A study
on

Maximal Intersecting Families of finite sets

THESIS

by

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Indian Statistical Institute

AUGUST 2015

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SUBMITTED IN PARTIAL FULFILMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

IN MATHEMATICS

BY

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Dedicated

to

My parents

Acknowledgement

It is needless to say that without the support, encouragement and friendly care of some people this research work would not have been possible. I am happy to acknowledge the help I received from various quarters.

It has been a great privilege to spend few years in the Theoretical Statistics and Mathematics Unit at Indian Statistical Institute, Bangalore Centre. I express my deepest gratitude to Indian Statistical Institute, both academically and personally, for the financial support and facilities. I am also grateful for having a chance to meet so many wonderful people and professionals. I perceive this opportunity of pursuing Mathematics as a big milestone in my career development. I will strive to use the gained skills and knowledge in the best possible way.

It is my radiant sentiment to place on record my sincere gratitude to Professor Bhaskar Bagchi, for the guidance which was extremely valuable. His quest for perfection and passion for Mathematics has always inspired me. From finding an appropriate problem in the beginning to the process of writing thesis, he offers his unreserved and unconditional help. I am thankful for his invaluably constructive criticism and friendly advice during this journey.

I would like to express my gratitude and respect to Professor Siva Athreya, Professor B.V. Rajarama Bhat, Professor Mohan Delampady, Professor Basudeb Datta, Professor Jaydeb Sarkar and Professor B. Sury. My deep respect, hearty felt gratitude and gratefulness will always remain unaffected to them.

I express my deepest thanks to Dr. Ambily A.A., Dr. Shibananda Biswas, Mr. Suprio Bhar, Dr. Anirban Bose, Dr. Prateep Chakraborty, Dr. Arup Chattopadhyay, Dr. Bata Krishna Das, Mr. Shubhabrata Das, Mr. Soumen Dey, Dr. Sushil Gorai, Dr. Sumesh K., Mrs. Sonali Majumder, Dr. Mithun Mukherjee, Mr. Satyaki Mukherjee, Dr. Sarbeswar Pal, Dr. Sourav Pal, Mr. Subham Sarkar and Dr. Amit Tripathi for taking part in useful discussions, giving necessary advices and guidance and arranged all facilities to make life easier. I choose this moment to acknowledge their contributions gratefully. Finally, I like to acknowledge the most important persons of my life:- my parents and family. They have supported me endlessly throughout this journey.

I take full responsibility for any errors or inadequacies (that may remain) in my thesis work.

Place: Kolkata Kaushik Majumder

Date : August 14, 2015.

By a *family* we shall mean a family of finite sets. For a family $\mathcal{F}$, the members of $\mathcal{F}$ are called its *blocks* and the elements of the blocks are called its *points*. A family $\mathcal{F}$ is said to be *uniform* if all its blocks have the same size. If $\mathcal{F}$ is a uniform family we shall denote its common block size by $\mathrm{k}(\mathcal{F})$. A *blocking set* of a family $\mathcal{F}$ is a set which intersects every block of $\mathcal{F}$. We define a *transversal* of $\mathcal{F}$ to be a blocking set of $\mathcal{F}$ with the smallest possible size – in case $\mathcal{F}$ has a finite blocking set. In this case we denote by $\mathrm{tr}(\mathcal{F})$ the common size of its transversals. If $\mathcal{F}$ has no finite blocking set we may put $\mathrm{tr}(\mathcal{F}) = \infty$. (Please note that, many authors use the word transversal as a synonym for a blocking set.) If $\mathrm{tr}(\mathcal{F}) < \infty$, we denote the family of transversals of $\mathcal{F}$ by $\mathcal{F}^{\top}$. Note that $\mathcal{F}^{\top}$ is a uniform family with $\mathrm{k}(\mathcal{F}^{\top}) = \mathrm{tr}(\mathcal{F})$. Now we introduce:-

Definition. A family $\mathcal{F}$ is said to be a *maximal intersecting family* (in short MIF) if $\mathrm{tr}(\mathcal{F}) < \infty$ and $\mathcal{F} = \mathcal{F}^{\top}$. We use $\mathrm{MIF}(k)$ as a generic name for MIF's with $\mathrm{k}(\mathcal{F}) = k$.

We say that a family $\mathcal{F}$ is an *intersecting family* if any two blocks of $\mathcal{F}$ have non empty intersection. Clearly any $\mathrm{MIF}(k)$ is an intersecting family. Indeed, the $\mathrm{MIF}(k)$'s are characterised among all $k-$uniform intersecting families as those families which are maximal with respect to the property of being intersecting. Thus, an intersecting family $\mathcal{F}$ of $k-$sets is a $\mathrm{MIF}(k)$ if and only if there is no $k-$set outside $\mathcal{F}$ (anywhere in the universe of all sets!) which is a blocking set of $\mathcal{F}$. In the hypergraph literature these are known as the *maximal $k-$cliques*.

Paul Erdős and László Lovász proved in [7] that, for any positive integer k, up to isomorphism there are only finitely many *maximal intersecting families of $k-$sets*. So they posed the problem of determining or estimating the largest number $\mathrm{N}(k)$ of the points and the largest number $\mathrm{M}(k)$ of blocks in such a family. Today these two problems remain more or less where Paul Erdős, László Lovász, Zsolt Tuza, Péter Frankl, Katsuhiro Ota and Norihide Tokushige left them. For instance, it is not known for large k which $\mathrm{MIF}(k)$ has $\mathrm{N}(k)$ points and which $\mathrm{MIF}(k)$ has $\mathrm{M}(k)$ blocks. This thesis work mainly deals with these two problems of finding a $\mathrm{MIF}(k)$ with $\mathrm{N}(k)$ points and finding a $\mathrm{MIF}(k)$ with $\mathrm{M}(k)$ blocks. We put our best effort to keep this work self contained and self explanatory. We now outline the contents of the thesis briefly.

In **Chapter 1**, we present a review of the literature on *maximal intersecting families* along with some proofs and constructions.

Paul Erdős and László Lovász proved by means of an example that

$$\mathrm{N}(k) \geq 2k - 2 + \frac{1}{2}\binom{2k-2}{k-1}.$$

Much later, Zsolt Tuza proved that the bound is best possible up to a multiplicative constant by showing that asymptotically $\mathrm{N}(k)$ is at most 4 times this lower bound. In **Chapter 2**, we reduce the gap between the lower and upper bound by showing that asymptotically $\mathrm{N}(k)$ is at most 3 times the Erdős-Lovász lower bound.

We find that each maximal intersecting family has a "core" which generates it. We call this core a *closed intersecting family*. In **Chapter 3**, we introduce the notion of closed intersecting families, some of its properties and examples.

In **Chapter 4**, we classify all the maximal intersecting families of 3−sets. We prove that there are 8 non isomorphic maximal intersecting families of 3−sets.

In **Chapter 5**, we study constructions over the cyclic graph. Erdős and Lovász showed by means of an example that there exists a $\mathrm{MIF}(k)$ with approximately $(e-1)k!$ blocks. This example is constructed by a recursive procedure. Lovász conjectured in [17], that the $\mathrm{MIF}(k)$ thus constructed was the extremal one. In this chapter, we present simpler constructions (see $\mathbb{G}(k,t)$ and $\mathbb{F}(k,t)$ in Construction 5.2.1) to prove that there exist at least two $\mathrm{MIF}(k)$ with at least (approximately) $(\frac{k}{2})^{k-1}$ blocks. (More precisely, we present an alternative proof of [9, § 2, Theorem 1], see Corollary 5.3.6 below). In [9], Frankl et al. conjectured that the maximal intersecting family of $k-$sets constructed by them has the largest number of blocks, and it is the only such family (up to isomorphism) with these many blocks. We use the theory developed in this chapter to prove that both these conjectures are false, at least for small k. Specifically, the uniqueness part is incorrect for $k = 4$, while the optimality part is incorrect for $k = 5$. We close this chapter by posing some interesting conjectures.

In the **appendices**, we study two extremal questions about $\mathrm{ISP}(k,t)$. The first is called the Bollobás Inequality. It deals with the problem of finding maximum number of pairs in an $\mathrm{ISP}(k,t)$. The second theorem mentioned here is based on the problem of finding maximum number of points in an $\mathrm{ISP}(k,t)$. These two theorems played an essential role in Chapter 2. In the final appendix, we re-investigate the transversal size of the family $\mathbb{F}(k,t)$ as described in Chapter 5, with a different approach. Here we are able to show that $\mathrm{tr}(\mathbb{F}(k,t)) = t$ for $t \leq 10$. Finally, we present a new proof of the fact that $\mathbb{F}(k,2)$ is the unique intersecting family of $k-$sets with the maximum number of transversals of size 2. This result was originally proved by Frankl et al. in [8].

Chapter 1

Introduction to Maximal Intersecting Families of finite sets

In this chapter, we survey the literature on *maximal intersecting families* and recall some important theorems and examples. An effort has been given to keep it self contained. The definitions and notations given here will be used throughout this work.

1.1 Introduction

By a *family* we mean a family (set) of finite sets. Such a family is called *intersecting* if any two of its members have non empty intersection. A *maximal intersecting family of $k-$sets* is an intersecting family which can not be embedded properly into any larger intersecting family of $k-$sets.

Definition. Let $\mathcal{G}$ be a non empty family of non empty sets. Any $B \in \mathcal{G}$ is called a *block* of $\mathcal{G}$. The *point set* of the family $\mathcal{G}$ is defined as $\underset{B\in\mathcal{G}}{\cup} B$ and is denoted by $\mathrm{P}_{\mathcal{G}}$. Any $x \in \mathrm{P}_{\mathcal{G}}$ is called a *point* of $\mathcal{G}$. In case $\mathcal{G}$ is finite, we denote its number of points (size of the point set) by $\mathrm{v}(\mathcal{G})$. A family $\mathcal{G}$ is said to be *uniform* if all its blocks have the same size. If $\mathcal{G}$ is a uniform family we denote its common block size by $\mathrm{k}(\mathcal{G})$.

Definition. A family $\mathcal{G}$ is said to be *isomorphic* to the family $\mathcal{H}$ if there exists a one-to-one and onto function $\phi : \mathrm{P}_{\mathcal{G}} \to \mathrm{P}_{\mathcal{H}}$ such that $\phi(B) \in \mathcal{H}$ if and only if $B \in \mathcal{G}$.

Definition. A *blocking set* of a family $\mathcal{G}$ is a set C which intersects every block of $\mathcal{G}$. In case $\mathcal{G}$ has a finite size blocking set, a blocking set of $\mathcal{G}$ of the smallest possible size is called a *transversal* of $\mathcal{G}$. In this case, we denote the common size of its transversals by $\mathrm{tr}(\mathcal{G})$. If $\mathcal{G}$ has no finite blocking set, we may put $\mathrm{tr}(\mathcal{G}) = \infty$. If $\mathrm{tr}(\mathcal{G}) < \infty$, we denote the family of transversals of $\mathcal{G}$ by $\mathcal{G}^{\top}$. Note that $\mathcal{G}^{\top}$ is a uniform family with $\mathrm{k}(\mathcal{G}^{\top}) = \mathrm{tr}(\mathcal{G})$.

Warning: Many authors use the word transversal as a synonym for a blocking set.

Definition. A family $\mathcal{F}$ is said to be a *maximal intersecting family* (in short MIF) if $\mathrm{tr}(\mathcal{F}) < \infty$ and $\mathcal{F} = \mathcal{F}^{\top}$. We use $\mathrm{MIF}(k)$ as a generic name for uniform MIF's with $\mathrm{k}(\mathcal{F}) = k$.

Clearly any $\mathrm{MIF}(k)$ is an intersecting family. In fact, the $\mathrm{MIF}(k)$'s are characterised among all $k-$uniform intersecting families as those families which are maximal with respect to the property of being intersecting. In the hypergraph literature the intersecting families of $k-$sets are called the *$k-$cliques* and the MIF(k)'s are known as the *maximal $k-$cliques.*

Let us give the following characterisation. In this thesis work we mostly use (c) and (d) of this proposition.

Proposition 1.1.1. *Let $\mathcal{F}$ be an intersecting family of $k-$sets with* $\mathrm{tr}(\mathcal{F}) \leq k$. *Then the following statements are equivalent:*

(a) *There is no k-set outside $\mathcal{F}$ (anywhere in the universe of all sets!) which can be added to the family without violating the property that $\mathcal{F}$ is an intersecting family of $k-$sets.*

(b) *There is no $k-$set outside $\mathcal{F}$ (anywhere in the universe of all sets!) which is a blocking set of $\mathcal{F}$.*

(c) *For any set C, consisting of at most k points of $\mathcal{F}$, if C is a blocking set of $\mathcal{F}$, then C itself is a block of $\mathcal{F}$.*

(d) $\mathcal{F} = \mathcal{F}^{\top}$.

`Proof` : Firstly we prove (a) $\Leftrightarrow$ (b), secondly we prove (b) $\Leftrightarrow$ (c) and finally we prove (c) $\Leftrightarrow$ (d).

Suppose (a) holds. Let C be a blocking $k-$set of $\mathcal{F}$ but $C \notin \mathcal{F}$. Then $\mathcal{F} \sqcup \{C\}$ is an intersecting family of $k-$sets. It contradicts the assumption (a). Hence (b) holds. Conversely suppose (b) holds. Let C be a $k-$set but $C \notin \mathcal{F}$ and it can be added to the family $\mathcal{F}$ without violating the property that $\mathcal{F}$ is an intersecting family of $k-$sets. It means that C is a blocking $k-$set of $\mathcal{F}$. It contradicts the assumption (b). Hence (a) holds.

Suppose (b) holds. Let C be a blocking set of $\mathcal{F}$, with $|C| \leq k$. Suppose C is not a block of $\mathcal{F}$, i.e. $C \notin \mathcal{F}$. Let us choose a $k-$set $C^{'}$, with $C \subsetneqq C^{'}$ such that $(C^{'} \smallsetminus C)$ is disjoint from $\mathrm{P}_{\mathcal{F}}$. Then $C^{'}$ is a blocking $k-$set of $\mathcal{F}$ but $C^{'} \notin \mathcal{F}$. It contradicts the assumption (b). Hence (c) holds. Conversely suppose (c) holds, if (b) is false, then there exists at least one blocking $k-$set of $\mathcal{F}$ stays outside $\mathcal{F}$, call it C. But by assumption (c) we have $C \in \mathcal{F}$, a contradiction. Hence (b) holds.

Suppose (c) holds. Since $\mathcal{F}$ is an intersecting family of $k-$sets, $\mathrm{tr}(\mathcal{F}) \leq k$. Suppose $\mathrm{tr}(\mathcal{F}) \leq k-1$, then $T \in \mathcal{F}^{\top}$. By assumption (c) we have $T \in \mathcal{F}$, with $|T| \leq k-1$, a contradiction. Therefore $\mathrm{tr}(\mathcal{F}) = k$, consequently $\mathcal{F} \subsetneqq \mathcal{F}^{\top}$. Let $T \in \mathcal{F}^{\top}$, then $|T| = k$, by assumption (c) we have $T \in \mathcal{F}$. Therefore, $\mathcal{F}^{\top} \subsetneqq \mathcal{F}$. Hence (d) holds. Conversely suppose (d) holds, let C be a blocking set of $\mathcal{F}$, with $|C| \leq k$. Now (d) implies $\mathrm{tr}(\mathcal{F}) = k$, therefore $|C| = k$, consequently $C \in \mathcal{F}$. Hence (c) holds. $\square$

Construction 1.1.2. Let k be positive integers and P be a $(2k-1)-$set. Construct the family $\mathcal{F} := \binom{P}{k}$, i.e. $\mathcal{F}$ is the family of all possible $k-$subsets of P. We denote this family by $\beta(k)$.

Theorem 1.1.3. *$\beta(k)$ is a* MIF(k).

`Proof` : Let B, $B^{'} \in \mathcal{F}$ then B intersects $B^{'}$ for otherwise $|P| \geq |B \sqcup B^{'}| \geq 2k$, a contradiction. Hence $\mathcal{F}$ is an intersecting family of $k-$sets. Let $C \subset P$ with $|C| \leq k$ and C is not a block of $\mathcal{F}$, hence $|C| \leq k-1$. We show that there exists a block of $\mathcal{F}$, which is disjoint from C. But $|P \setminus C| \geq k$ and any $k-$subset of $P \setminus C$ is the required block of $\mathcal{F}$ which is disjoint from C. $\square$

Theorem 1.1.4. *Any* MIF(k) *has at least* $2k-1$ *points.* $\beta(k)$ *is the only* MIF(k) *with* $2k-1$ *points.*

`Proof` : Let $\mathcal{A}$ be a MIF(k). Since $\mathcal{A}$ is non empty, it has at least one block say B. We choose any $(k-1)-$subset C of B. Then there is a block $B^{'}$ disjoint from C. So $B^{'} \sqcup C$ contains $2k-1$ points. Suppose $\mathcal{A}$ has exactly $2k-1$ points. We show that each set C of k points of $\mathcal{A}$ is a block of $\mathcal{A}$. If not, then exists a block B disjoint from C. Then $B \sqcup C$ contains $2k > (2k-1)$ points, a contradiction. $\square$

A *projective plane* of order k is a family of $(k+1)-$sets so that (a) for any two points there exists a unique block containing those two points and (b) any two blocks intersect in a unique point. It can be shown that, every point is in $k+1$ blocks and every block contains $k+1$ points. It has k^2+k+1 blocks and k^2+k+1 points.

Theorem 1.1.5. *Any projective plane of order k (if it exists) is a* MIF$(k+1)$.

`Proof` : Let $\mathcal{P}$ be a projective plane of order k. Since any two blocks intersect in a unique point, $\mathcal{P}$ is an intersecting family. Let C be a blocking set of $\mathcal{P}$, with $|C| \leq k+1$. We show that C is a block (or line) of $\mathcal{P}$. We observe that there are at least k^2 points not in C, we choose one such point $x \notin C$. The $k+1$ blocks through x are pairwise disjoint outside x and each of them intersects C in at least one point. Hence $|C| \geq k+1$. Then $|C| = k+1$ and each block through x meets C in a unique point. Now take two points $y \neq z$ in C (as $|C| = k+1 \geq 3$). Let l be the block joining y and z.

Claim : $l \subset C$

`Proof of claim` : Suppose not, then there exists $p \in (l \setminus C)$. Then $p \notin C$, but the block l through p meets C in at least two points y and z. This contradicts what we discussed in the previous paragraph. Hence the claim is established.

But $|l| = k+1 = |C|$. So $C = l$ is a block . $\square$

1.2 Examples

In this section we present some examples. These examples are important because they shed light on some extremal questions.

Construction 1.2.1 (§ 3, Construction (b), [7])**.** Let k be positive integers with $k \geq 2$. Let P be a $(2k-2)-$set. For each bi-partition $(A, P \setminus A)$ of P with $|A| = |P \setminus A| = k-1$, we introduce a new symbol x_a. We consider the family of all $k-$subsets of P together with all $k-$sets of the forms $\{x_a\} \sqcup A$ and $\{x_a\} \sqcup (P \setminus A)$. We denote this family of $k-$sets by $\beta_g(k)$.

Theorem 1.2.2. *$\beta_g(k)$ is a* MIF(k).

`Proof` : Clearly $\beta_g(k)$ is an intersecting family of $k-$sets. Let C be a blocking $k-$set of $\beta_g(k)$. We show that $C \in \beta_g(k)$. If $|C \cap P| \leq k-2$, then any $k-$subset of $P \setminus C$ is disjoint from C, a contradiction. Hence $|C \cap P| \geq k-1$. If $|C \cap P| = k$ then $C \in \beta_g(k)$ and we are done this case. Otherwise, we assume $|C \cap P| = k-1$. Since $|C| = k$ and $|C \cap P| = k-1$, C contains exactly one new symbol. Since $|C \cap P| = k-1$, it induces a natural bi-partition of P, namely $(A, P \setminus A)$, with $A = C \cap P$. If $C \neq A \sqcup \{x_a\}$, where x_a is the new symbol corresponds to the bi-partition $(A, P \setminus A)$, then $(P \setminus A) \sqcup \{x_a\}$ is disjoint from C. Therefore $C \neq A \sqcup \{x_a\}$ is not possible. So $C = A \sqcup \{x_a\}$, i.e. $C \in \beta_g(k)$. $\square$

An *affine plane* of order k is a family of $k-$sets so that (a) for any two points there exists a unique block containing those two points; (b) given a point x and a block B with $x \notin B$, there exists a unique block B', containing x and disjoint from B and (c) there are three distinct points not on a block.

Let B and B' be two blocks of an affine plane. We say B is parallel to B' if and only if $B = B'$ or $B \cap B' = \emptyset$. This relation is an equivalence relation. Any equivalence class with respect to this relation is called a parallel class. In an affine plane of order k there are k^2 points, k^2+k blocks and $k+1$ parallel classes. Each such parallel class contains k blocks and each block contains k points.

NOTATION : Let $\mathcal{G}$ and $\mathcal{H}$ be two non empty families of non empty sets. Suppose $\mathrm{P}_{\mathcal{G}}$ and $\mathrm{P}_{\mathcal{H}}$ are disjoint. Then $\mathcal{G} \circledast \mathcal{H}$ denotes the collection of all sets of the form $A \sqcup B$, where $A \in \mathcal{G}$ and $B \in \mathcal{H}$. If $\mathcal{G}$ consists of a single $k-$set B, then we denote $\mathcal{G} \circledast \mathcal{H}$ by $B \circledast \mathcal{H}$. If $\mathcal{G}$ consists of a single $1-$set $\{\alpha\}$, then we denote $\mathcal{G} \circledast \mathcal{H}$ by $\alpha \circledast \mathcal{H}$.

Lemma 1.2.3. *Let $\mathcal{F}$ be an affine plane of order k (assuming it exists) and $\mathcal{F}_0$, $\mathcal{F}_1, \ldots,$ $\mathcal{F}_k$ be its parallel classes. Fix an n_0, with $0 \leq n_0 \leq k$. If C is a blocking $k-$set of $\mathcal{F}_n$ for each $n \neq n_0$, with $0 \leq n \leq k$, then $C \in \mathcal{F}_{n_0}$.*

`Proof` : Let $L_\infty := \{\alpha_i : 0 \leq i \leq k\}$ be a set disjoint from $\mathrm{P}_\mathcal{F}$. Then

$$\mathcal{P}(\mathcal{F}) := \{L_\infty\} \sqcup (\bigsqcup_{i=0}^{k} (\alpha_i \circledast \mathcal{F}_i))$$

is a projective plane of order k and $C \sqcup \{\alpha_{n_0}\}$ is a blocking set of $\mathcal{P}(\mathcal{F})$. Hence by Theorem 1.1.5, $C \sqcup \{\alpha_{n_0}\}$ is a block of $\mathcal{P}(\mathcal{F})$. Consequently by the construction of $\mathcal{P}(\mathcal{F})$, we have $C \in \mathcal{F}_{n_0}$. □

Construction 1.2.4 (§ 2, [11])**.** Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be affine planes of order k (provided they exist) with pairwise disjoint point sets. Let $\mathcal{F}_0$, $\mathcal{F}_1, \ldots, \mathcal{F}_k$; $\mathcal{G}_0$, $\mathcal{G}_1, \ldots, \mathcal{G}_k$ and $\mathcal{H}_0$, $\mathcal{H}_1, \ldots, \mathcal{H}_k$ be the parallel classes of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ respectively. Let $\mathcal{F}_0 = \{F_i : 1 \leq i \leq k\}$, $\mathcal{G}_0 = \{G_i : 1 \leq i \leq k\}$ and $\mathcal{H}_0 = \{H_i : 1 \leq i \leq k\}$. Let $\mathbb{A}(k)$ be the union of all families of the form $F_n \circledast \mathcal{G}_n$, $G_n \circledast \mathcal{H}_n$ and $H_n \circledast \mathcal{F}_n$, where $1 \leq n \leq k$.

Theorem 1.2.5. *Let k be a positive integer such that affine planes of order k exist. Then $\mathbb{A}(k)$ is a* MIF$(2k)$.

`Proof` : Clearly $\mathbb{A}(k)$ is an intersecting family of $2k-$sets. Let C be a blocking set of $\mathbb{A}(k)$, with $|C| \leq 2k$. We show that $C \in \mathbb{A}(k)$. Suppose $|C \cap \mathrm{P}_\mathcal{F}| \leq k-1$. Then there exists $F_{n_0} \in \mathcal{F}_0$ for some n_0, with $1 \leq n_0 \leq k$, disjoint from $C \cap \mathrm{P}_\mathcal{F}$. But C is a blocking set of $F_{n_0} \circledast \mathcal{G}_{n_0}$. Since $\mathcal{G}_{n_0}$ consists of k mutually disjoint $k-$sets, $|C \cap \mathrm{P}_\mathcal{G}| \geq k$. Similarly, $|C \cap \mathrm{P}_\mathcal{G}| \leq k-1$ implies $|C \cap \mathrm{P}_\mathcal{H}| \geq k$ and $|C \cap \mathrm{P}_\mathcal{H}| \leq k-1$ implies $|C \cap \mathrm{P}_\mathcal{F}| \geq k$. Together we have if $|C \cap \mathrm{P}_\mathcal{F}| \leq k-1$, then $|C \cap \mathrm{P}_\mathcal{G}| \geq k$ and $|C \cap \mathrm{P}_\mathcal{H}| \geq k$. Since $|C| \leq 2k$, $|C \cap \mathrm{P}_\mathcal{F}| \leq k-1$ implies $|C \cap \mathrm{P}_\mathcal{G}| = k$ and $|C \cap \mathrm{P}_\mathcal{H}| = k$, consequently $C \cap \mathrm{P}_\mathcal{F}$ is empty. But C is a blocking set of $\mathbb{A}(k)$ and $C \cap \mathrm{P}_\mathcal{F}$ is empty, so $C \cap \mathrm{P}_\mathcal{G}$ is a blocking set of $F_n \circledast \mathcal{G}_n$ for each n, with $1 \leq n \leq k$. By using Lemma 1.2.3, we have $C \cap \mathrm{P}_\mathcal{G} \in \mathcal{G}_0$ say $C \cap \mathrm{P}_\mathcal{G} = G_{m_0}$ for some m_0, with $1 \leq m_0 \leq k$. Also $C \cap \mathrm{P}_\mathcal{H}$ is a blocking set of $H_n \circledast \mathcal{F}_n$ for each n, with $1 \leq n \leq k$. Since $C \cap \mathrm{P}_\mathcal{G} = G_{m_0}$, $C \cap \mathrm{P}_\mathcal{H}$ is a blocking set of $G_n \circledast \mathcal{H}_n$ for each $n \neq m_0$, with $1 \leq n \leq k$. In other words, $C \cap \mathrm{P}_\mathcal{H}$ is a blocking set of H_n for each $n \neq m_0$, with $0 \leq n \leq k$. Therefore, by using Lemma 1.2.3, we have $C \cap \mathrm{P}_\mathcal{H} \in \mathcal{H}_{m_0}$. Hence $C \in G_{m_0} \circledast \mathcal{H}_{m_0} \subset \mathbb{A}(k)$. Similarly, $|C \cap \mathrm{P}_\mathcal{G}| \leq k-1$ implies $C \in \mathbb{A}(k)$ and $|C \cap \mathrm{P}_\mathcal{H}| \leq k-1$ implies $C \in \mathbb{A}(k)$. Since $|C| = 2k$, either $|C \cap \mathrm{P}_\mathcal{F}| \leq k-1$ or $|C \cap \mathrm{P}_\mathcal{G}| \leq k-1$ or $|C \cap \mathrm{P}_\mathcal{H}| \leq k-1$. Therefore in any case $C \in \mathbb{A}(k)$. □

1.3 Extremal questions on the number of points and blocks

The *chromatic number* of a family is the smallest number of colours needed to colour the points in such a way that no monochromatic block occurs. It is trivial to see that the chromatic number of any uniform intersecting family (clique) $\mathcal{F}$ is at most 3. (Let us choose $x \in B \in \mathcal{F}$. Assign the first colour to x, the second colour to the other points of B and the third colour to the remaining points.) Thus any such family is either $2-$chromatic or $3-$chromatic. Every $k-$uniform $3-$chromatic intersecting family is a *maximal intersecting family of $k-$sets*. But the converse is not true. Any projective plane of order $q \geq 3$ (if it exists) is an example of a $2-$chromatic maximal intersecting family of $(q+1)-$sets. Also, it is easy to see that a $k-$uniform intersecting family is $3-$chromatic if and only if its blocks are the only minimal (as opposed to just minimum sized) blocking sets. Paul Erdős and László Lovász [7] were mainly concerned with $k-$uniform $3-$chromatic intersecting families. This is a proper subclass of the class of maximal intersecting families of $k-$sets. They proved the surprising result that any $\mathrm{MIF}(k)$ is finite; indeed it has at most k^k blocks. Therefore, we define the integers $\mathrm{N}(k)$, $\mathrm{M}(k)$ and $\mathrm{m}(k)$ as follows. Here we present a brief sketch of the results from the literature.

$$\mathrm{N}(k) := \max\left\{|\mathrm{P}_{\mathcal{F}}| : \mathcal{F} \text{ is a } \mathrm{MIF}(k)\right\},$$
$$\mathrm{M}(k) := \max\left\{|\mathcal{F}| : \mathcal{F} \text{ is a } \mathrm{MIF}(k)\right\},$$
$$\mathrm{m}(k) := \min\left\{|\mathcal{F}| : \mathcal{F} \text{ is a } \mathrm{MIF}(k)\right\}.$$

Theorem 1.3.1 ([5]). *Let $k \geq 4$ be a positive integer. Then $\mathrm{m}(k) \geq 3k$.*

`Proof` : Fix a block B of a $\mathrm{MIF}(k)$ $\mathcal{A}$. Note that through each point $x \in B$ there is a block B' of $\mathcal{A}$ which is tangent to B (i.e., such that $B \cap B' = \{x\}$). For $i \geq 1$, let m_i be the number of points of B through which exactly i tangents to B pass. Thus at least $1 \cdot m_1 + 2 \cdot m_2 + 3(k - m_1 - m_2) = 3k - 2m_1 - m_2$ blocks are tangent to B. If x is one of the m_1 points of B through which a unique tangent B' to B passes, then, for each $y \in B' \smallsetminus \{x\}$, $(B \smallsetminus \{x\}) \sqcup \{y\}$ is (a transversal and hence) a block of $\mathcal{A}$. So we get $(k-1)m_1$ blocks intersecting B in $k-1$ points.

