) \cong (L \times L, \epsilon)$, where $\epsilon(x, y) = (y, x)$ for all $(x, y) \in L \times L$. Hence $T \cong \mathbf{L}^{(1)}$. By Theorem 10.2.3, $Oct(G)$ splits and hence $K \subseteq Oct(G)$. When $K$ is a field extension, base changing to $K$ we see that $Oct(G) \otimes K$ splits. Hence $K \subseteq Oct(G)$ ([5], Lemma 5). □

We now prove a factorization result for the mod-2 invariant $f_5(G)$ associated to an algebraic group $G$ of type $F_4$ defined over $k$, given an embedding of a rank-2 $K$-unitary torus in $G$. Let $L, K$ be étale algebras of dimension $3, 2$ resp. and let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Recall that with $T$, we associate the quadratic form $q_T :=< 1, -\alpha\delta >$, where $Disc(L) = k(\sqrt{\delta})$ and $K = k(\sqrt{\alpha})$.

**Theorem 10.3.6** *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $L$ be a cubic étale $k$-algebra with discriminant $\delta$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose $T \hookrightarrow G$ over $k$. Then $f_5(A) = q_T \otimes \gamma$ for some 4-fold Pfister form $\gamma$ over $k$.*

Proof. Let $G = \mathbf{Aut}(A)$ be as in the hypothesis and let $T \hookrightarrow G$ over $k$.

**Claim**: $D_M(q_T \otimes M) \subseteq D_M(f_5(A) \otimes M)$ for all field extensions $M$ of $k$.

Let $F = k(\sqrt{\alpha\delta})$. Then $N_{F/k} = q_T$. Note that, $N_{F/k} \otimes M = N_{F\otimes M/M}$ and $\mathrm{f}_5(\mathrm{A}) \otimes \mathrm{M} = \mathrm{f}_5(\mathrm{A} \otimes \mathrm{M})$. If $N_{F\otimes M/M}$ is hyperbolic over $M$, then $\alpha = \delta\ {M^*}^2$ and hence $T \otimes M$ is a distinguished torus. Therefore, by Theorem 10.1.6, $f_5(A\otimes M) = 0$ and the claim follows trivially. We may therefore assume both $N_{F\otimes M/M}$ and $f_5(A\otimes M)$ are anisotropic. Hence $K' := F \otimes M$ is a field extension of $M$. Now further base changing to $K' \cong M\otimes_M K'$ we get,

$$(T\otimes M)\otimes_M K' \cong T\otimes_M K' \hookrightarrow \mathbf{Aut}((A\otimes_k M)\otimes_M K'). \qquad (*)$$

Since $\alpha = \delta\ {K'^*}^2$, $T \otimes_M K'$ is a distinguished torus. Taking $K'$ as the base field and applying Theorem 10.1.6 to the embedding $(*)$ we get, $f_5((A \otimes_k M) \otimes_M K') = 0$. Now, since $K'$ over $M$ is a finite field extension and $f_5(A \otimes_k M) \otimes_M K'$ is split, we have, by Theorem ([20], Chap. VII, Cor. 4.4), $N_{K'/M}(K'^*) \subseteq D_M(f_5(A \otimes_k M))$. Since $N_{K'/M}(K'^*) = D_M(q_T \otimes M)$, we have $D_M(q_T \otimes M) \subseteq D_M(f_5(A) \otimes M)$ for all extensions $M$ of $k$. Hence by Theorem 1.1.9, $N_{F/k}$ is isometric to a subform of $f_5(A)$ and we have, $f_5(A) = q_T \otimes \gamma$, for some 4-fold Pfister form $\gamma$ over $k$. $\square$

**Remark 10.3.7** **1)** *Note that the converse of the above theorem fails to hold.*

*Let $C$ denote the octonion division algebra represented by the 3-fold (anisotropic) Pfister form $< 1, -x > \otimes < 1, -y > \otimes < 1, -z >$ over $k = \mathbb{C}(x, y, z, w)$. Let $F \subset C$ be a* *quadratic subfield and let $h = diag(h_1, h_2, h_3)$ denote the hermitian form on $F^{\perp} \subset C$ induced by the norm bilinear from (see [12], §5, cf. Prop. 1.3.4). Consider the Albert algebra $A := J(M_3(F), *_h, 1, \mu)$ where $*_h(X) = h^{-1}\overline{X}^t h$ and $\mu \in F$ satisfies $\mu\overline{\mu} = 1$. Let $G = \mathbf{Aut}(A)$. Then $Oct(G) = C$ ([34], §1, Theorem 1.1). By Lemma 9.1.5, $f_5(A) = n_C \otimes << -1, -1 >>$ . Since $-1$ is a square in $k$, we have $f_5(A) = 0$. Let $K = k(\sqrt{w})$ and let $L$ be any cubic cyclic field extension of $k$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Since $-w$ is not represented by $n_C$, $K \not\subset C$. Hence, by Theorem 10.3.5, $T$ cannot embed in $G$ over $k$, however $q_T$ divides $f_5(A)$.*

**2)** *Let $q_T$ be as in the hypothesis of Theorem 10.3.6. Note that $q_T$ does not divide $f_3(G)$ in general. We use the construction as in the case* **(2)** *of Remark 10.3.4. Let $C$ be an octonion division algebra. Let $\Gamma = diag(1, -1, -1) \in GL_3(k)$. Consider the reduced Albert algebra $A := \mathcal{H}_3(C, \Gamma)$. Let $G = \mathbf{Aut}(A)$. Note that $Oct(G) = C$. Let $F \subseteq C$ be a quadratic subfield and $L = k \times F$. Let $T$ be the $F$-unitary algebra associated with the pair $(L, F)$. Then, as in the case* **(2)** *of Remark 10.3.4, $T \hookrightarrow G$ over $k$. Since $Disc(L) = F$, we have $\alpha = \delta \bmod k^{*2}$. Hence the Pfister form $q_T =< 1, -\alpha\delta >\cong< 1, -1 >$ and $q_T$ does not divide $f_3(G)$, since $C$ is a division algebra.*

On exactly similar lines we can derive a necessary condition for a rank-2 unitary torus to embed in a connected simple algebraic group of type $A_2$ or $G_2$:

**Theorem 10.3.8** *Let $G$ be a simple, simply connected $k$-group of type $A_2$ or $G_2$. Let $C := Oct(G)$ and $n_C$ denote the norm form of $C$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $L$ be a cubic étale $k$-algebra with discriminant $\delta$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose there exists a $k$-embedding $T \hookrightarrow G$. Then $n_C = q_T \otimes \gamma$ for some two fold Pfister form $\gamma$ over $k$.*

Proof. Recall that $q_T =< 1, -\alpha\delta >$. By Theorems 10.1.4, 10.1.5, one sees that if $T$ is distinguished then $C$ splits. Now using same arguments as in the proof of Theorem 10.3.6, we get the desired result. □

**Remark 10.3.9** *Note that the converse of the above theorem fails to hold.*

**1)** *Let $*$ denote the unitary involution $*(X) = \overline{X}^t$ on $M_3(\mathbb{C})$ and let $G = \mathbf{SU}(M_3(\mathbb{C}), *)$. Let $C = Oct(G)$. Then $n_C =< 1, 1 > \otimes < 1, 1 > \otimes < 1, 1 >$ ( see §2.3). Hence $C$ is the unique octonion an division algebra over $\mathbb{R}$. Take $K = \mathbb{R} \times \mathbb{R}$ and $L = \mathbb{R} \times \mathbb{C}$. Let*

*$T$ be the $K$-unitary torus associated with the pair $(L, K)$. Since $C$ is a division algebra, $K = \mathbb{R} \times \mathbb{R} \not\subset C$. Hence, by Theorem 10.3.3, $T$ does not embed in $G$ over $\mathbb{R}$ but the quadratic form $q_T =< 1, 1 >$ associated with $T$, is a factor of $n_C$.*

**2)** *Let $G$ be a group of group of type $G_2$ over $k$ arising from an octonion division algebra $C$. Let $K_0 = k(\sqrt{\delta}) \subset C$ be a fixed quadratic subfield. Note that $< 1, -\delta >$ is a factor of $n_C$. Take $K = k \times k$ and $L = k \times K_0$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Since $C$ is a division algebra, we have $K \not\subset C$. Hence, by Theorem 10.3.3, $T$ does not embed in $G$ over $k$, but the quadratic form $q_T =< 1, -\delta >$ associated with $T$, is a factor of $n_C$*

Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Let $L, K$ be étale algebras of dimension $3, 2$ resp. and $T$ be the $K$-unitary torus associated with the pair $(L, K)$. By case **(2)** of Remark 10.3.4, if there is a $k$-embedding $T \hookrightarrow G$, then $K$ need not embed in $Oct(G)$, i.e. if $K = k(\sqrt{\alpha})$ then $< 1, -\alpha >$ is not a factor of $f_3(G)$ in general. However,

**Theorem 10.3.10** *Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. Let $K = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $L$ be a cubic étale $k$-algebra. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Suppose there exists a $k$-embedding $T \hookrightarrow G$. Then $f_5(A) =< 1, -\alpha > \otimes \gamma$ for some 4-fold Pfister form $\gamma$ over $k$.*

Proof. We first assume that $K \cong k \times k$. If $L$ is not a field, then $L = k \times K_0$, where $K_0$ is a quadratic étale $k$-algebra. By Lemma 10.3.1, $T \cong R_{K_0/k}(\mathbb{G}_m) \hookrightarrow G$. Therefore $T$ is $k$-isotropic and $k$-rank of $G \geq 1$. Hence by Lemma 10.1.2, $f_5(A) = 0$. Let $L$ be a field extension. Base changing to $L$ we have, $L \otimes L \cong L \times K_0$, where $K_0 = L \otimes \Delta$ and $\Delta$ is the discriminant algebra of $L$ over $k$. Hence by the above argument, $T \otimes L$ is $L$-isotropic and $f_5(A \otimes L) = 0$. By Springer's theorem $f_5(A) = 0$. Therefore, if $\alpha \in k^{*2}$ (i.e, $K = k \times k$) and $T \hookrightarrow G$ over $k$, then $f_5(A) = 0$. Using the same arguments as in Theorem 10.3.6, with Pfister form $< 1, -\alpha >$ instead of $q_T$, we get the desired result.

□

**Theorem 10.3.11** *Let $G$ be a simple, simply connected algebraic group defined over $k$. Let $L$ be a cubic étale $k$-algebra with discriminant $K_0$. Suppose there exists an $k$-embedding $\mathbf{L}^{(1)} \hookrightarrow G$. We then have:*

**(a)** *if $G$ is of type $G_2$ or $A_2$ then $Oct(G)$ splits.*

**(b)** *if $G$ is of type $F_4$ then $f_5(G) = 0$ and $K_0 \subset Oct(G)$.*

Proof. Let $L$ be as in the hypothesis and $K = k \times k$. Let $(E, \tau)$ and $T$ be the $K$-unitary algebra and torus resp. associated with the pair $(L, K)$. We have $(E, \tau) \cong (L \times L, \epsilon)$, where $\epsilon(x, y) = (y, x)$ for all $(x, y) \in L \times L$. Hence $T \cong \mathbf{L}^{(1)}$. Therefore $T \hookrightarrow G$ over $k$.

**(a)** Let $G$ be a simply connected, simple group of type $G_2$ or $A_2$. Then, by Theorem 10.3.3, $K \subseteq Oct(G)$. Since $K$ is split, $Oct(G)$ splits.

**(b)** Let $G$ be a $k$-group of type $F_4$. Then by Theorem 10.3.10, $f_5(A) =< 1, -\alpha > \otimes\gamma$ for some 4-fold Pfister form $\gamma$ over $k$ where $K = k(\sqrt{\alpha})$. Since $K = k \times k$, we have $\alpha \in k^{*2}$ and hence $f_5(A) = 0$. If $K_0$ is split then $L$ has trivial discriminant. Hence, by Theorem 10.2.3, $f_3(G) = 0$ and $Oct(G)$ splits. Therefore $K_0 \subset Oct(G)$. If $K_0$ is a field extension, base changing to $K_0$ we see that $L \otimes K_0$ is a cubic étale algebra over $K_0$ of trivial discriminant. Applying Theorem 10.2.3 to the $K_0$-embedding $\mathbf{L}^{(1)} \otimes K_0 \hookrightarrow G \otimes K_0$, we get $f_3(G \otimes K_0) = f_3(G) \otimes K_0 = 0$. Hence $Oct(G)$ splits over $K_0$ and thus $K_0 \subseteq Oct(G)$.

$\square$

**Remark 10.3.12** *Let $G$ be a group of type $F_4$ over $k$. Let $L$ be a cubic étale $k$-algebra. Suppose there exists an $k$-embedding $\mathbf{L}^{(1)} \hookrightarrow G$. Then $f_3(G)$ may not be zero. We use the construction as in the case* **(2)** *of Remark 10.3.4. Let $C$ be an octonion division algebra. Let $\Gamma = diag(1, -1, -1) \in GL_3(k)$. Consider the reduced Albert algebra $A := \mathcal{H}_3(C, \Gamma)$. Let $G = \mathbf{Aut}(A)$. Note that $Oct(G) = C$. Let $F \subseteq C$ be a quadratic subfield and $L = k \times F$. Let $T$ be the $F$-unitary torus associated with the pair $(L, F)$. As in the case* **(2)** *of Remark 10.3.4, $T \hookrightarrow G$. Note that $T \cong \mathbf{SU}((k \times F) \otimes F,\ 1 \otimes ^-) \cong R_{F/k}(\mathbb{G}_m) \cong \mathbf{L}^{(1)}$ (Theorem 10.1.3, case* **(ii)***). Hence $\mathbf{L}^{(1)} \hookrightarrow G$ but $f_3(G) \neq 0$.*

# Chapter 11

# Generation results

This chapter reports the work done in [10], [9] on generation of $k$-groups of type $F_4$ by $k$-subgroups of type $A_2$ and $D_4$ and $k$-groups of type $G_2$ by $k$-subgroups of type $A_1$ and $A_2$. In chapter 9 we studied the factorization of the mod-2 invariants of the groups of type $F_4$ and $G_2$ in terms of the mod-2 invariants of the embedded group of type $A_1$ and $A_2$. Let $G$ be an simple group of type $F_4$ (resp. $G_2$) defined over a prefect (infinite) field $k$. We prove, as an easy consequence of simplicity of $G$ that it is generated by all $k$-subgroups of type $A_2$ and similarly $A_1$. In this chapter we answer the following question: What is the number of $k$-subgroups of type $A_2$ and similarly $A_1$ required to generate $G$? We prove that if $k$ is a perfect (infinite) field and $G$ is an algebraic group of type $F_4$ defined over $k$, arising from an Albert division algebra, then $G$ is generated by two $k$-subgroups of type $D_4$ and three $k$-subgroups of type $A_2$. Similarly, if $G$ is an algebraic group of type $G_2$ defined over $k$, arising from an octonion division algebra, then $G$ is generated by two $k$-subgroups of type $A_2$ and three $k$-subgroups of type $A_1$.

In chapter 10 we discussed conditions necessary for a rank-2 unitary $k$-torus to embed in simple groups of type $A_2$, $G_2$ and $F_4$ in terms of the mod-2 Galois cohomological invariants attached with these groups. One knows that any algebraic group $G$ of the above types is generated by its maximal tori (hence by its rank-2 $k$-tori). In this section we calculate the number of rank-2 $k$-tori required (in fact exhibit such tori explicitly) for the generation of groups of type $A_2$, $G_2$ and $F_4$ arising from division algebras and subgroups of type $D_4$ of $\mathbf{Aut}(A)$ for $A$ an Albert division algebra over perfect fields.
By $< H_1, H_2 >$ we will denote the algebraic subgroup of $G$ generated by $H_i$, $i = 1, 2$. We will often use the Borel-De Siebenthal algorithm (see §3.5). Let $X$ and $Y$ be types

of root systems. If $X$ is a subsystem of $Y$, we write $X \subseteq Y$. We fix a perfect field $k$ of characteristic different from 2 and 3. We first record a lemma which will be used through the section.

**Lemma 11.0.1** *Let $G$ be a $k$-anisotropic, connected, reductive algebraic group over a perfect (infinite) field $k$. Let $H$ be a connected subgroup of $G$. Then $H$ is a reductive, $k$-anisotropic subgroup.*

PROOF. Since $G$ is a $k$-anisotropic, by Prop. 3.2.2, $G(k)$ has no non-trivial unipotents. Hence $H(k)$ has no non-trivial unipotents and $R_u(H)(k) = \{1\}$. Since $k$ is perfect, by density of $k$-points it follows that, $R_u(H) = \{1\}$. □

## 11.1 Generation of $A_2$ by rank-2 tori

Let $G$ be a simple, simply connected group of type $A_2$ over $k$. We show that the minimum number of maximal $k$-tori required to generate $G$ is 2.

**Theorem 11.1.1** *Let $k$ be a perfect infinite field and $F$ be a quadratic étale $k$-algebra. Let $(B, \sigma)$ be a degree 3 central division algebra over $F$ with an involution $\sigma$ of the second kind. Let $G = \mathbf{SU}(B, \sigma)$. Let $E_1, E_2 \subset B$ be $F$-unitary subalgebras of $B$ such that $\sigma$ restricts to $E_1$ and $E_2$. Let $\sigma_i = \sigma|_{E_i}$. Assume that $\mathbf{SU}(E_1, \sigma_1) \neq \mathbf{SU}(E_2, \sigma_2)$. Then*

$$G = < \mathbf{SU}(E_1, \sigma_1), \mathbf{SU}(E_2, \sigma_2) > .$$

PROOF. Let $H = < \mathbf{SU}(E_1, \sigma_1), \mathbf{SU}(E_2, \sigma_2) >$. Then $H$ is a connected $k$-subgroup of $G$. Since $B$ is a division algebra, $G$ is a $k$-anisotropic group (see Theorem 10.2.1). Notice that since $\mathbf{SU}(E_i, \sigma_i)$, $i = 1, 2$ are maximal tori of $G$, $H$ is a non-toral subgroup. By Lemma 11.0.1, $H$ is a connected, reductive, $k$-anisotropic, non-toral subgroup of $G$. Since $G$ has absolute rank-2, $[H, H]$ is a semisimple group of absolute rank 1 or 2. Hence $[H, H]$ must be of type $A_2$, $A_1$, $A_1 \times A_1$, $G_2$ or $B_2 = C_2$. By the Borel-De Siebenthal algorithm, $A_1 \times A_1, B_2 \nsubseteq A_2$ (see Example 3.5.7). Notice that $G_2 \nsubseteq A_2$ (since Lie algebra of $G_2$ has dimension 14 whereas the dimension of Lie algebra of $A_2$ is 8). If $[H, H]$ is of type $A_1$, then $G$ has a $k$-torus $S$ of absolute rank 1, $S \subseteq [H, H]$. Necessarily, $S = \mathbf{M}^{(1)}$, the norm torus of a quadratic extension $M/k$ ([64], Chap.II, §IV, Example

6). But then, $S$ splits over $M$ and hence $G$ becomes isotropic over $M$. By Prop. 5.2.4, $B$ remains a division algebra over $M$. Hence by Theorem 10.2.1, $G$ remains anisotropic over $M$, a contradiction. Therefore $[H, H]$ cannot have type $A_1$. Hence $[H, H]$ must be of type $A_2$. Now $H \subseteq G = [G, G] = [H, H] \subseteq H$. Therefore $H = G$. $\square$

## 11.2 Generation of $F_4$ by $A_1$, $A_2$

Let $G$ be a group of type $F_4$ defined over $k$. We now discuss the question of generation of $G$ by $k$-subgroups of type $A_2$ (resp. $A_1$). Theorem 11.2.1 below was communicated to us by Maneesh Thakur; Theorem 11.2.3 is a refinement of ([35], Prop. 6.1).

**Theorem 11.2.1** *Let $A$ be an Albert division algebra over a perfect (infinite) field $k$. Let $L \subseteq A$ be a cubic subfield. Let $G = \mathbf{Aut}(A)$ and $H = \mathbf{Aut}(A/L)$. Then there exist 9-dimensional subalgebras $S_1$ and $S_2$ of $A$ such that $S_1 \cap S_2 = L$ and $H = H_1.H_2 \cdots .H_r$ as varieties over $k$, where $H_i = \mathbf{Aut}(A/S_1)$ or $H_i = \mathbf{Aut}(A/S_2)$, $1 \leq i \leq r$, for some $r$. Note that the $H_i$'s here are of type $A_2$.*

Proof. Let $0 \neq x_1 \in L^{\perp}$ and $S_1$ be the subalgebra generated by $L$ and $x_1$. Let $0 \neq x_2 \in S_1^{\perp}$ and $S_2$ be the subalgebra generated by $L$ and $x_2$. Then $S_1 \cap S_2 = L$ and $Dim(S_i) = 9$, since the subalgebras of an Albert division algebra can only have dimension $1, 3, 9, 27$ and any two elements generate a subalgebra of dimension at most 9 ([40], §2.10, Pg. 9). Moreover, since $S_1 \neq S_2$, the subalgebra generated by $S_1$ and $S_2$ equals $A$, since this subalgebra has dimension at least 10. Let $G_i = \mathbf{Aut}(A/S_i)$, $i = 1, 2$. Then $G_1 \cap G_2 = 1$, since any element in the intersection must restrict to identity on $S_1$ as well as on $S_2$ and therefore must be identity on $A$. Let $H$ be as in the hypothesis. Now $L \subset S_i$, hence $G_i \subset H$, $i = 1, 2$ and the subgroup $H'$ of $H$ generated by $G_1$ and $G_2$ is a closed connected subgroup defined over $k$. Hence by Lemma 11.0.1, $H'$ is a connected reductive, $k$-anisotropic, non-toral subgroup and contains $G_i$, $i = 1, 2$, properly. If $H'$ is of type $A_2$, consider $[G_1, G_1] \subseteq [H', H']$. Both these groups are connected simple of type $A_2$. Since $G_1$ is connected simple of type $A_2$ (see Thm. 6.2.4), we get $G_1 = [G_1, G_1] = [H', H']$. Arguing symmetrically we get, $G_2 = [H', H']$. Hence $G_1 = G_2$. But $G_1 \cap G_2 = 1$ doesn't allow this possibility. Therefore, by Theorem 11.2.3 stated below, $H'$ must be of type $A_2 \times A_2$ or $D_4$. One sees that $A_2 \times A_2 \nsubseteq D_4$ by applying the Borel-De Siebenthal algorithm ( [2]) (see Example 3.5.9). Hence $H'$ must be of type $D_4$ and therefore $H' = H$. That $H = H_1 \cdots H_r$ with $H_i = \mathbf{Aut}(A/S_1)$ or $\mathbf{Aut}(A/S_2)$, follows from a standard theorem in algebraic group theory ([11], Chap. II, Prop. 7.5). $\square$

From the proof of Theorem 11.2.1, one easily sees the following:

**Theorem 11.2.2** *Let $A$ be an Albert division algebra over a perfect (infinite) field $k$ and $G = \mathbf{Aut}(A)$. Let $L \subseteq A$ be a cubic subfield and $H = \mathbf{Aut}(A/L)$. Then $H$ is generated by two $k$-subgroups of type $A_2$.*

**Theorem 11.2.3** *Let $A$ be an Albert division algebra over a field $k$. Let $H \subset G = \mathbf{Aut}(A)$ be a proper connected reductive non-toral subgroup defined over $k$. Then $[H, H]$ is of type $A_2$, $A_2 \times A_2$ or $D_4$.*

Proof. Let $H$ be as in the hypothesis. In ([35], Prop. 6.1), it was shown that $[H, H]$ is of type $A_1$, $A_2 \times A_2$ or $D_4$. We will rule out type $A_1$. If $[H, H]$ is of type $A_1$, then $G$ has a $k$-torus $S$ of absolute rank 1, $S \subseteq [H, H]$. Necessarily, $S = K^{(1)}$, the norm torus of a quadratic extension $K/k$ ([64], Chap.II, §IV, Example 6). But then $S$ splits over $K$, which in turn implies that $G$ becomes isotropic over $K$. In particular, $A \otimes K$ is reduced (see Prop. 7.2.1). Since $A$ is an Albert division algebra,by (Proposition 5.2.4) no extension of degree $2^l$ can reduce it. Therefore $[H, H]$ cannot have type $A_1$. $\square$

**Lemma 11.2.4** *Let $A$ be an Albert division algebra over a field $k$. Let Let $H$ be a subgroup of $G$ of type $D_4$ and $H_0 \subseteq H$ be a non-toral reductive $k$-subgroup. Then $[H_0, H_0]$ is of type $A_2$ or $D_4$.*

Proof. By Theorem 11.2.3, $[H_0, H_0]$ is of type $A_2$, $A_2 \times A_2$ or $D_4$. By the Borel-De Siebenthal algorithm, $A_2 \times A_2 \nsubseteq D_4$ (see Example 3.5.9) and hence $[H_0, H_0]$ must be of type $A_2$ or $D_4$. $\square$

**Lemma 11.2.5** *Let $G$ be a simple algebraic group over a perfect (infinte) field $k$ and $X$ be a fixed Cartan-Killing type. Suppose $G$ contains a $k$-subgroup of type-$X$. Then $G$ is generated by all $k$-subgroups of type-$X$. Moreover if $G(k)$ is simple then $G(k)$ is generated by the groups of $k$-points of type-$X$ subgroups.*

PROOF. Let $H'$ be the algebraic subgroup of $G$ generated by the $k$-subgroups of type-$X$. Then $H'$ is a non-trivial closed connected subgroup defined over $k$. Also note that $gH'g^{-1} = H'$ for all $g \in G(k)$. Hence $G(k) \subseteq N_G(H')$ where $N_G(H')$ denotes the normalizer of $H'$ in $G$. Therefore by density of $k$ points, $G = N_G(H')$. Hence $H'$ is a normal closed connected subgroup of $G$. Since $G$ is simple, $G = H'$. Now suppose that $G(k)$ is simple. Let $H''$ be the subgroup of $G(k)$, generated by the groups of $k$-points of type-$X$ $k$-subgroups. Then $H''$ is non-trivial closed connected normal in $G(k)$. Hence $H'' = G(k)$. □

As a immediate consequence of Lemma 11.2.5, we have the following:

**Theorem 11.2.6** *Let $A$ be an Albert algebra over a perfect (infinte) field $k$. Then $G = \mathbf{Aut}(A)$ is generated by subgroups of type $A_2$, defined over $k$. Similarly, $G$ is generated by subgroups of type $A_1$, defined over $k$.*

Let $A$ be an Albert algebra over $k$ and $G = \mathbf{Aut}(A)$. In view of Theorem 11.2.6, it is of interest to find the number subgroups of type $A_2$, defined over $k$ needed to generate $G$. We discuss this below:

**Theorem 11.2.7** *Let $A$ be an Albert division algebra over a perfect (infinite) field $k$ and $G = \mathbf{Aut}(A)$. Let $H_i := \mathbf{Aut}(A/L_i) \subseteq G$, $i = 1, 2$, where $L_1 \neq L_2$ are cubic subfields of $A$. Then $G$ is generated by $H_i$, $i = 1, 2$.*