If $x_1 \neq x_2$ are two of the m_2 points of B through each of which two tangents pass, then either there is a block intersecting B in $\{x_1, x_2\}$ or (letting C_1 and C_2 be the tangents through x_1, D_1 and D_2 be the tangents through x_2, and choosing $y_1 \in C_1 \cap D_1$ and $y_2 \in C_2 \cap D_2$) the set $(B \smallsetminus \{x_1, x_2\}) \sqcup \{y_1, y_2\}$ is a (transversal and hence) block of $\mathcal{A}$ intersecting B in $k-2$ points. Thus $\binom{m_2}{2}$ distinct blocks intersect B in 2 or $k-2$ points. Thus including B we get $1 + 3k - 2m_1 - m_2 + m_1(k-1) + \binom{m_2}{2}$ distinct blocks of $\mathcal{A}$. So $\mathrm{m}(k)$

is at least the minimum of this expression (over all $m_1 \geq 0$, $m_2 \geq 0$ with $m_1 + m_2 \leq k$) which is $3k$. $\square$

In [23], it was proved that $\mathrm{m}(4) = 12$ and there is a unique $\mathrm{MIF}(4)$ with 12 blocks. We note that, Jeff Kahn [16, Conjecture 5.1] made a conjecture on the numbers $\mathrm{m}(k)$:

Conjecture 1.3.2 (Kahn)**.** $\lim\limits_{k\to\infty} \frac{\mathrm{m}(k)}{k}$ exists.

In [11], it is shown that if an affine plane of order k exists, then $\mathrm{m}(2k) \leq 3k^2$. This upper bound is produced through an example (namely, $\mathbb{A}(k)$ in Construction 1.2.4), which we discussed in Theorem 1.2.5. In [6] and [3], it is established respectively that if a projective plane of order k exists, then $\mathrm{m}(k^n + k^{n-1}) \leq k^{2n} + k^{2n-1} + k^{2n-2}$ for every positive integer n, and if $k-1$ is a prime power, then $\mathrm{m}(k) \leq \frac{k^2}{2} + 5k + o(k)$. However in [1], we have the strongest known result which is applicable to all positive integers k, it states that $\mathrm{m}(k) \leq k^5$.

This thesis work mainly deals with the problems of finding or estimating the numbers $\mathrm{N}(k)$ and $\mathrm{M}(k)$. Here we briefly discuss the previous and new results on $\mathrm{N}(k)$ and $\mathrm{M}(k)$.

Theorem 1.3.3 (Theorem 8, Erdős-Lovász, [7])**.**

$$2k - 2 + \frac{1}{2}\binom{2k-2}{k-1} \leq \mathrm{N}(k) \leq \frac{k}{2}\binom{2k-1}{k}.$$

From Theorem 1.2.2, we have $\beta_g(k)$ is a $\mathrm{MIF}(k)$ with $2k-2+\frac{1}{2}\binom{2k-2}{k-1}$ points. Hence the lower bound follows. In [25], Tuza improved the upper bound by showing:

Theorem 1.3.4. $\mathrm{N}(k) \leq \binom{2k-1}{k} - \binom{2k-3}{k} + \frac{3}{2}\sum\limits_{i=1}^{t-1}\binom{2i}{i}$.

However, we prove in Chapter 2 that

$$\limsup_{k\to\infty} \frac{\mathrm{N}(k)}{\binom{2k-2}{k-1}} \leq \frac{3}{2}.$$

Theorem 1.3.5 (§ 3, Construction (c), [7])**.** *Let $k \geq 2$ be a positive integer. Let $\mathcal{A}$ be a* $\mathrm{MIF}(k-1)$ *and let B be a k-set disjoint from* $\mathrm{P}_{\mathcal{A}}$*. Define*

$$\widehat{\mathcal{A}} = \{B\} \cup \{A \sqcup \{x\} : A \in \mathcal{A}, x \in B\}.$$

Then $\widehat{\mathcal{A}}$ is a $\mathrm{MIF}(k)$*.*

`Proof` : Clearly $\widehat{\mathcal{A}}$ is an intersecting family of $k-$sets. Let C be a blocking set of $\widehat{\mathcal{A}}$, with $|C| \leq k$. We show that $C \in \widehat{\mathcal{A}}$, i.e. we show that $C = B$ or $C = A \sqcup \{x\}$ for some $x \in B$

and $A \in \mathcal{A}$. Suppose $C \neq B$, then $B \setminus C$ is non empty and we choose $x_0 \in B \setminus C$. Since C intersects B, we have the natural decomposition $C = (C \cap \mathrm{P}_{\mathcal{A}}) \sqcup (C \cap B)$ and $C \cap \mathrm{P}_{\mathcal{A}}$ is a block of $\mathcal{A}$. (If not, then there exists $A \in \mathcal{A}$ disjoint from $C \cap \mathrm{P}_{\mathcal{A}}$. Consequently $A \sqcup \{x_0\}$ is disjoint from C.) Hence $|C \cap \mathrm{P}_{\mathcal{A}}| = k-1$ and $|C \cap B| = 1$. Therefore $C \in \widehat{\mathcal{A}}$. □

In this context, we mention that an alternative proof of Theorem 1.3.5 can be found by applying Theorem 3.2.3 (taking $\mathcal{F} = \{B\}$).

Theorem 1.3.6 (Theorem 7, Erdős-Lovász, [7])**.**

$$\lfloor (e-1)k! \rfloor \leq \mathrm{M}(k) \leq k^k.$$

Starting with $\beta(2)$ and applying repeatedly Theorem 1.3.5, we construct a MIF(k) with approximately $(e-1)k!$ blocks and this gives the lower bound of the Theorem 1.3.6. However, using Construction 5.2.1 (namely, $\mathbb{G}(k, k-1)$ and $\mathbb{F}(k, k-1)$), we construct at least two MIF(k)s with at least (approximately) $(\frac{k}{2})^{k-1}$ blocks. The upper bound of Theorem 1.3.6 is a special case of Theorem 2.2.2 (namely, case $t = k$).

Chapter 2

Maximum number of points in a Maximal Intersecting Family of finite sets

Paul Erdős and László Lovász proved in a landmark article that, for any positive integer k, up to isomorphism there are only finitely many *maximal intersecting families of* $k-$*sets* (maximal $k-$cliques). So they posed the problem of determining or estimating the largest number $\mathrm{N}(k)$ of the points in such a family. They also proved by means of an example that $\mathrm{N}(k) \geq 2k-2+\frac{1}{2}\binom{2k-2}{k-1}$. Much later, Zsolt Tuza proved that the bound is best possible up to a multiplicative constant by showing that asymptotically $\mathrm{N}(k)$ is at most 4 times this lower bound. In this chapter we reduce the gap between the lower and upper bound by showing that asymptotically $\mathrm{N}(k)$ is at most 3 times the Erdős-Lovász lower bound. A related conjecture of Zsolt Tuza, if proved, would imply that the explicit upper bound obtained in this chapter is only double the Erdős-Lovász lower bound. Most of the results in this chapter are from [19].

2.1 Introduction

In [7] Erdős and Lovász proved the surprising result that any $\mathrm{MIF}(k)$ is finite; indeed it has at most k^k blocks. In Theorem 2.2.2 we point out that, more generally, for any $k-$uniform family $\mathcal{F}$ with finite transversal size $\mathrm{tr}(\mathcal{F}) = t$, the family $\mathcal{F}^{\top}$ is finite. Indeed, $|\mathcal{F}^{\top}| \leq k^t$.

In view of the result of Erdős and Lovász quoted above, we see that, for any fixed $k \geq 1$, there are only finitely many $\mathrm{MIF}(k)$'s, up to isomorphism. This led Erdős and Lovász to ask for the determination of the maximum possible number $\mathrm{N}(k)$ of points among all $\mathrm{MIF}(k)$'s. By means of an explicit construction in [7] (see Constriction 1.2.1), it was proved that

$$\mathrm{N}(k) \geq 2k-2+\frac{1}{2}\binom{2k-2}{k-1}. \tag{2.1.1}$$

Note that the lower bound in (2.1.1) is asymptotically $\frac{1}{2}\binom{2k-2}{k-1}$. In 1985, Tuza [25] proved that, up to a multiplicative constant, this is best possible. In order to explain Tuza's contribution, we recall

Definition. An *intersecting set pair system* (in short ISP) is a collection $\{(A_i, B_i) : 1 \leq i \leq l\}$ of pairs of finite sets with the property that, for $1 \leq i, j \leq l$, $A_i \cap B_j = \emptyset$ if and only if $i = j$. Clearly, in such a system, the sets A_i (as well as the sets B_i) are distinct. The set $\overset{l}{\underset{i=1}{\cup}}(A_i \sqcup B_i)$ is called the *point set* of the ISP. We denote by $\mathrm{v}(\mathbb{I})$ the number of points of an ISP $\mathbb{I}$. If in $\mathbb{I}$, $|A_i| = k$ and $|B_i| = t$ for $1 \leq i \leq l$, then we say that $\mathbb{I}$ is an ISP with parameter (k, t). We use $\mathrm{ISP}(k, t)$ as a generic name for an ISP with parameter (k, t).

In [2], Bollobás proved the following inequality for arbitrary ISP's. If $\{(A_i, B_i) : 1 \leq i \leq l\}$ is an ISP, then

$$\sum_{i=1}^{l} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1. \tag{2.1.2}$$

In particular, for any $\mathrm{ISP}(k, t)$ consisting of l pairs, we have Bollobás's inequality

$$l \leq \binom{k+t}{k}. \tag{2.1.3}$$

This inequality shows that, for any two positive integers k and t, there are only finitely many $\mathrm{ISP}(k, t)$, up to isomorphism. This raises the question of determining or estimating the number

$$\mathrm{n}(k, t) := \max \{\mathrm{v}(\mathbb{I}) : \mathbb{I} \text{ is an } \mathrm{ISP}(k, t)\}.$$

Notice that we have $\mathrm{n}(k, t) = \mathrm{n}(t, k)$.

In Theorem 6(a) of [25], Tuza used an extremely elegant argument to deduce the following bound from Inequality (2.1.2). (The sum here is a simplification of the sum given by Tuza. See Theorem A.3.1 in Appendix A.)

$$\text{For } k \geq t, \quad \mathrm{n}(k, t) \leq \binom{k+t}{t+1} - \binom{2t-1}{t+1} + \frac{3}{2}\sum_{i=1}^{t-1}\binom{2i}{i}. \tag{2.1.4}$$

A family $\mathcal{F}$ is $1-$critical if for any $x \in B \in \mathcal{F}$, there is a $B' \in \mathcal{F}$ such that $B \cap B' = \{x\}$ (see [25, $\nu-$critical family]). Notice that any $\mathrm{MIF}(k)$ is $1-$critical (else $B \setminus \{x\}$ would be a blocking set). In Corollary 12 of [25], Tuza observes that $\mathrm{n}(k, k-1)$ is an upper bound on the number of points in any $k-$uniform $1-$critical family. In particular this applies to $\mathrm{MIF}(k)$'s. So we have

$$\mathrm{N}(k) \leq \mathrm{n}(k, k-1). \tag{2.1.5}$$

Substituting $t = k - 1$ in (2.1.4) we therefore get

$$\mathrm{N}(k) \leq \frac{3}{2}\sum_{i=1}^{k-1}\binom{2i}{i} \sim 2\binom{2k-2}{k-1}, \tag{2.1.6}$$

where the asymptotics is determined by Lemma 2.2.1 below. Thus, as $k \to \infty$, Tuza's upper bound is 4 times the lower bound given by Erdős and Lovász.

The main objective of this chapter is to improve the estimate (2.1.6) on $\mathrm{N}(k)$. The method adopted here is inspired by that of Tuza [25]. We introduce the problem of finding or estimating the number

$$\mathrm{N}^{\top}(k,t) := \max\left\{|\mathrm{P}_{\mathcal{F}^{\top}}| : \mathcal{F} \text{ is a uniform family with } \mathrm{k}(\mathcal{F}) = k \text{ and } \mathrm{tr}(\mathcal{F}) = t\right\}.$$

(Note that we are trying to maximise the size of the point set of the family of transversals of $\mathcal{F}$, which in general is a subset of the point set of $\mathcal{F}$.) This number is finite in view of Theorem 2.2.2 below. In Theorem 2.2.5 we prove:

$$\mathrm{N}^{\top}(k,t) \leq \mathrm{n}(k,t-1). \tag{2.1.7}$$

In Theorem 2.2.7, we show that, given any MIF(k) $\mathcal{F}$ that has two points α and β such that $\{\alpha,\beta\}$ is not contained in any block of $\mathcal{F}$, one can construct another MIF(k), denoted $\mathcal{F}[\beta \mapsto \alpha]$, with one less point. Among the blocks of the new MIF(k) are included the sets $\{\alpha\} \sqcup (B \smallsetminus \{\beta\})$, $\beta \in B \in \mathcal{F}$; hence the name. One might imagine that a method to reduce the number of points in a MIF(k) can not have much to do with the problem of estimating the largest possible number of points in a MIF(k). However, our final result (Theorem 2.2.8) is a new upper bound on $\mathrm{N}(k)$ obtained by combining Theorem 2.2.5 and Theorem 2.2.7 with Inequality (2.1.3). Here we prove

$$\mathrm{N}(k) \leq \frac{1}{2}\binom{2k-2}{k-1} + \mathrm{n}(k,k-2). \tag{2.1.8}$$

In view of Tuza's inequality (2.1.4), this yields the bound

$$\mathrm{N}(k) \leq \frac{3}{2}\sum_{i=1}^{k-1}\binom{2i}{i} - \frac{1}{2}\binom{2k-2}{k-1} \sim \frac{3}{2}\binom{2k-2}{k-1}. \tag{2.1.9}$$

Again, the asymptotic value here follows from Lemma 2.2.1. Thus as $k \to \infty$, $\mathrm{N}(k)$ is at most 3 times the lower bound (2.1.1) of Erdős and Lovász.

In [13], Hanson and Toft proved that, actually, $\mathrm{N}(k) = 2k-2+\frac{1}{2}\binom{2k-2}{k-1}$ for $2 \leq k \leq 4$. In conjunction with Tuza's bound (2.1.6) and its improvement (2.1.9), this result leads us to pose:

Conjecture 2.1.1. For $k \geq 2$, $\mathrm{N}(k) = 2k-2+\frac{1}{2}\binom{2k-2}{k-1}$.

It may be noted that Tuza constructed ([25, Construction 11]) a $k-$uniform $1-$critical family with $2k-4+2\binom{2k-4}{k-2}$ points. This number is larger than $2k-2+\frac{1}{2}\binom{2k-2}{k-1}$ for $k \geq 5$. However, as already noted, the class of $1-$critical uniform families is larger than that of MIF's. Indeed, the families constructed by Tuza are not MIF's. So this construction does not disprove the above conjecture.

Finally, we note that, Tuza [25] made a precise conjecture on the numbers $\mathrm{n}(k,t)$:

Conjecture 2.1.2 (Tuza)**.** For $k \geq t+2$,

$$\mathrm{n}(k,t) = \left\lceil \frac{k}{t+1} \right\rceil \binom{\lfloor \frac{kt}{t+1} \rfloor + t}{t} + \left\lfloor \frac{kt}{t+1} \right\rfloor + t.$$

If this is correct, then, in particular, $\mathrm{n}(k,k-2) = 2k-4+2\binom{2k-4}{k-2}$ for $k \geq 3$, so that our bound (2.1.8) becomes

$$\mathrm{N}(k) \leq \frac{1}{2}\binom{2k-2}{k-1} + 2\binom{2k-4}{k-2} + 2k-4 \sim \binom{2k-2}{k-1},$$

which is asymptotically double the precise value of $\mathrm{N}(k)$ we conjectured above.

2.2 On the maximum number of points in a MIF(k)

Recall that, for any finite uniform family $\mathcal{F}$, $\mathrm{k}(\mathcal{F})$ is its common block size. $\mathcal{F}^\top$ is the family of transversals of $\mathcal{F}$ and $\mathrm{tr}(\mathcal{F})$ is the common size of the transversals. $\mathrm{N}(k)$ is the maximum of $|\mathrm{P}_\mathcal{F}|$ over all MIF(k) $\mathcal{F}$. $\mathrm{N}^\top(k,t)$ is the maximum of $|\mathrm{P}_{\mathcal{F}^\top}|$ over all $\mathcal{F}$ with $\mathrm{k}(\mathcal{F}) = k$ and $\mathrm{tr}(\mathcal{F}) = t$. Also $\mathrm{n}(k,t)$ is the maximum of $\mathrm{v}(\mathbb{I})$ over all ISP(k,t) $\mathbb{I}$.

We begin by establishing the asymptotic values claimed in (2.1.6) and (2.1.9).

Lemma 2.2.1. *As $m \to \infty$,*

$$\sum_{i=1}^{m} \binom{2i}{i} \sim \frac{4}{3}\binom{2m}{m}.$$

`Proof` : Stirling's famous asymptotic formula for the factorial implies the following equally well known result:

$$\binom{2m}{m} \sim \frac{1}{\sqrt{\pi}} \cdot \frac{4^m}{\sqrt{m}}.$$

Therefore we get

$$\sum_{i=1}^{m} \binom{2i}{i} \sim \frac{1}{\sqrt{\pi}} \sum_{i=1}^{m} \frac{4^i}{\sqrt{i}}.$$

So, to complete the proof, it suffices to show that

$$\sum_{i=1}^{m} \frac{4^i}{\sqrt{i}} \sim \frac{4}{3} \cdot \frac{4^m}{\sqrt{m}}. \tag{2.2.1}$$

But we have:

$$\sum_{i=1}^{m} \frac{4^i}{\sqrt{i}} = \frac{1}{3}\sum_{i=1}^{m} \frac{4^{i+1}-4^i}{\sqrt{i}}$$

$$= \frac{4}{3} \cdot \frac{4^m}{\sqrt{m}} - \frac{4}{3} + \frac{1}{3} \left\{ \sum_{i=2}^{m} 4^i \left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{i}} \right) \right\}.$$

We observe that $0 \leq \frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{i}} \leq \frac{1}{2} \cdot \frac{1}{(i-1)^{\frac{3}{2}}}$. Therefore,

$$0 \leq \sum_{i=2}^{m} 4^i \left(\frac{1}{\sqrt{i-1}} - \frac{1}{\sqrt{i}} \right) \leq \frac{1}{2} \sum_{i=2}^{m} \frac{4^i}{(i-1)^{\frac{3}{2}}}.$$

An elementary estimate shows that the right hand sum is of smaller order of growth than $\frac{4^m}{\sqrt{m}}$. Hence (2.2.1) follows. □

Theorem 2.2.2. *If* $\mathrm{k}(\mathcal{F}) = k$ *and* $\mathrm{tr}(\mathcal{F}) = t$ *, then* $|\mathcal{F}^\top| \leq k^t$*.*

`Proof` : This is the $s = 0$ case of the following.

Claim : For $0 \leq s \leq t$, any set of s points of $\mathcal{F}$ are together contained in at most k^{t-s} transversals of $\mathcal{F}$.

`Proof of claim` : We prove this claim by backward induction on s. It is trivial for $s = t$. So suppose the claim holds for some s, with $1 \leq s \leq t$. Take any set A of $s-1$ points. Since $\mathrm{tr}(\mathcal{F}) = t > |A|$, A is not a blocking set of $\mathcal{F}$. So there is a block $B \in \mathcal{F}$ disjoint from A. Therefore each transversal containing A contains at least one of the k sets $A \sqcup \{x\}$, $x \in B$. By induction hypothesis, $A \sqcup \{x\}$ is contained in at most k^{t-s} transversals for each $x \in B$. Therefore A is contained in at most $k \cdot k^{t-s} = k^{t-(s-1)}$ transversals. This completes the induction. □

Corollary 2.2.3. *Let k, t be positive integers. Then up to isomorphism, there are only finitely many families $\mathcal{G}$ with $\mathrm{k}(\mathcal{G}) = t$ such that $\mathcal{G}$ is isomorphic to $\mathcal{F}^\top$ for some uniform family $\mathcal{F}$ with $\mathrm{k}(\mathcal{F}) = k$.*

`Proof` : By Theorem 2.2.2, any such $\mathcal{G}$ has at most k^t blocks; hence it has at most $t.k^t$ points. Therefore up to isomorphism, we may assume that all such families $\mathcal{G}$ are contained in the power set of a fixed set of size tk^t. So there are only finitely many $\mathcal{G}$'s. □

This corollary is the central attraction to study intersecting families with finite transversal size. It shows that $\mathrm{N}(k)$ and $\mathrm{N}^\top(k,t)$ are both finite.

Construction 2.2.4. Let $2 \leq t \leq k-1$ and let S be a set of $k+t-2$ symbols. Let $\binom{S}{i}$ denote the family consisting of all $i-$subsets of S. Take a new symbol x_A (from outside S) for each $A \in \binom{S}{k-1}$. Let

$$\mathcal{F} = \binom{S}{k} \sqcup \left\{ \{x_A\} \sqcup A : A \in \binom{S}{k-1} \right\}.$$

We shall discuss this construction in detail in Chapter 3 [Construction 3.1.5]. In brief, we have $\mathrm{tr}(\mathcal{F}) = t$ and

$$\mathcal{F}^{\top} = \binom{S}{t} \sqcup \left\{ \{x_A\} \sqcup (S \smallsetminus A) : A \in \binom{S}{k-1} \right\}.$$

Theorem 2.2.5. *For* $2 \leq t \leq k-1$*,*

$$k + t - 2 + \binom{k+t-2}{t-1} \leq \mathrm{N}^{\top}(k,t) \leq \mathrm{n}(k, t-1).$$

`Proof` : Construction 2.2.4 yields a $k-$uniform family $\mathcal{F}$ such that $\mathrm{tr}(\mathcal{F}) = t$ and $\mathcal{F}^{\top}$ has $k + t - 2 + \binom{k+t-2}{t-1}$ points. Hence we get the lower bound.

Let $\mathcal{F}$ be a $k-$uniform family with $\mathrm{tr}(\mathcal{F}) = t$. We show that $|\mathrm{P}_{\mathcal{F}^{\top}}| \leq \mathrm{n}(k, t-1)$. Let

$$\mathcal{E} = \{B_i : 1 \leq i \leq n\}$$

be a minimal subfamily of $\mathcal{F}$ such that $\mathrm{tr}(\mathcal{E}) = t$. Then, for $1 \leq i \leq n$, $\mathcal{E}_i := \mathcal{E} \smallsetminus \{B_i\}$ has $\mathrm{tr}(\mathcal{E}_i) = t - 1$. Let us choose a transversal T_i of $\mathcal{E}_i$, where $1 \leq i \leq n$. Since $\mathrm{tr}(\mathcal{E}) = t$, it follows that $T_i \cap B_i = \emptyset$. Thus

$$\mathbb{I} = \{(B_i, T_i) : 1 \leq i \leq n\}$$

is an ISP$(k, t-1)$. Therefore, to complete the proof, it suffices to show that each point x of $\mathcal{F}^{\top}$ is a point of $\mathbb{I}$. Let us choose a transversal T of $\mathcal{F}$ such that $x \in T$. Then T intersects all the B_i's. If x was not a point of $\mathcal{E}$, then $T \smallsetminus \{x\}$ would be a blocking set of $\mathcal{E}$, of size $t-1$, contradicting the choice of $\mathcal{E}$. So x is a point of $\mathcal{E}$ and hence of $\mathbb{I}$. □

Since, clearly, $\mathrm{N}(k) \leq \mathrm{N}^{\top}(k,k)$, Theorem 2.2.5 includes Tuza's upper bound (2.1.5) on $\mathrm{N}(k)$.

Construction 2.2.6. Let $\mathcal{F}$ be a MIF(k) and suppose $\alpha \neq \beta$ are two points of $\mathcal{F}$ such that no block of $\mathcal{F}$ contains $\{\alpha, \beta\}$. Let $\mathcal{G} := \{B \in \mathcal{F} : \alpha \notin B, \beta \notin B\}$. Put

$$\mathcal{F}[\beta \mapsto \alpha] := \mathcal{G} \sqcup \{T \sqcup \{\alpha\} : T \in \mathcal{G}^{\top}\}.$$

Theorem 2.2.7. *Let* α, β *be two points of a* MIF(k) $\mathcal{F}$ *such that no block of* $\mathcal{F}$ *contains both* α *and* β*. Then the family* $\mathcal{F}[\beta \mapsto \alpha]$ *(given by* Construction 2.2.6*) is a* MIF(k) *with point set* $\mathrm{P}_{\mathcal{F}} \smallsetminus \{\beta\}$*.*

`Proof` : Let $\mathcal{G}$ be as in Construction 2.2.6. If T is a transversal of $\mathcal{G}$ with $|T| \leq k-2$, then $T \sqcup \{\alpha, \beta\}$ is a blocking set of $\mathcal{F}$ of size at most k. Since $\mathcal{F}$ is a MIF(k), it follows that $T \sqcup \{\alpha, \beta\}$ is a block of $\mathcal{F}$. This is a contradiction since no block of $\mathcal{F}$ contains both α

and β. Thus $\mathrm{tr}(\mathcal{G}) \geq k-1$. Since, for $\beta \in B \in \mathcal{F}$, $B \smallsetminus \{\beta\}$ is a blocking set of $\mathcal{G}$, it follows that $\mathrm{tr}(\mathcal{G}) = k-1$. Thus $\widehat{\mathcal{F}} := \mathcal{F}[\beta \mapsto \alpha]$ is uniform with $\mathrm{k}(\widehat{\mathcal{F}}) = k$. This argument also shows that if $\beta \notin B \in \mathcal{F}$, then B is a block of $\widehat{\mathcal{F}}$. Also if $\beta \in B \in \mathcal{F}$, then $\{\alpha\} \sqcup (B \smallsetminus \{\beta\})$ is a block of $\widehat{\mathcal{F}}$. We have the following.

Claim : For each $T \in \mathcal{G}^\top$ there exists $T' \in \mathcal{G}^\top$ such that $T \cap T' = \emptyset$.

`Proof of claim` : Suppose the claim is false. Then there exists $T \in \mathcal{G}^\top$ such that T is a blocking set of $\mathcal{G}^\top$. So T is a blocking set of $\mathcal{G} \sqcup \mathcal{G}^\top$, and hence of $\mathcal{F}$. This means $\mathrm{tr}(\mathcal{F}) \leq |T| = k-1$, a contradiction.

Let C be a blocking set of $\widehat{\mathcal{F}}$. Then in particular it is a blocking set of $\mathcal{G}$. Since $\mathrm{tr}(\mathcal{G}) = k-1$, it follows that $|C| \geq k-1$. If $|C| = k-1$, then $C \in \mathcal{G}^\top$, so that $\alpha \notin C$. By the above claim, there exists a $T \in \mathcal{G}^\top$ such that $T \cap C = \emptyset$. Hence C is disjoint from $T \sqcup \{\alpha\} \in \widehat{\mathcal{F}}$, a contradiction. Hence $|C| \geq k$. Therefore $\mathrm{tr}(\widehat{\mathcal{F}}) = k$. Since $\mathcal{F}$ is an intersecting family, the construction of $\widehat{\mathcal{F}}$ shows that $\widehat{\mathcal{F}}$ is an intersecting family. Consequently $\widehat{\mathcal{F}} \subseteqq (\widehat{\mathcal{F}})^\top$. If T is a transversal of $\widehat{\mathcal{F}}$ and $\alpha \in T$, then $T \smallsetminus \{\alpha\}$ is a transversal of $\mathcal{G}$, so that $T = (T \smallsetminus \{\alpha\}) \sqcup \{\alpha\} \in \widehat{\mathcal{F}}$. If T is a transversal of $\widehat{\mathcal{F}}$ and $\alpha \notin T$, then (as all the blocks of $\mathcal{F}$ with $\beta \notin B$ are blocks of $\widehat{\mathcal{F}}$ and for $\beta \in B \in \mathcal{F}$, $(B \smallsetminus \{\beta\}) \sqcup \{\alpha\}$ is a block of $\widehat{\mathcal{F}}$) T is a transversal of $\mathcal{F}$. Hence $T \in \mathcal{F}$ and $\beta, \alpha \notin T$, so that $T \in \mathcal{G} \subseteq \widehat{\mathcal{F}}$. Thus $(\widehat{\mathcal{F}})^\top \subseteqq \widehat{\mathcal{F}}$, so that $\widehat{\mathcal{F}}$ is a MIF(k).

Clearly the point set of $\widehat{\mathcal{F}}$ is contained in $\mathrm{P}_{\mathcal{F}} \smallsetminus \{\beta\}$. Take any $\gamma \in \mathrm{P}_{\mathcal{F}} \smallsetminus \{\beta\}$. Take a block B of $\mathcal{F}$ such that $\gamma \in B$. If $\beta \notin B$, then we have $\gamma \in B \in \widehat{\mathcal{F}}$ and hence γ is a point of $\widehat{\mathcal{F}}$. If $\beta \in B$, then, as $|B| = k = \mathrm{tr}(\mathcal{F})$, there is a block B' of $\mathcal{F}$ such that $B \cap B' = \{\gamma\}$. Then $\gamma \in B' \in \widehat{\mathcal{F}}$, hence again γ is a point of $\widehat{\mathcal{F}}$. Thus the point set of $\widehat{\mathcal{F}}$ is $\mathrm{P}_{\mathcal{F}} \smallsetminus \{\beta\}$. $\square$

Theorem 2.2.8. *For $k \geq 2$,*

$$\mathrm{N}(k) \leq \frac{1}{2}\binom{2k-2}{k-1} + \mathrm{n}(k, k-2).$$

`Proof` : Let $\mathcal{F}$ be a MIF(k). We show that $|\mathrm{P}_{\mathcal{F}}| \leq \frac{1}{2}\binom{2k-2}{k-1} + \mathrm{n}(k, k-2)$. Fix a point α of $\mathcal{F}$. We inductively define two finite sequences: a sequence $\{\beta_n : 0 \leq n \leq M-1\}$ of distinct points of $\mathcal{F}$ and a sequence $\{\mathcal{F}_n : 1 \leq n \leq M\}$ of MIF(k)'s. Define $\beta_0 = \alpha$, $\mathcal{F}_1 = \mathcal{F}$. Suppose we have already defined β_m for $0 \leq m \leq n-1$, and $\mathcal{F}_m$ for $1 \leq m \leq n$. If for each point β of $\mathcal{F}_n$ there is a block of $\mathcal{F}_n$ containing both α and β, then put $n = M$ and terminate the construction. Otherwise, we choose a point β_n of $\mathcal{F}_n$ such that no block of $\mathcal{F}_n$ contains both α and β_n and construct $\mathcal{F}_{n+1} := \mathcal{F}_n[\beta_n \mapsto \alpha]$. By construction and Theorem 2.2.7, for $n \geq 1$ each $\mathcal{F}_{n+1}$ is a MIF(k) with $\mathrm{P}_{\mathcal{F}_{n+1}} = \mathrm{P}_{\mathcal{F}_n} \smallsetminus \{\beta_n\}$.

Notice that this construction must end in finitely many steps, since by Theorem 2.2.2, $\mathcal{F}_1 = \mathcal{F}$ is finite. Since induction has terminated at the $M-$th step, $\mathcal{F}_M$ has the property that for each point β of $\mathcal{F}_M$ there is a block of $\mathcal{F}_M$ containing both α and β. Put

$$\mathcal{G} = \{B \in \mathcal{F}_M : \alpha \notin B\}.$$

For $\alpha \in B \in \mathcal{F}_M$, $B \smallsetminus \{\alpha\}$ is a blocking set of $\mathcal{G}$ of size $k-1$. So $\mathrm{tr}(\mathcal{G}) \leq k-1$. If T is a transversal of $\mathcal{G}$ with $|T| \leq k-1$, then $T \sqcup \{\alpha\}$ is a blocking set of $\mathcal{F}_M$ of size at most k. Since $\mathcal{F}_M$ is a MIF(k), it follows that $T \sqcup \{\alpha\}$ is a block of $\mathcal{F}_M$. Thus $\mathrm{tr}(\mathcal{G}) = k-1$ and consequently, $\mathcal{G}^{\top} = \{B \smallsetminus \{\alpha\} : \alpha \in B \in \mathcal{F}_M\}$. As $M-$th step is the terminal step so $\mathrm{P}_{\mathcal{G}} = \mathrm{P}_{\mathcal{G}^{\top}} = \mathrm{P}_{\mathcal{F}} \smallsetminus \{\beta_n : 0 \leq n \leq M-1\}$. Therefore, by Theorem 2.2.5,

$$|\mathrm{P}_{\mathcal{F}}| = M + |\mathrm{P}_{\mathcal{G}^{\top}}| \leq M + \mathrm{N}^{\top}(k, k-1) \leq M + \mathrm{n}(k, k-2). \tag{2.2.2}$$

Let us choose two blocks B_0, $B_0^{'}$ of $\mathcal{F} = \mathcal{F}_1$ such that $B_0 \cap B_0^{'} = \{\beta_0\}$. Also, for $1 \leq n \leq M-1$, we choose two blocks B_n, $B_n^{'}$ of $\mathcal{F}_n$ such that $B_n \cap B_n^{'} = \{\beta_n\}$. (As already remarked, any point of a MIF(k) lies in such a pair of blocks.) Put $T_n = B_n \smallsetminus \{\beta_n\}$, $T_n^{'} = B_n^{'} \smallsetminus \{\beta_n\}$. Thus $T_n \cap T_n^{'} = \emptyset$ for $0 \leq n \leq M-1$.

Claim : For $0 \leq m < n \leq M-1$, $T_m \sqcup \{\alpha\}$ and $T_m^{'} \sqcup \{\alpha\}$ are blocks of $\mathcal{F}_n$.

`Proof of claim :` This claim may be proved by finite induction on n.
If $n = m+1$, then $\mathcal{F}_n = \mathcal{F}_m[\beta_m \mapsto \alpha]$ and $T_m \sqcup \{\beta_m\}$, $T_m^{'} \sqcup \{\beta_m\} \in \mathcal{F}_m$ implies $T_m \sqcup \{\alpha\}$, $T_m^{'} \sqcup \{\alpha\} \in \mathcal{F}_{m+1} = \mathcal{F}_n$. If $m < n \leq M-1$, and the claim is correct for this value of n, then $T_m \sqcup \{\alpha\}$, $T_m^{'} \sqcup \{\alpha\} \in \mathcal{F}_n$ and $\mathcal{F}_{n+1} = \mathcal{F}_n[\beta_n \mapsto \alpha]$ implies $T_m \sqcup \{\alpha\}$, $T_m^{'} \sqcup \{\alpha\} \in \mathcal{F}_{n+1}$.

Now for $0 \leq m < n \leq M-1$, $T_m \sqcup \{\alpha\}$, $T_m^{'} \sqcup \{\alpha\}$, $T_n \sqcup \{\beta_n\}$ and $T_n^{'} \sqcup \{\beta_n\}$ are blocks of the intersecting family $\mathcal{F}_n$. Therefore these four sets intersect pairwise. Since $\beta_n \neq \alpha$, it follows that $T_m \cap T_n \neq \emptyset$, $T_m^{'} \cap T_n \neq \emptyset$, $T_m \cap T_n^{'} \neq \emptyset$ and $T_m^{'} \cap T_n^{'} \neq \emptyset$ for $0 \leq m < n \leq M-1$. Therefore,

$$\mathbb{I} := \Big\{(T_n, T_n^{'}) : 0 \leq n \leq M-1\Big\} \sqcup \Big\{(T_n^{'}, T_n) : 0 \leq n \leq M-1\Big\}$$

is an ISP($k-1, k-1$) containing $2M$ pairs. Therefore by Inequality (2.1.3), we get

$$M \leq \frac{1}{2}\binom{2k-2}{k-1}. \tag{2.2.3}$$

From (2.2.2) and (2.2.3), we conclude that $|\mathrm{P}_{\mathcal{F}}| \leq \frac{1}{2}\binom{2k-2}{k-1} + \mathrm{n}(k, k-2)$. $\square$

Corollary 2.2.9.

$$\mathrm{N}(k) \leq \frac{3}{2}\sum_{i=1}^{k-1}\binom{2i}{i} - \frac{1}{2}\binom{2k-2}{k-1} \sim \frac{3}{2}\binom{2k-2}{k-1}.$$

`Proof` : Follows from Theorem 2.2.8 and Theorem A.3.1 (with $t = k - 2$). The asymptotics is now immediate from Lemma 2.2.1. $\square$

Chapter 3

Closed Intersecting Families of finite sets

This chapter introduces closed intersecting families, some of its properties and examples. We present results of an ab initio study on closed intersecting families. Most of the results in this chapter are from [20].

3.1 Introduction

Our idea is to decompose a maximal intersecting family into some suitable subfamilies and study these subfamilies to gain a better understanding of the original family. Using this idea we are able to locate a similarity between the recursive Erdős-Lovász construction in [7, Construction (c), Page 620] and non recursive Frankl-Ota-Tokushige constructions in [9, § 2]. We find that each maximal intersecting family has a "core" which generates it. We call this core a *closed intersecting family.* In [9], Frankl et al. conjectured that the maximal intersecting family of $k-$sets constructed by them has the largest number of blocks, and it is the only such family (up to isomorphism) with these many blocks. We use the theory developed here to prove that both these conjectures are false, at least for small k (see Example 5.3.1 and Example 5.3.2). Before going into the technicalities let us recall some notations. Let $\mathcal{G}$ and $\mathcal{H}$ be two non empty families of non empty sets. $A \sqcup B$ denotes the union of two disjoint sets A and B. $\mathcal{G} \sqcup \mathcal{H}$ denotes the union of two disjoint families $\mathcal{G}$ and $\mathcal{H}$. For any set A, $|A|$ will denote the cardinality of A. Suppose $\mathrm{P}_{\mathcal{G}}$ and $\mathrm{P}_{\mathcal{H}}$ are disjoint, then $\mathcal{G} \circledast \mathcal{H}$ denotes the collection of all sets of the form $A \sqcup B$, where $A \in \mathcal{G}$ and $B \in \mathcal{H}$. If $\mathcal{G}$ consists of a single $k-$set B, then we denote $\mathcal{G} \circledast \mathcal{H}$ by $B \circledast \mathcal{H}$. If $\mathcal{G}$ consists of a single $1-$set $\{\alpha\}$, then we denote $\mathcal{G} \circledast \mathcal{H}$ by $\alpha \circledast \mathcal{H}$.

Definition. Let $\mathcal{F}$ be a uniform family with $\mathrm{k}(\mathcal{F}) = k$ and $\mathrm{tr}(\mathcal{F}) = t$. $\mathcal{F}$ is said to be a *closed intersecting family* (in short CIF) if $\mathrm{tr}(\mathcal{F}) \leq \mathrm{k}(\mathcal{F}) - 1$ and $\mathcal{F} = (\mathcal{F} \sqcup \mathcal{F}^{\top})^{\top}$. We use $\mathrm{CIF}(k,t)$ as a generic name for CIF's $\mathcal{F}$ with $\mathrm{k}(\mathcal{F}) = k$ and $\mathrm{tr}(\mathcal{F}) = t$. Note that any closed intersecting family is necessarily an intersecting family.

We have the following characterisation.

Proposition 3.1.1. *Let $\mathcal{F}$ be an intersecting family of $k-$sets with $\mathrm{tr}(\mathcal{F}) \leq k-1$. Then the following statements are equivalent:*

(a) *Any $k-$set which is a blocking set of $\mathcal{F} \sqcup \mathcal{F}^{\top}$ is a block of $\mathcal{F}$.*

(b) *If a $k-$set is a blocking set of $\mathcal{F}$, but not a block of $\mathcal{F}$, then it is not a blocking set of $\mathcal{F}^\top$.*

(c) $\mathcal{F} = (\mathcal{F} \sqcup \mathcal{F}^\top)^\top$.

`Proof` : Firstly we prove (a) $\Leftrightarrow$ (b) and then we prove (c) $\Leftrightarrow$ (a).

Let C be a $k-$set which is a blocking set of $\mathcal{F}$ and $C \notin \mathcal{F}$. Suppose C is a blocking set of $\mathcal{F}^\top$, then by (a) $C \in \mathcal{F}$, a contradiction. Hence C is not a blocking set of $\mathcal{F}^\top$. Conversely, let C be a $k-$set which is a blocking set of $\mathcal{F} \sqcup \mathcal{F}^\top$. Suppose $C \notin \mathcal{F}$, then by (b) C is a not blocking set of $\mathcal{F}^\top$, a contradiction to the assumption. So our supposition $C \notin \mathcal{F}$ was wrong. Hence $C \in \mathcal{F}$.

From (c) it follows that $\mathrm{tr}(\mathcal{F} \sqcup \mathcal{F}^\top) = k$. Let C be a blocking $k-$set of $\mathcal{F} \sqcup \mathcal{F}^\top$. Then C is transversal of the family $\mathcal{F} \sqcup \mathcal{F}^\top$. Hence $C \in \mathcal{F}$. Conversely, let C be a transversal of $\mathcal{F} \sqcup \mathcal{F}^\top$. Suppose $|C| \leq k-1$. Consider a set X, of size $k - |C|$, disjoint from $\mathrm{P}_{\mathcal{F}}$. Then $X \sqcup C$ is a blocking $k-$set of $\mathcal{F} \sqcup \mathcal{F}^\top$ and it is not a block of $\mathcal{F}$, a contradiction to (a). So $|C| = k$ and hence by (a) $C \in \mathcal{F}$, which proves (c). $\square$

Henceforth, by *closure property* we refer any one of (a), (b) and (c) in our study.

Construction 3.1.2. Let k, t be positive integers with $t \leq k-1$. Fix a $(k+t-1)-$set and let $\beta(k,t)$ denote all $k-$subsets of the set. So any two $k-$sets in $\beta(k,t)$ has non empty intersection. Therefore $\beta(k,t)$ is an intersecting family of $k-$sets.

Theorem 3.1.3. $\mathrm{tr}(\beta(k,t)) = t$ *and* $\beta(k,t)$ *is a* $\mathrm{CIF}(k,t)$*. Its transversals are all $t-$subsets of $\mathrm{P}_{\beta(k,t)}$. $\beta(k,t)$ has $k+t-1$ points, $\binom{k+t-1}{k}$ blocks and $\binom{k+t-1}{t}$ transversals.*

`Proof` : Let C be a set of size at most $t-1$. Therefore $|C \cap \mathrm{P}_{\beta(k,t)}| \leq t-1$ and consequently $|\mathrm{P}_{\beta(k,t)} \setminus C| \geq k$. If we choose a $k-$set $B \subset \mathrm{P}_{\beta(k,t)} \setminus C$, then B is disjoint from C. Since C is chosen arbitrarily, $\mathrm{tr}(\beta(k,t)) \geq t$. We observe that any $t-$subset of $\mathrm{P}_{\beta(k,t)}$ is a blocking set of $\beta(k,t)$. Therefore $\mathrm{tr}(\beta(k,t)) = t$.

Let C be a $k-$set with the property that $C \notin \beta(k,t)$ but C is a blocking set of $\beta(k,t)$. Then $|C \cap \mathrm{P}_{\beta(k,t)}| \leq k-1$, as all $k-$sets from $\mathrm{P}_{\beta(k,t)}$ are blocks of $\beta(k,t)$. So any $t-$set from $\mathrm{P}_{\beta(k,t)} \setminus C$ does not intersect C; i.e the closure property is satisfied. $\square$

Theorem 3.1.4. *Any* $\mathrm{CIF}(k,t)$ *has at least $k+t-1$ points. Moreover, $\beta(k,t)$ is the only* $\mathrm{CIF}(k,t)$ *which contains exactly $k+t-1$ points.*

`Proof` : Let $\mathcal{F}$ be such a family. Let $T \in \mathcal{F}^{\top}$. We observe that for each $x \in T$ there exists $B \in \mathcal{F}$ disjoint from $T \setminus \{x\}$. Hence $|\mathrm{P}_{\mathcal{F}}| \geq |B| + |T \setminus \{x\}| = k + t - 1$.

Let $\mathcal{F}$ be a such a family with exactly $k + t - 1$ points. Suppose there exists a $k-$set $B \subset \mathrm{P}_{\mathcal{F}}$ but $B \notin \mathcal{F}$.

`Case A` : B is not a blocking set of $\mathcal{F}$.

In this case there exists $B' \in \mathcal{F}$ disjoint from B. Hence $|\mathrm{P}_{\mathcal{F}}| \geq 2k$, a contradiction.

`Case B` : B is a blocking set of $\mathcal{F}$.

In this case, using the closure property of $\mathcal{F}$ there exists $T \in \mathcal{F}^{\top}$ disjoint from B. Hence $|\mathrm{P}_{\mathcal{F}}| \geq k + t$, a contradiction.

Since both cases lead to a contradiction, our supposition that there exists a $k-$set $B \subset \mathrm{P}_{\mathcal{F}}$ such that $B \notin \mathcal{F}$ was wrong. Consequently, each $k-$set from $\mathrm{P}_{\mathcal{F}}$ is a block. Hence $\mathcal{F}$ is isomorphic to $\beta(k, t)$. $\square$

Construction 3.1.5. Let k, t be positive integers with $2 \leq t \leq k - 1$. Let P be a $(k + t - 2)-$set. For each bi-partition $(A, P \setminus A)$ of P with $|A| = t - 1$, we introduce a new symbol x_A. We consider the family of all $k-$subsets of P together with all $k-$sets of the form $\{x_A\} \sqcup (P \setminus A)$. We denote this family of $k-$sets by $\beta_g(k, t)$.

Theorem 3.1.6. $\mathrm{tr}(\beta_g(k,t)) = t$ *and* $\beta_g(k, t)$ *is a* $\mathrm{CIF}(k, t)$. *Its transversals are all* $t-$*subsets of* P *and all* $t-$*sets of the form* $\{x_A\} \sqcup A$, *for any bi-partition* $(A, P \setminus A)$ *of* P *with* $|A| = t - 1$. *It has* $k + t - 2 + \binom{k+t-2}{k-1}$ *points,* $\binom{k+t-2}{k} + \binom{k+t-2}{k-1}$ *blocks and* $\binom{k+t-2}{t} + \binom{k+t-2}{t-1}$ *transversals.*

`Proof` : Since $|P| = k + t - 2$ and $2 \leq t \leq k - 1$, therefore $\beta_g(k, t)$ is an intersecting family of $k-$sets. Let C be a set of size at most $t - 1$. We show that there exists at least one $B \in \beta_g(k, t)$, which is disjoint from C. If any one of the new symbols $x_A \in C$, where x_A corresponds to the bi-partition $(A, P \setminus A)$, then $|C \cap P| \leq t - 2$. So any $k-$subset of $P \setminus C$ (note that, any such $k-$subset is a block of $\beta_g(k, t)$) is disjoint from C. Without loss of generality we assume that C does not contain any such new symbols. Again if $|C \cap P| \leq t - 2$, then again any $k-$subset of $P \setminus C$ is disjoint from C. So if $|C \cap P| = t - 1$, then we note that $\{x_A\} \sqcup (P \setminus C)$, where $x_A \notin C$ corresponds to the bi-partition $(A, P \setminus A)$ of P with $A = C \cap P$ and $P \setminus A = P \setminus C$, is the required block of $\beta_g(k, t)$, which is disjoint from C. Therefore $\mathrm{tr}(\beta_g(k, t)) = t$.

Let C be a blocking $k-$set of $\beta_g(k, t)$ such that $C \notin \beta_g(k, t)$. We show that there exists at least one $T \in \beta_g^{\top}(k, t)$, which is disjoint from C. If $|C \cap P| \leq k - 2$, then any $t-$subset of $P \setminus C$ (note that, any such $t-$subset is a transversal of $\beta_g(k, t)$) is disjoint from C. Without loss of generality we assume that $k-1 \le |C \cap P| \le k$. Since $C \notin \beta_g(k,t)$, $|C \cap P| \neq k$. Hence $|C \cap P| = k-1$. Then, there exists a bi-partition $(A, P \setminus A)$, with $|A| = t-1$ and $P \setminus A = C \cap P$. Now $\{x_A\} \sqcup (C \cap P)$, where x_A is the new symbol corresponds to the bi-partition $(A, P \setminus A)$ of P, is a block of $\beta_g(k,t)$. Since we assume that $C \notin \beta_g(k,t)$, $x_A \notin C$. Hence C is disjoint from $\{x_A\} \sqcup A$, which is the required transversal. □

We present some immediate properties of closed intersecting families.

Proposition 3.1.7. *Let $\mathcal{F}$ be a* CIF(k,t).

(a) *For each $T \in \mathcal{F}^{\top}$ there exists at least one $T' \in \mathcal{F}^{\top}$ disjoint from T.*

(b) $2 \le \mathrm{tr}(\mathcal{F}^{\top}) \le k$.

(c) *If $\mathrm{tr}(\mathcal{F}^{\top}) = k$, then $\mathcal{F} \subset \mathcal{F}^{\top\top}$.*

(d) *For each $x \in \mathrm{P}_{\mathcal{F}^{\top}}$, there exists $T \in \mathcal{F}^{\top}$ and $B \in \mathcal{F}$ such that $T \cap B = \{x\}$.*

(e) *If $\mathrm{P}_{\mathcal{F}^{\top}} \neq \mathrm{P}_{\mathcal{F}}$, then $\mathrm{tr}(\mathcal{F}^{\top}) \le k-1$.*

(f) *If $\mathrm{tr}(\mathcal{F}^{\top}) \le k-1$, then for each $A \in \mathcal{F}^{\top\top}$ there exists $B \in \mathcal{F}$ disjoint from A.*

(g) $k \le \mathrm{tr}(\mathcal{F}) + \mathrm{tr}(\mathcal{F}^{\top})$.

`Proof` : Let $T \in \mathcal{F}^{\top}$. Then by assumption $|T| \le k-1$. Since $\mathrm{tr}(\mathcal{F} \sqcup \mathcal{F}^{\top}) = k$, there exists $T' \in \mathcal{F} \sqcup \mathcal{F}^{\top}$ disjoint from T. Since $T \in \mathcal{F}^{\top}$ such $T' \notin \mathcal{F}$. Hence $T' \in \mathcal{F}^{\top}$. This immediately implies (a) and (b). We observe that each $B \in \mathcal{F}$ is a blocking set of $\mathcal{F}^{\top}$. So if $\mathrm{tr}(\mathcal{F}^{\top}) = k$, then $B \in \mathcal{F}^{\top\top}$ and the part (c) follows.

Let $x \in \mathrm{P}_{\mathcal{F}^{\top}}$. Then there exists a block $T \in \mathcal{F}^{\top}$ such that $x \in T$. Since $|T \setminus \{x\}| = t-1$, $T \setminus \{x\}$ is not a blocking set of $\mathcal{F}$. In other words, there exists a block $B \in \mathcal{F}$ disjoint from $T \setminus \{x\}$. But T intersects B; hence $T \cap B = \{x\}$ and the part (d) follows.

To establish (e), we observe that $\mathrm{P}_{\mathcal{F}^{\top}} \subset \mathrm{P}_{\mathcal{F}}$. So $\mathrm{P}_{\mathcal{F}^{\top}} \neq \mathrm{P}_{\mathcal{F}}$ means that there exists a point $\alpha \in \mathrm{P}_{\mathcal{F}} \setminus \mathrm{P}_{\mathcal{F}^{\top}}$ and consequently there exists a block B such that $\alpha \in B$. We observe that $B \setminus \{\alpha\}$ is a blocking set of $\mathcal{F}^{\top}$. Hence $\mathrm{tr}(\mathcal{F}^{\top}) \le k-1$.

Let $\mathrm{tr}(\mathcal{F}^{\top}) \le k-1$ and $A \in \mathcal{F}^{\top\top}$. Suppose A is a blocking set of $\mathcal{F}$. Then by the closure property of $\mathcal{F}$, we have $|A| \ge k$, which is a contradiction to the assumption. Hence the part (f) follows.

To establish (g), we observe that the result is obvious if $\mathrm{tr}(\mathcal{F}^{\top}) = k$. Without loss of generality let $\mathrm{tr}(\mathcal{F}^{\top}) \le k-1$. Now let $A \in \mathcal{F}^{\top\top}$ and construct the following non-empty subfamily of $\mathcal{F}$,

$$\mathcal{F}_A = \{B \in \mathcal{F} : A \cap B = \emptyset\}.$$

Let $C \in \mathcal{F}_A^\top$. Then $A \sqcup C$ is a blocking set of $\mathcal{F}$ and $\mathcal{F}^\top$. Also we observe that $|C| \leq \mathrm{tr}(\mathcal{F})$. Hence by the closure property of $\mathcal{F}$, $k \leq |A| + |C| = \mathrm{tr}(\mathcal{F}^\top) + |C| \leq \mathrm{tr}(\mathcal{F}^\top) + \mathrm{tr}(\mathcal{F})$. $\square$

3.2 Correspondence between closed and maximal intersecting families

Theorem 3.2.1. *Let $\mathcal{F}$ be a subfamily of a* MIF(k) *$\mathcal{X}$ such that $t := \mathrm{tr}(\mathcal{F}) \leq k-1$ and $\mathcal{X} \smallsetminus \mathcal{F} = \mathcal{A} \circledast \mathcal{F}^\top$ for some family $\mathcal{A}$. Then $\mathcal{F}$ is a* CIF(k,t) *if and only if $\mathcal{A}$ is a* MIF$(k-t)$.

`Proof` : Let $\mathcal{F}$ be a CIF(k,t) and let $T \in \mathcal{F}^\top$. Then, by (a) of Proposition 3.1.7 there exists at least one $T' \in \mathcal{F}^\top$ disjoint from T. Since $\mathcal{A} \circledast \mathcal{F}^\top$ is an intersecting family of $k-$sets, it follows that $\mathcal{A}$ is an intersecting family of $(k-t)-$sets.

Let $C \in \mathcal{A}^\top$. Then, for each $T \in \mathcal{F}^\top$, $C \sqcup T$ is a blocking set of $\mathcal{X}$ and hence $|C \sqcup T| \geq k$. Thus $|C| \geq k - t$, with equality if $C \sqcup T \in \mathcal{X}$ for all $T \in \mathcal{F}^\top$. Since each block of $\mathcal{A}$ is a blocking set of size $k-t$, it follows that $\mathrm{tr}(\mathcal{A}) = k - t$ and $\mathcal{A} \subseteqq \mathcal{A}^\top$. Also if $C \in \mathcal{A}$ and $T \in \mathcal{F}^\top$, then $C \sqcup T \in \mathcal{X}$. The argument in the previous paragraph shows that $C \sqcup T$ is not a blocking set of $\mathcal{F}^\top$ and hence $C \sqcup T \notin (\mathcal{F} \sqcup \mathcal{F}^\top)^\top = \mathcal{F}$. Thus $C \sqcup T \in \mathcal{X} \smallsetminus \mathcal{F} = \mathcal{A} \circledast \mathcal{F}^\top$. Hence $C \in \mathcal{A}$. Thus $\mathcal{A}^\top \subseteqq \mathcal{A}$ and hence $\mathcal{A} = \mathcal{A}^\top$. Thus $\mathcal{A}$ is a MIF$(k-t)$.

Conversely, suppose $\mathcal{A}$ is a MIF$(k-t)$. Since $\mathcal{F}$ is an intersecting family of $k-$sets, every block of $\mathcal{F}$ is a blocking $k-$set of $\mathcal{F} \sqcup \mathcal{F}^\top$ and hence $\mathrm{tr}(\mathcal{F} \sqcup \mathcal{F}^\top) \leq k$. We show that $\mathcal{F}$ is a CIF(k,t). It suffices to show that if C is a blocking set of size k for $\mathcal{F}$ which is not a block of $\mathcal{F}$, then C is not a blocking set of $\mathcal{F}^\top$. If $C \notin \mathcal{A} \circledast \mathcal{F}^\top$, then C is not a block of $\mathcal{X}$ and hence there is a block $B \in \mathcal{X}$ disjoint from C. But C is a blocking set of $\mathcal{F}$. So $B \in \mathcal{A} \circledast \mathcal{F}^\top$. Then $B \cap \mathrm{P}_\mathcal{F}$ is a block of $\mathcal{F}^\top$ disjoint from C, so that C is not a blocking set of $\mathcal{F}^\top$ in this case. On the other hand, if $C \in \mathcal{A} \circledast \mathcal{F}^\top$, then we choose a point $\alpha \in C \cap \mathrm{P}_\mathcal{A}$. (It exists since $k = |C| > t = \mathrm{k}(\mathcal{F}^\top)$.) Since C is a block of a MIF(k) $\mathcal{X}$, there exists at least one $B \in \mathcal{X}$ such that $B \cap C = \{\alpha\}$. Since $\alpha \notin \mathrm{P}_\mathcal{F}$, it follows that $B \in \mathcal{A} \circledast \mathcal{F}^\top$ and hence $B \cap \mathrm{P}_\mathcal{F}$ is a block of $\mathcal{F}^\top$ disjoint from C. So C is not a blocking set of $\mathcal{F}^\top$ in this case also. $\square$

The following immediate corollary of Theorem 3.2.1 shows the existence of a closed intersecting family inside any maximal intersecting family.

Corollary 3.2.2. *Let $\mathcal{X}$ be a* $\mathrm{MIF}(k)$*, then there exists at least one* $\mathrm{CIF}(k, k-1)$ $\mathcal{F}$ *and* $\alpha \in \mathrm{P}_{\mathcal{X}} \setminus \mathrm{P}_{\mathcal{F}}$ *such that* $\mathcal{X} = \mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^{\top})$.

`Proof` : Let $\alpha \in \mathrm{P}_{\mathcal{X}}$. Define $\mathcal{F} = \{B \in \mathcal{X} : \alpha \notin B\}$. Then $\mathcal{F}^{\top} = \{B \setminus \{\alpha\} : \alpha \in B \in \mathcal{X}\}$. The conclusion follows as an application of Theorem 3.2.1. □

The following theorem is a sort of converse to Theorem 3.2.1. Together, Theorem 3.2.1 and Theorem 3.2.3 show that closed intersecting families are the cores which may be used to obtain maximal intersecting families via recursive construction.

Theorem 3.2.3. *Let $\mathcal{A}$ and $\mathcal{F}$ be a* $\mathrm{MIF}(k-t)$ *and a* $\mathrm{CIF}(k, t)$ *respectively where $\mathcal{A}$ and $\mathcal{F}$ have disjoint point sets. Then* $\mathcal{F} \sqcup (\mathcal{A} \circledast \mathcal{F}^{\top})$ *is a* $\mathrm{MIF}(k)$.

`Proof` : Let C be a blocking $k-$set of $\mathcal{F} \sqcup (\mathcal{A} \circledast \mathcal{F}^{\top})$. It is enough to prove $C \in \mathcal{F} \sqcup (\mathcal{A} \circledast \mathcal{F}^{\top})$. If $C \in \mathcal{F}$ we are done. Assume $C \notin \mathcal{F}$. By the closure property of $\mathcal{F}$, C is not a blocking set of $\mathcal{F} \sqcup \mathcal{F}^{\top}$. This implies C is not a blocking set of $\mathcal{F}^{\top}$. Hence there exists at least one $T \in \mathcal{F}^{\top}$ which is disjoint from C. Since C is a blocking set of $T \circledast \mathcal{A}$, it follows that $C \cap \mathrm{P}_{\mathcal{A}}$ is a blocking set of $\mathcal{A}$. So $|C \cap \mathrm{P}_{\mathcal{A}}| \geq k - t$. Also $C \cap \mathrm{P}_{\mathcal{F}}$ is a blocking set of $\mathcal{F}$, and hence $|C \cap \mathrm{P}_{\mathcal{F}}| \geq t$. But $|C| = k$, so $|C \cap \mathrm{P}_{\mathcal{A}}| = k - t$ and $|C \cap \mathrm{P}_{\mathcal{F}}| = t$. Hence $C \cap \mathrm{P}_{\mathcal{A}} \in \mathcal{A}$ and $C \cap \mathrm{P}_{\mathcal{F}} \in \mathcal{F}^{\top}$. So $C \in \mathcal{A} \circledast \mathcal{F}^{\top}$. This shows that every blocking $k-$set of the family $\mathcal{F} \sqcup (\mathcal{A} \circledast \mathcal{F}^{\top})$ is a block of that family. □

We recall that $\mathrm{M}(k)$ is the maximum of $|\mathcal{F}|$ over all $\mathrm{MIF}(k)$ $\mathcal{F}$. The following immediate corollary of Theorem 3.2.3 helps to estimate $\mathrm{M}(k)$.