PROOF. Let $H =< H_1, H_2 >$. By Lemma 11.0.1, $H$ is a connected, reductive, $k$-anisotropic, non-toral subgroup of $G$. By Theorem 11.2.3, $[H, H]$ is of type $A_2$, $A_2 \times A_2$, $D_4$ or $F_4$. Since $D_4 \nsubseteq A_2, A_2 \times A_2$, $[H, H]$ is of type $D_4$ or $F_4$. If $[H, H]$ is of type $D_4$, then $H_i = [H_i, H_i] \subseteq [H, H]$, $i = 1, 2$ and $H_i$, $i = 1, 2$, is of type $D_4$, hence $H_i = [H, H]$, $i = 1, 2$, a contradiction since $H_1 \neq H_2$ . Therefore $[H, H]$ is of type $F_4$. Hence $H = G$. □

From Theorem 11.2.7 and Theorem 11.2.1, we can immediately deduce that when $A$ is an Albert division algebra over a perfect (infinite) field $k$, then $G = \mathbf{Aut}(A)$ is generated by four $k$-subgroups of type $A_2$. However, we can do better:

**Theorem 11.2.8** *Let $A$ be an Albert division algebra over $k$ and $G = \mathbf{Aut}(A)$. Then $G$ is generated by three $k$-subgroups of type $A_2$.*

PROOF. By Theorem 11.2.7, there exists 3-dimensional subalgebras $L_1, L_2$ of $A$ such that $G$ is generated by $H_i$, $i = 1, 2$ where $H_i := \mathbf{Aut}(A/L_i)$. Choose a $k$-subgroup $S \subseteq H_2$ of type $A_2$ such that $S \nsubseteq H_1$. Such a choice is possible since by Theorem 11.2.2 $H_i$, $i = 1, 2$ are generated by $k$-subgroups of type $A_2$. Let $H' :=< H_1, S >$. By Lemma 11.0.1, $H'$ is a connected, reductive, $k$-anisotropic, non-toral subgroup of $G$. By Theorem 11.2.3, the possible types of $[H', H']$ are $A_2$, $A_2 \times A_2$, $D_4$ or $F_4$. Now $H_1 = [H_1, H_1] \subseteq [H', H']$. Note that $H'$ contains $H_1$ properly. Since $D_4 \nsubseteq A_2$ and $D_4 \nsubseteq A_2 \times A_2$ (since Lie algebra of $D_4$ has dimension 28 whereas the dimension of Lie algebra of $A_2 \times A_2$ is 16), $[H', H']$ cannot be of type $A_2$ or $A_2 \times A_2$. Suppose $[H', H']$ is of type $D_4$. Then $H_1 = [H', H']$. Now $H' = [H', H'].Z(H')^o = H_1.Z(H')^o$. Since the rank of maximal tori of $H'$ and $H_1$ is four we have, $Z(H')^o = \{1\}$. Hence $H' = H_1$, a contradiction. Therefore $[H', H']$ is of type $F_4$ and $H' = G$. By Theorem 11.2.2, $H_1$ is generated by two $k$-subgroups of type $A_2$. Hence $G$ is generated by three $k$-subgroups of type $A_2$. □

## 11.3 Generation of $F_4$ by rank-2 tori

Let $G$ be a group of type $F_4$ defined over $k$. We now calculate the number of rank-2 $k$-tori required to generate $G$. In Theorem 11.2.6 we proved that a group of type $F_4$ is generated by its $k$-subgroups of type $A_2$. The results below are continuation of that. We first prove that a group of type $F_4$ is also generated by two $k$-subgroups of type $D_4$. Using this we deduce that a group of type $F_4$ is generated by four rank-2 $k$-tori.

**Theorem 11.3.1** *Let $A$ be an Albert division algebra over a perfect (infinite) field $k$ and $G = \mathbf{Aut}(A)$. Let $H = \mathbf{Aut}(A/L)$ where $L$ is a 3-dimensional subalgebra of $A$. Then $H$ is generated by three rank-2 tori over $k$.*

Proof. By Theorem 11.2.1, $H =< H_1, H_2 >$, where $H_i = \mathbf{Aut}(A/S_i)$ where $S_i$ are 9-dimensional subalgebras of $A$ with $S_1 \cap S_2 = L$. Note that $H_1 \cap H_2 = \{1\}$. By Theorem 6.2.4, $H_i$, $i = 1, 2$, is simple, simply connected subgroup of type $A_2$. Also $H_i$, $i = 1, 2$, arise from division algebras.

**Claim:** We can choose a maximal torus $S \subseteq H_1$ such that $S \nsubseteq \mathbf{Aut}(A, S_2)$.

If not, then $H_1 \subseteq \mathbf{Aut}(A, S_2)$ (since $H_1$ is generated by its maximal $k$-tori). Note that $H_2 = \mathbf{Aut}(A/S_2) \subseteq \mathbf{Aut}(A, S_2)$. Hence $H \subseteq \mathbf{Aut}(A, S_2)$, a contradiction, since $D_4 \nsubseteq A_2 \times A_2$. Thus we can choose a maximal $k$-torus $S \subseteq H_1$ such that $S \nsubseteq \mathbf{Aut}(A, S_2)$. Let $H_0 :=< S, H_2 >\subseteq H$. Then, by Theorem 11.1.1, $H_0$ is generated by three rank 2

$k$-tori. We will prove that $H_0 = H$. By Lemma 11.0.1, $H_0$ is a connected reductive, $k$-anisotropic, non-toral subgroup of $G$ containing $S$ and $H_2$ properly. By Lemma 11.2.4, $[H_0, H_0]$ is of type $A_2$ or $D_4$. If $[H_0, H_0]$ is of type $A_2$, then $H_2 = [H_2, H_2] = [H_0, H_0]$ (since $H_2$ is of type $A_2$). This shows that $H_2$ is a normal subgroup of $H_0$. Also $S \cap H_2 = \{1\}$, hence $H_0 =< S, H_2 >= S.H_2$. Now

$$H_0 = [H_0, H_0].Z(H_0)^o = H_2.Z(H_0)^o.$$

Consider the projection maps $\tau$ and $\tau'$ given by,

$$Z(H_0)^o \subseteq H_0 = S.H_2 \xrightarrow{\tau} S, H_0 = S.H_2 \xrightarrow{\tau'} H_2.$$

Since $H_0 \neq H_2$, we have $Z(H_0)^o \neq \{1\}$. Since $A$ is a division algebra, $\mathbf{Aut}(A)$ does not have rank-1 $k$-tori (Theorem 10.2.1). Hence $Z(H_0)^o$ is a rank-2 $k$-torus. Since $\tau(Z(H_0)^o)$ is connected, $\tau(Z(H_0)^o) = S$ or $\{1\}$. If $\tau(Z(H_0)^o) = \{1\}$, then $Z(H_0)^o \subseteq H_2$, hence $H_0 = H_2$, a contradiction, since $S \cap H_2 = \{1\}$. Therefore $\tau(Z(H_0)^o) = S$.
Let $H' = \tau'(Z(H_0)^o)$. Note that $1 \in H'$. If $H' = \{1\}$ then $Z(H_0)^o = S$. Since $Z(H_0)^o$ centralizes $H_2$, we see that $Z(H_0)^o$ stabilizes $A^{H_2}$. Therefore $S \subseteq \mathbf{Aut}(A, S_2)$, a contradiction. Hence $H' \neq \{1\}$.

**Claim:** $H'$ is a rank-2 $k$-torus of $H_2$.

We have, for $s_i h_i \in Z(H_0)^o$, $(s_1h_1)(s_2h_2) = s_2(s_1h_1)h_2 = (s_1s_2)(h_1h_2)$. Hence $\tau'$ is a homomorphism. It follows that $H' = \tau'(Z(H_0)^o)$ is a $k$-torus. Now since $H_2$ does not have any rank-1 $k$-tori (Theorem 10.2.1) and $H' \neq \{1\}$, $H'$ is a rank-2 $k$-torus of $H_2$.

**Claim:** $S$ centralizes $H'$.

Let $s \in S$ and $h \in H'$. Since $h \in H'$, there exists $s_0 \in S$ such that $s_0 h \in Z(H_0)^o$. Since $s_0 h \in Z(H_0)^o$, we have,

$$shs^{-1} = ss_0^{-1}s_0hs^{-1} = s_0hss_0^{-1}s^{-1} = s_0hs_0^{-1} = h.$$

Hence $S$ centralizes $H'$ and therefore $S$ stabilizes $A^{H'}$. Since $A^{H'} = S_2$, we have $S \subseteq \mathbf{Aut}(A, S_2)$, a contradiction. Hence $[H_0, H_0]$ cannot be of type $A_2$. Therefore $[H_0, H_0]$ is of type $D_4$. Now $H_0 \subseteq H = [H, H] = [H_0, H_0] \subseteq H_0$. Therefore $H = H_0$ and $H$ is generated by three rank-2 tori over $k$. $\square$

**Theorem 11.3.2** *Let $A$ be an Albert division algebra over a perfect (infinite) field $k$. Then $G = \mathbf{Aut}(A)$ is generated by four rank-2 tori over $k$.*

Proof. By Theorem 11.2.7, $G =< H_1, H_2 >$, $H_i = \mathbf{Aut}(A/L_i)$ where $L_i$, $i = 1, 2$, are three dimensional subalgebras. Choose a rank-2 $k$-torus $T \subseteq H_1$ such that $T \nsubseteq H_2$ (otherwise $H_1 = H_2$ since $H_i$'s are generated by their rank-2 $k$-tori). Let $H =< T, H_2 >$. By Lemma 11.0.1, $H$ is a connected, reductive, $k$-anisotropic, non-toral subgroup of $G$. By Theorem 11.2.3, the possible types of $[H, H]$ are $A_2$, $A_2 \times A_2$, $D_4$ or $F_4$. Now $H_2 = [H_2, H_2] \subseteq [H, H]$. Since $H$ contains $H_2$ properly, $[H, H]$ cannot be of type $A_2$ or $A_2 \times A_2$. Suppose $[H, H]$ is of type $D_4$. Then $H_2 = [H, H]$. Now $H = [H, H].Z(H)^o = H_2.Z(H)^o$. Since the rank of maximal tori of $H$ and $H_2$ is four we have, $Z(H)^o = \{1\}$. Hence $H = H_2$, a contradiction. Therefore $[H, H]$ is of type $F_4$ and $H = G$. □

## 11.4 Generation of $G_2$ by $A_1$, $A_2$

Let $G$ be a group of type $G_2$ defined over $k$. We now discuss the question of generation of $G$ by $k$-subgroups of type $A_1$ and $A_2$ analogous to those in section 11.2. Be begin with,

**Proposition 11.4.1** *Let $C$ be an octonion division algebra over $k$. Let $G = \mathbf{Aut}(C)$. Let $H$ be a proper connected reductive non-toral subgroup of $G$ defined over $k$. Then $[H, H]$ is of type $A_1$, $A_1 \times A_1$ or $A_2$.*

Proof. The algebraic group $G = \mathbf{Aut}(C)$ is a connected simple algebraic group of type $G_2$, in particular, $G$ has absolute rank 2. Let $H \subseteq G$ be as in the hypothesis. Then, since $H$ is not a torus, $[H, H]$ is a semisimple subgroup having absolute rank 1 or 2. Therefore the possible types for $[H, H]$ are $A_1$, $A_1 \times A_1 = D_2$, $B_2 = C_2$ or $A_2$. We therefore have to rule out $B_2 = C_2$. This follows from an application of the Borel-De Siebenthal algorithm ([2]), which shows that $B_2 \nsubseteq G_2$ (see Example 3.5.8). □

**Theorem 11.4.2** *Let $k$ be a perfect (infinite) field. Let $G = \mathbf{Aut}(C)$, $C$ an octonion division algebra over $k$. Let $K \subseteq C$ be a quadratic subfield. Then there exists quaternion* *subalgebras $Q_1$ and $Q_2$ of $C$, $Q_1 \cap Q_2 = K$ and $\mathbf{Aut}(C/K) = H_1H_2...H_r$ as varieties over $k$, where $H_i = \mathbf{Aut}(C/Q_1)$ or $\mathbf{Aut}(C/Q_2)$ for $i = 1, 2$. Note that the $H_i$'s here are of type $A_1$.*

Proof. Since $C$ is a division algebra, its norm form is $k$-anisotropic. Choose a non-zero element $x_1 \in K^\perp$ and let $Q_1$ be the subalgebra of $C$ generated by $K$ and $x_1$. Then $Q_1$ is a quaternion subalgebra. Again choose a non-zero element $x_2 \in Q_1^\perp$ and let $Q_2$ be the subalgebra generated by $K$ and $x_2$. Then $Q_2$ is a quaternion subalgebra and $Q_1 \cap Q_2 = K$. Moreover, since $Q_1 \neq Q_2$, the subalgebra generated by $Q_1$ and $Q_2$ equals $C$. Let $G_i = \mathbf{Aut}(C/Q_i)$, $i = 1, 2$. Then $G_1 \cap G_2 = 1$, since any element in the intersection must restrict to identity on $Q_1$ as well as on $Q_2$ and therefore must be identity on $C$. Now $K \subset Q_i$, $i = 1, 2$, hence $G_i \subset \mathbf{Aut}(C/K)$, $i = 1, 2$ and the subgroup $H$ of $\mathbf{Aut}(C/K)$ generated by $G_1$ and $G_2$ is a closed connected subgroup defined over $k$. Since $\mathbf{Aut}(C/K)$ is anisotropic, by Lemma 11.0.1 $H$ itself is a $k$-anisotropic, reductive, non-toral subgroup. Hence $H$ is connected, reductive, non-toral and contains $G_i$ properly. By Proposition 11.4.1, $[H, H]$ must be of type $A_1$, $A_1 \times A_1$ or $A_2$. If $[H, H]$ is of type $A_1$, then consider $[G_1, G_1] \subset [H, H]$. Both these groups are connected simple of type $A_1$. Since $G_1$ is connected simple of type $A_1$ (see Thm. 6.2.5) , we get $G_1 = [G_1, G_1] = [H, H]$. Arguing symmetrically we get, $G_2 = [H, H]$. Hence $G_1 = G_2$. But $G_1 \cap G_2 = 1$ doesn't allow this possibility. Hence $[H, H]$ must be of type $A_1 \times A_1$ or $A_2$. But $[H, H] \subset \mathbf{Aut}(C/K)$ and $\mathbf{Aut}(C/K)$ is semisimple of type $A_2$. By an application of the Borel-De Siebenthal algorithm ([2]), we see that $A_1 \times A_1 \not\subseteq A_2$ (see Example 3.5.7). Hence $[H, H]$ must be of type $A_2$. Now

$$H \subseteq \mathbf{Aut}(C/K) = [\mathbf{Aut}(C/K), \mathbf{Aut}(C/K)] = [H, H] \subseteq H.$$