Corollary 3.2.4. *For each integer $k \geq t + 1$, if $\mathcal{F}$ is a* $\mathrm{CIF}(k, t)$ *with b blocks and $b^{\top}$ transversals, then* $\mathrm{M}(k) \geq b + b^{\top}\mathrm{M}(k - t)$.

`Proof` : We choose a $\mathrm{MIF}(k-t)$ with $\mathrm{M}(k-t)$ blocks so that its point set is disjoint from $\mathrm{P}_{\mathcal{F}}$. Call it $\mathcal{A}$. Hence by Theorem 3.2.3, $\mathcal{F} \sqcup (\mathcal{A} \circledast \mathcal{F}^{\top})$ is a $\mathrm{MIF}(k)$ with $b + b^{\top}\mathrm{M}(k - t)$ blocks. □

In the following theorem, we use three copies of closed intersecting families in a circular way to obtain a maximal intersecting family.

Theorem 3.2.5. *Let $k \leq 2t - 1$ and let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be three* $\mathrm{CIF}(k, t)$*'s with mutually disjoint point sets and* $\mathrm{tr}(\mathcal{F}^{\top}) = \mathrm{tr}(\mathcal{G}^{\top}) = \mathrm{tr}(\mathcal{H}^{\top}) = k$. *Then*

$$(\mathcal{F} \circledast \mathcal{G}^{\top}) \sqcup (\mathcal{G} \circledast \mathcal{H}^{\top}) \sqcup (\mathcal{H} \circledast \mathcal{F}^{\top})$$

is a $\mathrm{MIF}(k + t)$.

`Proof` : Let $\mathcal{A} := (\mathcal{F} \circledast \mathcal{G}^\top) \sqcup (\mathcal{G} \circledast \mathcal{H}^\top) \sqcup (\mathcal{H} \circledast \mathcal{F}^\top)$. Let $C \subset \mathrm{P}_\mathcal{F} \sqcup \mathrm{P}_\mathcal{G} \sqcup \mathrm{P}_\mathcal{H}$ such that $|C| \leq k+t$ and C is a blocking set of $\mathcal{A}$. We show that $C \in \mathcal{A}$. Since $k \leq 2t-1$ we have $|C| \leq k+t \leq 3t-1$, hence at least one of $C \cap \mathrm{P}_\mathcal{F}$, $C \cap \mathrm{P}_\mathcal{G}$ and $C \cap \mathrm{P}_\mathcal{H}$ contains less than t points.

Claim 1 : If $|C \cap \mathrm{P}_\mathcal{F}| \leq t-1$, then $|C \cap \mathrm{P}_\mathcal{G}| \geq k$. Similarly, if $|C \cap \mathrm{P}_\mathcal{G}| \leq t-1$, then $|C \cap \mathrm{P}_\mathcal{H}| \geq k$ and if $|C \cap \mathrm{P}_\mathcal{H}| \leq t-1$, then $|C \cap \mathrm{P}_\mathcal{F}| \geq k$.

`Proof of claim` : Due to similarity, it is enough to prove only the first statement. Suppose $|C \cap \mathrm{P}_\mathcal{F}| \leq t-1$ and $\mathrm{tr}(\mathcal{F}) = t$. Then there exists at least one $B_\mathcal{F} \in \mathcal{F}$ disjoint from $C \cap \mathrm{P}_\mathcal{F}$. But C is a blocking set of $B_\mathcal{F} \circledast \mathcal{G}^\top$. This implies C is a blocking set of $\mathcal{G}^\top$. Hence $|C \cap \mathrm{P}_\mathcal{G}| = |C \cap \mathrm{P}_{\mathcal{G}^\top}| \geq k$ and the claim is established.

Claim 2 : If $|C \cap \mathrm{P}_\mathcal{F}| \leq t-1$, then $C \in \mathcal{G} \circledast \mathcal{H}^\top$. Similarly, if $|C \cap \mathrm{P}_\mathcal{G}| \leq t-1$, then $C \in \mathcal{H} \circledast \mathcal{F}^\top$ and if $|C \cap \mathrm{P}_\mathcal{H}| \leq t-1$, then $C \in \mathcal{F} \circledast \mathcal{G}^\top$.

`Proof of claim` : Due to similarity, it is enough to prove only the first statement. Suppose $|C \cap \mathrm{P}_\mathcal{F}| \leq t-1$. Therefore $|C \cap \mathrm{P}_\mathcal{H}| \geq t$. (If not, then $|C \cap \mathrm{P}_\mathcal{H}| \leq t-1$ and by the above claim $|C \cap \mathrm{P}_\mathcal{F}| \geq k \geq t+1$, a contradiction.) Again by the above claim $|C \cap \mathrm{P}_\mathcal{G}| \geq k$. Since $|C| \leq k+t$, we have $|C \cap \mathrm{P}_\mathcal{F}| = 0$, $|C \cap \mathrm{P}_\mathcal{H}| = t$ and $|C \cap \mathrm{P}_\mathcal{G}| = k$. Since C is a blocking set of $\mathcal{H} \circledast \mathcal{F}^\top$ and $C \cap \mathrm{P}_\mathcal{F}$ is empty therefore $C \cap \mathrm{P}_\mathcal{H}$ is a blocking set of $\mathcal{H}$. Since $|C \cap \mathrm{P}_\mathcal{H}| = t$ we have $C \cap \mathrm{P}_\mathcal{H} \in \mathcal{H}^\top$. Since $C \cap \mathrm{P}_\mathcal{F}$ is empty, $C \cap \mathrm{P}_\mathcal{G}$ is a blocking $k-$set of $\mathcal{G} \sqcup \mathcal{G}^\top$. Therefore, by the closure property of $\mathcal{G}$ we have $C \cap \mathrm{P}_\mathcal{G} \in \mathcal{G}$. Consequently, $C = (C \cap \mathrm{P}_\mathcal{G}) \sqcup (C \cap \mathrm{P}_\mathcal{H}) \in \mathcal{G} \circledast \mathcal{H}^\top$ and the claim is established.

From the above two claims, it follows that $C \in \mathcal{A}$. □

Remark 3.2.6. The inequality in the statement of Theorem 3.2.5 is necessary. For $k \geq 2t$, we conclude with a similar proof that

$$\mathrm{tr}(\mathcal{F} \circledast \mathcal{G}^\top) \sqcup (\mathcal{G} \circledast \mathcal{H}^\top) \sqcup (\mathcal{H} \circledast \mathcal{F}^\top) = 3t.$$

3.3 Construction of maximal intersecting families using closed intersecting families

Proposition 3.3.1. *Let $\mathcal{F}$ be a* CIF(k,t). *Suppose for each i, with $1 \leq i \leq n$, $\mathcal{A}_i$ is a* MIF$(k-t)$ *and $\mathcal{C}_i$ is a subfamily of $\mathcal{F}^\top$ with the following properties:*

(a) *each $\mathcal{A}_i$ and $\mathcal{F}$ have disjoint point sets;*

(b) $\mathcal{F}^{\top} = \bigsqcup_{i=1}^{n} \mathcal{C}_i$;

(c) *each $t-$set of $\mathcal{C}_i$ is a blocking set of $\mathcal{F}^{\top} \setminus \mathcal{C}_i$.*

Then $\mathcal{F} \sqcup (\bigsqcup_{i=1}^{n} \mathcal{A}_i \circledast \mathcal{C}_i)$ is a MIF(k). *Moreover, $n \leq \frac{1}{2}\binom{2t}{t}$.*

`Proof` : Let $\mathcal{G} := \mathcal{F} \sqcup (\bigsqcup_{i=1}^{n} \mathcal{A}_i \circledast \mathcal{C}_i)$. Clearly it is an intersecting family of $k-$sets. Let C be a blocking set of $\mathcal{G}$ with size at most k. To prove C is a block of $\mathcal{G}$. If $C \in \mathcal{F}$ we are done. So assume $C \notin \mathcal{F}$. By the closure property of $\mathcal{F}$ there exists at least one $T \in \mathcal{F}^{\top}$ such that $C \cap \mathrm{P}_{\mathcal{F}}$ is disjoint from T and $T \in \mathcal{C}_i$ for a unique i. Since C is a blocking set of $T \circledast \mathcal{A}_i$, we have $|C \cap \mathrm{P}_{\mathcal{A}_i}| \geq k-t$. Also $C \cap \mathrm{P}_{\mathcal{F}}$ is a blocking set of $\mathcal{F}$. Hence $|C \cap \mathrm{P}_{\mathcal{F}}| \geq t$. This implies $|C \cap \mathrm{P}_{\mathcal{A}_i}| = k-t$ and $|C \cap \mathrm{P}_{\mathcal{F}}| = t$, hence $C \in \mathcal{A}_i \circledast \mathcal{C}_i$.

For the next part, by assumption (c) we observe that, for each i with $1 \leq i \leq n$ there exists at least one pair $(T_i, T_i^{'})$, where T_i, $T_i^{'} \in \mathcal{C}_i$ with $T_i \cap T_i^{'} = \emptyset$. Also for each i, j with $1 \leq i < j \leq n$, we have $T_i \cap T_j^{'} \neq \emptyset$ and $T_i^{'} \cap T_j \neq \emptyset$. Hence by using (A.2.3) of Theorem A.2.1, we have $\{(T_i, T_i^{'}) : 1 \leq i \leq n\}$ is an ISP(t,t). Therefore $n \leq \frac{1}{2}\binom{2t}{t}$. □

Remark 3.3.2. The proof of Theorem 3.2.3 follows from Proposition 3.3.1 corresponding to the case $n = 1$. The above proposition is of interest, since there is no restriction on the choice of $\mathcal{A}_i$, where $1 \leq i \leq n$.

Proposition 3.3.3. *Let $\mathcal{F}$ be a* CIF$(k, k-n)$. *Suppose $\mathcal{F}^{\top} = \bigsqcup_{i=1}^{n+1} \mathcal{C}_i$, where for each i, with $1 \leq i \leq n+1$, the subfamily $\mathcal{C}_i$ satisfies the following properties.*

(a) *Each $\mathcal{C}_i$ is an intersecting family of $(k-n)-$sets.*

(b) *If $i \neq j$, then for each $T \in \mathcal{C}_i$ there exists at least one $T^{'} \in \mathcal{C}_j$ disjoint from T.*

Let $\mathcal{A}_1$, $\mathcal{A}_2$, …, $\mathcal{A}_{n+1}$ be the $(n+1)-$parallel classes of an affine plane of order n (assuming it exists), whose point set is disjoint from $\mathrm{P}_{\mathcal{F}}$. Then $\mathcal{F} \sqcup (\bigsqcup_{i=1}^{n+1} \mathcal{A}_i \circledast \mathcal{C}_i)$ is a MIF(k).

`Proof` : Let $\mathcal{G} := \mathcal{F} \sqcup (\bigsqcup_{i=1}^{n+1} \mathcal{A}_i \circledast \mathcal{C}_i)$. Clearly it is an intersecting family of $k-$sets. Let P be the point set of this affine plane. Let C be a blocking set of $\mathcal{G}$ with size at most k. To prove, C is a block of $\mathcal{G}$. If $C \in \mathcal{F}$ we are done. So assume $C \notin \mathcal{F}$. By the closure property of $\mathcal{F}$ there exists at least one $T \in \mathcal{F}^{\top}$ such that $C \cap \mathrm{P}_{\mathcal{F}}$ is disjoint from T. Then there exists at least one i, with $1 \leq i \leq n+1$, such that $T \in \mathcal{C}_i$. But C is a blocking set of $T \circledast \mathcal{A}_i$ hence $|C \cap P| \geq n$. Also $C \cap \mathrm{P}_{\mathcal{F}}$ is a blocking set of $\mathcal{F}$; hence $|C \cap \mathrm{P}_{\mathcal{F}}| \geq k-n$. This implies $|C \cap P| = n$ and $|C \cap \mathrm{P}_{\mathcal{F}}| = k-n$ and hence $C \cap \mathrm{P}_{\mathcal{F}} \in \mathcal{F}^{\top}$. So by assumption (a) and (b), $C \cap \mathrm{P}_{\mathcal{F}} \in \mathcal{C}_j$ for some $j \neq i$. Then again by assumption (b) there exists at least one $T_l \in \mathcal{C}_l$ such that $C \cap \mathrm{P}_{\mathcal{F}}$ is disjoint from T_l, for each l with $l \neq j$. So $C \cap P$ is a blocking set of each such $T_l \circledast \mathcal{A}_l$. Hence, by using Lemma 1.2.3 we have, $C \cap P$ is a line of $\mathcal{A}_j$. Therefore $C \in \mathcal{A}_j \circledast \mathcal{C}_j$. □

3.4 Recursive constructions of closed intersecting families

Theorem 3.4.1. *Let $\mathcal{A}$ be a* $\mathrm{MIF}(l)$ *and let $\mathcal{F}_x$, $x \in \mathrm{P}_{\mathcal{A}}$, be uniform families with pairwise disjoint point sets. Suppose* $\mathrm{k}(\mathcal{F}_x) = k$ *and* $\mathrm{tr}(\mathcal{F}_x) = t$ *for all x. Put*

$$\mathcal{G} = \left\{ \bigsqcup_{x \in A} F_x : A \in \mathcal{A}, F_x \in \mathcal{F}_x \text{ for all } x \in A \right\}.$$

Then we have the following.

(a) $\mathcal{G}^{\top} = \left\{ \bigsqcup_{x \in A} T_x : A \in \mathcal{A}, T_x \in \mathcal{F}_x^{\top} \text{ for all } x \in A \right\}$. *In particular,* $\mathrm{k}(\mathcal{G}) = kl$ *and* $\mathrm{tr}(\mathcal{G}) = tl$.

(b) *If, further, each $\mathcal{F}_x$ is a* $\mathrm{CIF}(k, t)$ *with* $\mathrm{tr}(\mathcal{F}_x^{\top}) = k$*, then $\mathcal{G}$ is a* $\mathrm{CIF}(kl, tl)$.

`Proof` : If $A \in \mathcal{A}$ and $T_x \in \mathcal{F}_x^{\top}$ for all $x \in A$, then clearly $\bigsqcup_{x \in A} T_x$ is a blocking set of $\mathcal{G}$ of size tl. Thus, $\mathrm{tr}(\mathcal{G}) \leq tl$. Let B be a blocking set of $\mathcal{G}$ of size at most tl. For $x \in \mathrm{P}_{\mathcal{A}}$, put $T_x = B \cap \mathrm{P}_{\mathcal{F}_x}$. Let $A = \{x \in \mathrm{P}_{\mathcal{A}} : |T_x| \geq t\}$. We have

$$\sum_{x \in \mathrm{P}_{\mathcal{A}}} |T_x| = |B| \leq tl \tag{3.4.1}$$

and hence $|A| \leq l$. If A is not a block of the $\mathrm{MIF}(l)$ $\mathcal{A}$, then there is a block A' of $\mathcal{A}$ disjoint from A. Hence $|T_x| \leq t-1$ for all $x \in A'$. So, for each $x \in A'$ there is a block F_x of $\mathcal{F}_x$ disjoint from T_x. Hence $\bigsqcup_{x \in A'} F_x$ is a block of $\mathcal{G}$ disjoint from B, a contradiction. So $A \in \mathcal{A}$ and $|A| = l$. Then (3.4.1) implies that $|T_x| = 0$ for $x \notin A$ and $|T_x| = t$ for $x \in A$. Thus, $|B| = tl$, so that $\mathrm{tr}(\mathcal{G}) = tl$ and $B \in \mathcal{G}^{\top}$. Since $B = \bigsqcup_{x \in A} T_x \in \mathcal{G}^{\top}$ and $|T_x| = t$, it follows that $T_x \in \mathcal{F}_x^{\top}$ for all $x \in A$. This proves part (a).

Now we assume each $\mathcal{F}_x$ is a $\mathrm{CIF}(k, t)$. Since $\mathcal{A}$, as well as each $\mathcal{F}_x$, is an intersecting family it follows that $\mathcal{G}$ is an intersecting family. Using the description of $\mathcal{G}^{\top}$ from part (a) and applying part (a) to the families $\mathcal{F}_x^{\top}$, $x \in \mathrm{P}_{\mathcal{A}}$, we see that

$$\mathcal{G}^{\top\top} = \left\{ \bigsqcup_{x \in A} S_x : A \in \mathcal{A}, S_x \in \mathcal{F}_x^{\top\top} \text{ for all } x \in A \right\}.$$

Thus $\mathrm{tr}(\mathcal{G} \sqcup \mathcal{G}^{\top}) \geq \mathrm{tr}(\mathcal{G}^{\top}) = kl$. Since all the blocks of $\mathcal{G}$ are blocking sets of $\mathcal{G} \sqcup \mathcal{G}^{\top}$ of size kl, it follows that $\mathrm{tr}(\mathcal{G} \sqcup \mathcal{G}^{\top}) = kl$ and $\mathcal{G} \subseteq (\mathcal{G} \sqcup \mathcal{G}^{\top})^{\top}$. Let C be a transversal of

$\mathcal{G} \sqcup \mathcal{G}^{\top}$. Then $C \in \mathcal{G}^{\top\top}$ and hence $C = \bigsqcup_{x \in A} S_x$, for some $A \in \mathcal{A}$ and $S_x \in \mathcal{F}_x^{\top\top}$ for all $x \in A$. If we can show that $C \in \mathcal{G}$, then we are done. Otherwise, there exists at least one $y \in A$ such that $S_y \notin \mathcal{F}_y = (\mathcal{F}_y \sqcup \mathcal{F}_y^{\top})^{\top}$. Since $S_y \in \mathcal{F}_y^{\top\top}$ it follows that S_y is not a blocking set of $\mathcal{F}_y$. So there exists at least one $U_y \in \mathcal{F}_y$ disjoint from S_y. Since $y \in A \in \mathcal{A}$ and $\mathcal{A}$ is a $\mathrm{MIF}(l)$, there is a $B \in \mathcal{A}$ such that $A \cap B = \{y\}$. For each $x \in B \smallsetminus \{y\}$, we choose arbitrary $U_x \in \mathcal{F}_x$. Then $\bigsqcup_{x \in B} U_x$ is a block of $\mathcal{G}$ disjoint from the blocking set C, a contradiction. Thus $C \in \mathcal{G}$. Hence $(\mathcal{G} \sqcup \mathcal{G}^{\top})^{\top} \subseteq \mathcal{G}$. Therefore $\mathcal{G} = (\mathcal{G} \sqcup \mathcal{G}^{\top})^{\top}$ and this proves part (b). $\square$

Theorem 3.4.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be two uniform families with disjoint point sets. Let $\mathrm{k}(\mathcal{F}) = k$, $\mathrm{k}(\mathcal{G}) = k + t$, $\mathrm{tr}(\mathcal{F}) = t'$ and $\mathrm{tr}(\mathcal{G}) = t$. Suppose $\mathrm{tr}(\mathcal{G}^{\top}) > t + t'$. Let $\mathcal{H} = \mathcal{G} \sqcup (\mathcal{F} \circledast \mathcal{G}^{\top})$. Then,*

(a) $\mathcal{H}^{\top} = \mathcal{F}^{\top} \circledast \mathcal{G}^{\top}$. *In particular* $\mathrm{k}(\mathcal{H}) = k + t$, $\mathrm{tr}(\mathcal{H}) = t + t'$.

(b) *If, further, both $\mathcal{F}$ and $\mathcal{G}$ are closed intersecting families, then $\mathcal{H}$ is a closed intersecting family.*

`Proof` : Since every member of $\mathcal{F}^{\top} \circledast \mathcal{G}^{\top}$ is a blocking set of $\mathcal{H}$ of size $t + t'$, $\mathrm{tr}(\mathcal{H}) \leq t + t'$. Let C be a transversal of $\mathcal{H}$. Then $|C| \leq t + t'$. If $C \cap \mathrm{P}_{\mathcal{F}}$ is not a blocking set of $\mathcal{F}$, then there is a block $A \in \mathcal{F}$ disjoint from $C \cap \mathrm{P}_{\mathcal{F}}$. Since $|C| \leq t + t' \leq \mathrm{tr}(\mathcal{G}^{\top}) - 1$, there is a $B \in \mathcal{G}^{\top}$ disjoint from $C \cap \mathrm{P}_{\mathcal{G}}$. Then $A \sqcup B \in \mathcal{H}$ is disjoint from the blocking set C, a contradiction. Thus, $C \cap \mathrm{P}_{\mathcal{F}}$ is a blocking set of $\mathcal{F}$. Clearly $C \cap \mathrm{P}_{\mathcal{G}}$ is a blocking set of $\mathcal{G}$. Therefore $|C \cap \mathrm{P}_{\mathcal{F}}| \geq t'$ and $|C \cap \mathrm{P}_{\mathcal{G}}| \geq t$. Since $|C| \leq t + t'$ and $\mathrm{P}_{\mathcal{F}}$, $\mathrm{P}_{\mathcal{G}}$ are disjoint, it follows that $|C \cap \mathrm{P}_{\mathcal{F}}| = t'$ and $|C \cap \mathrm{P}_{\mathcal{G}}| = t$. Therefore $C \cap \mathrm{P}_{\mathcal{F}} \in \mathcal{F}^{\top}$ and $C \cap \mathrm{P}_{\mathcal{G}} \in \mathcal{G}^{\top}$. Thus $C \in \mathcal{F}^{\top} \circledast \mathcal{G}^{\top}$. This proves part (a).

Now suppose $\mathcal{F}$ and $\mathcal{G}$ are closed intersecting families. In particular they are intersecting families. Hence $\mathcal{H}$ is an intersecting family. Thus the blocks of $\mathcal{H}$ are blocking sets of $\mathcal{H} \sqcup \mathcal{H}^{\top}$ of size $k + t$. So $\mathrm{tr}(\mathcal{H} \sqcup \mathcal{H}^{\top}) \leq k + t$. Let C be a transversal of $\mathcal{H} \sqcup \mathcal{H}^{\top}$. Thus $|C| \leq k + t$. If we can show that $C \in \mathcal{H}$, then $\mathcal{H}$ is a closed intersecting family and we are done. If $C \in \mathcal{G}$ we are done. So suppose $C \notin \mathcal{G}$. But C is a blocking set of $\mathcal{G}$. Since $\mathcal{G}$ is a closed intersecting family, it follows that there is a $T \in \mathcal{G}^{\top}$ disjoint from C. Since C is a blocking set of $\mathcal{F} \circledast \mathcal{G}^{\top} \subseteq \mathcal{H}$ and also of $\mathcal{F}^{\top} \circledast \mathcal{G}^{\top} = \mathcal{H}^{\top}$, it follows that $C \cap \mathrm{P}_{\mathcal{F}}$ is a blocking set of $\mathcal{F} \sqcup \mathcal{F}^{\top}$. Since $\mathcal{F}$ is a closed intersecting family with $\mathrm{k}(\mathcal{F}) = k$, we get $|C \cap \mathrm{P}_{\mathcal{F}}| \geq k$. Also, as $C \cap \mathrm{P}_{\mathcal{G}}$ is a blocking set of $\mathcal{G}$ and $\mathrm{tr}(\mathcal{G}) = t$, $|C \cap \mathrm{P}_{\mathcal{G}}| \geq t$. Since $\mathrm{P}_{\mathcal{F}}$ and $\mathrm{P}_{\mathcal{G}}$ are disjoint, $|C| \geq k + t$. But $|C| \leq k + t$. Therefore $|C| = k + t$, $C \cap \mathrm{P}_{\mathcal{F}} \in \mathcal{F}$, $C \cap \mathrm{P}_{\mathcal{G}} \in \mathcal{G}^{\top}$. Consequently, $C \in \mathcal{F} \circledast \mathcal{G}^{\top} \subseteq \mathcal{H}$. Hence $C \in \mathcal{H}$. This proves part (b). $\square$

Chapter 4

Classification of Maximal Intersecting Families of 3−sets

4.1 Introduction

This chapter is meant to classify maximal intersecting families of 3−sets. We prove that there are 8 non isomorphic maximal intersecting families of 3−sets. The elementary constructions and various extremal bounds are given in [7], where the authors studied 3−chromatic intersecting families of k−sets. Any intersecting family of k−sets is either 2−chromatic or 3−chromatic. Any 3−chromatic intersecting family of k−sets is a $\mathrm{MIF}(k)$ but the converse is not true in general. But the converse is true for $k = 3$; i.e. any $\mathrm{MIF}(3)$ is a 3−chromatic intersecting family of 3−sets. In this chapter we list all the $\mathrm{MIF}(3)$s. We observe that [14, Theorem 5] gives a classification theorem for $\mathrm{MIF}(3)$. This chapter provides an independent proof of that theorem. The results in this chapter are from [21].

We fix the following notations, which are used throughout this chapter. Let $\mathcal{G}$ be a family of k−sets. For each $x \in \mathrm{P}_{\mathcal{G}}$, the number $|\{B \in \mathcal{G} : x \in B\}|$ is defined as $\deg_{\mathcal{G}}(x)$. xyz denotes the 3−set $\{x, y, z\}$ and xy denotes the 2−set $\{x, y\}$. Let $\mathcal{A}$ be an $\mathrm{MIF}(3)$ and α be a point of $\mathcal{A}$. By using Theorem 3.2.3, we decompose $\mathcal{A}$ in the form $\mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^{\top})$, where $\mathcal{F}$ is $\mathrm{CIF}(3, 2)$. Here $\mathcal{F}^{\top}$ is realised as a graph with vertex set $\mathrm{P}_{\mathcal{F}}$ and edge set $\mathcal{F}^{\top}$. In fact, the vertex set is $\mathrm{P}_{\mathcal{F}^{\top}}$ but by property (d) of Proposition 3.1.7 we have $\mathrm{P}_{\mathcal{F}^{\top}} \subset \mathrm{P}_{\mathcal{F}}$ so we can assume the vertices from $\mathrm{P}_{\mathcal{F}} \setminus \mathrm{P}_{\mathcal{F}^{\top}}$ remain isolated. We denote this graph by F. Let $T \in \mathcal{F}^{\top}$. Then $\{\alpha\} \sqcup T$ is a block of $\mathrm{MIF}(3)$ $\mathcal{A}$. Therefore there exists at least one block $B \in \mathcal{A}$ such that $B \cap (\{\alpha\} \sqcup T) = \{\alpha\}$. Hence there exists at least one transversal of $\mathcal{F}$, namely $T' := B \setminus \{\alpha\}$, disjoint from T. This induces the following property on the graph F.

Lemma 4.1.1. *For each edge e of the graph F there exists another edge e' disjoint from e.*

4.2 Classification of $\mathrm{MIF}(3)$s with 6 points

This section classifies $\mathrm{MIF}(3)$s with 6 points through Theorem 4.2.8. Here we let $\mathcal{A}$ to be an $\mathrm{MIF}(3)$ with 6 points and α to be a point of $\mathcal{A}$ such that $\deg_{\mathcal{A}}(\alpha)$ is minimum. So here F has 5 vertices and $\deg_{\mathcal{A}}(\alpha)$ edges. The proof of Theorem 4.2.8 is dependent on $\deg_{\mathcal{A}}(\alpha)$.

Lemma 4.2.1. $\mathcal{F} = \left\{ \mathrm{P}_{\mathcal{F}} \setminus T : T \in \binom{\mathrm{P}_{\mathcal{F}}}{2} \setminus \mathcal{F}^{\top} \right\}$.

`Proof` : Let $T \in \binom{\mathrm{P}_{\mathcal{F}}}{2} \setminus \mathcal{F}^{\top}$. Then T is not a blocking set of $\mathcal{F}$. So there exists at least one $B \in \mathcal{F}$ disjoint from T. Since $|\mathrm{P}_{\mathcal{F}}| = 5$ and $\mathrm{tr}(\mathcal{F}) = |T| = 2$, so $B = \mathrm{P}_{\mathcal{F}} \setminus T$ is the unique block disjoint from T. Therefore, $\left\{ \mathrm{P}_{\mathcal{F}} \setminus T : T \in \binom{\mathrm{P}_{\mathcal{F}}}{2} \setminus \mathcal{F}^{\top} \right\} \subseteq \mathcal{F}$. Since we have a unique association $T \mapsto \mathrm{P}_{\mathcal{F}} \setminus T$ from $\binom{\mathrm{P}_{\mathcal{F}}}{2} \setminus \mathcal{F}^{\top}$ to $\mathcal{F}$, this proves the other inclusion $\mathcal{F} \subseteq \left\{ \mathrm{P}_{\mathcal{F}} \setminus T : T \in \binom{\mathrm{P}_{\mathcal{F}}}{2} \setminus \mathcal{F}^{\top} \right\}$. □

Lemma 4.2.2. *For each* $x \in \mathrm{P}_{\mathcal{F}}$*,* $\deg_{\mathcal{A}}(\alpha) - 3 \leq \deg_{\mathsf{F}}(x) \leq 3$*. Moreover,* $2 \leq \deg_{\mathcal{A}}(\alpha) \leq 5$*.*

`Proof` : Let $x \in \mathrm{P}_{\mathcal{F}^{\top}}$. Then there exists at least one block of $\mathcal{A}$ which contains both α and x. Since $\mathrm{tr}(\mathcal{A}) = 3$, there exists at least one block $B \in \mathcal{A}$ disjoint from $\{\alpha, x\}$. Hence there are at most 3 blocks which contain both α and x. Consequently $\deg_{\mathsf{F}}(x) \leq 3$. By using the unique association $T \mapsto \mathrm{P}_{\mathcal{F}} \setminus T$ from $\binom{\mathrm{P}_{\mathcal{F}}}{2} \setminus \mathcal{F}^{\top}$ to $\mathcal{F}$ in Lemma 4.2.1 we have the following.

$$\begin{aligned}
\deg_{\mathcal{F}}(x) &= |\{B \in \mathcal{F} : x \in B\}| \\
&= |\{\mathrm{P}_{\mathcal{F}} \setminus T : x \notin T \notin \mathcal{F}^{\top}\}| \\
&= |\{T \notin \mathcal{F}^{\top} : x \notin T\}| \\
&= \left| \left\{ T \in \binom{\mathrm{P}_{\mathcal{F}}}{2} : T \notin \mathcal{F}^{\top} \right\} \right| - \left| \left\{ T \in \binom{\mathrm{P}_{\mathcal{F}}}{2} : T \notin \mathcal{F}^{\top} \text{ and } x \in T \right\} \right| \\
&= \binom{5}{2} - |\mathcal{F}^{\top}| - \{4 - \deg_{\mathsf{F}}(x)\}. \qquad (4.2.1)
\end{aligned}$$

We already assumed that $\alpha \in \mathrm{P}_{\mathcal{A}}$ is a point such that $\deg_{\mathcal{A}}(\alpha)$ is minimum. Therefore for each $x \in \mathrm{P}_{\mathcal{A}} \setminus \{\alpha\}$ we have

$$\begin{aligned}
|\mathcal{F}^{\top}| = \deg_{\mathcal{A}}(\alpha) &\leq \deg_{\mathcal{F}}(x) + \deg_{\mathsf{F}}(x) \\
&= \binom{5}{2} - |\mathcal{F}^{\top}| - \{4 - \deg_{\mathsf{F}}(x)\} + \deg_{\mathsf{F}}(x) \\
&= 6 - |\mathcal{F}^{\top}| + 2\deg_{\mathsf{F}}(x). \qquad (4.2.2)
\end{aligned}$$

Thus we have the lower bound since $\deg_{\mathcal{A}}(\alpha) = |\mathcal{F}^{\top}|$. This completes the first part.