Hence $H = \mathbf{Aut}(C/K)$. The last assertion is a standard result in algebraic groups. $\square$

From the proof of Theorem 11.4.2, one easily sees the following:

**Theorem 11.4.3** *Let $C$ be an octonion division algebra over a perfect (infinite) field $k$ and $G = \mathbf{Aut}(C)$. Let $K \subseteq C$ be a quadratic subfield and $H = \mathbf{Aut}(C/K)$ be a $k$-subgroup of type $A_2$. Then $H$ is generated by two $k$-subgroups of type $A_1$.*

As an immediate consequence of Lemma 11.2.5 we have the following,

**Theorem 11.4.4** *Let $C$ be an octonion algebra over $k$, where $k$ is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by $k$-subgroups of type $A_1$. Similarly $G$ is also generated by $k$-subgroups of type $A_2$.*

In view of Theorem 11.4.4 it is of interest to find the number of $k$-subgroups of type $A_1$ (or $A_2$) required to generate $G$. We first prove that a group of type $G_2$ is generated by two $k$-subgroups of type $A_2$.

**Theorem 11.4.5** *Let $C$ be an octonion division algebra over $k$, where $k$ is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by two $k$-subgroups of type $A_2$.*

Proof. Choose quadratic subfields $K_1, K_2 \subset C$ such that $K_1 \cap K_2 = k$. Let $H_i = \mathbf{Aut}(C/K_i)$, $i = 1, 2$. By Theorem 6.2.5, $H_i$, $i = 1, 2$, are simple, simply connected subgroups of type $A_2$. Let $H$ denote the closed subgroup of $G$ generated by $H_i$, $i = 1, 2$. By Lemma 11.0.1, $H$ is a connected, reductive, $k$-anisotropic, non-toral subgroup of $G$ containing $H_i$, $i = 1, 2$ properly. By Prop. 11.4.1, $[H, H]$ is of type $A_1$, $A_1 \times A_1$, $A_2$ or $G_2$. Now $H_1 = [H_1, H_1] \subseteq [H, H]$. Since $H_1$ is of type $A_2$, by the Borel-De Siebenthal algorithm, $[H, H]$ cannot be of type $A_1$ or $A_1 \times A_1$ (see Example 3.5.7). Therefore $[H, H]$ must be of type $A_2$ or $G_2$. If $[H, H]$ is of type $A_2$ then $[H, H] = H_i$, $i = 1, 2$. Hence $\mathbf{Aut}(C/K_1) = \mathbf{Aut}(C/K_2)$ and hence $\mathbf{Aut}(C/K_1) = \mathbf{Aut}(C/Q)$ where $Q$ denotes the quaternion subalgebra of $C$ generated by $K_1$ and $K_2$. This is a contradiction since $\mathbf{Aut}(C/Q)$ is of type $A_1$ (Theorem 6.2.5) while $\mathbf{Aut}(C/K_1)$ is of type $A_2$. Hence $[H, H]$ is of type $G_2$. Now $H \subseteq G = [G, G] = [H, H] \subseteq H$. Therefore $H = G$. $\square$

From Theorem 11.4.3 and Theorem 11.4.5, we can immediately deduce that when $C$ is an octonion division algebra over a perfect (infinite) field $k$, then $G = \mathbf{Aut}(C)$ is generated by four $k$-subgroups of type $A_1$. However, we can do better:

**Theorem 11.4.6** *Let $C$ be an octonion division algebra over $k$, where $k$ is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by three $k$-subgroups of type $A_1$.*

Proof. By Theorem 11.4.2, there exists quadratic subfields $K_1, K_2$ of $C$ such that $K_1 \cap K_2 = k$ and $G$ is generated by $H_i$, $i = 1, 2$ where $H_i := \mathbf{Aut}(C/K_i)$. Choose a $k$-subgroup $S \subseteq H_2$ of type $A_1$ such that $S \not\subseteq H_1$. Such a choice is possible since by Theorem 11.4.3 $H_i$, $i = 1, 2$ are generated by $k$-subgroups of type $A_1$. Let $H :=< H_1, S >$. By Lemma 11.0.1, $H$ is a connected, reductive, $k$-anisotropic, non-toral subgroup of $G$. By Prop. 11.4.1, $[H, H]$ is of type $A_1$, $A_1 \times A_1$, $A_2$ or $G_2$. Now $H_1 = [H_1, H_1] \subseteq [H, H]$.

Since $H_1$ is of type $A_2$, by the Borel-De Siebenthal algorithm, $[H, H]$ cannot be of type $A_1$ or $A_1 \times A_1$ (see Example 3.5.7). Therefore $[H, H]$ must be of type $A_2$ or $G_2$. Suppose $[H, H]$ is of type $A_2$. Then $H_1 = [H, H]$. Now $H = [H, H].Z(H)^\circ = H_1.Z(H)^\circ$. Since the rank of maximal tori of $H$ and $H_1$ is two we have, $Z(H)^\circ = \{1\}$. Hence $H = H_1$ a contradiction. Hence $[H, H]$ is of type $G_2$. Now $H \subseteq G = [G, G] = [H, H] \subseteq H$. Therefore $H = G$. Since by Theorem 11.4.3, $H_1$ is generated by two $k$-subgroups of type $A_1$, $G$ is generated by three $k$-subgroups of type $A_1$. $\square$

## 11.5 Generation of $G_2$ by rank-2 tori

Let $G$ be a group of type $G_2$ defined over $k$. We now calculate the number of rank-2 $k$-tori required to generate $G$. From Theorems 11.1.1, 11.4.5, we can immediately deduce that when $C$ is an octonion division algebra over a perfect (infinite) field $k$, then $G = \mathbf{Aut}(C)$ is generated by four $k$-tori of rank-2. However, we can do better:

**Theorem 11.5.1** *Let $C$ be an octonion division algebra over $k$, where $k$ is a perfect (infinite) field. Then $G = \mathbf{Aut}(C)$ is generated by three $k$-tori of rank-2.*

Proof. The algebraic group $G = \mathbf{Aut}(C)$ is a connected, simple algebraic group of type $G_2$, in particular, $G$ has absolute rank-2. By Theorem 11.4.5, $G$ is generated by two subgroups $H_i$, $i = 1, 2$, of type $A_2$ with $H_1 \neq H_2$. Choose a maximal $k$-torus $T \subseteq H_1$ such that $T \nsubseteq H_2$. Let $H =< T, H_2 >$ be the (closed) subgroup generated by $T$ and $H_2$. Since $C$ is a division algebra, $G$ is $k$-anisotropic (Prop. 6.1.2). By Lemma 11.0.1, $H$ is a connected reductive $k$-anisotropic non-toral subgroup of $G$ containing $H_2$ properly. Using same arguments as in Theorem 11.4.5, it follows that $[H, H]$ must be of type $A_2$ or $G_2$. If $[H, H]$ is of type $A_2$, then $[H, H] = H_2$ (since $H_2 = [H_2, H_2] \subseteq [H, H]$ and both are of type $A_2$). Now $H = [H, H].Z(H)^o = H_2.Z(H)^o$ and $Z(H) = \cap T_i$, $T_i$'s are maximal tori of $H$ ([3], §13.17, Cor. 2). Since any maximal torus in $H_2$ is maximal in $H$ we have, $Z(H) \subset H_2$. Hence $H = H_2$ and $T \subseteq H_2$, contradicting the choice of $T$. Therefore $[H, H]$ is of type $G_2$. Now $H \subseteq G = [G, G] = [H, H] \subseteq H$. Hence $H = G$. The result now follows since $H_2$ is itself generated by two rank-2 $k$-tori. $\square$

We summarize these results in the tables below,

Table 11.1: Number of $k$-subgroups required for generation of groups

| Type of group | Type of $k$-subgroup | Number of $k$-subgroups required for generation |
|---|---|---|
| $F_4$ | $A_2$ | 3 |
| $F_4$ | $D_4$ | 2 |
| $D_4$ | $A_2$ | 2 |
| $G_2$ | $A_1$ | 3 |
| $G_2$ | $A_2$ | 2 |
| $A_2$ | $A_1$ | 2 |

Table 11.2: Number of $k$-tori required for generation of groups

| Type of group | Number of rank-2 $k$-tori required for generation |
|---|---|
| $A_2$ | 2 |
| $G_2$ | 3 |
| $D_4$ | 3 |
| $F_4$ | 4 |

# Chapter 12

# Cohomology of unitary tori and applications

In chapter 8, §8.2 we defined a unitary $k$-torus. The aim of this chapter is to study the first cohomology of such tori and discuss some applications to algebraic groups and étale Tits processes.

Let $L, K$ be étale $k$-algebras of dimension $n, 2$ resp. and let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. The first section of the chapter calculates $H^1(k, T)$. In the second section we study some applications of the cohomology compuation to étale Tits processes. We establish a relation between $H^1(k, T)$ and the set of $L$-isomorphism classes of étale Tits process algebras arising from $(L, K)$. In the final section we see some applications to algebaric groups. We study the effect of the presence of a unitary $k$-torus $T$ as above in groups of type $A_2, G_2$ and $F_4$ when $H^1(k, T) = 0$. Fix a field $k$ of characteristic different from 2 and 3.

## 12.1 Cohomology of unitary $k$-tori

In this section we will use some definitions and results from Chapter 4. We will specialize and adapt some of the computations done in ([19], §29.17), to the case of unitary algebras. Let $k$ be a field and $K$ be a quadratic étale $k$- algebra. Let $^-$ denote the non-trivial $k$-automorphism of $K$. Let $L$ be an étale algebra of dimension $n$ over $k$ and $E = L \otimes K$ be the associated $K$-unitary algebra with the involution $\tau = 1 \otimes \bar{}$.

We first calculate $H^1(k, \mathbf{SU}(E, \tau))$. Let $W = E \oplus K$. Define a representation $\rho$ :

$\mathbf{GL}_1(E) \longrightarrow \mathbf{GL}(W)$ by $\rho(b)(x,y) = (bx\tau(b), N_{E\otimes k_s/K\otimes k_s}(b)y)$ for all $b \in \mathbf{GL}_1(E)$, $x \in E$, $y \in K$.

Let $w_0 = (1,1) \in W$. Note that $\mathbf{GL}_1(E)(k_s) = (E \otimes k_s)^*$.

**Claim**: $\mathbf{Aut}_{\mathbf{GL}_1(E)}(w_0) = \mathbf{SU}(E,\tau)$.

We have,

$$\begin{aligned}
\mathbf{Aut}_{\mathbf{GL}_1(E)}(w_0) &= \{g \in (E\otimes k_s)^* |\ \rho_{sep}(g)(1,1) = (1,1)\} \\
&= \{g \in (E\otimes k_s)^* |\ g\tau(g) = 1,\ N_{E\otimes k_s/K\otimes k_s}(g) = 1\} \\
&= \mathbf{SU}(E,\tau).
\end{aligned}$$

Hence, in view of Proposition 4.4.1, we have a bijection

$$\eta : Isom(A(\rho, w_0)) \leftrightarrow H^1(k, \mathbf{SU}(E,\tau)).$$

We define a product on $Isom(A(\rho, w_0))$ as follows:

$$\rho_{sep}(g)(w_0)\rho_{sep}(g')(w_0) := \rho_{sep}(gg')(w_0) \text{ for all } g, g' \in (E \otimes k_{sep})^*.$$

A routine calculation shows that this product is well defined. Since $\mathbf{SU}(E,\tau)$ is a torus, $H^1(k, \mathbf{SU}(E,\tau))$ is an abelian group. It is immediate that $\eta$ is a homomorphism of groups. Define

$$V := \{(s,z) \in L^* \times K^* |\ N_{L/k}(s) = z\overline{z}\}.$$

Given a twisted $\rho$-form $w'$ of $w_0$ which lies in $W$, there exists $b \in (E \otimes k_{sep})^*$ such that $w' = \rho_{sep}(b)(w_0)$. Now $\rho_{sep}(b)(w_0) = \rho_{sep}(b)(1,1) = (b\tau(b), N_{E\otimes k_{sep}/(K\otimes k_{sep})}(b))$. Along similar lines as in ([19], §29.17), we can show that $\rho_{sep}(b)(w_0) \in V$ and $V$ is precisely the set of twisted $\rho$-forms of $w_0$ which lie in $W$. Define an equivalence $\sim$ on $V$ as follows:

$$(s,z) \sim (s',z') \text{ if and only if } s' = bs\tau(b) \text{ and } z' = N_{E/K}(b)z \text{ for some } b \in E^*.$$

We will denote equivalence class of $(s,z) \in V$ by $[(s,z)]$. Note that $V$ is a subgroup of $L^* \times K^*$. It is easy to see that the product on $V$ induces a well defined product on $V/\sim$ as follows:

$$[(s,z)][(s',z')] = [(ss',zz')] \text{ for all } (s,z),(s',z') \in V.$$

Define $\xi : Isom(A(\rho, w_0)) \to V/\sim$ by $\xi([w']) = [(b\tau(b), N_{E\otimes k_{sep}/(K\otimes k_{sep})}(b))]$, where $w' = \rho_{sep}(b)(w_0)$, for some $b \in (E \otimes k_{sep})^*$. It follows that $\xi$ is a homomorphism of groups.