From the first part it follows that $|\mathcal{F}^{\top}| \leq 6$. If $|\mathcal{F}^{\top}| = 6$, then again using the first part of the lemma it follows that the graph F is regular of degree 3 and it has $\frac{5 \times 3}{2} = 7\frac{1}{2}$ edges, a contradiction. Hence $\deg_{\mathcal{A}}(\alpha) \leq 5$. By Lemma 4.2.1 we have there are at least 2 transversals of $\mathcal{F}$. Hence $\deg_{\mathcal{A}}(\alpha) \geq 2$. This completes the proof of the second part of the lemma. □

Lemma 4.2.3. *For each* $x \in \mathrm{P}_{\mathcal{F}}$*,* $\deg_{\mathsf{F}}(x) \leq 2$*.*

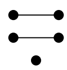
`Proof` : Let $x \in \mathrm{P}_{\mathcal{F}}$. Using Lemma 4.2.2 we have $\deg_{\mathsf{F}}(x) \leq 3$. Suppose for some $x \in \mathrm{P}_{\mathcal{F}}$ we get $\deg_{\mathsf{F}}(x) = 3$. Let $\mathrm{P}_{\mathcal{F}} = \{x_1, x_2, x_3, x_4, x_5\}$. Without loss of generality, let $x = x_1$ and x_1x_2, x_1x_3 and x_1x_4 be only edges through x_1 in the graph F. This implies $x_2x_3x_4 \in \mathcal{F}$. Suppose $\deg(x_5) = 0$, then from Lemma 4.2.1, it follows that F contains a complete sub graph on 4 vertices namely x_1, x_2, x_3 and x_4. Therefore by using Lemma 4.2.1, we have $\binom{4}{2} = |\mathcal{F}^{\top}| = \deg_{\mathcal{A}}(\alpha) \leq 5$, a contradiction. So $\deg_{\mathsf{F}}(x_5) \geq 1$. Thus there exists at least one edge of the form x_5x, where $x \in \{x_2, x_3, x_4\}$ without loss of generality let it be x_2x_5. But there exists at least one edge disjoint from x_1x_2 so it is either x_4x_5 or x_3x_4. Since $|\mathcal{F}^{\top}| \leq 5$, both of x_4x_5, x_3x_4 can not be edges. Thus the following two cases exhaust all the possibilities.

`Case A.` $\mathcal{F}^{\top} = \{x_1x_2, x_1x_3, x_1x_4, x_2x_5, x_4x_5\}$.

`Case B.` $\mathcal{F}^{\top} = \{x_1x_2, x_1x_3, x_1x_4, x_2x_5, x_3x_4\}$.

But in each of the above cases there exists at least one $x \in \mathrm{P}_{\mathcal{F}}$ such that $\deg_{\mathsf{F}}(x) = 1$. (For `Case A` $x = x_3$ and for `Case B` $x = x_5$.) Since from Lemma 4.2.1 we have $\mathcal{F} = \left\{\mathrm{P}_{\mathcal{F}} \smallsetminus T : T \in \binom{\mathrm{P}_{\mathcal{F}}}{2} \smallsetminus \mathcal{F}^{\top}\right\}$, $\deg_{\mathcal{F}}(x) = 2$. Hence $\deg_{\mathcal{A}}(x) = \deg_{\mathcal{F}}(x) + \deg_{\mathsf{F}}(x) = 3$. A contradiction arises since $\deg_{\mathcal{A}}(\alpha)$ is minimum and $5 = \deg_{\mathcal{A}}(\alpha) \leq \deg_{\mathcal{A}}(x) = 3$. □

Lemma 4.2.4. *If* $\deg_{\mathcal{A}}(\alpha) = 2$*, then the graph* F *is isomorphic to the following graph.*



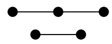
Consequently, $\mathcal{A}$ *is isomorphic to* $\{234, 235, 245, 246, 256, 345, 346, 356\} \sqcup \{123, 145\}$. *(Here* $\mathcal{A}$ *is expressed in the form* $\mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^{\top})$*.)*

`Proof` : Since $\deg_{\mathcal{A}}(\alpha) = 2$, F has two edges. By using Lemma 4.2.3 and the first part of Lemma 4.2.2 we have for each $x \in \mathrm{P}_{\mathcal{F}}$, $0 \leq \deg_{\mathsf{F}}(x) \leq 2$. But there does not exist any $x \in \mathrm{P}_{\mathcal{F}}$ with $\deg_{\mathsf{F}}(x) = 2$. If for some $x \in \mathrm{P}_{\mathcal{F}}$ $\deg_{\mathsf{F}}(x) = 2$, then it follows that the graph F is isomorphic to

.

It contradicts Lemma 4.1.1. So for each $x \in \mathrm{P}_{\mathcal{F}}$, $0 \leq \deg_{\mathsf{F}}(x) \leq 1$. It proves the first part of this result. Using Lemma 4.2.1 we get the consequent part. □

Lemma 4.2.5. *If* $\deg_{\mathcal{A}}(\alpha) = 3$*, then the graph* F *is isomorphic to the following graph.*

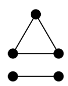


Consequently, $\mathcal{A}$ *is isomorphic to* $\{235, 236, 245, 246, 345, 346, 356\} \sqcup \{123, 134, 156\}$*. (Here* $\mathcal{A}$ *is expressed in the form* $\mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^{\top})$*.)*

`Proof` : Since $\deg_{\mathcal{A}}(\alpha) = 3$, F has three edges. By using Lemma 4.2.3 and the first part of Lemma 4.2.2 we have for each $x \in \mathrm{P}_{\mathcal{F}}$, $0 \leq \deg_{\mathsf{F}}(x) \leq 2$. But there does not exist any $x \in \mathrm{P}_{\mathcal{F}}$ with $\deg_{\mathsf{F}}(x) = 0$. If for some $x \in \mathrm{P}_{\mathcal{F}}$ $\deg_{\mathsf{F}}(x) = 0$, then it follows that the graph F is isomorphic to

It contradicts Lemma 4.1.1. So for each $x \in \mathrm{P}_{\mathcal{F}}$, $1 \leq \deg_{\mathsf{F}}(x) \leq 2$. It proves the first part of this result. Using Lemma 4.2.1 we get the consequent part. □

Lemma 4.2.6. *If* $\deg_{\mathcal{A}}(\alpha) = 4$*, then the graph* F *is isomorphic to either of the following graphs.*



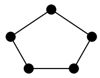
Consequently, $\mathcal{A}$ *is isomorphic to*

$$\{235, 236, 245, 246, 345, 346\} \sqcup \{123, 124, 134, 156\} \text{ and}$$
$$\{235, 245, 246, 345, 346, 356\} \sqcup \{123, 134, 145, 156\}$$

respectively. (Here $\mathcal{A}$ *is expressed in the form* $\mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^{\top})$*.)*

`Proof` : Since $\deg_{\mathcal{A}}(\alpha) = 4$, F has four edges. By using Lemma 4.2.3 and the first part of Lemma 4.2.2 we have for each $x \in \mathrm{P}_{\mathcal{F}}$, $1 \leq \deg_{\mathsf{F}}(x) \leq 2$. It proves the first part of this result. Using Lemma 4.2.1 we get the consequent part. □

Lemma 4.2.7. *If* $\deg_{\mathcal{A}}(\alpha) = 5$*, then the graph* F *is isomorphic to the following graph.*



Consequently, $\mathcal{A}$ is isomorphic to $\{235, 245, 246, 346, 356\} \sqcup \{123, 126, 134, 145, 156\}$.

`Proof` : Since $\deg_{\mathcal{A}}(\alpha) = 5$, F has five edges. By using Lemma 4.2.3 and the first part of Lemma 4.2.2 we have for each $x \in \mathrm{P}_{\mathcal{F}}$, $\deg_{\mathsf{F}}(x) = 2$. It proves the first part of this result. Using Lemma 4.2.1 we get the consequent part. $\square$

Combining Lemma 4.2.4, Lemma 4.2.5, Lemma 4.2.6 and Lemma 4.2.7, we prove the following classification theorem.

Theorem 4.2.8. *Any* MIF(3) *with* 6 *points* $\mathcal{A}$ *is isomorphic to one of the following:*

(a) $\{123, 145, 234, 235, 245, 246, 256, 345, 346, 356\}$,

(b) $\{123, 134, 156, 235, 236, 245, 246, 345, 346, 356\}$,

(c) $\{123, 124, 134, 156, 235, 236, 245, 246, 345, 346\}$,

(d) $\{123, 134, 145, 156, 235, 245, 246, 345, 346, 356\}$,

(e) $\{123, 126, 134, 145, 156, 235, 245, 246, 346, 356\}$.

4.3 Classification of MIF(3)s with at least 7 points

In this section, we classify all MIF(3)s with at least 7 points in Theorem 4.3.5. Here F has at least 6 vertices and $\deg_{\mathcal{A}}(\alpha)$ edges. Suppose that the graph F has N vertices, i.e. $\mathrm{P}_{\mathcal{F}} = \mathrm{P}_{\mathcal{A}} \smallsetminus \{\alpha\} = \{x_i : 1 \leq i \leq N\}$, where $N \geq 6$.

Lemma 4.3.1. *For each* $x \in \mathrm{P}_{\mathcal{F}}$, $\deg_{\mathsf{F}}(x) \leq 3$. *Moreover,* $\deg_{\mathsf{F}}(x) \neq 2$ *for each* $x \in \mathrm{P}_{\mathcal{F}}$.

`Proof` : Let $x \in \mathrm{P}_{\mathcal{F}}$. If $x \notin \mathrm{P}_{\mathcal{F}^{\top}}$, then $\deg_{\mathsf{F}}(x) = 0$ and we are done this case.
Now let $x \in \mathrm{P}_{\mathcal{F}^{\top}}$ i.e. there exists at least one block B in $\mathcal{A}$ such that $\{\alpha, x\} \subset B$. Therefore there exists at least one block B' disjoint from $\{\alpha, x\}$. Hence there exists at most three blocks B in $\mathcal{A}$ such that $\{\alpha, x\} \subset B$, namely $\{\alpha, x, y\}$ where $y \in B'$. Since $|B'| = 3$, $\deg_{\mathsf{F}}(x) \leq 3$. This proves the first part of this result.

Suppose, if possible, $\deg_{\mathsf{F}}(x) = 2$ for some $x \in \mathrm{P}_{\mathcal{F}}$. Without loss of generality, let $x = x_1$ and let x_1x_2, x_1x_3 be only edges through x_1. So $x_1x_2, x_1x_3 \in \mathcal{F}^{\top}$ and $x_1x \notin \mathcal{F}^{\top}$

for each $x \in \{x_i : 4 \leq i \leq N\}$. Since $\mathcal{A}$ is a MIF(3), there exists at least one block $B \in \mathcal{A}$ disjoint from $\{\alpha, x_1\}$ but $\{x_2, x_3\} \subset B$. Hence $B \in \mathcal{F}$.

Claim : $|\{B \in \mathcal{F} : x_2, x_3 \in B, x_1 \notin B\}| = 2$

`Proof of claim :` Using the same argument as in the previous paragraph, we conclude that $|\{B \in \mathcal{F} : x_2, x_3 \in B, x_1 \notin B\}| \leq 3$. Suppose $|\{B \in \mathcal{F} : x_2, x_3 \in B, x_1 \notin B\}| = 3$. So let

$$\{B \in \mathcal{F} : x_2, x_3 \in B, x_1 \notin B\} = \{x_2x_3x, x_2x_3y, x_2x_3z\}$$

where $x, y, z \in \{x_i : 4 \leq i \leq N\}$, this implies xyz is a block of $\mathcal{F}$ but $x_1x_2 \in \mathcal{F}^\top$ does not intersect it, a contradiction. Hence $|\{B \in \mathcal{F} : x_2, x_3 \in B, x_1 \notin B\}| \leq 2$. Now let for some $x \in \{x_i : 4 \leq i \leq N\}$, $x_2x_3x \in \mathcal{F}$. Since $x_1x \notin \mathcal{F}^\top$, there exists at least one block $B \in \mathcal{F}$ disjoint from it. Again since x_1x_2, $x_1x_3 \in \mathcal{F}^\top$, such a B is of the form x_2x_3y, where $y \in \{x_i : 4 \leq i \leq N\} \setminus \{x\}$. Hence the claim is established.

We assume without loss of generality that $x_2x_3x_4, x_2x_3x_5 \in \mathcal{F}$. Using the same argument as before there exists at least one block $B \in \mathcal{A}$ such that $x_4, x_5 \in B$ and $x_2, x_3 \notin B$. We observe that such a B is in either of the form αx_4x_5 or of the form xx_4x_5, where $x = x_1$ or $x \in \{x_i : 6 \leq i \leq N\}$. Since x_1x_2 and $x_1x_3 \in \mathcal{F}^\top$, therefore we have either $x_4x_5 \in \mathcal{F}^\top$ or $x_1x_4x_5 \in \mathcal{F}$.

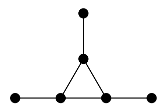
`Case` $x_4x_5 \in \mathcal{F}^\top$. Here we observe that for each $x \in \{x_i : 6 \leq i \leq N\}$ there exist at least 2 blocks $B_1^x, B_2^x \in \mathcal{A}$ with $B_1^x \cap B_2^x = \{x\}$. So at most one of them contains α and hence at least one of B_1^x or B_2^x belongs to $\mathcal{F}$. So without loss of generality let $B_1^x \in \mathcal{F}$. Hence x_1x_2, x_1x_3 and x_4x_5 intersects B_1^x, and this implies $B_1^x = xx_1y$ where $y \in \{x_4, x_5\}$. But this block does not intersect both blocks $x_2x_3x_4$ and $x_2x_3x_5$ of $\mathcal{F}$, a contradiction.

`Case` $x_1x_4x_5 \in \mathcal{F}$. For this case also we observe that for each $x \in \{x_i : 6 \leq i \leq N\}$ there exist at least 2 blocks $B_1^x, B_2^x \in \mathcal{A}$ with $B_1^x \cap B_2^x = \{x\}$ so at most one of them contains x_1 and hence at least one of B_1^x or B_2^x does not contain x_1. Without loss of generality let B_1^x do not contain x_1. Hence B_1^x is of the form xyz where $y \in \{x_4, x_5\}$ and z is a common point among the blocks αx_1x_2, αx_1x_3, $x_2x_3x_5$ or αx_1x_2, αx_1x_3, $x_2x_3x_4$, according as $y = x_4$ or $y = x_5$ respectively, a contradiction, since there is no such common point.

Since we are lead to contradictions in both cases, the supposition $\deg_{\mathsf{F}}(x) = 2$ for some $x \in \mathrm{P}_{\mathcal{F}}$ was wrong. This completes the proof. □

Lemma 4.3.2. *If for some* $x \in \mathrm{P}_{\mathcal{F}}$ $\deg_{\mathsf{F}}(x) = 3$, *then*

(a) F *contains a sub graph which is isomorphic to the following graph,*


,

(b) $\mathcal{F}$ *is isomorphic to* $\{123, 234, 246, 345\}$*,*

(c) $\mathcal{A}$ *is isomorphic to* $\{123, 234, 246, 345\} \sqcup \{147, 237, 247, 257, 347, 367\}$*. (Here* $\mathcal{A}$ *is expressed in the form* $\mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^{\top})$*.)*

`Proof` : Without loss of generality let $\deg_{\mathsf{F}}(x_1) = 3$; also let x_1x_2, x_1x_3 and x_1x_4 be the only edges through x_1. This immediately implies that $x_2x_3x_4 \in \mathcal{F}$.

Claim : Exactly one of x_2x_3, x_2x_4 and x_3x_4 is a transversal of $\mathcal{F}$.

`Proof of claim` : Suppose all the above three belong to $\mathcal{F}^{\top}$. Since all the above three form a MIF(2), $\mathrm{tr}(\mathcal{F}^{\top}) = 3$. As a result we have $x_1x_2x_3$, $x_1x_3x_4$, $x_1x_2x_4 \in \mathcal{F}$. So $\mathcal{A}$ contains $x_1x_2x_3$, $x_1x_3x_4$, $x_1x_2x_4$, $x_2x_3x_4$, αx_1x_2, αx_1x_3, αx_1x_4, αx_2x_3, αx_2x_4, αx_3x_4. It means $\mathcal{A}$ is isomorphic to the MIF(3) $\beta(3)$. But $\mathcal{A}$ is a MIF(3) with at least 7 points, a contradiction.

So suppose two the above belong to $\mathcal{F}^{\top}$. Without loss of generality let x_2x_3 and $x_2x_4 \in \mathcal{F}^{\top}$. This implies $x_1x_3x_4 \in \mathcal{F}$. Using Lemma 4.3.1 there exists another transversal of $\mathcal{F}$ through x_3 other than x_1x_3, x_2x_3, similarly through x_4 other than x_1x_4, x_2x_4. Again using Lemma 4.3.1 we let x_3x_5, $x_4x \in \mathcal{F}^{\top}$, where $x \in \{x_i : 5 \leq i \leq N\}$. If possible suppose x_3x_5, $x_4x_5 \in \mathcal{F}^{\top}$. Then by using Lemma 4.3.1 there exists at least one $x_5x \in \mathcal{F}^{\top}$ where $x \in \{x_i : 6 \leq i \leq N\}$. This is impossible since x_5x does not intersect $x_1x_3x_4 \in \mathcal{F}$. So without loss of generality we let x_3x_5, $x_4x_6 \in \mathcal{F}^{\top}$ and consequently $x_1x_2x_5$, $x_1x_2x_6 \in \mathcal{F}$. This is impossible since $x_3x_5 \in \mathcal{F}^{\top}$ does not intersect $x_1x_2x_6 \in \mathcal{F}$ and similarly $x_4x_6 \in \mathcal{F}^{\top}$ does not intersect $x_1x_2x_5 \in \mathcal{F}$.

So suppose none of the above belongs to $\mathcal{F}^{\top}$. From Lemma 4.1.1 there exists at least one edge e disjoint from x_1x_2 and intersects $x_2x_3x_4$. Hence e contains at least one of x_3 or x_4. By assumption $e \neq x_3x_4$. So let $e = x_3x$, where $x \in \{x_i : 5 \leq i \leq N\}$. Without loss of generality we assume $e = x_3x_5$. By Lemma 4.3.1 there exists at least one edge through x_3 other than x_1x_3 and x_3x_5. Without loss of generality suppose it is x_3x_6. This implies $x_1x_5x_6 \in \mathcal{F}$. This is impossible since $x_2x_3x_4 \in \mathcal{F}$ and it does not intersect $x_1x_5x_6$. Hence only remaining possibility is the statement of this claim, so the claim is established.

Therefore without loss of generality let $x_3x_4 \in \mathcal{F}^\top$. But by Lemma 4.3.1 there exists at least one edge through x_3 other than x_1x_3 and x_3x_4. Without loss of generality let it be x_3x_5, similarly there exists at least one edge through x_4 other than x_1x_4 and x_3x_4. Let it be x_4x, where $x \in \{x_i : 5 \leq i \leq N\}$. But such an $x \neq x_5$, if so, then by Lemma 4.3.1 there exists at least one edge x_5y other than x_3x_5 and x_4x_5. Hence $yx_3x_4 \in \mathcal{F}$ and intersects $x_1x_2 \in \mathcal{F}^\top$. Thus $y \in \{x_1, x_2\}$. But x_1x_2, x_1x_3 and x_1x_4 are only edges through x_1. Hence $y = x_2$ and $x_2x_3x_4 \in \mathcal{F}$. Now from Lemma 4.3.1 there exists at least one edge through x_2 other than x_1x_2 and x_2x_5. By the above claim x_2z, where $z \in \{x_i : 6 \leq i \leq N\}$, is only possible edge. This implies $x_1x_5z \in \mathcal{F}$ which is disjoint from $x_2x_3x_4 \in \mathcal{F}$, a contradiction. Thus $x \in \{x_i : 6 \leq i \leq N\}$. Without loss of generality let it be x_4x_6. Hence

$$\mathcal{F}^\top \supseteq \{x_1x_2, x_1x_3, x_1x_4, x_3x_4, x_3x_5, x_4x_6\}$$

and (a) follows. This immediately implies $x_1x_3x_6$, $x_1x_4x_5$, $x_2x_3x_4 \in \mathcal{F}$. Only remain to show $x_1x_3x_4 \in \mathcal{F}$. We decompose $\mathcal{A} = \mathcal{G} \sqcup (x_1 \circledast \mathcal{G}^\top)$, where $\mathcal{G}$ is a CIF$(3,2)$ and $x_1 \notin \mathrm{P}_{\mathcal{G}}$. Thus from the previous conclusions this implies

$$\{\alpha x_2, \alpha x_3, \alpha x_4, x_3x_6, x_4x_5\} \subset \mathcal{G}^\top.$$

By using the similar claim as above, we have exactly one of x_2x_3, x_2x_4 and x_3x_4 is a transversal of $\mathcal{G}$. But x_2x_3 is disjoint from $\alpha x_4x_6 \in \mathcal{G}$ and x_2x_4 is disjoint from $\alpha x_3x_5 \in \mathcal{G}$. This implies $x_3x_4 \in \mathcal{G}^\top$. Hence $x_1x_3x_4 \in \mathcal{A}$ and therefore $x_1x_3x_4 \in \mathcal{F}$. This proves (b).

Now from (a) and (b) we get $\mathcal{A}$ contains a subfamily $\mathcal{H}$ isomorphic to

$$\{123, 234, 246, 345\} \sqcup \{147, 237, 247, 257, 347, 367\}.$$

But $\mathcal{H}$ is a MIF(3). Hence $\mathcal{A} = \mathcal{H}$ and this proves (c). $\square$

Lemma 4.3.3. *For each $x \in \mathrm{P}_{\mathcal{F}}$, if $\deg_{\mathsf{F}}(x) \neq 0$ or $\deg_{\mathsf{F}}(x) \neq 3$, then $\mathcal{A}$ is a finite projective plane of order 2.*

`Proof` : We first observe that if for each $x \in \mathrm{P}_{\mathcal{F}}$, $0 \leq \deg_{\mathsf{F}}(x) \leq 1$, then any two transversal of $\mathcal{F}$ are mutually disjoint. (If possible, suppose there exist T and $T' \in \mathcal{F}^\top$ such that $T \cap T' = \{x\}$ for some $x \in \mathrm{P}_{\mathcal{F}}$, so $\deg_{\mathsf{F}}(x) \geq 2$, a contradiction.) As $\mathcal{F}$ is a $3-$uniform family, hence there are at most 3 transversals of $\mathcal{F}$. Here using Lemma 4.3.1 we have for each $x \in \mathrm{P}_{\mathcal{F}}$, $\deg_{\mathsf{F}}(x) = 1$. Hence there exists exactly 3 transversals of $\mathcal{F}$. Without loss of generality let

$$\mathcal{F}^\top = \{x_1x_2, x_3x_4, x_5x_6\}. \tag{4.3.1}$$

Therefore any block of $\mathcal{F}$ is of the form xyz where $x \in \{x_1, x_2\}$, $y \in \{x_3, x_4\}$ and $z \in \{x_5, x_6\}$. Without loss of generality let $x_1x_3x_5 \in \mathcal{F}$. We decompose $\mathcal{A} = \mathcal{G} \sqcup (x_1 \circledast \mathcal{G}^\top)$ where $\mathcal{G}$ is a CIF$(3,2)$ and $x_1 \notin \mathrm{P}_\mathcal{G}$. We observe that $\{\alpha x_2, x_3x_5\} \subset \mathcal{G}^\top$.

Claim 1 : $x_4x_6 \in \mathcal{G}^\top$ and x_3x_6, $x_4x_5 \notin \mathcal{G}^\top$.

`Proof of claim` : If $x_4x_6 \notin \mathcal{G}^\top$, then $x_1x_4x_6 \notin \mathcal{F}$. Since $\mathcal{F}$ is a CIF$(3,2)$, there exists at least one $T \in \mathcal{F}^\top$ disjoint from $x_1x_4x_6$. It is impossible by (4.3.1). So $x_4x_6 \in \mathcal{G}^\top$. Suppose, if possible, $x_3x_6 \in \mathcal{G}^\top$. Consequently $x_1x_3x_6 \in \mathcal{F}$. Since $x_1x_3 \notin \mathcal{F}^\top$, there exists at least one block $B \in \mathcal{F}$ disjoint from $\{x_1, x_3\}$. Hence by (4.3.1), $B = x_2x_4x$ where $x \in \{x_5, x_6\}$. But such a B does not intersect both $x_1x_3x_5$ and $x_1x_3x_6$, a contradiction. By a similar reasoning $x_4x_5 \notin \mathcal{G}^\top$. Hence the claim is established.

Claim 2 : $\mathcal{G}^\top = \{\alpha x_2, x_3x_5, x_4x_6\}$.

`Proof of claim` : From Claim 1 we have $\{\alpha x_2, x_3x_5, x_4x_6\} \subseteqq \mathcal{G}^\top$. Since $x_3x_4 \in \mathcal{F}^\top$, $\alpha x_3x_4 \in \mathcal{G}$. Let $pq \in \mathcal{G}^\top \smallsetminus \{\alpha x_2, x_3x_5, x_4x_6\}$, then $p \in \{\alpha, x_3, x_4\}$.

Suppose $p = \alpha$. Since $x_1 \notin \mathrm{P}_{\mathcal{G}^\top}$ and $\alpha x_2 \in \mathcal{G}^\top$, so $q \in \{x_i : 3 \leq i \leq N\}$, hence $\alpha x_1 q \in \mathcal{A}$ and consequently $x_1 q \in \mathcal{F}^\top$, which violates (4.3.1). Now suppose $p = x_3$. Then from the previous arguments we have $q \neq \alpha$. But $\alpha x_5x_6 \in \mathcal{G}$ and $x_3x_5 \in \mathcal{G}^\top$. Hence $q = x_6$, which violates Claim 1. Similarly suppose $p = x_4$, then from the previous arguments we have $q \neq \alpha$ and $q \neq x_3$. But $\alpha x_5x_6 \in \mathcal{G}$ and $x_4x_6 \in \mathcal{G}^\top$. Hence $q = x_5$, which violates Claim 1. This proves Claim 2.

Claim 3 : $x_2x_3x_6$, $x_2x_4x_5 \in \mathcal{G}$

`Proof of claim` : By Claim 1 we have $x_4x_5 \notin \mathcal{G}^\top$. So there exists at least one block $B \in \mathcal{G}$ such that B disjoint from x_4x_5. Hence using Claim 2 either $B = \alpha x_3x_6$ or $B = x_2x_3x_6$. Due to (4.3.1) we have $B \neq \alpha x_3x_6$. Therefore $B = x_2x_3x_6$. By a similar argument $x_2x_4x_5 \in \mathcal{G}$ and the claim is established.

Thus $\mathcal{A}$ contains a subfamily $\mathcal{H}$ isomorphic to

$$\{\alpha x_1x_2, \alpha x_3x_4, \alpha x_5x_6, x_1x_3x_5, x_1x_4x_6, x_2x_3x_6, x_2x_4x_5\}.$$

We observe that $\mathcal{H}$ is a MIF(3) with 7 points which is also the example of finite projective plane of order 2. Hence $\mathcal{A}$ is a MIF(3) and is a finite projective plane of order 2. □

Lemma 4.3.4. *Suppose $\mathcal{A}$ is not isomorphic to a finite projective plane of order* 2 *and for each $x \in \mathrm{P}_\mathcal{F}$ $0 \leq \deg_\mathsf{F}(x) \leq 1$. Then there exists at least one point $\beta (\neq \alpha) \in \mathrm{P}_\mathcal{A}$ such* *that if* $\mathcal{A} = \mathcal{G} \sqcup (\beta \circledast \mathcal{G}^\top)$*, where* $\mathcal{G}$ *is a* CIF(3,2) *and* $\beta \notin \mathrm{P}_\mathcal{G}$*, then there exists at least one vertex* y *in the graph* $\mathsf{G} := (\mathrm{P}_\mathcal{G}, \mathcal{G}^\top)$ *such that* $\deg_\mathsf{G}(y) = 3$.

`Proof` : Using an argument in Lemma 4.3.3, we observe that if for each $x \in \mathrm{P}_\mathcal{F}$, $0 \leq \deg_\mathsf{F}(x) \leq 1$, then any two transversals of $\mathcal{F}$ are disjoint. As $\mathcal{F}$ is a $3-$uniform family, there exists at most 3 transversals of $\mathcal{F}$. But if there exists exactly 3 mutually disjoint transversals, then $N \geq 6$.