We have proved the following,

**Theorem 12.1.1** *Let $L, K$ be étale $k$-algebras of dimension $n, 2$ resp. and $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Then there exists a natural isomorphism: $H^1(k, T) \mapsto V/\sim$ of groups.*

Henceforth we will identity $H^1(k, T)$ with $V/\sim$ and write elements in $H^1(k, T)$ as equivalence classes $[(s, z)] \in V/\sim$.

**Theorem 12.1.2** *Let $L, K$ be étale $k$-algebras of dimension $n, 2$ resp. Let $E$ be the $K$-unitary algebra and $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Then,*

$$\frac{H^1(k,\ T)}{K_0^{(1)}} \cong \frac{S}{N_{E/L}(E^*)},$$

*where*

$$S := \{u \in L^* |\ N_{L/k}(u) \in N_{K/k}(K^*)\} \text{ and } \mathrm{K}_0^{(1)} := \{[(1, \mu)] \in \mathrm{H}^1(\mathrm{k},\ \mathrm{T}) |\ \mu\overline{\mu} = 1\}.$$

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes {}^-)$ is the $K$-unitary algebra associated with the pair $(L, K)$. Define $\phi : H^1(k, \mathbf{SU}(E, \tau)) \longrightarrow \frac{S}{N_{E/L}(E^*)}$ by $\phi([(s, z)]) = sN_{E/L}(E^*)$. If $(s, z) \sim (s', z')$ then $s = s'b\tau(b)$ for some $b \in E$. Hence $s = s'N_{E/L}(E^*)$ and $\phi$ is well defined. We now check that $\phi$ is surjective. Let $s \in S$. By definition, there exists $z \in K^*$ such that $N_{L/k}(s) = z\overline{z}$, for some $z \in K$. Hence $\phi([(s, z)]) = s$, showing that $\phi$ is onto. Clearly $\phi$ is a homomorphism. Now, $Ker\ \phi = \{[(s, z)]|\ s \in N_{E/L}(E^*)\}$. Clearly, $K_0^{(1)} \subseteq Ker\ \phi$. Let $[(s, z)] \in Ker\ \phi$. Then $s \in N_{E/L}(E^*) = e\tau(e)$ for some $e \in E$. Let $\mu = zN_{E/K}(e^{-1})$. Then

$$N_{L/k}(s) = z\overline{z} = N_{E/K}(s) = N_{E/K}(e\tau(e)) = N_{E/K}(e)\overline{N_{E/K}(e)}.$$

Hence $z\overline{z} = N_{E/K}(e)\overline{N_{E/K}(e)}$. Therefore $\mu\overline{\mu} = 1$. It follows that $(s, z) = (e\tau(e), z) \sim (1, \mu)$. Hence, $Ker\phi = K_0^{(1)}$. □

We obtain below an explicit expression for $H^1(k, \mathbf{SU}(E, \tau))$. Consider the the exact sequence,

$$1 \longrightarrow K_0^{(1)} \xrightarrow{q} H^1(k, \mathbf{SU}(E,\tau)) \xrightarrow{\phi} \frac{S}{N_{E/L}(E^*)} \longrightarrow 1,$$

where $q$ denotes the inclusion map and $\phi$ is as above. We provide a splitting of this sequence when dimension of $L$ is odd. We will from here on assume that the $k$-dimension $n$ of $L$ is odd. Let $n = 2r+1$.

Define $t : H^1(k, \mathbf{SU}(E,\tau)) \longrightarrow K_0^{(1)}$ by,

$$t([(u,\mu)]) = [(1, \mu^{-r}\overline{\mu}^r)].$$

We first check that this map is well defined. Let $w \in E^*$.

Then $(u,\mu) \sim (wu\tau(w), N_{E/K}(w)\mu)$. Now $w^{-r}\tau(w)^r \in \mathbf{U}(E,\tau)$. Hence

$$(1, \mu^{-r}\overline{\mu}^r) \sim (1, N_{E/K}(w^{-r}\tau(w)^r)\mu^{-r}\overline{\mu}^r) = (1, (N_{E/K}(w)\mu)^{-r}\overline{(N_{E/K}(w)\mu)^r}).$$

Therefore, $t$ is well defined. It is immediate that $t$ is a homomorphism. We have

$$t \circ\ q[(1,\ \mu)] = [(1, \mu^{-r}\overline{\mu}^r)] = [(1,\mu)]$$

(Since $\mu^{2r+1} = N_{E/K}(\mu)$ and $\mu\overline{\mu} = 1$). Hence, $t \circ q = Id_{K_0^{(1)}}$. Therefore there exists a homomorphism $\psi : \frac{S}{N_{E/L}(E^*)} \longrightarrow H^1(k, \mathbf{SU}(E,\tau))$ such that $\phi \circ \psi = Id$. In fact $\psi$ is given by,

$$\psi([u]) := [(u,\ \mu)]q(t([(u,\ \mu)])^{-1}) = [(u,\ \mu^{r+1}\overline{\mu}^{-r})],$$

where $N_{L/k}(u) = \mu\overline{\mu}$, $\mu \in K$. We now make some observations based on the above exact sequence. We have,

$$\begin{aligned} Ker\ t &= \{[(u,\ \mu)] \mid (1,\ \mu^{-r}\overline{\mu}^r) \sim (1,1)\} = \{[(u,\ \mu)] \mid \mu^{-r}\overline{\mu}^r \in N_{E/K}(\mathbf{U}(E,\tau))\}. \\ &= Image\ \psi = \{[(u,\ \mu^{r+1}\overline{\mu}^{-r})] \mid N_{L/k}(u) = \mu\overline{\mu}\}. \end{aligned}$$

Since $\psi$ is an injective homomorphism, we have $Image\ \psi \cong \frac{S}{N_{E/L}(E^*)}$. Hence

$$\frac{S}{N_{E/L}(E^*)} \cong \{[(u,\ \mu)] \mid \mu^{-r}\overline{\mu}^r \in N_{E/K}(\mathbf{U}(E,\tau))\} \cong \{[(u,\ \mu^{r+1}\overline{\mu}^{-r})] \mid N_{L/k}(u) = \mu\overline{\mu}\}.$$

Owing to the splitting of the exact sequence above we have, $H^1(k, \mathbf{SU}(E,\tau)) = Image\ q \times Ker\ t$. We have already seen that $Ker\ t \cong \frac{S}{N_{E/L}(E^*)}$. Now, $Image\ q = K_0^{(1)}$. Let $K^{(1)}$

denote the norm 1 elements of $K$. Define a map

$$\chi : K^{(1)} \longrightarrow K_0^{(1)}$$

by $\chi(\mu) := [(1, \mu)]$ for all $\mu \in K^{(1)}$. This map is clearly a surjective homomorphism. Now,

$$\begin{aligned} Ker\ \chi &= \{\mu \in K^{(1)} |\ [(1,\ \mu)] = [(1,\ 1)]\}. \\ &= \{\mu \in K^{(1)} |\ \mu = N_{E/K}(w),\ w\tau(w) = 1,\ w \in E\}. \\ &= N_{E/K}(\mathbf{U}(E, \tau)). \end{aligned}$$

Hence, $\frac{K^{(1)}}{N_{E/K}(\mathbf{U}(E,\tau))} \cong K_0^{(1)}$. We summarize this as:

**Theorem 12.1.3** *Let $K$ be a quadratic étale $k$-algebra and $L$ be an étale $k$-algebra of dimension $n = 2r + 1$. Let $E$ be the $K$-unitary algebra and $T$ the $K$-unitary torus associated with the pair $(L, K)$. Then,*

$$H^1(k, T) \cong \frac{K^{(1)}}{N_{E/K}(\mathbf{U}(E, \tau))} \times \frac{S}{N_{E/L}(E^*)}.$$

In fact, an explicit isomorphism is as follows:

$$\phi : H^1(k, T) \longrightarrow K_0^{(1)} \times \frac{S}{N_{E/L}(E^*)}$$

$$\phi([(u,\ \mu)]) = ([(1,\ \mu^{-r}\overline{\mu}^r)],\ [u]).$$

We now prove a somewhat analogous result to Theorem 12.1.2, for the cohomology of a unitary torus.

**Theorem 12.1.4** *Let $L, K$ be étale $k$-algebras of dimension $n, 2$ resp. Let $E$ be the associated $K$-unitary algebra and $T$ the $K$-unitary torus associated with the pair $(L, K)$. Then,*

$$\frac{H^1(k, T)}{L_0^{(1)}} \cong \frac{M}{N_{E/K}(E^*)},$$

*where*

$$M = \{\mu \in K^* |\ \mu\overline{\mu} \in N_{L/k}(L^*)\} \text{ and } L_0^{(1)} = \{[(u,\ 1)] |\ N_{L/k}(u) = 1\}.$$

PROOF. By definition, $T = \mathbf{SU}(E,\tau)$, where $(E,\tau) = (L \otimes K, 1 \otimes \bar{\ })$ is the $K$-unitary algebra associated with the pair $(L,K)$. We define a map

$$\phi : H^1(k, \mathbf{SU}(E,\tau)) \longrightarrow \frac{M}{N_{E/K}(E^*)}$$

by $[(s,z)] \mapsto zN_{E/K}(E^*)$. It is easy to see that $\phi$ is a well defined surjective homomorphism and $Ker\ \phi = \{[(s,z)] |\ z \in N_{E/K}(E^*)\}$. Clearly, $L_0^{(1)} \subseteq Ker\ \phi$. Let $[(s,z)] \in Ker\ \phi$. Then $z = N_{E/K}(w)$, for some $w \in E^*$. Let $u = w^{-1}s\tau(w^{-1})$. Now, $N_{E/K}(u) = N_{E/K}(w^{-1}\tau(w^{-1})s)$ and

$$N_{E/K}(s) = N_{L/k}(s) = z\overline{z} = N_{E/K}(w\tau(w)).$$

Hence $N_{E/K}(u) = 1$. Also $(u,1) = (w^{-1}s\tau(w^{-1}),1) \sim (s,\ z)$. Therefore we have $Ker\ \phi = L_0^{(1)}$ □

We now provide a decomposition of $H^1(k, \mathbf{SU}(E,\tau))$ analogous to that in Theorem 12.1.3. Consider the exact sequence,

$$1 \longrightarrow L_0^{(1)} \xrightarrow{q} H^1(k, \mathbf{SU}(E,\tau)) \xrightarrow{\phi} \frac{M}{N_{E/K}(E^*)} \longrightarrow 1,$$

where the maps $q$ and $\phi$ are as above. We provide a splitting of this sequence when dimension of $L$ is odd. We will from here on assume that the $k$-dimension $n$ of $L$ is odd. Let $n = 2r + 1$.

We define a map

$$t : H^1(k, \mathbf{SU}(E,\tau)) \longrightarrow L_0^{(1)}$$

by $t([(u,\mu)]) := [(u^nN_{L/k}(u^{-1}),\ 1)]$. We first check that this map is well defined. Let $w \in E^*$. Then $(u,\mu) \sim (wu\tau(w),\ N_{E/K}(w)\mu)$. Since $N_{E/K}(w^nN_{E/K}(w^{-1})) = 1$ and $N_{E/K}(w^{-1}\tau(w^{-1})) = N_{L/k}(w^{-1}\tau(w^{-1}))$ we have,

$$\begin{aligned}(u^nN_{L/k}(u^{-1}),1) &\sim (w^nN_{E/K}(w^{-1})u^nN_{L/k}(u^{-1})\tau(w^n)N_{E/K}(\tau(w^{-1})),1)\\ &= ((wu\tau(w))^nN_{L/k}((wu\tau(w))^{-1}),\ 1)).\end{aligned}$$

Therefore, $t$ is well defined. It is easily checked that $t$ is a homomorphism. Since $u \in L$ and $N_{E/K}(u) = N_{L/k}(u) = u\tau(u) = 1$, we have,

$$t \circ\ q[(u,\ 1)] = [(u^{2r+1},\ 1)] = [(u^{2r-1}(u\tau(u)), 1)] = [(u^{2r-1}, 1)].$$

By a similar calculation, $[(u^{2r-1}, 1)] = [(u^{2r-3}, 1)] = ... = [(u, 1)]$. Hence $[(u^n, 1)] = [(u, 1)]$ and therefore $t \circ q = Id_{L_0^{(1)}}$.
Hence there exists a homomorphism $\psi : \frac{M}{N_{E/K}(E^*)} \longrightarrow H^1(k, \mathbf{SU}(E, \tau))$ such that $\phi \circ \psi = Id$. Explicitly, this map is given by

$$\psi([\mu]) := [(u,\ \mu)]q(t([(u^{-1},\ \mu^{-1})])) = [(u^{-n+1}N_{L/k}(u),\ \mu)]$$

where $N_{L/k}(u) = \mu\overline{\mu}$, $\mu \in K^*$. We now make some observations based on this exact sequence. We have,

$$\begin{aligned} Ker\ t &= \{[(u,\ \mu)] |\ [(u^n N_{L/k}(u^{-1}),\ 1)] \sim (1,1)\} \\ &= Image\ \psi \\ &= \{[(u^{-n+1}N_{L/k}(u),\ \mu)] |\ N_{L/k}(u) = \mu\overline{\mu}\}. \end{aligned}$$

Since $\psi$ is an injective group homomorphism, $Image\ \psi \cong \frac{M}{N_{E/K}(E^*)}$. Hence

$$\frac{M}{N_{E/K}(E^*)} \cong \{[(u,\ \mu)] |\ [(u^n N_{L/k}(u^{-1}),\ 1)] \sim (1,1)\} \cong \{[(u^{-n+1}N_{L/k}(u),\ \mu)] |\ N_{L/k}(u) = \mu\overline{\mu}\}.$$

Since the above sequence is split exact, we have $H^1(k, \mathbf{SU}(E, \tau)) = Image\ q \times Ker\ t$. We have already seen that $Ker\ t \cong \frac{M}{N_{E/K}(E^*)}$ and $Image\ q = L_0^{(1)}$. Let $L^{(1)}$ denote norm 1 elements of $L$ and $E^{(1)} = \{x \in E |\ N_{E/K}(x) = 1\}$. Now define $\phi : L^{(1)} \longrightarrow L_0^{(1)}$ by $u \mapsto [(u, 1)]$. It is easily checked that $Ker\ \phi = N_{E/L}(E^{(1)})$. Hence $\frac{L^{(1)}}{N_{E/L}(E^{(1)})} \cong L_0^{(1)}$. We summarize this as,

**Theorem 12.1.5** *Let $K$ be a quadratic étale $k$-algebra and $L$ be an étale $k$-algebra of dimension $n = 2r + 1$. Let $E$ be the $K$-unitary algebra and $T$ the $K$-unitary torus associated to the pair $(L, K)$. Then,*

$$H^1(k, T) \cong \frac{L^{(1)}}{N_{E/L}(E^{(1)})} \times \frac{M}{N_{E/K}(E^*)}.$$

We now discuss the special case, when $L = k \times K_0$, where $K_0$ is a quadratic étale extension of $k$.

**Theorem 12.1.6** *Let $K$ and $K_0$ be a quadratic étale extensions of $k$. Let $L = k \times K_0$ and $T$ be the $K$- unitary torus associated with the pair $(L, K)$. Then $H^1(k, T) = K_0^*/N_{K\otimes K_0/K_0}(K_0^*)$.*

Proof. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes {}^-)$ is the $K$-unitary algebra associated with the pair $(L, K)$. Let $M := K\otimes K_0$. Then $L\otimes K \cong K\times(K\otimes K_0) = K \times M$. Now,

$$\begin{aligned} SU(E,\tau) &\cong \{(a,x) \in K \times M|\ a\overline{a} = 1, x\tau(x) = 1, aN_{M/K}(x) = 1\} \\ &= \{(N_{M/K}(x^{-1}), x)|\ x\tau(x) = 1, x \in M^*\}. \end{aligned}$$

It follows that $\mathbf{SU}(E, \tau) \cong R^{(1)}_{K_0/k}(M)$. By Shapiro's Lemma ([19], Lemma 29.6), $H^1(k, R^{(1)}_{K_0/k}(M)) = H^1(K_0, \mathbf{M}^{(1)})$. Hence, $H^1(k, T) = K_0^*/N_{K\otimes K_0/K_0}(K_0^*)$. □

**Corollary 12.1.7** *Let $K = k(\sqrt{\alpha})$ and $K_0$ be quadratic field extensions of $k$. Let $L = k\times K_0$ and $T$ be the $K$- unitary torus associated with the pair $(L, K)$. Then $H^1(k, T) = 0$ if and only if the quadratic form $< 1, -\alpha >$ becomes universal over $K_0$.*

**Corollary 12.1.8** *Let $K$ be a quadratic étale extension of $k$ and $L = k \times K_0$, where $K_0$ is a quadratic étale extension of $k$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Let $H^1(k, T) = 0$. Then any composition algebra of dimension $\geq 4$ which contains $K$ contains $K_0$.*

Proof. By Theorem 12.1.6, $H^1(k, T) = 0$ if and only if $N_{K\otimes K_0/K_0}(K \otimes K_0)^* = K_0^*$. Let $C$ be a composition algebra properly containing $K$. Then $K \otimes K_0 \subseteq C \otimes K_0$. By doubling, $C \otimes K_0 = (K \otimes K_0) \oplus (K \otimes K_0).x$, for some $x \in (K \otimes K_0)^\perp$, $N_{C\otimes K_0}(x) \neq 0$. But since $K_0^* = N_{K\otimes K_0/K_0}(K\otimes K_0)^*$, we have $N_{C\otimes K_0}(x) \in N_{K\otimes K_0/K_0}(K\otimes K_0)^*$. Hence $C \otimes K_0$ is split and $K_0 \subseteq C$ ([5], Lemma 5). □

One may be tempted to believe that for a distinguished $K$-unitary torus $T$, $H^1(k, T) = 0$. We give below an example to show that this is false. We also produce an example of a non-distinguished ($k$-anisotropic) torus $T$ such that $H^1(k, T) = 0$.

**Example 12.1.9** *Let $k = \mathbb{R}(x)$ and $\delta = -1$. Choose $\alpha \in k^*$ such that $\alpha \notin k^{*2}$ and $\alpha \neq \delta \bmod k^{*2}$. Let $K = k(\sqrt{\alpha})$ and $K_0 = k(\sqrt{\delta})$. Then $K_0$ and $K$ are fields. Also note that $K_0 = \mathbb{C}(x)$ is a $C_1$ field. Let $L = k \times K_0$, and $T$ be the $K$-unitary torus associated with the pair $(L, K)$. Then, as in the proof of Theorem 12.1.6, $T \cong R^{(1)}_{K_0/k}(K \otimes K_0)$. Also by Theorem 12.1.6, $H^1(k, T) = {K_0}^*/N_{K \otimes K_0/K_0}(K \otimes K_0)$. Since $\alpha \neq \delta \bmod k^{*2}$, $T$ is not distinguished. By Corollary 12.1.7, $H^1(k, T) = 0$ if and only if the binary form $< 1, -\alpha >$ becomes universal over $K_0$. Since $K_0$ is a $C_1$ field, all binary forms over $K_0$ are universal, in particular $< 1, -\alpha >$ becomes universal over $K_0$. Hence $H^1(k, T) = 0$. Note that since $\alpha \neq \delta \bmod k^{*2}$, $K \otimes K_0$ is a field. Hence by ([46], Example on Pg. 54), $T$ is $k$-anisotropic. This also gives an example of a $k$-anisotropic $K$-unitary torus $T$ such that $H^1(k, T) = 0$.*

**Example 12.1.10** *Let $K = k \times k$ and $L$ be a cyclic cubic field extension of $k$. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$. By definition, $T = \mathbf{SU}(E, \tau)$, where $(E, \tau) = (L \otimes K, 1 \otimes \bar{\ })$. Note that $(E, \tau) = (L \times L, \epsilon)$. Hence $T \cong \mathbf{L}^{(1)}$. Now*

$$H^1(k, T) \cong H^1(k, \mathbf{L}^{(1)}) \cong \frac{k^*}{N_{L/k}(L^*)}.$$

*Let $p(X) = X^3 - 3X - 1 \in \mathbb{Q}[X]$, then $p(X)$ is irreducible over $\mathbb{Q}$. Let $L' := \mathbb{Q}[X]/ < p(X) >$. Then $L'$ is a cyclic cubic extension of $\mathbb{Q}$ such that $N_{L'/\mathbb{Q}}(L'^*) \neq \mathbb{Q}^*$ ([52], Pg. 186). Let $T$ be the $K$-unitary torus associated with the pair $(L', K)$. Then $T$ is a distinguished torus such that $H^1(\mathbb{Q}, T) \neq 0$.*

**Example 12.1.11** *Let $T$ be a distinguished $k$-torus arising from a pair $(L, K)$ where $L$ is a cubic étale $k$-algebra which is not a field extension and $K$ is a quadratic étale $k$-algebra. By Theorem 10.1.3, $T$ is either $\mathbb{G}_m \times \mathbb{G}_m$ or $R_{K/k}(\mathbb{G}_m)$. In either case, by Hilbert theorem 90 and Shapiro's Lemma ([19], Lemma 29.6), $H^1(k, T) = 0$.*

**Example 12.1.12** *Let $L = k \times k \times k$ and $K$ be a quadratic étale extension of $k$. Then*

$$SU(E, \tau) = \{(x, y, z) \in K \times K \times K | x\overline{x} = y\overline{y} = z\overline{z} = 1, xyz = 1\} \cong K^{(1)} \times K^{(1)}.$$

*Thus $\mathbf{SU}(E, \tau) \cong \mathbf{K}^{(1)} \times \mathbf{K}^{(1)}$ and $H^1(k, \mathbf{SU}(E, \tau)) = k^*/N_{K/k}(K^*) \times k^*/N_{K/k}(K^*)$. Hence $H^1(k, \mathbf{SU}(E, \tau)) = 0$ if and only if $k^* = N_{K/k}(K^*)$. Let $A$ be an Albert algebra. Let $T$ be the $K$-unitary torus associated with the pair $(L, K)$ such that $H^1(k, T) = 0$.*

*Let $T \hookrightarrow \mathbf{Aut}(A)$ over $k$. Since $L$ has trivial discriminant over $k$, by Theorem 10.3.5, $K \subset Oct(A)$. Since $H^1(k,T) = 0$, $k^* = N_{K/k}(K^*)$. Hence $f_3(A) = 0$.*

**Theorem 12.1.13** *Let $K$ be a quadratic étale extension of $k$ and $L = k \times K_0$, where $K_0$ is a quadratic étale algebra over $k$. Let $T$ be the $K$-unitary torus associated with the pair $(L,K)$ with $H^1(k,T) = 0$. Let $A$ be an Albert algebra over $k$. If there exists an $k$-embedding $T \hookrightarrow \mathbf{Aut}(A)$, then $K_0 \subset Oct(A)$.*

Proof. If $K_0 = k \times k$, then by Example 12.1.12, $Oct(A)$ splits. Hence $K_0 \subset Oct(A)$. Let $K_0$ be a field extension. Base changing to $K_0$ we have,

$$T \otimes K_0 = \mathbf{SU}((L \otimes K_0) \otimes_{K_0} (K_0 \otimes K), \tau \otimes 1) \hookrightarrow \mathbf{Aut}(A) \otimes K_0.$$

Since $L \otimes K_0$ has trivial discriminant over $K_0$, by Theorem 10.3.5, $K_0 \otimes K \subset Oct(A) \otimes K_0$. By doubling, $Oct(A) \otimes K_0 = (K \otimes K_0) \oplus (K \otimes K_0).x$, for some $x \in (K \otimes K_0)^{\perp}$, $N_{Oct(A)\otimes K_0}(x) \neq 0$. But since $H^1(k,T) = 0$, by Theorem 12.1.6 we have, $K_0^* = N_{K\otimes K_0/K_0}(K \otimes K_0)^*$. Hence $N_{Oct(A)\otimes K_0}(x) \in N_{K\otimes K_0/K_0}(K \otimes K_0)^*$. Therefore $Oct(A) \otimes K_0$ is split and by ([5], Lemma 5), $K_0 \subseteq Oct(A)$. $\square$

## 12.2 Application to Tits processes

In this section, we develop some results on étale Tits processes, in the context of unitary tori. For the results on étale Tits processes used in this section, we refer to §5.3. Recall that two étale Tits processes $J_1$ and $J_2$ arising from étale algebras $L$ and $K$ of dimensions $3, 2$ resp., are defined to be $L$-isomorphic, denoted by $J_1 \cong_L J_2$, if there exists a $k$-isomorphism $J_1 \to J_2$, which restricts to an automorhism of the subalgebra $L$ of $J_1$ and $J_2$.

**Lemma 12.2.1** *Let $L, K$ be étale $k$-algebras of dimension $3, 2$ resp. Let $(E,\tau)$ be the $K$-unitary algebra associated with the pair $(L,K)$. Suppose $\phi : J(E,\tau,u,\mu) \to J(E,\tau,v,\nu)$ is an $L$-isomorphism. Then there exists $w \in E$ such that $u = \phi^{-1}(v)w\tau(w)$ and $\mu = N_{E/K}(w)\nu$ or $\mu = N_{E/K}(w)\overline{\nu}$.*

Proof. By definition, $(E,\tau) = (L \otimes K, 1 \otimes \bar{\ })$. By Theorem 5.3.5, we may assume $N_{L/k}(v) = \nu\overline{\nu} = 1$. Let $\phi : J(E,\tau,u,\mu) \to J(E,\tau,v,\nu)$ be an $L$-isomorphism. Define

$h : L \oplus E \to L$ and $g : L \oplus E \to E$ by

$$\phi(a, b) = (h(a, b), g(a, b)),$$

for $a \in L, b \in E$. Since $\phi$ is an isomorphism, one can easily check that $g$ and $h$ are $k$-linear maps. Since $\phi$ is an isomorphism of Jordan algebras, it preserves the trace forms on both the algebras. Note that $L^{\perp}$ in $J(E, \tau, u, \mu)$ with respect to the trace form, is the $k$-subspace $\{(0, e)|\ e \in E\}$, and similarly for $J(E, \tau, \upsilon, \nu)$. Since $\phi$ restricts to $L$, $\phi$ maps $L^{\perp}$ in $J(E, \tau, u, \mu)$ to $L^{\perp}$ in $J(E, \tau, \upsilon, \nu)$. Hence for $b \in E$, $\phi(0, b) = (0, b')$ for some $b' \in E$. It follows that $h(0, b) = 0$ for all $b \in E$. Therefore $h(a, b) = h(a, 0)$ for all $a \in L,\ b \in E$. We will now on write simply $h(a)$ for $h(a, b)$. Since $\phi$ is an isomorphism of Jordan algebras, it is easy to check that $h : L \to L$ is an automorphism. Since $\phi$ restricts to $L$, $\phi(a, 0) = (h(a), 0)$ for all $a \in L$. Hence $g(a, 0) = 0$ for all $a \in L$. It follows that $g(a, b) = g(0, b)$ for all $a \in L, b \in E$. We will now on write simply $g(b)$ for $g(0, b)$. Again since $\phi$ is an isomorphism of Jordan algebras, it is easy to check that $g : E \to E$ is a bijection. Since $\phi$ preserves norms, $N(a, b) = N(h(a), g(b))$. Expanding norms we get,

$$\begin{aligned} & N_{L/k}(a) + \mu N_{E/k}(b) + \overline{\mu N_{E/k}(b)} - t_{L/k}(abu\tau(b)) \\ = \quad & N_{L/k}(h(a)) + \nu N_{E/k}(g(b)) + \overline{\nu N_{E/k}(g(b))} - t_{L/k}(h(a)g(b)\upsilon\tau(g(b))). \end{aligned}$$