If $N \geq 7$, then there exists at least one $x \in \mathrm{P}_\mathcal{F} \setminus \mathrm{P}_{\mathcal{F}^\top}$. So there exists at least one $B \in \mathcal{F}$ such that $x \in B$ and $B \setminus \{x\}$ is a blocking set of $\mathcal{F}^\top$ with size 2. It contradicts that $\mathrm{tr}(\mathcal{F}^\top) = 3$. Hence $N = 6$ and consequently all the conditions of Lemma 4.3.3 are satisfied. Therefore, $\mathcal{A}$ is isomorphic to a finite projective plane of order 2 which contradicts the assumption. Thus $|\mathcal{F}^\top| \leq 2$. Since $\mathcal{A}$ is a MIF(3), each point belongs to at least 2 blocks hence $|\mathcal{F}^\top| = 2$. Without loss of generality, let $\mathcal{F}^\top = \{x_1x_2, x_3x_4\}$. Note that $\mathcal{A}$ is a MIF(3) and x_1x_2 intersects all the members of $\mathcal{A}$ except αx_3x_4. Hence $x_1x_2x \in \mathcal{A}$ where $x \in \{x_3, x_4\}$. By a similar reasoning $x_3x_4y \in \mathcal{F}$ where $y \in \{x_1, x_2\}$. Therefore

$$\{x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4\} \sqcup (\alpha \circledast \{x_1x_2, x_3x_4\}) \subset \mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^\top) \subset \mathcal{A}.$$

So we choose $\beta = x_1$. Then $\{\alpha x_2, x_2x_3, x_2x_4, x_3x_4\} \subset \mathcal{G}^\top$ and using Lemma 4.3.1 we have $\deg_\mathsf{G}(x_2) = 3$, i.e. required vertex y is x_2. □

Theorem 4.3.5. *Any* MIF(3) *with at least* 7 *points is isomorphic to one of the following:*

(a) $\{123, 145, 167, 246, 257, 347, 356\}$ *i.e. finite projective plane of order* 2*,*

(b) $\{123, 124, 127, 145, 147, 167, 246, 247, 257, 347\}$.

`Proof` : Let $\mathcal{A}$ be a MIF(3) with at least 7 points. We decompose $\mathcal{A}$ in the form $\mathcal{F} \sqcup (\alpha \circledast \mathcal{F}^\top)$, where $\mathcal{F}$ is a CIF(3,2) and $\alpha \notin \mathrm{P}_\mathcal{F}$. Let $\mathsf{F} := (\mathrm{P}_\mathcal{F}, \mathcal{F}^\top)$ be the graph as we mentioned in the introduction. By using Lemma 4.3.1, we have for each $x \in \mathrm{P}_\mathcal{F}$, $\deg_\mathsf{F}(x) \leq 3$. If for some $x \in \mathrm{P}_\mathcal{F}$ we get $\deg_\mathsf{F}(x) = 3$, then by using part (c) of Lemma 4.3.2 we have $\mathcal{A}$ is isomorphic to $\{123, 124, 127, 145, 147, 167, 246, 247, 257, 347\}$.

Now we assume that for each $x \in \mathrm{P}_\mathcal{F}$, $\deg_\mathsf{F}(x) \neq 3$. Again by using Lemma 4.3.1 we have for each $x \in \mathrm{P}_\mathcal{F}$, $0 \leq \deg_\mathsf{F}(x) \leq 1$. If for each $x \in \mathrm{P}_\mathcal{F}$ we get $\deg_\mathsf{F}(x) = 1$, then by using Lemma 4.3.3 we have $\mathcal{A}$ is isomorphic to $\{123, 145, 167, 246, 257, 347, 356\}$ (i.e. finite projective plane of order 2).

Now we assume $\mathcal{A}$ is not isomorphic to finite projective plane of order 2 and for each $x \in \mathrm{P}_\mathcal{F}$, $0 \leq \deg_\mathsf{F}(x) \leq 1$. Therefore all the conditions of Lemma 4.3.4 are satisfied.

Lemma 4.3.4 ensures that there exists another decomposition of $\mathcal{A}$ in the form $\mathcal{G} \sqcup (\beta \circledast \mathcal{G}^\top)$, where $\mathcal{G}$ ($\neq \mathcal{F}$) is a CIF$(3,2)$ and $\beta \notin \mathrm{P}_{\mathcal{G}}$. With respect to this (new) decomposition there exists at least one vertex y in the graph $\mathsf{G} := (\mathrm{P}_{\mathcal{G}}, \mathcal{G}^\top)$ such that $\deg_{\mathsf{G}}(y) = 3$. Therefore by using (c) of Lemma 4.3.2, we have $\mathcal{A}$ is isomorphic to $\{123, 124, 127, 145, 147, 167, 246, 247, 257, 347\}$. $\square$

We observe that Theorem 4.3.5 ensures, any MIF(3) with at least 7 points has exactly 7 points. Therefore as a consequence we have the following corollary.

Corollary 4.3.6. *There does not exist any* MIF(3) *with v points, where $v \geq 8$.*

4.4 Conclusion: The classification result

Any MIF(3) contains at least 5 points and $\beta(3)$ is the only MIF(3)s with 5 points, so the complete list of MIF(3) is the following.

(i) Up to isomorphism there is a unique MIF(3) with 5 points, namely $\beta(3)$.

(ii) Up to isomorphism there are five MIF(3)s with 6 points, namely

- (a) $\{123, 145, 234, 235, 245, 246, 256, 345, 346, 356\}$,
- (b) $\{123, 134, 156, 235, 236, 245, 246, 345, 346, 356\}$,
- (c) $\{123, 124, 134, 156, 235, 236, 245, 246, 345, 346\}$,
- (d) $\{123, 134, 145, 156, 235, 245, 246, 345, 346, 356\}$,
- (e) $\{123, 126, 134, 145, 156, 235, 245, 246, 346, 356\}$.

(iii) Up to isomorphism there are two MIF(3)s with 7 points, namely

- (a) $\{123, 145, 167, 246, 257, 347, 356\}$,
- (b) $\{123, 124, 127, 145, 147, 167, 246, 247, 257, 347\}$.

(iv) There does not exist any MIF(3) with 8 or more points.

Chapter 5

Constructions over the Cyclic Graph and their applications

In this chapter, we study constructions over the cyclic graph. In Section 5.3 it is shown that Example 5.3.1 and Example 5.3.2 are counter examples to [9, § 3, Conjecture 4] in certain cases. In the final section we close this chapter by stating some conjectures. Most of the results in this chapter are from [20, 22].

5.1 Introduction

Erdős and Lovász established, in their landmark article [7], that any MIF(k) has at most k^k blocks. They showed by means of an example that there exists a MIF(k) with approximately $(e-1)k!$ blocks. They constructed it by a recursive procedure [7, Construction (c), Page 620] starting with the unique MIF(1). Lovász conjectured in [17], that the MIF(k) thus constructed was the extremal one. Later in [9], an extremely elegant and complicated example was given to show that there exists a MIF(k) with at least (approximately) $(\frac{k}{2})^{k-1}$ blocks (i.e. it has more blocks) and it disproves Lovász conjecture. In this chapter, we present two comparatively simpler constructions (see $\mathbb{G}(k,t)$ and $\mathbb{F}(k,t)$ in Construction 5.2.1) to prove that there exists at least two MIF(k)'s with at least (approximately) $(\frac{k}{2})^{k-1}$ blocks. (More precisely, we present an alternative proof of [9, § 2, Theorem 1], see Corollary 5.3.6 below). In [9], it is conjectured that the construction of Frankl et al. yields the unique MIF(k) with the largest number of blocks. Here we show that both parts of this conjecture are false. Specifically, the uniqueness part is incorrect for $k = 4$, while the optimality part is incorrect for $k = 5$.

5.2 Constructions over the Cyclic Graph

Construction 5.2.1. Let k and t be positive integers with $t \leq k$. Let X_n, $0 \leq n \leq t-1$, be t pairwise disjoint sets with

$$|X_n| = \begin{cases} k - \lfloor \frac{t}{2} \rfloor & \text{if} \quad 0 \leq n \leq \lfloor \frac{t-1}{2} \rfloor \\ k - \lfloor \frac{t-1}{2} \rfloor & \text{if} \quad \lfloor \frac{t-1}{2} \rfloor + 1 \leq n \leq t-1 \end{cases}$$

say $X_n = \{x_p^n : 0 \leq p \leq |X_n| - 1\}$.

(a) Let $\mathbb{G}(k,t)$ be the family of all the $k-$sets of the form

$$X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq k - |X_n|\right\},$$

where $0 \leq n \leq t-1$, $0 \leq p_i \leq |X_{n+i}| - 1$ and addition in the superscript is modulo t.

(b) Let $\mathbb{F}(k,t)$ be the family of all the $k-$sets of the form

$$X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq k - |X_n|\right\},$$

where $0 \leq n \leq t-1$, addition in the superscript is modulo t and $\{p_m : m \geq 0\}$ varies over all finite sequences of non negative integers satisfying,

$$p_0 = 0 \text{ and for } m \geq 1, p_m = p_{m-1} \text{ or } 1 + p_{m-1}. \tag{$\star$}$$

Clearly, both the families $\mathbb{F}(k,t)$ and $\mathbb{G}(k,t)$ are examples of intersecting families of $k-$sets (since the $t-$cycle is a graph with diameter $\lfloor \frac{t}{2} \rfloor$). Also the family $\mathbb{F}(k,t)$ is a subfamily of $\mathbb{G}(k,t)$. In this context, we mention that there are similar type of families, namely $\mathscr{G}$ in [9, § 2]. However, the compact description given here is amenable to rigorous arguments.

Theorem 5.2.2. $\operatorname{tr}(\mathbb{G}(k,t)) = t$.

`Proof` : We prepare a $t-$set B by choosing one element from each X_n, with $0 \leq n \leq t-1$, then B is a blocking set of $\mathbb{G}(k,t)$. Therefore $\operatorname{tr}(\mathbb{G}(k,t)) \leq t$. Let C be an arbitrary but fixed set of size $t-1$. To show that $\operatorname{tr}(\mathbb{G}(k,t)) \geq t$, it is enough to show there exists a block of $\mathbb{G}(k,t)$ which is disjoint from C. We divide our arguments in the following two exhaustive cases.

`Case A` : For each n, with $0 \leq n \leq t-1$, $|C \cap X_n| \leq |X_n| - 1$.

Since $|C| = t-1$, there exists a set X_n, with $0 \leq n \leq t-1$, which is disjoint from C. Without loss of generality let $n = 0$. In this case we have, for each m, with $1 \leq m \leq k - |X_0|$, $X_m \smallsetminus C$ is non empty. Now we choose one element namely $x_{p_m}^m \in X_m \smallsetminus C$. Therefore, $X_0 \sqcup \{x_{p_m}^m \in X_m \smallsetminus C : 1 \leq m \leq k - |X_0|\}$ is the required block of $\mathbb{G}(k,t)$, which is disjoint from C.

`Case B` : For some n, with $0 \leq n \leq t-1$, $C \cap X_n = X_n$. (This case will arise for k, with $t \leq k \leq t - 1 + \lfloor \frac{t-1}{2} \rfloor$.)

Since $|C| = t-1$, there exists at most one n, with $0 \leq n \leq t-1$, such that $C \cap X_n = X_n$. We observe that

$$|C \smallsetminus X_n| = t - 1 - |X_n| \leq t + \left\lfloor \frac{t}{2} \right\rfloor - 1 - k.$$

Since $k \geq t$ we have,

$$|C \smallsetminus X_n| = t - 1 - |X_n| \leq \left\lfloor \frac{t}{2} \right\rfloor - 1.$$

So there exists at least one m, with $n+1 \leq m \leq n + \lfloor \frac{t}{2} \rfloor$, such that $(C \smallsetminus X_n) \cap X_m$ is empty; call such an $m = m_0$. Therefore, for $1 \leq i \leq k - |X_{m_0}|$, we have $X_{m_0+i} \smallsetminus C$ is non empty and choose one element say $x^{m_0+i}_{p_{m_0+i}} \in X_{m_0+i} \smallsetminus C$. Therefore,

$$X_{m_0} \sqcup \{x^{m_0+i}_{p_{m_0+i}} \in X_{m_0+i} \smallsetminus C : 1 \leq i \leq k - |X_{m_0}|\}$$

is the required block of $\mathbb{G}(k,t)$, which is disjoint from C. □

Theorem 5.2.3. *For $k \geq t+1$, $\mathbb{G}(k,t)$ is a* CIF(k,t)*. Moreover, each transversal of $\mathbb{G}(k,t)$ intersects each X_n in exactly one point, where $0 \leq n \leq t-1$.*

`Proof` : Let C be a $k-$set. If for each n, with $0 \leq n \leq t-1$, $C \cap X_n \subsetneq X_n$ then $X_n \smallsetminus C$ is non empty and $T(C) := \{x_n \in X_n \smallsetminus C : 0 \leq n \leq t-1\}$ is a transversal of $\mathbb{G}(k,t)$, which is disjoint from C. Now suppose for some n, with $0 \leq n \leq t-1$, $C \cap X_n = X_n$; since $|C| = k$ and $k \geq t+1$, there exists at most one such n; call it n_0. Therefore $|C \smallsetminus X_{n_0}| = k - |X_{n_0}|$. We observe that for each $m \neq n_0$, with $0 \leq m \leq t-1$, $C \cap X_m \subsetneq X_m$, hence $X_m \smallsetminus C$ is non empty and choose $x^m_{q_m} \in X_m \smallsetminus C$. If for some m, with $n_0 + 1 \leq m \leq n_0 + \lfloor \frac{t}{2} \rfloor$, $|X_m \cap C| \geq 2$, then there exists m_0, with $n_0 + 1 \leq m_0 \leq n_0 + \lfloor \frac{t}{2} \rfloor$ such that X_{m_0} is disjoint from C. Consequently, $X_{m_0} \sqcup \{x^{m_0+i}_{q_{m_0+i}} \in X_{m_0+i} \smallsetminus C : 1 \leq i \leq k - |X_{m_0}|\}$ is disjoint from C. So for each m, with $n_0 + 1 \leq m \leq n_0 + \lfloor \frac{t}{2} \rfloor$, $|X_m \cap C| = 1$. Therefore in such a case C is a block of $\mathbb{G}(k,t)$ containing X_{n_0}. This implies that, for an arbitrary $k-$set C which is not a block of $\mathbb{G}(k,t)$, there exists a transversal $T(C)$ of $\mathbb{G}(k,t)$ which is disjoint from C.

Let T be a transversal of $\mathbb{G}(k,t)$. Here $k \geq t+1$. Arguing similarly as in Case B, while proving Theorem 5.2.2, we have for each n, with $0 \leq n \leq t-1$, $X_n \cap T \neq X_n$. Therefore $X_n \smallsetminus T$ is non empty and we choose $x^n_{q_n} \in X_n \smallsetminus T$. If for some m, with $0 \leq m \leq t-1$, X_m is disjoint from T, then $X_m \sqcup \{x^{m+i}_{q_{m+i}} \in X_{m+i} \smallsetminus T : 1 \leq i \leq k - |X_m|\}$ is disjoint from T, a contradiction. Therefore for each n, with $0 \leq n \leq t-1$, $|X_n \cap T| \geq 1$. Hence the second part of the result follows from pigeonhole principle. □

The following remarkable lemma is essentially the case $n = 1$ of [24, Theorem 2.1]. Since the original proof is obscured by many hypotheses and technical terms, we include a simpler proof for the sake of completeness.

Recall that, for any finite sequence $(r_1, \ldots, r_t)$ its cyclic shifts are the t sequences $(r_{i+1}, \ldots, r_{i+t})$ where $0 \leq i \leq t-1$ and the addition in the subscripts is modulo t.

Lemma 5.2.4 (Raney). *Let $(r_1, r_2, \ldots, r_t)$ be a finite sequence of integers such that $\sum_{i=1}^{t} r_i = 1$. Then, exactly one of the t cyclic shifts of this sequence has all its partial sums strictly positive.*

`Proof` : For $1 \leq n \leq t$, let $s_n = r_1 + \ldots + r_n - \frac{n}{t}$. Let μ be an index such that s_μ is the minimum of these t numbers. Now, for $\mu + 1 \leq m \leq t$,

$$r_{\mu+1} + \ldots + r_m = (s_m - s_\mu) + \frac{m - \mu}{t} > 0$$

and for $1 \leq m \leq \mu$,

$$\begin{aligned} r_{\mu+1} + \ldots + r_t + r_1 + \ldots + r_m &= 1 - (s_\mu + \frac{\mu}{t}) + (s_m + \frac{m}{t}) \\ &= (s_m - s_\mu) + 1 - \frac{\mu - m}{t} > 0. \end{aligned}$$

Thus, the partial sums of $(r_{\mu+1}, \ldots, r_{\mu+t})$ are all strictly positive. This proves the existence.

Conversely, let μ be an index for which the partial sums of $(r_{\mu+1}, \ldots, r_{\mu+t})$ are all strictly positive. Then each of these partial sums is at least 1, so that if we subtract a proper fraction from one of them, then the result remains positive. For $\mu + 1 \leq m \leq t$,

$$s_m - s_\mu = (r_{\mu+1} + \ldots + r_m) - \frac{m - \mu}{t} > 0$$

and for $1 \leq m < \mu$,

$$s_m - s_\mu = (r_{\mu+1} + \ldots + r_t + r_1 + \ldots + r_m) - (1 - \frac{\mu - m}{t}) > 0$$

Thus μ is the unique index for which $s_\mu = \min\{s_i : 1 \leq i \leq t\}$. This proves the uniqueness. □

Theorem 5.2.5. $\mathrm{tr}(\mathbb{F}(k,t)) = t$.

`Proof` : If C is any $t-$set which intersects each X_n in a singleton, then C is a blocking set of $\mathbb{F}(k,t)$. So $\mathrm{tr}(\mathbb{F}(k,t)) \leq t$. So, it suffices to show that $\mathbb{F}(k,t)$ has no blocking set C of size $t-1$. Assume the contrary. For $0 \leq n \leq t-1$, $|C \cap X_n|$ is a non negative integer and $\sum_{i=0}^{t-1} |C \cap X_i| = t - 1$. Therefore, if we define the integers $r_{n+1} = 1 - |C \cap X_n|$, where $0 \leq n \leq t-1$, then $\sum_{i=1}^{t} r_i = 1$. So applying Lemma 5.2.4 to this sequence, we get a unique $0 \leq \mu \leq t-1$ such that $\sum_{i=0}^{n} r_{\mu+i} \geq 1$, i.e. $|C \cap (\bigsqcup_{i=0}^{n} X_{\mu+i})| \leq n$, for $0 \leq n \leq t-1$. In particular, C is disjoint from X_μ. For $1 \leq n \leq k - |X_\mu|$, let $l_n = n - \sum_{i=1}^{n} |C \cap X_{\mu+i}|$. Thus

$l_n \geq 0$. Let P_n be the set of all integers $p \geq 0$ for which there is a sequence $(p_1, \ldots, p_n)$ satisfying $(\star)$ such that $p_n = p$ and for $1 \leq i \leq n$, $x_{p_i}^{\mu+i} \notin C$.

Claim : $|P_n| \geq 1 + l_n$ for $1 \leq n \leq k - |X_\mu|$.

`Proof of Claim` : We prove it by finite induction on n. When $n = 1$,

$$\begin{aligned} |P_n| &= 2 - |C \cap X_{\mu+n}| \\ &= 1 + l_n. \end{aligned}$$

So the claim is true for $n = 1$.

Now let $1 \leq m \leq k - 1 - |X_\mu|$ and suppose that the claim is true for m. Since $|C \cap X_{\mu+m+1}| = 1 + l_m - l_{m+1}$ and clearly

$$P_{m+1} \supseteqq (P_m \cup \{1 + p : p \in P_m\}) \setminus (C \cap X_{\mu+m+1}),$$

we have

$$\begin{aligned} |P_{m+1}| &\geq |P_m \cup \{1 + p : p \in P_n\}| - |C \cap X_{\mu+m+1}| \\ &\geq 1 + |P_m| - |C \cap X_{\mu+m+1}| \\ &\geq 2 + l_m - (1 + l_m - l_{m+1}) \\ &= 1 + l_{m+1} \end{aligned}$$

This completes the induction and proves the claim.

By the case $n = k - |X_\mu|$ of the claim, $P_{k-|X_\mu|}$ is non empty. Hence there is a sequence $\{p_1, \ldots, p_{k-|X_\mu|}\}$ satisfying $(\star)$ and such that $\{x_{p_i}^{\mu+i} : 1 \leq i \leq k - |X_\mu|\}$ is disjoint from C. Therefore, the block $X_\mu \sqcup \{x_{p_i}^{\mu+i} : 1 \leq i \leq k - |X_\mu|\}$ is disjoint from C. Thus C is not a blocking set of $\mathbb{F}(k, t)$. Since C is an arbitrary set of size $t-1$, this shows $\mathrm{tr}(\mathbb{F}(k, t)) \geq t$. □

Theorem 5.2.6. *For $k \geq t + 1$, $\mathbb{F}(k, t)$ is a* CIF(k, t)*.*

`Proof` : Let C be a blocking k–set of $\mathbb{F}(k, t)$ such that $C \notin \mathbb{F}(k, t)$. It is enough to show that there exists at least one $T \in \mathbb{F}^\top(k, t)$ disjoint from C. If for each integer n, with $0 \leq n \leq t-1$, there exists at least one $x_n \in X_n$ such that $x_n \notin C$, then $\{x_n : 0 \leq n \leq t-1\}$ is the required T and we are done in this case. Suppose there exists at least one integer n, with $0 \leq n \leq t - 1$, such that $X_n \subsetneqq C$. Notice that for each m with $m \neq n$ and $0 \leq m \leq t - 1$, there exists at least one $x_m \in X_m$ such that $x_m \notin C$. (If not, then there exists at least one such integer m with $X_m \subsetneqq C$. This implies that $X_n \sqcup X_m \subset C$; a contradiction arises since $k \geq t + 1$.) When $t = 2r - 1$, then without loss of generality we can assume $X_0 \subset C$. When $t = 2r$, then without loss of generality we can assume either $X_0 \subset C$ or $X_{\lfloor \frac{t-1}{2} \rfloor + 1} = X_r \subset C$.

`Case A` : Let $X_0 \subsetneqq C$.

Here $C = X_0 \sqcup Y$. We observe that if Y is disjoint from $T_n := \{x_i^n : 0 \leq i \leq n\}$, for some n with $1 \leq n \leq \lfloor \frac{t}{2} \rfloor$, then $T_n \sqcup \{x_i : x_i \in X_i \setminus C, n+1 \leq i \leq t-1\}$ is the required transversal disjoint from C and we are done. So we assume that $Y \cap T_n \neq \emptyset$ for each n with $1 \leq n \leq \lfloor \frac{t}{2} \rfloor$. Since $|Y| = \lfloor \frac{t}{2} \rfloor$ and T_n, $1 \leq n \leq \lfloor \frac{t}{2} \rfloor$, are $\lfloor \frac{t}{2} \rfloor$ pairwise disjoint sets, so Y intersects T_n in exactly one point for each n. Since $C \notin \mathbb{F}(k,t)$ so Y is not of the form $\{x_{p_i}^i : 1 \leq i \leq \lfloor \frac{t}{2} \rfloor\}$. In the next paragraph, under these assumptions on Y, we produce a transversal $T \in \mathbb{F}^{\top}(k,t)$ which is disjoint from both Y and X_0. (Consequently, such a T is disjoint from both C and X_0.)

We have $|Y \cap \{x_0^1, x_1^1\}| = 1$ suppose $x_{\epsilon_1}^1 \in Y$ and $x_{1-\epsilon_1}^1 \notin Y$, where $\epsilon_1 \in \{0,1\}$. Set $c_1 = \epsilon_1$. If Y is disjoint from $\{x_{c_1}^2, x_{1+c_1}^2\}$, then

$$\{x_{1-\epsilon_1}^1, x_{c_1}^2, x_{1+c_1}^2\} \sqcup \{x_i : x_i \in X_i \setminus C, 3 \leq i \leq t-1\}$$

is the required transversal and we are done. So let $|Y \cap \{x_{c_1}^2, x_{1+c_1}^2\}| = 1$. Suppose $x_{c_1+\epsilon_2}^2 \in Y$ and $x_{c_1+1-\epsilon_2}^2 \notin Y$, where $\epsilon_2 \in \{0,1\}$. Set $c_2 = c_1 + \epsilon_2$. In general our construction procedure is as follows: suppose we have already constructed a sequence $c_1, c_2, \ldots, c_m$ with the following properties.

(a) For each n with $1 \leq n \leq m$, $c_n = c_{n-1} + \epsilon_n$ and $\epsilon_n \in \{0,1\}$.

(b) $\{x_{c_n}^n : 1 \leq n \leq m\} \subset Y$.

(c) $S_m := \{x_{1-\epsilon_1}^1\} \sqcup \{x_{c_{n-1}+1-\epsilon_n}^n : 2 \leq n \leq m\}$ is disjoint from Y.

Now we construct c_{m+1} if necessary. If Y is disjoint from $\{x_{c_m}^{m+1}, x_{1+c_m}^{m+1}\}$, then

$$S_m \sqcup \{x_{c_m}^{m+1}, x_{1+c_m}^{m+1}\} \sqcup \{x_i : x_i \in X_i \setminus C, m+2 \leq i \leq t-1\}$$

is the required transversal and we are done. Now let $|Y \cap \{x_{c_m}^{m+1}, x_{1+c_m}^{m+1}\}| = 1$, suppose $x_{c_m+\epsilon_{m+1}}^{m+1} \in Y$ and $x_{c_m+1-\epsilon_{m+1}}^{m+1} \notin Y$, where $\epsilon_{m+1} \in \{0,1\}$. Set $c_{m+1} = c_m + \epsilon_{m+1}$. This yields $\{x_{c_n}^n : 1 \leq n \leq m+1\} \subset Y$ and S_{m+1} is disjoint from Y. Since Y is not of the form $\{x_{p_i}^i : 1 \leq i \leq \lfloor \frac{t}{2} \rfloor\}$, this sequence contains at most $\lfloor \frac{t}{2} \rfloor - 1$ terms. If this sequence contains exactly M terms, then Y is disjoint from $\{x_{c_M}^{M+1}, x_{1+c_M}^{M+1}\}$. Consequently,

$$S_M \sqcup \{x_{c_M}^{M+1}, x_{1+c_M}^{M+1}\} \sqcup \{x_i : x_i \in X_i \setminus C, M+2 \leq i \leq t-1\}$$

is the required transversal.

`Case B` : Let $X_{\lfloor\frac{t-1}{2}\rfloor+1} \subsetneqq C$.

Here $C = X_{\lfloor\frac{t-1}{2}\rfloor+1} \sqcup Y$. This case is similar to the above case. (Precisely, we need to replace $\lfloor\frac{t}{2}\rfloor$ by $\lfloor\frac{t-1}{2}\rfloor$, $x_p^{(\bullet)}$ by $x_p^{\lfloor\frac{t-1}{2}\rfloor+1+(\bullet)}$ and $x_{(\bullet)}$ by $x_{\lfloor\frac{t-1}{2}\rfloor+1+(\bullet)}$.) $\square$

Now we investigate the transversal size of $\mathbb{G}^\top(k,t)$ and $\mathbb{F}^\top(k,t)$. The answers are given in Theorem 5.2.9 and Theorem 5.2.8.

Lemma 5.2.7. *Let $\mathcal{G}$ and $\mathcal{H}$ be two families with finite transversal size. Let $\operatorname{tr}(\mathcal{H}) = 1$ and suppose that no transversal of $\mathcal{G}$ is a blocking set of $\mathcal{H}$. Then $\operatorname{tr}(\mathcal{G}\cup\mathcal{H}) = \operatorname{tr}(\mathcal{G}) + \operatorname{tr}(\mathcal{H})$. Consequently, $\mathcal{G}^\top \circledast \mathcal{H}^\top \subseteqq (\mathcal{G}\cup\mathcal{H})^\top$.*

`Proof` : Let $T_1 \in \mathcal{G}^\top$ and $T_2 \in \mathcal{H}^\top$. Then $T_1 \cup T_2$ is a blocking set of $\mathcal{G}\cup\mathcal{H}$ with size at most $|T_1| + |T_2| \le \operatorname{tr}(\mathcal{G}) + \operatorname{tr}(\mathcal{H})$. Hence $\operatorname{tr}(\mathcal{G}\cup\mathcal{H}) \le \operatorname{tr}(\mathcal{G}) + \operatorname{tr}(\mathcal{H}) = \operatorname{tr}(\mathcal{G}) + 1$. Note that, if we prove equality here then it follows that $|T_1 \cup T_2| = |T_1| + |T_2|$. Hence T_1 and T_2 are disjoint. Therefore $T_1 \cup T_2 \in \mathcal{G}^\top \circledast \mathcal{H}^\top$, showing that $\mathcal{G}^\top \circledast \mathcal{H}^\top \subseteqq (\mathcal{G}\cup\mathcal{H})^\top$. So, to complete the proof, it is enough to show that $\operatorname{tr}(\mathcal{G}\cup\mathcal{H}) \nleq \operatorname{tr}(\mathcal{G})$. Otherwise, if $T \in \mathcal{G}^\top$, then any transversal of $\mathcal{G}\cup\mathcal{H}$ is a blocking set of $\mathcal{H}$, contrary to assumption. $\square$

Theorem 5.2.8. $\operatorname{tr}(\mathbb{G}^\top(k,t)) = k$.

`Proof` : We establish this result by using induction on k. From Theorem 5.2.2 we have $\operatorname{tr}(\mathbb{G}^\top(t,t)) = t$ i.e. the result is true for $k = t$. Suppose the result is true for $k = n \ge t$, i.e. $\operatorname{tr}(\mathbb{G}^\top(n,t)) = n$. We show that the result is true for $k = n+1$, i.e. $\operatorname{tr}(\mathbb{G}^\top(n+1,t)) = n+1$.

We construct the following $t-$sets.

$$|T_n| := \begin{cases} \{x_{n-r}^i : 0 \le i \le 2r-2\} & \text{if} \quad t = 2r-1 \\ \{x_{n-r}^i : 0 \le i \le r-1\} \sqcup \{x_{n-r+1}^j : r \le j \le 2r-1\} & \text{if} \quad t = 2r. \end{cases}$$

We observe that T_{n+1} is a transversal of $\mathbb{G}(n+1,t)$ and it consists of the "new points" from each X_i, where $0 \le i \le t-1$. Therefore it is disjoint from $\mathrm{P}_{\mathbb{G}^\top(n,t)}$. Let $\mathcal{G} := \mathbb{G}^\top(n,t) \sqcup \{T_{n+1}\}$. Then by using Lemma 5.2.7 we have $\operatorname{tr}(\mathcal{G}) = n+1$. By definition of transversal each $B \in \mathbb{G}(n+1,t)$ is a blocking set of $\mathbb{G}^\top(n+1,t)$, therefore $\operatorname{tr}(\mathbb{G}^\top(n+1,t)) \le |B| = n+1$. But $\mathcal{G} \subset \mathbb{G}^\top(n+1,t)$. Hence $\operatorname{tr}(\mathbb{G}^\top(n+1,t)) = n+1$. $\square$

We use the same approach as in Theorem 5.2.8, to prove the same result for $\mathbb{F}^\top(k,t)$. But we prove it by combining the previous results.