Putting $a = 0$, we get

$$\mu N_{E/k}(b) + \overline{\mu N_{E/k}(b)} = \nu N_{E/k}(g(b)) + \overline{\nu N_{E/k}(g(b))},\ b \in E.$$

Since $h$ is an automorphism of $L$, we have $N_{L/k}(a) = N_{L/k}(h(a))$, $a \in L$. Hence we get,

$$t_{L/k}(abu\tau(b)) = t_{L/k}(h(a)g(b)\upsilon\tau(g(b))),\ a \in L,\ b \in E.$$

Putting $b = 1$, we get,

$$t_{L/k}(au) = t_{L/k}(h(a)g(1)\upsilon\tau(g(1))),\ a \in L.$$

Since $g(1)v\tau(g(1)) \in L$, there exist $b \in L$ such that $h(b) = g(1)v\tau(g(1))$. Hence

$$t_{L/k}(au) = t_{L/k}(h(a)h(b)) = t_{L/k}(h(ab)) = t_{L/k}(ab)$$

for all $a \in L$. Since the trace bilinear form $T(a,b) := T_{L/k}(ab)$ on $L/k$ is non-degenerate, we have $u = b$. Let $\hat{h} : E \to E$ be defined by $\hat{h} = h \otimes 1$. Then $\hat{h}$ is the extension of $h$ to a $K$-automorphism of $E$. In particular, $\hat{h}$ commutes with $\tau$. We have,

$$u = \hat{h}^{-1}(g(1))\hat{h}^{-1}(v)\hat{h}^{-1}\tau(g(1)) = \hat{h}^{-1}(g(1))h^{-1}(v)\tau(\hat{h}^{-1}(g(1))).$$

Hence $u = wh^{-1}(v)\tau(w) = \phi^{-1}(v)w\tau(w)$, where $w = \hat{h}^{-1}(g(1)) \in E$. This proves the first assertion in the Lemma.

Now we prove the assertion on $\mu$. Let $h^{-1}(v) = v_0 \in L$. Then $N_{L/k}(v_0) = N_{L/k}(v) = 1$. Let $u_1, u_2 \in E$. Define,

$$< u_1 > \cong < u_2 > \text{ over } E \text{ if and only if } u_1 = wu_2\tau(w), \text{ for some } w \in E.$$

Hence , $< u > \cong < v_0 >$ over $E$.

We now introduce an equivalence on the set of admissible pairs in $L^* \times K^*$ as follows: $(u_1, \mu_1) \sim (u_2, \mu_2)$ if and only if there exists $w \in E$ such that $u_2 = wu_1\tau(w)$ and $\mu_2 = N_{E/K}(w)\mu_1$ or $\mu_2 = N_{E/K}(w)\overline{\mu_1}$.

**Claim** $(u, \mu) \sim (v_0, \nu)$.

Since $J(E, \tau, u, \mu) \cong J(E, \tau, v, \nu)$, we have $J(E, \mu) \cong J(E, \nu)$ over $K$ and by ([36], Prop. 4.3), $\mu \in \nu N_{E/K}(E^*)$ or $\mu \in \overline{\nu} N_{E/K}(E^*)$. Let $\mu = \nu N_{E/K}(w)$ or $\mu = \overline{\nu} N_{E/K}(w)$ as is the case accordingly, for some $w \in E$. Let $v' = w^{-1}u\tau(w^{-1})$. Then $N_{L/k}(v') = \nu\overline{\nu} = 1$ and $(v', \nu) \sim (u, \mu)$. Now $< u > \cong < v' >$ and $< u > \cong < v_0 >$ over $E$. Hence $< v' > \cong < v_0 >$ over $E$.

Therefore, there exists $w'' \in E$ such that $v_0 = w''v'\tau(w'')$. Let $\lambda = N_{E/K}(w'')$. Since $N_{L/k}(v_0) = N_{L/k}(v') = 1$, we have $\lambda\overline{\lambda} = 1$. Hence by ([36], Lemma 4.5), there exists $w_1 \in E$ such that $\lambda = N_{E/K}(w_1)$ and $w_1\tau(w_1) = 1$.

Therefore,

$$(v', \nu) \sim (v_0, \lambda\nu) \sim (v_0, \nu).$$

Thus $(u, \mu) \sim (v_0, \nu) = (u, \mu) \sim (h^{-1}(v), \nu) = (\phi^{-1}(v), \nu)$. Hence, by the definition of the equivalence, $\mu = N_{E/K}(w)\nu$ or $\mu = N_{E/K}(w)\overline{\nu}$. This completes the proof. $\square$

**Remark 12.2.2** *As a converse to the above lemma, if there exists $w \in E^*$ such that $u = \phi^{-1}(v)w\tau(w)$ and $\mu = N_{E/K}(w)\nu$ (or $\mu = N_{E/K}(w)\overline{\nu}$), where $\phi \in Gal(L/k)$, then $J(E,\tau,u,\mu)$ is $L$-isomorphic to $J(E,\tau,v,\nu)$ (resp. $J(E,\tau,v,\overline{\nu})$). To see this, suppose $\mu = N_{E/K}(w)\nu$. By Theorem 5.3.3,*

$$J(E,\tau,u,\mu) = J(E,\tau,\phi^{-1}(v)w\tau(w), N_{E/K}(w)\nu) \cong_L J(E,\tau,\phi^{-1}(v),\nu).$$

*Extend $\phi$ to an automorphism $\hat{\phi}$ of $E$, defined as $\hat{\phi} = \phi\otimes 1$. Note that $\hat{\phi}$ commutes with $\tau$. Consider the map $\psi : J(E,\tau,\phi^{-1}(v),\nu) \longrightarrow J(E,\tau,v,\nu)$ given by $(a,x) \mapsto (\phi(a),\hat{\phi}(x))$ for $a \in L$, $x \in E$. Clearly $\psi((1,0)) = (1,0)$. We have,*

$$\begin{aligned}
N(\phi(a),\hat{\phi}(x)) &= N_{L/k}(\phi(a)) + \mu N_{E/k}(\hat{\phi}(x)) + \overline{\mu N_{E/k}(\hat{\phi}(x))} - t_{L/k}(\phi(a)v\hat{\phi}(x)\tau(\hat{\phi}(x))) \\
&= N_{L/k}(a) + \mu N_{E/k}(x) + \overline{\mu N_{E/k}(x)} - t_{L/k}(\hat{\phi}(a\phi^{-1}(v)x\tau(x))) \\
&= N_{L/k}(a) + \mu N_{E/k}(x) + \overline{\mu N_{E/k}(x)} - t_{L/k}(a\phi^{-1}(v)x\tau(x)) = N(a,x).
\end{aligned}$$

*Hence $\psi$ is a $k$-linear bijection preserving norms and identities. Therefore, by Theroem 5.3.7, $\psi$ is an isomorphism of Jordan algebras. Also $\psi$ restricts to $L$. Hence $\psi$ is an $L$-isomorphism. When $\mu = N_{E/K}(w)\overline{\nu}$, a similar argument completes the proof.*

**Corollary 12.2.3** *Let $L, K$ be étale $k$-algebras of dimension $3, 2$ resp. Let $(E,\tau)$ be the $K$-unitary torus associated with the pair $(L,K)$. There exists an $L$-isomorphism $J(E,\tau,u,\mu) \cong J(E,\tau,1,1)$ if and only if there exists $w \in E$ such that $u = w\tau(w)$ and $\mu = N_{E/K}(w)$.*

**Theorem 12.2.4** *There exists a surjective map from $H^1(k,\mathbf{SU}(E,\tau))$ to the set of $L$-isomorphism classes of étale Tits process algebras arising from $(L,K)$.*

Proof. Let $X$ denote the set of $L$-isomorphism classes of étale Tits process algebras arising from $(L,K)$. Given an étale Tits process $J$, let $[J]$ denote the $L$-isomorphism class of $J$. Let $\phi : H^1(k,\mathbf{SU}(E,\tau)) \to X$ be defined by $\phi([(u,\mu)]) := [J(E,\tau,u,\mu)]$. Let $[(u,\mu)] \in H^1(k,\mathbf{SU}(E,\tau))$ and $[(u,\mu)] = [(v,\nu)]$. Then $u = vw\tau(w)$ and $\mu = N_{E/K}(w)\nu$ for some $w \in E$. Hence by Theorem 5.3.3, $J(E,\tau,u,\mu)$ is $L$-isomorphic to $J(E,\tau,v,\nu)$. Therefore $\phi$ is well defined. Clearly $\phi$ is onto. $\square$

As an easy consequence of the above theorem, we note that if $H^1(k, \mathbf{SU}(E,\tau)) = 0$, then all étale Tits process algebras arising from $(L,K)$ are isomorphic. More precisely, we have,

**Theorem 12.2.5** *Let $L, K$ be a étale $k$-algebras of dimension $3, 2$ resp. and $(E,\tau)$ be the $K$-unitary algebra and $T$ the $K$-unitary torus associated with the pair $(L,K)$. Then $H^1(k,T) = 0$ if and only $J(E,\tau,u,\mu) \cong_L J(E,\tau,1,1)$, for all admissible pairs $(u,\mu) \in L^* \times K^*$.*

Proof. Suppose $J(E,\tau,u,\mu) \cong_L J(E,\tau,1,1)$ for all admissible pairs $(u,\mu) \in L^* \times K^*$. Let $S$ be as in Theorems 12.1.2.

**Claim:** $S = N_{E/L}(E^*)$ and $K^{(1)} = N_{E/K}(U(E,\tau))$.

Let $u \in S$. Since $J(E,\tau,u,\mu) \cong_L J(E,\tau,1,1)$, by Corollary 12.2.3, $u = N_{E/L}(w) = w\tau(w)$ and $\mu = N_{E/K}(w)$ for some $w \in E$. Hence $S = N_{E/L}(E^*)$. Let $\mu_0 \in K^{(1)}$. Since $J(E,\tau,1,\mu_0) \cong_L J(E,\tau,1,1)$, by Corollary 12.2.3, $\mu_0 = N_{E/K}(w)$ where $w\tau(w) = 1$. Hence $K^{(1)} = N_{E/K}(U(E,\tau))$, and by Theorem 12.1.3, $H^1(k,T) = 0$. The converse follows immediately from Theorem 12.2.4. $\square$

**Corollary 12.2.6** *Let $L$ be a cubic étale $k$-algebra with discriminant $\delta$ and $K = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra. Let $T$ be the $K$-unitary torus associated with the pair $(L,K)$ and $H^1(k,T) = 0$. Let $B$ be any degree 3 central simple algebra over $k(\sqrt{\alpha\delta})$ with an involution $\sigma$ of the second kind such that $L \subseteq (B,\sigma)_+$. Then $B \cong M_3(k(\sqrt{\alpha\delta}))$ and $\sigma$ is distinguished.*

Proof. By Theorem 5.3.2, there exists an admissible pair $(u,\mu) \in L^* \times K^*$ such that $\phi : (B,\sigma)_+ \cong J(E,\tau,u,\mu)$, where the isomorphism $\phi$ restricts to the identity of $L$. Since $H^1(k,T) = 0$, by Theorem 12.2.5, $(B,\sigma)_+ \cong J(E,\tau,u,\mu) \cong_L J(E,\tau,1,1)$. Since $J(E,\tau,1,1)$ is reduced (see Remark 5.3.4), by Theorem 5.3.1, $B \cong M_3(k(\sqrt{\alpha\delta}))$. By Lemma 12.3.1, there exists $v \in L$ such that $Int(v) \circ \sigma$ is distinguished. Since $\phi$ restricts to the identity of $L$, taking isotopes with respect to $v$ on both sides, we have $(B, Int(v) \circ \sigma)_+ \cong J(E,\tau,u,\mu)^{(v)}$ (see [38], Prop. 3.9). By ([38], Prop. 3.9) $J(E,\tau,u,\mu)^{(v)} \cong J(E,\tau,uv^{\#},N(v)\mu) \cong J(E,\tau,1,1)$. Hence $(B,\sigma)_+ \cong (B, Int(v)\circ\sigma)_+$. By ([19], Prop. 37.6), we have $f_3((B,\sigma)) = f_3((B, Int(v)\circ\sigma)) = 0$ and $\sigma$ is distinguished. $\square$

**Corollary 12.2.7** *Let the hypothesis be as in Corollary 12.2.6. Let $B$ be any degree 3 central simple algebra over $k(\sqrt{\alpha\delta})$ with an involution of the second kind such that $L \subseteq B$. Then*

$$B \cong M_3(k(\sqrt{\alpha\delta})).$$

Proof. By ([7], Prop.17, cf. [19], Cor. 19.30), there exists an involution $\sigma$ on $B$ such that $L \subseteq (B,\sigma)_+$. Hence by the above corollary $B$ splits. □

In view of Corollary 12.2.7, when $L$ is a cubic étale algebra over $k$ with trivial discriminant, we have the following

**Corollary 12.2.8** *Let $L$ be a cubic étale algebra over $k$ with trivial discriminant and $K$ be a quadratic étale $k$-algebra. Let $T$ be the $K$-unitary torus associated with the pair $(L,K)$ and $H^1(k,T)=0$. Let $B$ be any degree 3 central simple algebra over $K$ with an involution $\sigma$ of the second kind such that $L \subseteq B$. Then $B \cong M_3(K)$.*

Proof. Let $L$ be as in the hypothesis. When $L = k \times k \times k$, it is immediate that $B \cong M_3(K)$. When $L$ is a cubic cyclic field extension of $k$, by Corollary 12.2.7, we get the desired result. □

## 12.3 Application to algebraic groups

In this section we deduce that a group of type $G_2$ splits if and only if it contains a maximal torus whose first cohomology vanishes. A weaker result holds for groups of type $A_2$. We prove these results via explicit computations of cohomology done in section §12.1. We shall consider cohomology of maximal tori in groups of type $A_2$ and $G_2$. These tori arise from six dimensional unitary algebras, hence we can compute their cohomology using Theorems 12.1.5, 12.1.3 with $n=3$. We need a variant of ([7], Prop. 17) for our purpose.

**Lemma 12.3.1** *Let $F = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $F$ with an involution $\sigma$ of the second kind. Let $L$ be a cubic étale algebra such that $L \subseteq (B,\sigma)_+$. Then there exists $l \in L$ with $N_{L/k}(l) \in k^{*2}$ such that $Int(l) \circ \sigma$ is distinguished.*

Proof. Since $L \subseteq (B,\sigma)_+$, by ([7], Proposition 11), there exists $\mu \in L^*$ with $N_{L/k}(\mu) \in k^{*2}$ such that

$$Q_\sigma =< 1,2,2\delta > \perp < 2 > . << \alpha\delta >> . \, t_{L/k}(< \mu >).$$

Let $\lambda_0 \in L^*$ be such that $t_{L/k}(\lambda_0) = 0$ and let $\lambda := \frac{\lambda_0}{N_{L/k}(\lambda_0)}$. Then $\lambda \in L^*$ and $N_{L/k}(\lambda) \in k^{*2}$. Hence there exists $\xi \in k^*$ such that $N_{L/k}(\lambda\mu^{-1}) = \xi^2$. Consider $\psi := Int(\xi\lambda^{-1}\mu) \circ \sigma = Int(\lambda^{-1}\mu) \circ \sigma$.

**Claim:** $\psi$ is a distinguished involution.

We will use the proofs of ([7] Prop. 17, Corollary 14). Since $\lambda^{-1}\mu \in L$, we have $L \subseteq (B,\psi)_+$. Let $q : L \to L$ be defined by, $lq(l) = n_{L/k}(l)$. By ([7], Proposition 13), we have

$$Q_\psi =< 1,2,2\delta > \perp < 2 > . << \alpha\delta >> . \, t_{L/k}(< q(\xi\lambda^{-1}\mu)\mu >).$$

It is easy to check that $q(\xi\lambda^{-1}\mu) = \lambda\mu^{-1}$. Hence

$$\begin{aligned} Q_\psi &= \; < 1,2,2\delta > \perp < 2 > . << \alpha\delta >> . \, t_{L/k}(< \lambda >) \\ &= \; < 1,1,1 > \perp < 2\delta > . << \alpha >> . \, t_{L/k}(< \lambda >). \end{aligned}$$

Let $(B,\sigma)^\circ_+ = \{x \in (B,\sigma)_+ | \, T_B(x) = 0\}$ and $Q_\psi{}^\circ$ denote the restriction of $Q_\psi$ to $(B,\sigma)^\circ_+$. Then $Q_\psi{}^\circ =< 2 > .(< 1,3 > \perp < \delta > . << \alpha >> . \, t_{L/k}(< \lambda >)$. Since $t_{L/k}(\lambda) = 0$, the form $t_{L/k}(< \lambda >)$ is isotropicover $k$ and the Witt index of $<< \alpha >> . \, t_{L/k}(< \lambda >)$ is at least two. Hence by ([19], Theorem 16 (c)), $\psi$ is distinguished. □

**Theorem 12.3.2** *Let $F = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $B$ be a degree 3 central simple algebra over $F$ with an involution $\sigma$ of the second kind. Let $T$ be a maximal torus of $\mathbf{SU}(B,\sigma)$. If $H^1(k,T) = 0$ then $\sigma$ is distinguished.*

Proof. Let $T$ be a maximal torus of $\mathbf{SU}(B,\sigma)$. By Theorem 8.1.1, $T \cong \mathbf{SU}(E,\sigma)$, where $(E,\sigma) \subseteq (B,\sigma)$ is an $F$-unitary algebra. Let $L = E^\sigma$ and $Disc(L) = \delta$. By Lemma 8.1.3, $E = L \otimes F$. By Lemma 12.3.1, there exists $l \in L$, $N_{L/k}(l) \in k^{*2}$ such that $Int(l) \circ \sigma$ is distinguished. Let $\psi := Int(l) \circ \sigma$ and $S = \{u \in L^* | \, N_{L/k}(u) \in N_{F/k}(F^*)\}$. Since $H^1(k,T) = 0$, by Theorem 12.1.3 $\frac{S}{N_{E/L}(E^*)} = \{1\}$. Let $u \in S$. Then $u = w\sigma(w)$ for some $w \in E$ and $N_{L/k}(u) = \gamma\overline{\gamma}$ for some $\gamma \in F$. Consider the Albert algebra

$A := J(B,\sigma,u,\gamma)$. By ([19] Lemma 39.2), $J(B,\sigma,u,\gamma) \cong J(B,\sigma,w'u\sigma(w'),N_B(w')\gamma)$ for all $w' \in B^*$. Hence for $w' = w^{-1}$, we have $A = J(B,\sigma,u,\gamma) \cong J(B,\sigma,w\sigma(w),\gamma) \cong J(B,\sigma,1,\rho)$, where $\rho = N_B(w)^{-1}\gamma$. Therefore, $f_3(A) = f_3(B, Int(u)\circ\sigma) = f_3(B,\sigma)$ for all $u \in S$. Taking $u = l \in S$, we get $f_3(A) = f_3(B,\sigma) = 0$. Hence $\sigma$ is distinguished. Therefore $G$ is distinguished. □

**Remark 12.3.3** *A converse of the above theorem holds when $B$ is split. Let $F = k(\sqrt{\alpha})$ be a quadratic étale $k$-algebra and $\sigma$ be a distinguished involution on $M_3(F)$. Let $L = k\times F$. Note that $L \hookrightarrow M_3(F)$ as a $k$-subalgebra (via the embedding $(\gamma,x) \to diag(\gamma,x,x)$, $\gamma \in k$, $x \in F$). Since $\sigma$ is distinguished, by ([7], Cor 18), there exists a $k$-embedding $L \hookrightarrow (M_3(F),\sigma)_+$. Let $T$ be the $F$-unitary torus associated with the pair $(L,F)$. By Lemma 8.2.1, $T \hookrightarrow G$ over $k$. Then by case* **(ii)** *of the proof of Theorem 10.1.3, $T \cong R_{F/k}(\mathbb{G}_m)$. Hence $T \hookrightarrow \mathbf{SU}(B,\sigma)$ is a maximal $k$-torus with $H^1(k,T) = 0$.*

**Theorem 12.3.4** *Let $G$ be a group of type $G_2$ over $k$. Then $G$ splits over $k$ if and only if there exist a maximal $k$-torus $T \subset G$ such that $H^1(k,T) = 0$.*

Proof. Let $T \subset G$ be a maximal $k$-torus such that $H^1(k,T) = 0$. As in §2.8, there exists a quadratic étale $k$-algebra $K$ and $h \in GL_3(k)$ such that $T \subseteq \mathbf{SU}(M_3(K),*_h) \subseteq G$, where $*_h$ denotes the involution on $M_3(K)$ given by $*_h(X) = h^{-1}\overline{X}^t h$. Since $H^1(k,T) = 0$, $*_h$ is a distinguished involution (see Theorem 12.3.2). Hence by Theorem 9.2.3, $G$ splits over $k$. For the converse, we choose $T$ to be a split maximal $k$-torus in $G$, then $H^1(k,T) = 0$. □

**Remark 12.3.5** *Let $T = \mathbf{SU}(E,\tau)$ be a $k$-maximal torus in a simple, simply connected group $G$ of type $A_2$ or $G_2$. Recall that $H^1(k,T) \cong \frac{K^{(1)}}{N_{E/K}(U(E,\tau))} \times \frac{S}{N_{E/L}(E^*)}$ (see Theorem 12.1.3). In the proofs of Theorem 12.3.2 and Theorem 12.3.4, we do not require the hypothesis $H^1(k,T) = 0$ in its full force. The proofs use only the vanishing of the factor $\frac{S}{N_{E/L}(E^*)}$ of $H^1(k,T)$.*

### The Real Case

Let $G$ be a group of type $F_4$ over $\mathbb{R}$. Let $L, K$ be étale algebras over $\mathbb{R}$ of dimension $3, 2$ resp. and $T$ be the $K$-unitary torus associated with the pair $(L,K)$. Suppose $T \hookrightarrow G$

over $\mathbb{R}$. If $H^1(\mathbb{R}, T) = 0$ then $f_5(G) = 0$. Note that $K = \mathbb{R} \times \mathbb{R}$ or $\mathbb{C}$. If $K = \mathbb{R} \times \mathbb{R}$, then by Theorem 10.3.10, $f_5(A) = 0$. Suppose $K = \mathbb{C}$. Note that $L = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or $L = \mathbb{R} \times \mathbb{C}$. If $L = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, then by case **(i)** of proof of Theorem 10.1.3, $T \cong \mathbb{G}_m \times \mathbb{G}_m$ over $\mathbb{R}$. Hence $\mathbb{R}$-rank of $G \geq 2$ and by Lemma 10.1.2, $f_3(G) = f_5(G) = 0$. Suppose $L = \mathbb{R} \times \mathbb{C}$. Then by case **(ii)** of proof of Theorem 10.1.3, $T \cong R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ over $\mathbb{R}$. Hence $\mathbb{R}$-rank of $G \geq 1$ and by Lemma 10.1.2, $f_5(G) = 0$.

**Remark 12.3.6** *The real case along with Example 12.1.12, leads us to raise the following question: Let $L, K$ be étale algebras of dimension $3, 2$ resp. and $T$ be the $K$-unitary torus associated to the pair $(L, K)$. Let $G$ be a group of type $F_4$ defined over $k$ and $T \hookrightarrow G$ over $k$, then does $H^1(k, T) = 0$ imply $f_5(G) = 0$? Though we have not been able to settle this over an arbitrary field, we can prove a weaker result.*

**Theorem 12.3.7** *Let $L, K$ be étale algebras over $k$ of dimension $3, 2$ resp. and $E$ be the $K$-unitary algebra and $T$ the $K$-unitary torus associated with the pair $(L, K)$. Let $G$ be a group of type $F_4$ (resp. $G_2$ or a simple simply connected group of type $A_2$). Assume there is a $k$-embedding $T \hookrightarrow G$. If $H^1(\mathbf{U}(E, \tau)) = 0$ then $f_5(A) = 0$ (resp. $Oct(G)$ splits).*

Proof. Consider the exact sequence

$$1 \longrightarrow \mathbf{U}(E, \tau) \longrightarrow E^* \xrightarrow{N_{E/L}} L^* \longrightarrow 1.$$

The long exact cohomology sequence yields the exact sequence,

$$\mathbf{U}(E, \tau) \longrightarrow E^* \xrightarrow{N_{E/L}} L^* \longrightarrow H^1(\mathbf{U}(E, \tau)) \longrightarrow 1.$$

Hence

$$H^1(\mathbf{U}(E, \tau)) = L^*/N_{E/L}(E^*).$$

Since $H^1(\mathbf{U}(E, \tau)) = 0$, $L^* = N_{E/L}(E^*)$. Let $K = k(\sqrt{\alpha})$ and $q =< 1, -\alpha >$. If $K = k \times k$ then by Theorem 10.3.10, $f_5(G) = 0$. Hence we may assume that $K$ is a field extension. If $L = k \times K_0$, for some quadratic étale algebra $K_0$ over $k$, then $H^1(\mathbf{U}(E, \tau)) = k^*/N_{K/k}(K^*) \times K_0^*/N_{K_0 \otimes K/K_0}(K_0 \otimes K)^*$. Since $H^1(\mathbf{U}(E, \tau)) = 0$, $k^* = N_{K/k}(K^*)$. Hence $q$ is universal over $k$. By Theorem 10.3.10, $q$ divides $f_5(G)$ and is a subform of $f_5(G)$ (see Theorem 1.1.9). Hence we have $f_5(G) = 0$. Suppose $L$ is a field extension. Now $q_L = N_{E/L}(E^*)$. Since $q_L$ splits over $E$, by Knebusch norm principal (see Theorem 1.1.8), $N_{E/L}(E^*) = L^* \subseteq D_L(q_L)$. Hence $q$ is universal over an odd degree extension $L$ of $k$. By Springer's theorem, $q$ is universal over $k$. Let $G$ be a group of type $F_4$. By Theorem 10.3.10, $q$ divides $f_5(G)$, hence $f_5(G) = 0$. Let $G$ be a group of type $G_2$ or $A_2$. By Theorem 10.3.3, $q$ divides $f_3(G)$, hence $f_3(G) = 0$. Thus $Oct(G)$ splits. $\square$

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## List of Publications

1. Neha Hooda:

   Invariants Mod-2 and subgroups of $G_2$ and $F_4$.

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2. Neha Hooda:

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