Theorem 5.2.9. $\operatorname{tr}(\mathbb{F}^\top(k,t)) = k$.

`Proof` : We observe firstly that each $B \in \mathbb{F}(k,t)$ is a blocking set of $\mathbb{F}^\top(k,t)$. Therefore $\mathrm{tr}(\mathbb{F}^\top(k,t)) \leq |B| = k$. Secondly, $\mathbb{F}(k,t) \subsetneqq \mathbb{G}(k,t)$. So by using Theorem 5.2.2 and Theorem 5.2.5 we have $\mathbb{G}^\top(k,t) \subset \mathbb{F}^\top(k,t)$. Consequently $\mathrm{tr}(\mathbb{G}^\top(k,t)) \leq \mathrm{tr}(\mathbb{F}^\top(k,t))$. Finally, the result follows from Theorem 5.2.8. □

5.3 Some applications

In this section, it is shown that Example 5.3.1 and Example 5.3.2 are counter examples to [9, § 3, Conjecture 4] in special cases. In the following examples we continue with the notations of Construction 5.2.1.

Example 5.3.1. By using Theorem 5.2.6, we have for $k \geq 2$, $\mathrm{tr}(\mathbb{F}(k,2)) = 2$. So by Theorem 5.2.6 we have, for $k \geq 3$, $\mathbb{F}(k,2)$ is a $\mathrm{CIF}(k,2)$. We observe that, the transversals of $\mathbb{F}(k,2)$ are $\{x^0_p, x^1_q\}$; $\{x^1_0, x^1_1\}$, where $0 \leq p \leq k-2$ and $0 \leq q \leq k-1$. Hence there are $k^2 - k + 1$ transversals and 3 blocks in $\mathbb{F}(k,2)$. So if we take $k = 4$ we have a $\mathrm{CIF}(4,2)$ with 3 blocks and 13 transversals. Let $\mathcal{A}$ be the unique $\mathrm{MIF}(2)$ isomorphic to $\{\{a,b\},\{b,c\},\{a,c\}\}$ and $\mathrm{P}_{\mathcal{A}} \cap \mathrm{P}_{\mathbb{F}(4,2)} = \emptyset$. Therefore by Theorem 3.2.3, $\mathbb{F}(4,2) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(4,2))$ is a $\mathrm{MIF}(4)$ with 42 blocks and 10 points. In this $\mathrm{MIF}(4)$ there are 3 points in 26 blocks, 5 points in 14 blocks and 2 points in 10 blocks.

Example 5.3.2. By using Theorem 5.2.6, we have for $k \geq 3$, $\mathrm{tr}(\mathbb{F}(k,3)) = 3$. So by Theorem 5.2.6 we have, for $k \geq 4$, $\mathbb{F}(k,3)$ is a $\mathrm{CIF}(k,3)$. We observe that the transversals of $\mathbb{F}(k,3)$ are $\{x^0_p, x^1_q, x^2_r\}$; $\{x^0_0, x^0_1, x^1_p\}$; $\{x^1_0, x^1_1, x^2_p\}$ and $\{x^2_0, x^2_1, x^0_p\}$, where $0 \leq p,q,r \leq k-2$. Hence there are $(k-1)^3 + 3(k-1)$ transversals and 6 blocks in $\mathbb{F}(k,3)$. So if we take $k = 4$ and $k = 5$ respectively, we have a $\mathrm{CIF}(4,3)$ and $\mathrm{CIF}(5,3)$ with 6 blocks and 36 & 76 transversals respectively. Let $\mathcal{A}$ be the unique $\mathrm{MIF}(1)$ (respectively, unique $\mathrm{MIF}(2)$ isomorphic to $\{\{a,b\},\{b,c\},\{a,c\}\}$) and $\mathrm{P}_{\mathcal{A}} \cap \mathrm{P}_{\mathbb{F}(4,3)} = \emptyset$(respectively, $\mathrm{P}_{\mathcal{A}} \cap \mathrm{P}_{\mathbb{F}(5,3)} = \emptyset$). By Theorem 3.2.3, $\mathbb{F}(4,3) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(4,3))$ is a $\mathrm{MIF}(4)$ with 42 blocks (respectively, $\mathbb{F}(5,3) \sqcup (\mathcal{A} \circledast \mathbb{F}^\top(5,3))$ is a $\mathrm{MIF}(5)$ with 234 blocks). In this $\mathrm{MIF}(4)$ there are 1 point in 36 blocks, 6 points in 16 blocks and 3 points in 12 blocks.

Remark 5.3.3. Example 5.3.1 and Example 5.3.2 proves the existence of two non isomorphic $\mathrm{MIF}(4)$ with 42 blocks. It disproves a special case (case $k = 4$) of Conjecture 4 in [9], which claims such $\mathrm{MIF}(4)$ is unique up to isomorphism.

Remark 5.3.4. Example 5.3.2 proves the existence of a $\mathrm{MIF}(5)$ with 234 blocks. So we have $\mathrm{M}(5) \geq 234$. It disproves a special case (case $k = 5$) of Conjecture 4 in [9], which claims $\mathrm{M}(5) = 228$.

Lovász conjectured that, for large positive integer k, $\mathrm{M}(k)$ is asymptotic to $(e-1)k!$. Disproving this was the prime object of article [9]. Here we present an alternative and simpler construction to prove $\mathrm{M}(k)$ is at least $(\frac{k}{2})^{k-1}$.

Theorem 5.3.5. *Let $k \geq t+1$. Then*

$$\mathrm{M}(k) \geq \begin{cases} (2r-1)(k-r+1)^{r-1} + (k-r+1)^{2r-1}\mathrm{M}(k-2r+1) & \text{if} \quad t = 2r-1 \\ 2r(k-r)^{r-1} + (k-r)^r(k-r+1)^r\mathrm{M}(k-2r) & \text{if} \quad t = 2r. \end{cases} \tag{5.3.1}$$

`Proof` : Let $\mathcal{A}$ be a MIF$(k-t)$ with $\mathrm{M}(k-t)$ blocks. By Theorem 5.2.3 and Theorem 3.2.3, it follows that $\mathbb{G}(k,t) \sqcup (\mathcal{A} \circledast \mathbb{G}^\top(k,t))$ is a MIF(k). Here we observe that any block of $\mathbb{G}(k,t)$ is of the form

$$X_n \sqcup \{x_p^{n+i} \in X_{n+i} : 1 \leq i \leq k - |X_n|\},$$

where $0 \leq n \leq t-1$. It means that for each X_n, with $0 \leq n \leq t-1$, and for each i, with $1 \leq i \leq k - |X_n|$, there are $|X_{n+i}|$ number of choices for x_p^{n+i}. Therefore there are $\prod_{i=1}^{k-|X_n|} |X_{n+i}|$ choices for such blocks. Hence,

$$|\mathbb{G}(k,t)| \geq \begin{cases} (2r-1)(k-r+1)^{r-1} & \text{if} \quad t = 2r-1 \\ 2r(k-r)^{r-1} & \text{if} \quad t = 2r. \end{cases}$$

Also by using Theorem 5.2.3, we have

$$|\mathbb{G}^\top(k,t)| = \begin{cases} (k-r+1)^{2r-1} & \text{if} \quad t = 2r-1 \\ (k-r)^r(k-r+1)^r & \text{if} \quad t = 2r. \end{cases}$$

Therefore the results follow from Corollary 3.2.4. □

If we take $t = k-1$ in (5.3.1), we obtain the following corollary (Theorem 1 of [9, § 2]), which shows that $\mathrm{M}(k)$ grows like at least $(\frac{k}{2})^{k-1}$ and it disproves Lovász Conjecture.

Corollary 5.3.6 (Frankl-Ota-Tokushige)**.**

$$\mathrm{M}(k) \geq \begin{cases} (\frac{k}{2}+1)^{k-1} & \text{if} \quad k \text{ is even} \\ (\frac{k+1}{2})^{\frac{k-1}{2}}(\frac{k+3}{2})^{\frac{k-1}{2}} & \text{if} \quad k \text{ is odd .} \end{cases}$$

The problem of interest is to find a MIF(k) with $\mathrm{M}(k)$ blocks. Using Theorem 3.2.3, we observe that this problem actually boils down to find a CIF(k,t) $\mathcal{F}$ and a MIF$(k-t)$ $\mathcal{A}$, so that $|\mathcal{F}|+|\mathcal{A}||\mathcal{F}^\top|$ is maximum for some suitable choice of $t \leq k-1$. So we formulate the following conjecture.

Conjecture 5.3.7. For any large positive integer k, any $\mathrm{MIF}(k)$ with $\mathrm{M}(k)$ blocks contains a $\mathrm{CIF}(k,t)$, for some $t \leq k-1$, which is isomorphic to a subfamily of $\mathbb{G}(k,t)$.

Our future interest is to resolve Conjecture 5.3.7. Currently, we do not have any approach to solve it. But we feel that there are a lot of intermediate questions which need to be addressed first. Our prediction is the following.

Conjecture 5.3.8.

$$\mathrm{M}(k) = |\mathbb{F}(k,k-1)| + |\mathbb{F}^{\top}(k,k-1)| \text{ for every large positive integer } k. \tag{5.3.2}$$

But $\mathbb{F}(k,k-1)$ is not the unique $\mathrm{CIF}(k,k-1)$ for which (5.3.2) holds. So we formulate the following conjectures and close this chapter.

Conjecture 5.3.9. For every sufficiently large positive integer t and every integer k with $k \geq t+1$,

(a) There exist at least two non-isomorphic subfamilies of $\mathbb{G}(k,t)$, say $\mathcal{F}_1$ and $\mathcal{F}_2$ such that

$$|\mathcal{F}_1^{\top}| = |\mathcal{F}_2^{\top}| = |\mathbb{F}^{\top}(k,t)|.$$

(b) We define the integer

$$\mathrm{M}^{\top}(k,t) := \max\left\{|\mathcal{F}^{\top}| : \mathcal{F} \text{ is an intersecting family with } \mathrm{k}(\mathcal{F}) = k \text{ and } \mathrm{tr}(\mathcal{F}) = t\right\}.$$

Then

$$\mathrm{M}^{\top}(k,t) = |\mathbb{F}^{\top}(k,t)|.$$

Acknowledgement. Theorem 5.2.5 is a joint work with Mr. Satyaki Mukherjee.

A

Appendix

A.1 Introduction

In this chapter we study two extremal questions about ISP$(k,t)s$. The first is known as Bollobás Inequality. It deals with the problem of finding the maximum number of members (pairs) in an ISP(k,t). The second theorem mentioned here is based on the problem of the finding maximum number of points in an ISP(k,t). These two theorems played an essential role to solve the problem of getting the maximum number of points in a maximum intersecting family in Chapter 2. Before going into the theorems let us quickly recall the definitions of an intersecting set pair system and an ISP(k,t).

Definition.

(a) An *intersecting set pair system* $\mathbb{S}$ (in short ISP) is a collection of pairs of sets of the form (B,T), with the property that if (B_1,T_1), $(B_2,T_2) \in \mathbb{S}$, then $B_i \cap T_j = \emptyset$ if and only if $i = j$.

(b) Let k, t be positive integers, with $t \leq k$. An ISP(k,t) is an intersecting set pair system $\mathbb{S}$, where each pair (B,T) has the property that $|B| = k$ and $|T| = t$.

Example A.1.1. Let k, t be positive integers. Let P be a set of $k+t$ symbols. All pairs of sets of the form $(B, P \setminus B)$, where B is a $k-$set from P, form an ISP(k,t). Denote this ISP(k,t) by $\mathbb{B}(k,t)$, which contains $\binom{k+t}{k}$ pairs.

A.2 Bollobás inequality

In this section we prove Theorem A.2.1. The proof idea is due to Lubell [18] and it is based on counting permutation.

Theorem A.2.1 (Bollobás Inequality). *Let $\mathbb{S} = \{(B_i,T_i) : 1 \leq i \leq n\}$ be an intersecting set pair system. Suppose $|B_i| = k_i$ and $|T_i| = t_i$, where $1 \leq i \leq n$. Then the following inequality holds:*

$$\sum_{i=1}^{n} \frac{1}{\binom{k_i+t_i}{k_i}} \leq 1. \tag{A.2.1}$$

Moreover the following inequalities hold.

(a) *If* $k = \max\{k_i : 1 \leq i \leq n\}$ *and* $t = \max\{t_i : 1 \leq i \leq n\}$, *then*

$$n \leq \binom{k+t}{k}. \tag{A.2.2}$$

Equality in (A.2.2) *is uniquely attained by* $\mathbb{B}(k,t)$.

(b) *If both* $\{B : (B,T) \in \mathbb{S}\}$, $\{T : (B,T) \in \mathbb{S}\}$ *are intersecting families, then*

$$\sum_{i=1}^{n} \frac{1}{\binom{k_i+t_i}{k_i}} \leq \frac{1}{2}. \tag{A.2.3}$$

(i) *Moreover in this case, if* $k = \max\{k_i : 1 \leq i \leq n\}$ *and* $t = \max\{t_i : 1 \leq i \leq n\}$, *then*

$$n \leq \frac{1}{2}\binom{k+t}{k}.$$

(ii) *Let* $\mathbb{S}$ *be an* ISP(k,k) *with the property that both* $\{B : (B,T) \in \mathbb{S}\}$, $\{T : (B,T) \in \mathbb{S}\}$ *are intersecting families. If* $\mathbb{S}$ *contains exactly* $\frac{1}{2}\binom{2k}{k}$ *pairs then* $\mathbb{S}$ *is isomorphic to* $\mathbb{B}(k,k)$.

`Proof :` Suppose $|\mathrm{P}_{\mathbb{S}}| = N$ and let P be the set of all linear orders on $\mathrm{P}_{\mathbb{S}}$. Hence $|P| = N!$. Let A and B be mutually disjoint subsets of $\mathrm{P}_{\mathbb{S}}$ and $\leq$ be a linear order on it. By $A \leq B$ we mean (with respect to the linear order $\leq$) for each $a \in A$ and $b \in B$ $a \leq b$. Fix $(B,T) \in \mathbb{S}$. We define the following sets

$$Q(B,T) = \{\leq \in P : B \leq T\},$$

$$R(B,T) = \{\leq \in P : T \leq B\},$$

$$Q(B,T,x,i) = \left\{ \leq \in Q(B,T) \;\middle|\; \begin{array}{l} x \in B, \\ x \text{ occurs in the } i^{th} \text{ position and} \\ y \leq x \text{ for each } y \in B \end{array} \right\}.$$

Claim 1 : $Q(B,T) = \bigsqcup_{x \in B} \bigsqcup_{i=1}^{n} Q(B,T,x,i)$ and $Q(B,T)$ contains exactly $\frac{N!}{\binom{k+t}{k}}$ linear orders, where $|B| = k$ and $|T| = t$.

`Proof of claim :` It is easy to see that

$$Q(B,T,x,i) \cap Q(B,T,x,j) = \emptyset$$

for $i \neq j$ and $Q(B,T,x,i) \cap Q(B,T,y,i) = \emptyset$ for $x \neq y$. So let $i \neq j$ and $x \neq y$. Suppose $\leq \in Q(B,T,x,i) \cap Q(B,T,y,j)$, then x occurs in the i^{th} position and y occurs in the j^{th} position with respect to $\leq$. Since $i \neq j$, either $x \leq y$ or $y \leq x$. Without loss of generality assume that $x \leq y$. Here $\leq \in Q(B,T,x,i)$ so $y \leq x$. This implies that $y = x$, a contradiction arises since we assume $y \neq x$. Hence the first part of the claim is established.

For the next part we observe that,

$$\begin{aligned}|Q(B,T,x,i)| &= \binom{i-1}{k-1}(k-1)!\binom{N-i}{t}t!(N-k-t)! \\ &= (i-1)!(N-i)!\binom{N-k-t}{i-k}.\end{aligned}$$

From the first part we have,

$$\begin{aligned}|Q(B,T)| &= \sum_{x\in B}\sum_{i=1}^{N}(i-1)!(N-i)!\binom{N-k-t}{i-k} \\ &= \left\{\sum_{i=1}^{N}(i-1)!(N-i)!\binom{N-k-t}{i-k}\right\}\left(\sum_{x\in B}1\right) \\ &= \left\{N!\sum_{i=k}^{N-t}\binom{N-k-t}{i-k}\beta(i,N-i+1)\right\}(k) \\ &= k\left\{\frac{N!}{k\binom{k+t}{k}}\right\} \\ &= \frac{N!}{\binom{k+t}{k}}.\end{aligned}$$

Hence the claim is established.

Claim 2 : $Q(B,T) \cap Q(B^{'},T^{'}) = \emptyset$, for each distinct $(B,T),(B^{'},T^{'}) \in \mathbb{S}$.

`Proof of claim :` Suppose $\leq \in Q(B,T) \cap Q(B^{'},T^{'})$. Let $x \in B \cap T^{'}$ and $y \in B^{'} \cap T$. Then $x \leq y$ and $y \leq x$ since $B \leq T$ and $B^{'} \leq T^{'}$ respectively. Therefore $x = y$ and hence $x \in B \cap T$. This contradicts that $B \cap T = \emptyset$. Hence the claim is established.

From Claim 2 we have,

$$\bigsqcup_{(B,T)\in\mathbb{S}} Q(B,T) \subseteqq P. \tag{A.2.4}$$

Now using Claim 1 we get,

$$\sum_{i=1}^{n}\frac{N!}{\binom{k_i+t_i}{k_i}} \leq N!.$$

Hence the inequality (A.2.1) follows.

For the next part, we are associating for each i two mutually disjoint sets $B_i^{'}$ and $T_i^{'}$ of size $(k-k_i)$ and $(t-t_i)$ respectively, so that they are disjoint from $P_{\mathbb{S}}$. Then

$$\left\{(B_i \sqcup B_i^{'}, T_i \sqcup T_i^{'}) : 1 \leq i \leq n\right\}$$

is an ISP(k,t). Denote it by $\mathbb{S}^{'}$ and proceeding similarly for $\mathbb{S}^{'}$ as we did for $\mathbb{S}$ up to (A.2.4) in the above argument, we conclude that

$$\sum_{i=1}^{n} \frac{N!}{\binom{k+t}{k}} \leq N!.$$

Hence the inequality (A.2.2) follows.

Claim 3 : If the equality in (A.2.4) holds for some ISP(k,t) $\mathbb{S}$

$$\text{i.e.} \bigsqcup_{(B,T)\in\mathbb{S}} Q(B,T) = P,$$

then $\mathbb{S}$ is isomorphic to $\mathbb{B}(k,t)$.

`Proof of claim` : Fix a $k-$set C from $\mathrm{P}_{\mathbb{S}}$. Let $\leq$ be a linear order, which keeps the elements of C as first k terms of the order followed by the elements of $\mathrm{P}_{\mathbb{S}} \setminus C$. Then the equality in (A.2.4) implies that there exists $(B,T) \in \mathbb{S}$ such that $\leq \in Q(B,T)$, this also implies that $C = B$. It means that any $k-$set from $\mathrm{P}_{\mathbb{S}}$ is a member of $\{B : (B,T) \in \mathbb{S}\}$. Arguing similarly, we can conclude that any $t-$set from $\mathrm{P}_{\mathbb{S}}$ is a member of $\{T : (B,T) \in \mathbb{S}\}$.

Suppose $|\mathrm{P}_{\mathbb{S}}| \geq k+t+1$. Fix a $k-$set B from $\mathrm{P}_{\mathbb{S}}$. Then there exists at least two $t-$sets T_1 and T_2 disjoint from B. We deduced in the earlier paragraph that $\mathbb{S}$ has the following property.

$$\begin{aligned} &\text{Any } k-\text{set from } \mathrm{P}_{\mathbb{S}} \text{ is a member of } \{B : (B,T) \in \mathbb{S}\}. \\ &\text{Any } t-\text{set from } \mathrm{P}_{\mathbb{S}} \text{ is a member of } \{T : (B,T) \in \mathbb{S}\}. \end{aligned} \tag{A.2.5}$$

Therefore, $B \in \{B : (B,T) \in \mathbb{S}\}$. Since $\mathbb{S}$ is an ISP(k,t), exactly one of (B,T_1) or (B,T_2) is a pair of $\mathbb{S}$. It means at least one of T_1 and T_2 is not a member of $\{T : (B,T) \in \mathbb{S}\}$. It contradicts property (A.2.5) of $\mathbb{S}$. Hence $\mathrm{P}_{\mathbb{S}}$ is a $(k+t)-$set and the claim is established.

Claim 4 : Suppose both $\{B : (B,T) \in \mathbb{S}\}$ and $\{T : (B,T) \in \mathbb{S}\}$ are intersecting families; then for each $(B^{'},T^{'}) \in \mathbb{S}$,

$$R(B,T) \cap Q(B^{'},T^{'}) = \emptyset \text{ and } R(B,T) \cap R(B^{'},T^{'}) = \emptyset.$$

`Proof of claim` : Suppose $\leq \in R(B,T) \cap Q(B^{'},T^{'})$ for some $(B^{'},T^{'}) \in \mathbb{S}$. From the hypothesis of this claim, there exists $x \in B \cap B^{'}$ and $y \in T \cap T^{'}$. Now $T \leq B$ and $B^{'} \leq T^{'}$ imply $y \leq x$ and $x \leq y$ respectively. Therefore $x = y$ and hence $x \in B^{'} \cap T^{'}$ – Contradicts that $B^{'} \cap T^{'} = \emptyset$. The proof of the second part of this claim is similar to Claim 2. Hence the claim is established.

From Claim 4 we have,

$$\bigsqcup_{(B,T)\in\mathbb{S}} \{Q(B,T) \sqcup R(B,T)\} \subseteqq P.$$

Now using Claim 1 we get,

$$2\sum_{i=1}^{n} \frac{N!}{\binom{k_i+t_i}{k_i}} \le N!.$$

Hence the inequality (A.2.3) follows by taking $k_i = t_i = k$, for each i with $1 \le i \le n$.

Claim 5 : Let $\mathbb{S}$ be an ISP(k,k) with the property that both $\{B : (B,T) \in \mathbb{S}\}$, $\{T : (B,T) \in \mathbb{S}\}$ are intersecting families. Suppose $\mathbb{S}$ contains exactly $\frac{1}{2}\binom{2k}{k}$ pairs,

$$\text{i.e.} \quad \bigsqcup_{(B,T)\in\mathbb{S}} \{Q(B,T) \sqcup R(B,T)\} = P,$$

then $\mathbb{S}$ is isomorphic to $\mathbb{B}(k,k)$.

`Proof of claim` : Fix a $k-$set C from $\mathrm{P}_{\mathbb{S}}$. Let $\le$ be a linear order, which keeps the elements of C as first k terms of the order followed by the elements of $\mathrm{P}_{\mathbb{S}} \smallsetminus C$. Then the equality in (A.2.3) implies that there exists a pair $(B,T) \in \mathbb{S}$ such that $\le \in Q(B,T)$ or $\le \in R(B,T)$, this also implies that $C = B$ or $C = T$. It means that any $k-$set from $\mathrm{P}_{\mathbb{S}}$ is a member of $\{B : (B,T) \in \mathbb{S}\} \sqcup \{T : (B,T) \in \mathbb{S}\}$.

Suppose $|\mathrm{P}_{\mathbb{S}}| \ge 2k+1$. Fix a $k-$set Y from $\mathrm{P}_{\mathbb{S}}$. Then there exists at least two $k-$sets T_1 and T_2 disjoint from Y. By the property (A.2.5) of $\mathbb{S}$ we have

$$Y \in \{B : (B,T) \in \mathbb{S}\} \sqcup \{T : (B,T) \in \mathbb{S}\}.$$

Without loss of generality suppose $Y \in \{B : (B,T) \in \mathbb{S}\}$. Since $\mathbb{S}$ is an ISP(k,k), exactly one of (Y,T_1) or (Y,T_2) is a pair of $\mathbb{S}$. It means at least one of T_1 and T_2 is not a member of $\{T : (B,T) \in \mathbb{S}\}$. It contradicts the property (A.2.5) of $\mathbb{S}$. Hence $\mathrm{P}_{\mathbb{S}}$ is a $2k-$set and the claim is established.

□

A.3 On the number of points in an ISP(k,t)

Bollobás inequality shows that, for any two positive integers k and t, there are only finitely many ISP(k,t), up to isomorphism. This raises the question of determining or estimating the number

$$\mathrm{n}(k,t) := \max\{\mathrm{v}(\mathbb{I}) : \mathbb{I} \text{ is an ISP}(k,t)\}.$$

Notice that we have $\mathrm{n}(k,t) = \mathrm{n}(t,k)$.

Theorem A.3.1 (Theorem 6(a), [25]). *For $k \geq t$,*

$$\mathrm{n}(k,t) \leq \binom{k+t}{t+1} - \binom{2t-1}{t+1} + \frac{3}{2}\sum_{i=1}^{t-1}\binom{2i}{i}.$$

`Proof` : Let $\mathbb{S}$ be an ISP(k,t) with n pairs. Let $\mathbb{S}_1$ be a sub collection of $\mathbb{S}$ with respect to the minimality property $\mathrm{P}_{\mathbb{S}_1} = \mathrm{P}_{\mathbb{S}}$. Suppose it contains $m_1 \leq n$ pairs. Due to minimality property of $\mathbb{S}_1$, each pair $(B,T) \in \mathbb{S}_1$ contains a point $x_{(B,T)}$ such that $x_{(B,T)} \notin \mathrm{P}_{\mathbb{S}_1 \smallsetminus \{(B,T)\}}$. This implies that

$$\mathbb{S}_1^{'} = \{(B \smallsetminus x_{(B,T)}, T \smallsetminus x_{(B,T)}) : (B,T) \in \mathbb{S}_1\} \tag{A.3.1}$$

is an intersecting set pair system such that

$$|\mathrm{P}_{\mathbb{S}_1}| = m_1 + |\mathrm{P}_{\mathbb{S}_1^{'}}|. \tag{A.3.2}$$

Now from (A.2.1) of Lemma A.2.1, it follows that

$$\sum_{i=1}^{m_1} \frac{1}{\binom{k+t-1}{k_i}} \leq 1 \text{ where } k_i = k-1 \text{ or } t-1.$$

Proceeding inductively for $j \geq 2$, we let $\mathbb{S}_j$ be a minimal intersecting set pair subsystem of $\mathbb{S}_{j-1}^{'}$ with respect to the property $\mathrm{P}_{\mathbb{S}_j} = \mathrm{P}_{\mathbb{S}_{j-1}^{'}}$. Suppose that it contains $m_j \leq m_{j-1}$ pairs. Due to minimality property of $\mathbb{S}_j$, each pair $(B,T) \in \mathbb{S}_j$ contains a point $x_{(B,T)}$ such that $x_{(B,T)} \notin \mathrm{P}_{\mathbb{S}_j \smallsetminus \{(B,T)\}}$. This implies that

$$\mathbb{S}_{j+1}^{'} = \left\{(B \smallsetminus \{x_{(B,T)}\}, T \smallsetminus \{x_{(B,T)}\}) : (B,T) \in \mathbb{S}_j\right\}$$

is an intersecting set pair system such that

$$|\mathrm{P}_{\mathbb{S}_j}| = m_j + |\mathrm{P}_{\mathbb{S}_j^{'}}|.$$

Now from Bollobás inequality (A.2.1) it follows that for $j \geq 2$,

$$\sum_{i=1}^{m_j} \frac{1}{\binom{k+t-j}{k_i}} \leq 1 \text{ where } k_i \in \{k-j, \ldots, k-1\} \sqcup \{t-j, \ldots, t-1\},$$

We observe that, for $k_i \in \{k-j, \ldots, k-1\} \sqcup \{t-j, \ldots, t-1\}$

$$\binom{k+t-j}{k_i} \leq \begin{cases} \binom{k+t-j}{t} & \text{if} \quad j \leq k-t \\ \binom{k+t-j}{\lfloor \frac{k+t-j}{2} \rfloor} & \text{if} \quad j \geq k-t \end{cases}$$

Therefore, we have,

$$
\begin{aligned}
|\mathrm{P}_{\mathbb{S}}| \leq \sum_{j\geq 1} m_j &\leq \sum_{i=2t-1}^{k+t-1}\binom{i}{t} + \sum_{i=1}^{2t-2}\binom{i}{\lfloor \frac{i}{2}\rfloor} \\
&= \sum_{i=2t-1}^{k+t-1}\binom{i}{t} + \sum_{i=1}^{t-1}\left\{\binom{2i-1}{i-1} + \binom{2i}{i}\right\} \\
&= \sum_{i=2t-1}^{k+t-1}\binom{i}{t} + \sum_{i=1}^{t-1}\left\{\frac{1}{2}\binom{2i}{i} + \binom{2i}{i}\right\} \\
&= \sum_{i=t}^{k+t-1}\binom{i}{t} - \sum_{i=t}^{2t-2}\binom{i}{t} + \frac{3}{2}\sum_{i=1}^{t-1}\binom{2i}{i}.
\end{aligned}
$$

The result follows since for $n \geq t$ we have $\sum\limits_{i=t}^{n}\binom{i}{t} = \binom{n+1}{t+1}$. ☐

Construction A.3.2. Let k, t and $t^{'}$ be non negative integers, with $0 \leq t^{'} \leq t$.

(a) Let P be a set of size $k+t^{'}$. For $A \in \binom{P}{k}$, let X_A be a $(t-t^{'})-$set disjoint from P and for all A, $B \in \binom{P}{k}$ X_A is disjoint from X_B. Let $\mathbb{I}$ be the following collection.

$$\left\{(A, (P \setminus A) \sqcup X_A) : A \in \binom{P}{k}\right\}.$$

(b) Let Q be a set of size $k+t^{'}-1$. For $A \in \binom{Q}{k-1}$, let Y_A be a $(t-t^{'}+1)-$set disjoint from Q and for all A, $B \in \binom{Q}{k-1}$ Y_A is disjoint from Y_B. Let $\mathbb{J}$ be the collection of all pairs of the form

$$(A \sqcup \{y\}, (P \setminus A) \sqcup (Y_A \setminus \{y\})),$$

where $A \in \binom{Q}{k-1}$ and $y \in Y_A$.

We observe that $\mathbb{I}$ and $\mathbb{J}$ are examples of ISP(k,t) with $k+t^{'}+(t-t^{'})\binom{k+t^{'}}{k}$ points and $k+t^{'}-1+(t-t^{'}+1)\binom{k+t^{'}-1}{k-1}$ points respectively. In [25], Tuza has constructed the above two examples of ISP(k,t) and made a precise conjecture (see Conjecture 2.1.2) on the numbers $n(k,t)$. It states that for $k \geq t+2$,

$$n(k,t) = \left\lceil \frac{k}{t+1} \right\rceil \binom{\lfloor \frac{kt}{t+1} \rfloor + t}{t} + \left\lfloor \frac{kt}{t+1} \right\rfloor + t.$$

B

Appendix

B.1 Introduction

We start this chapter by recalling the construction of $\mathbb{F}(k,t)$ from Construction 5.2.1.

Construction. Let k, t be positive integers with $t \leq k$. Let X_n, $0 \leq n \leq t-1$, be t pairwise disjoint sets with

$$|X_n| = \begin{cases} k - \lfloor \frac{t}{2} \rfloor & \text{if} \quad 0 \leq n \leq \lfloor \frac{t-1}{2} \rfloor \\ k - \lfloor \frac{t-1}{2} \rfloor & \text{if} \quad \lfloor \frac{t-1}{2} \rfloor + 1 \leq n \leq t-1 \end{cases}$$

say $X_n = \{x_p^n : 0 \leq p \leq |X_n| - 1\}$. Let $\mathbb{F}(k,t)$ be the family of all the $k-$sets of the form

$$X_n \sqcup \left\{ x_{p_i}^{n+i} : 1 \leq i \leq k - |X_n| \right\},$$

where addition in the superscript is modulo t and $\{p_n\}$ varies over finite sequences of non-negative integers such that $p_0 = 0$ and for $n \geq 1$, $p_n = p_{n-1}$ or $1 + p_{n-1}$.

B.2 Stepwise Constructions

Purpose of this section is to present a second proof of Theorem 5.2.5. With this method, we prove that $\operatorname{tr}(\mathbb{F}(k,t)) = t$ for $t \leq 10$. We expect that this method may be completed to the cases $t \geq 11$.

B.2.1 Construction of $\mathcal{G}_m$

Construction B.2.1. We continue with the notations of the above construction. Let $\mathcal{G}_1 := \{X_0\}$ and $m \leq t$ be a positive integer. Suppose $\mathcal{G}_m$ is known. If $B \in \mathcal{G}_m$ with $|B| = k$, then $B \in \mathcal{G}_{m+1}$. Otherwise, if $B \in \mathcal{G}_m$ with $|B| \leq k-1$ then there exists a least integer p such that $x_p^{m-1} \in B$, consequently $B \sqcup \{x_p^m\}$ and $B \sqcup \{x_{1+p}^m\} \in \mathcal{G}_{m+1}$. Also $X_m \in \mathcal{G}_{m+1}$.

By constructing $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_t$ recursively, we can see that $\mathcal{G}_m$, for $1 \leq m \leq t$, is the family consisting of all sets of the form

$$X_n \sqcup \left\{ x_{p_i}^{n+i} : 1 \leq i \leq \min(m-n-1, k-|X_n|) \right\},$$

where $0 \leq n \leq m-1$.

Lemma B.2.2. *Let $1 \leq m \leq t-1$. Let B be a block of $\mathcal{G}_{m+1}$ such that $x_p^m \in B$ for some non-negative integer p. Then either $B = X_m$ or B contains exactly one of the two points x_p^{m-1} and x_{p-1}^{m-1}.*

`Proof` : Let such $B \neq X_m$. Then B is of the form

$$X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq \min(m-n, k-|X_n|)\right\}$$

for some non-negative integer n with $0 \leq n \leq m-1$. But $x_p^m \in B$; therefore B is of the form $X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq m-n\right\}$ for some non-negative integer n, with $0 \leq n \leq m-1$. Since $x_p^m \in B$, we have $p_{m-n} = p$. It implies $p-1 \leq p_{m-n-1} \leq p$. The result follows since $x_{p_{m-n-1}}^{m-1} \in B$. $\square$

Lemma B.2.3. *Let $1 \leq m \leq t-1$. Let B be a block of $\mathcal{G}_{m+1} \setminus \{X_m\}$ which contains x_p^m but does not contain x_p^{m-1} for some non-negative integer p. Then there exists a block B' of $\mathcal{G}_{m+1} \setminus \{X_m\}$ such that $B \setminus \{x_p^m\} = B' \setminus \{x_{p-1}^m\}$.*

`Proof` : First we observe that such a p must be positive integer. If B is such a block then by a similar argument as in the proof of Lemma B.2.2 we have, B is of the form $X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq m-n\right\}$ for some non-negative integer n with $0 \leq n \leq m-1$. By Lemma B.2.2 we have, $\{x_{p-1}^{m-1}, x_p^m\} \subset B$. So $p_{m-n-1} = p-1$ and $p_{m-n} = p$. But $X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq m-n-1\right\} \sqcup \{x_{p-1}^m\}$ is also a block of $\mathcal{G}_{m+1} \setminus \{X_m\}$, which is the required B'. $\square$

Lemma B.2.4. *Let C be a blocking set of $\mathcal{G}_{m+1}$ which contains a unique element of X_m say x_p^m, for some non-negative integer p. Then $C \setminus \{x_p^m\}$ is a blocking set of $\mathcal{G}_{m+1} \setminus \{X_m\}$.*

`Proof` : If B is a block of $\mathcal{G}_{m+1} \setminus \{X_m\}$ which does not contain x_p^m then $B \cap (C \setminus \{x_p^m\}) \neq \emptyset$. So let B be a block of $\mathcal{G}_{m+1} \setminus \{X_m\}$ which contains x_p^m. If B is such block then by a similar argument as in the proof of Lemma B.2.2 we have, B is of the form $X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq m-n\right\}$ for some non-negative integer n, with $0 \leq n \leq m-1$. So $p_{m-n} = p$. Therefore either $X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq m-n-1\right\} \sqcup \{x_{p-1}^m\} =: B_1$ (say) or $X_n \sqcup \left\{x_{p_i}^{n+i} : 1 \leq i \leq m-n-1\right\} \sqcup \{x_{p+1}^m\} =: B_2$ (say) is a block of $\mathcal{G}_{m+1} \setminus \{X_m\}$. None of the blocks contain x_p^m, so either $B_1 \cap (C \setminus \{x_p^m\}) \neq \emptyset$ or $B_2 \cap (C \setminus \{x_p^m\}) \neq \emptyset$. But clearly either $B_1 \setminus \{x_{p-1}^m\} = B \setminus \{x_p^m\}$ or $B_2 \setminus \{x_{p+1}^m\} = B \setminus \{x_p^m\}$. Since x_p^m is the unique element of $C \cap X_m$ therefore it follows that $B \cap (C \setminus \{x_p^m\}) \neq \emptyset$. This completes the proof. $\square$

Lemma B.2.5. *Let C be a blocking set of $\mathcal{G}_{m+1}$ and p be the least non-negative integer such that $x_p^m \in C$. Then $C' := (C \setminus \{x_p^m\}) \cup \{x_p^{m-1}\}$ is a blocking set of $\mathcal{G}_{m+1} \setminus \{X_m\}$. Consequently, $|C \cap X_m| - |C' \cap X_m| = 1$ and $|C'| \leq |C|$. Moreover, $|C'| = |C|$ if and only if $x_p^{m-1} \notin C$.*

`Proof` : We observe from Lemma B.2.2 the blocks of $\mathcal{G}_{m+1} \setminus \{X_m\}$, which contains x_p^m must also contain exactly one of the two points x_p^{m-1} and x_{p-1}^{m-1}. If B is a block of $\mathcal{G}_{m+1} \setminus \{X_m\}$ which does not contain x_p^m, then $(C \setminus \{x_p^m\}) \cap B \neq \emptyset$. So let B be a block of $\mathcal{G}_{m+1} \setminus \{X_m\}$ which contains x_p^m. In this case by Lemma B.2.3, either $x_p^{m-1} \in B$ or there exists $B' \in (\mathcal{G}_{m+1} \setminus \{X_m\})$ such that $B' \setminus \{x_{p-1}^m\} = B \setminus \{x_p^m\}$. Since p is the least non-negative integer such that $x_p^m \in C$, $x_{p-1}^m \notin C$. Consequently, $C \cap (B' \setminus \{x_{p-1}^m\}) \neq \emptyset$, which implies $C \cap (B \setminus \{x_p^m\}) \neq \emptyset$. Therefore in both the cases $C' \cap B \neq \emptyset$. Hence C' is a blocking set of $\mathcal{G}_{m+1} \setminus \{X_m\}$. The consequence part follows immediately from the construction of C'. □

Lemma B.2.6. *For $1 \leq m \leq t-1$, $\mathrm{tr}(\mathcal{G}_{m+1}) = 1 + \mathrm{tr}(\mathcal{G}_m)$.*

`Proof` : Let $\mathcal{F}_1 = \mathcal{G}_{m+1} \setminus \mathcal{F}_2$ where $\mathcal{F}_2$ consists only of X_m. Firstly we show that $\mathrm{tr}(\mathcal{F}_1) = \mathrm{tr}(\mathcal{G}_m)$. Finally we show that each $T \in \mathcal{F}_1^{\top}$ is disjoint from X_m.

Let $T \in \mathcal{G}_m^{\top}$. Then T is a blocking set of $\mathcal{F}_1$. Consequently, $\mathrm{tr}(\mathcal{F}_1) \leq \mathrm{tr}(\mathcal{G}_m)$. Suppose $\mathrm{tr}(\mathcal{F}_1) \leq \mathrm{tr}(\mathcal{G}_m) - 1$. With this assumption let $T \in \mathcal{F}_1^{\top}$. Then T can not be disjoint from X_m. (If so then T is a blocking set of $\mathcal{G}_m$. Hence $\mathrm{tr}(\mathcal{F}_1) = |T| \geq \mathrm{tr}(\mathcal{G}_m)$, a contradiction.) Therefore by Lemma B.2.5, there exists a transversal T' of $\mathcal{F}_1$, which contains exactly one element of X_m. So by Lemma B.2.4 a proper subset of T' is again a blocking set of $\mathcal{F}_1$, which violates the minimality property of the transversal T'. Hence $\mathrm{tr}(\mathcal{F}_1) = \mathrm{tr}(\mathcal{G}_m)$.

Let $T \in \mathcal{F}_1^{\top}$. If possible suppose T is not disjoint from X_m then (by the same argument as above) by Lemma B.2.5 there exists a transversal T' of $\mathcal{F}_1$, which contains exactly one element of X_m. So by Lemma B.2.4 a proper subset of T' is again a blocking set of $\mathcal{F}_1$, which violates the minimality property of the transversal T', a contradiction. Hence T is disjoint from X_m.

Let $T \in \mathcal{F}_1^{\top}$. Since $T \cap X_m = \emptyset$, therefore T is a blocking set of $\mathcal{G}_m$ and $\mathrm{tr}(\mathcal{F}_1) = \mathrm{tr}(\mathcal{G}_m)$. Hence $\mathcal{F}_1^{\top} = \mathcal{G}_m^{\top}$. Therefore the result follows due to Lemma 5.2.7. □

Therefore we get the following theorem.

Theorem B.2.7. $\mathrm{tr}(\mathcal{G}_t) = t$.

`Proof` : Since $\mathcal{G}_1 = \{X_0\}$, it has transversal size 1. By Lemma B.2.6, $\mathrm{tr}(\mathcal{G}_m) = m$ for $1 \leq m \leq t$. In particular, $\mathrm{tr}(\mathcal{G}_t) = t$. $\square$

B.2.2 Construction of $\mathcal{H}_m$

Construction B.2.8. Let $\mathcal{H}_0 := \mathcal{G}_t$ and m be a non-negative integer. If $B \in \mathcal{H}_m$ with $|B| = k$ then $B \in \mathcal{H}_{m+1}$. Otherwise, if $B \in \mathcal{H}_m$ with $|B| \leq k-1$, then there exists a least integer p such that $x_p^{m-1} \in B$; consequently $B \sqcup \{x_p^m\}$ and $B \sqcup \{x_{1+p}^m\} \in \mathcal{H}_{m+1}$.

Note that we constructed recursively, $\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{\lfloor \frac{t-1}{2} \rfloor}$, where $\mathcal{H}_{\lfloor \frac{t-1}{2} \rfloor} = \mathbb{F}(k,t)$. Any blocking set of $\mathcal{H}_m$ is a blocking set of $\mathcal{H}_{m+1}$. We identify all blocking sets of $\mathcal{H}_{m+1}$ interns of various conditions. As a consequence we finally obtain the transversal size of $\mathbb{F}(k,t) = \mathcal{H}_{\lfloor \frac{t-1}{2} \rfloor}$.

We observe that if C is a blocking set of $\mathcal{H}_{m+1}$ disjoint from X_m, then C is a blocking set of $\mathcal{H}_m$. Now we assume C is not disjoint from X_m and make the following definitions.

Definition. Let C be a blocking set of $\mathcal{H}_{m+1}$.

(a) A point $x_p^m \in C \cap X_m$ is said to be an *isolated point* of X_m in C if C is disjoint from $\{x_{p-1}^m, x_{p+1}^m\}$.

(b) Let $n \geq 2$ be a positive integer. A set of n consecutive points $\{x_{p+i}^m \in X_m : 0 \leq i \leq n-1\} \subsetneqq C \cap X_m$ is said to be an *isolated consecutive $n-$set* of X_m in C if C is disjoint from $\{x_{p-1}^m, x_{p+n}^m\}$.

(c) An isolated consecutive $n-$set $\{x_{p+i}^m \in X_m : 0 \leq i \leq n-1\} \subseteqq C \cap X_m$ is said to be *strongly isolated consecutive $n-$set* of X_m in C if C is disjoint from $\{x_{p-1}^{m-1}, x_{p+n-1}^{m-1}\}$.

Lemma B.2.9. *Let C be a blocking set of $\mathcal{H}_{m+1}$. If C contains only isolated points (or a unique point) of X_m, then C is a blocking set of $\mathcal{H}_m$.*

`Proof` : It is enough to show that C is a blocking set of $\mathcal{H}_m \setminus \mathcal{H}_{m+1}$. Let x_p^m be an isolated point (or the unique point) of X_m in C. We observe from the construction of $\mathcal{H}_{m+1}$ that all possible blocks B of $\mathcal{H}_m \setminus \mathcal{H}_{m+1}$ such that $B \sqcup \{x_p^m\} \in \mathcal{H}_{m+1}$ must contain either of x_{p-1}^{m-1} or x_p^{m-1}. Suppose B contains x_{p-1}^{m-1}, since C is a blocking set of $\mathcal{H}_{m+1}$ and $B \sqcup \{x_{p-1}^m\} \in \mathcal{H}_{m+1}$, so C intersects $B \sqcup \{x_{p-1}^m\}$. But x_p^m is an isolated point and hence C intersects B. Similarly suppose B contains x_p^{m-1}. Since C is a blocking set of $\mathcal{H}_{m+1}$ and $B \sqcup \{x_{p+1}^m\} \in \mathcal{H}_{m+1}$, C intersects $B \sqcup \{x_{p+1}^m\}$. But x_p^m is an isolated point and hence C intersects B. Since x_p^m is an arbitrarily chosen isolated point (or the unique point) in C, C is a blocking set of $\mathcal{H}_m$. $\square$

By Lemma B.2.9, we can assume if C is a finite blocking set of $\mathcal{H}_{m+1}$, then C does not contain any isolated points (or a unique point) of X_m.

Lemma B.2.10. *Let C be a blocking set of $\mathcal{H}_{m+1}$. Let p and $q \geq m+1$ be the least integer and the greatest integer respectively such that $p \neq q$ and $\{x_p^m, x_q^m\} \subset C$. Then $C^{'} := (C \setminus \{x_q^m\}) \cup \{x_{q-1}^{m-1}\}$ is a blocking set of $\mathcal{H}_{m+1}$.*

`Proof` : Note that $q \geq m+1$. Consider the blocks of $\mathcal{H}_{m+1}$ which contain x_q^m, except the blocks containing the set X_m. Such blocks are of the form $B^{'} \sqcup \{x_q^m\}$, where $B^{'} \in \mathcal{H}_m \setminus \mathcal{H}_{m+1}$ with either $x_{q-1}^{m-1} \in B^{'}$ or $x_q^{m-1} \in B^{'}$, but not both. If $x_{q-1}^{m-1} \in B^{'}$ then $(C \setminus \{x_q^m\}) \cup \{x_{q-1}^{m-1}\}$ intersects $B^{'}$ at least in the point x_{q-1}^{m-1}. On the other hand we have $x_q^{m-1} \in B^{'}$. For such a $B^{'}$ we observe that $B^{'} \sqcup \{x_{q+1}^m\} \in \mathcal{H}_{m+1}$. Since q is the greatest element, $x_{q+1}^m \notin C$. Therefore $C \setminus \{x_q^m\}$ intersects $B^{'}$. Consequently $C^{'}$ is a blocking set of $\mathcal{H}_{m+1}$. □

Suppose C is a finite blocking set of $\mathcal{H}_{m+1}$, with $0 \leq p \leq m$, $q \geq m+1$ and $\{x_p^m, x_q^m\} \subset C$. Then by repeated use of Lemma B.2.10, we have a blocking set $C^{'}$ of $\mathcal{H}_{m+1}$, with $|C^{'}| \leq |C|$, such that $C^{'} \cap X_m$ is a non empty subset of $\{x_i^m : 0 \leq i \leq m\}$. Therefore, we can assume that if C is a finite blocking set of $\mathcal{H}_{m+1}$, then C contains an isolated consecutive $n-$set, for some suitable positive integer $n \geq 2$ and $C \cap X_m \subsetneqq \{x_i^m : 0 \leq i \leq m\}$.

Lemma B.2.11. *Let C be a blocking set of $\mathcal{H}_{m+1}$ and $\{x_{p+i}^m \in X_m : 0 \leq i \leq n-1\} \subsetneqq C$ be an isolated consecutive $n-$set of X_m, but not strongly isolated consecutive $n-$set, where $n \geq 2$. Then*

$$C^{'} := \begin{cases} (C \setminus \{x_p^m\}) \cup \{x_p^{m-1}\} & \text{if } \; x_{p-1}^{m-1} \in C \\ (C \setminus \{x_{p+n-1}^m\}) \cup \{x_{p+n-2}^{m-1}\} & \text{if } \; x_{p+n-1}^{m-1} \in C \end{cases}$$

is a blocking set of $\mathcal{H}_{m+1}$.

`Proof` : Consider the blocks of $\mathcal{H}_{m+1}$ which contain x_p^m (respectively, x_{p+n-1}^m), except the blocks containing the set X_m. Such blocks contain either x_{p-1}^{m-1} or x_p^{m-1} (respectively, either x_{p+n-2}^{m-1} or x_{p+n-1}^{m-1}). Hence $(C \setminus \{x_p^m\}) \cup \{x_p^{m-1}\}$ (respectively, $(C \setminus \{x_{p+n-1}^m\}) \cup \{x_{p+n-2}^{m-1}\}$) intersects such blocks if $x_{p-1}^{m-1} \in C$ (respectively, if $x_{p+n-1}^{m-1} \in C$). Therefore, if $x_{p-1}^{m-1} \in C$ (respectively, if $x_{p+n-1}^{m-1} \in C$) then $C^{'} = (C \setminus \{x_p^m\}) \cup \{x_p^{m-1}\}$ (respectively,$C^{'} = (C \setminus \{x_{p+n-1}^m\}) \cup \{x_{p+n-2}^{m-1}\})$ is a blocking set of $\mathcal{H}_{m+1}$. □

By repeated use of Lemma B.2.11, we observe that there exists a blocking set $C^{'}$ of $\mathcal{H}_{m+1}$, with $|C^{'}| \leq |C|$, such that $C^{'}$ contains an isolated point of X_m (namely, either

x^m_{p+n-1} or x^m_p respectively). Therefore, we can assume that if C is a finite blocking set of $\mathcal{H}_{m+1}$, then C contains some strongly isolated consecutive $n-$set, for some suitable positive integer $n \geq 2$ and some isolated points of X_m.

Again, by using Lemma B.2.9, we observe that we can assume C does not contain any isolated points (or a unique point) of X_m. Therefore, we can assume that if C is a finite blocking set of $\mathcal{H}_{m+1}$, then C contains some strongly isolated consecutive $n-$set, for some suitable positive integer $n \geq 2$.

Now, by using Lemma B.2.10, we can assume that if C is a finite blocking set of $\mathcal{H}_{m+1}$, then C contains some strongly isolated consecutive $n-$set, for some suitable positive integer $n \geq 2$ and $C \cap X_m \subsetneqq \{x^m_i : 0 \leq i \leq m\}$.

Lemma B.2.12. *Let C be a blocking set of $\mathcal{H}_{m+1}$, with $C \cap X_m \subsetneqq \{x^m_i : 0 \leq i \leq m\}$ and $x^m_p \in C$, where $p \neq 0$ or $p \neq m$. Then*

$$C^{'} := \begin{cases} (C \setminus \{x^m_0\}) \cup \{x^{m-1}_0\} & \text{if} \quad x^m_0 \in C \\ (C \setminus \{x^m_m\}) \cup \{x^{m-1}_{m-1}\} & \text{if} \quad x^m_m \in C \end{cases}$$

is a blocking set of $\mathcal{H}_{m+1}$.

`Proof` : Let B be a block of $\mathcal{H}_{m+1}$ which do not contain X_m. If $x^m_0 \in B$, then $x^{m-1}_0 \in B$. Therefore if $x^m_0 \in C$ then $C^{'} = (C \setminus \{x^m_0\}) \cup \{x^{m-1}_0\}$ intersects all the blocks $B \in \mathcal{H}_{m+1}$, which do not contain X_m. Since $x^m_p \in C$, where $p \neq 0$, we have $C^{'} = (C \setminus \{x^m_0\}) \cup \{x^{m-1}_0\}$ is a blocking set of $\mathcal{H}_{m+1}$.

Let $x^m_m \in B \in \mathcal{H}_{m+1}$. If $|B| = k$ then $X_0 \subset B$ and if $|B| \leq k-1$, then either $x^{m-1}_m \in B$ or $x^{m-1}_{m-1} \in B$. We divide our arguments in two exhaustive cases.

`Case A` : Suppose $|B| = k$. Then $X_0 \subset B$.

We observe from the construction of $\mathcal{H}_{m+1}$, that such B is of the form

$$X_0 \sqcup \{x^i_i : 1 \leq i \leq m\} \sqcup \{x^{m+i}_{m+q_i} : 1 \leq i \leq k - |X_0| - m\},$$

where $\{q_n\}$ varies over finite sequences of non-negative integers such that $q_0 = 0$ and for $n \geq 1$, $p_n = p_{n-1}$ or $1 + p_{n-1}$. We note that all such blocks contain x^{m-1}_{m-1}. Hence $(C \setminus \{x^m_m\}) \cup \{x^{m-1}_{m-1}\}$ intersects these blocks.

`Case B` : Suppose $|B| \leq k-1$. Then either $x^{m-1}_m \in B$ or $x^{m-1}_{m-1} \in B$.

For such blocks we note that $C \cap X_m \subset \{x^m_i : 0 \leq i \leq m\}$. Hence C intersects all such blocks B, with $\{x^{m-1}_m, x^m_m\} \subset B$, other than at a point x^m_m. Therefore, $C^{'} = (C \setminus \{x^m_m\}) \cup \{x^{m-1}_{m-1}\}$ intersects such blocks. Otherwise any such B, with $\{x^{m-1}_{m-1}, x^m_m\} \subset B$, hence $C^{'} = (C \smallsetminus \{x_m^m\}) \cup \{x_{m-1}^{m-1}\}$ intersects all such blocks.

Since $x_p^m \in C$, where $p \neq m$, from the above two cases we conclude that $C^{'} = (C \smallsetminus \{x_m^m\}) \cup \{x_{m-1}^{m-1}\}$ is a blocking set of $\mathcal{H}_{m+1}$. $\square$

By using Lemma B.2.12, we can assume that if C is a finite blocking set of $\mathcal{H}_{m+1}$, then C contains some strongly isolated consecutive $n-$set, for some suitable positive integer $n \geq 2$ and $C \cap X_m \subsetneqq \{x_i^m : 1 \leq i \leq m-1\}$.

Lemma B.2.13. *Let C be a blocking set of $\mathcal{H}_{m+1}$. If C contains only strongly isolated consecutive $2-$sets, then C is a blocking set of $\mathcal{H}_m$.*

`Proof` : Let $\{x_p^m, x_{p+1}^m\}$ be a strongly isolated consecutive $2-$set. It is enough to show that C intersects $B^{'}$, where $B^{'} \in \mathcal{H}_m$ with $|B^{'}| \leq k-1$ and either $B^{'} \sqcup \{x_p^m\}$ or $B^{'} \sqcup \{x_{p+1}^m\} \in \mathcal{H}_{m+1}$. Now we divide our arguments in the following two exhaustive cases.

`Case A` : Let $B^{'} \in \mathcal{H}_m$ with $|B^{'}| \leq k-1$ and $B^{'} \sqcup \{x_p^m\} \in \mathcal{H}_{m+1}$.

From the construction of $\mathcal{H}_m$, either $x_{p-1}^{m-1} \in B^{'}$ or $x_p^{m-1} \in B^{'}$. If $x_{p-1}^{m-1} \in B^{'}$, then $B^{'} \sqcup \{x_{p-1}^m\} \in \mathcal{H}_{m+1}$. Since $\{x_p^m, x_{p+1}^m\}$ is a strongly isolated consecutive $2-$set in C, therefore x_{p-1}^{m-1}, $x_{p-1}^m \notin C$. Since C is a blocking set of $\mathcal{H}_{m+1}$, C intersects $B^{'}$ other than at a point x_{p-1}^{m-1}. If $x_p^{m-1} \in B^{'}$, then such a $B^{'}$ contains either x_{p-1}^{m-2} or x_p^{m-2}. We show that C intersects such a $B^{'}$ other than at a point x_p^{m-1}. If $\{x_{p-1}^{m-2}, x_p^{m-1}\} \subset B^{'}$, then $(B^{'} \smallsetminus \{x_p^{m-1}\}) \sqcup \{x_{p-1}^{m-1}\} \in \mathcal{H}_{m+1}$. So by the previous arguments we have C intersects $(B^{'} \smallsetminus \{x_p^{m-1}\}) \sqcup \{x_{p-1}^{m-1}\}$ other than at a point x_{p-1}^{m-1}. Hence C intersects such a $B^{'}$ other than at a point x_p^{m-1}. We deal with the particular case $\{x_p^{m-2}, x_p^{m-1}\} \subset B^{'}$ in Case B.

`Case B` : Let $B^{'} \in \mathcal{H}_m$ with $|B^{'}| \leq k-1$ and $B^{'} \sqcup \{x_{p+1}^m\} \in \mathcal{H}_{m+1}$.

From the construction of $\mathcal{H}_m$, either $x_{p+1}^{m-1} \in B^{'}$ or $x_p^{m-1} \in B^{'}$. If $x_{p+1}^{m-1} \in B^{'}$, then $B^{'} \sqcup \{x_{p+2}^m\} \in \mathcal{H}_{m+1}$. Since $\{x_p^m, x_{p+1}^m\}$ is a strongly isolated consecutive $2-$set in C, therefore x_{p+1}^{m-1}, $x_{p+2}^m \notin C$. Since C is a blocking set of $\mathcal{H}_{m+1}$, C intersects $B^{'}$ other than at a point x_{p+1}^{m-1}. If $x_p^{m-1} \in B^{'}$, then such a $B^{'}$ contains either x_{p-1}^{m-2} or x_p^{m-2}. In Case A we already dealt the case $\{x_{p-1}^{m-2}, x_p^{m-1}\} \subset B^{'}$. So let $\{x_p^{m-2}, x_p^{m-1}\} \subset B^{'}$. We show that C intersects $B^{'}$ other than at a point x_p^{m-1}. If $\{x_p^{m-2}, x_p^{m-1}\} \subset B^{'}$, then $(B^{'} \smallsetminus \{x_p^{m-1}\}) \sqcup \{x_{p+1}^{m-1}\} \in \mathcal{H}_{m+1}$. So by the previous arguments we have C intersects $(B^{'} \smallsetminus \{x_p^{m-1}\}) \sqcup \{x_{p+1}^{m-1}\}$ other than x_{p+1}^{m-1}. Hence C intersects $B^{'}$ other than at a point x_p^{m-1}. $\square$

We do not have any analogous results like Lemma B.2.13 for strongly isolated consecutive $n-$sets, where $n \geq 3$. Therefore this method stops here. However there are some answers due to Lemma B.2.9 and Lemma B.2.13. By using Lemma B.2.12, we can assume that if C is an arbitrary finite blocking set of $\mathcal{H}_{m+1}$, then C contains some strongly isolated consecutive $n-$set, for some suitable positive integer $n \leq 2$ and $C \cap X_m \subsetneqq \{x_i^m : 1 \leq i \leq m-1\}$. Therefore if $|\{x_i^m : 1 \leq i \leq m-1\}| = m-1 < 3$, i.e. if $m < 4$ then $\operatorname{tr}(\mathcal{H}_{m+1}) = t$. Since $\mathbb{F}(k,t) = \mathcal{H}_{\lfloor \frac{t-1}{2} \rfloor}$, we have if $m+1 \leq \lfloor \frac{t-1}{2} \rfloor$, i.e. $m \leq \lfloor \frac{t-1}{2} \rfloor - 1$, then $\operatorname{tr}(\mathcal{H}_{m+1}) = t$. Consequently if $\lfloor \frac{t-1}{2} \rfloor - 1 < 4$, i.e. $\lfloor \frac{t-1}{2} \rfloor < 5$, then $\operatorname{tr}(\mathcal{H}_{\lfloor \frac{t-1}{2} \rfloor}) = t$. Hence we have the following theorem.

Theorem B.2.14. $\operatorname{tr}(\mathbb{F}(k,t)) = t$, *for* $t \leq 10$.

B.3 An alternative proof of Raney's Lemma (Existence Part)

In this section we prove Lemma B.3.4 which is a crucial step in proving Theorem 5.2.5.

We consider the cyclic graph $\mathtt{G}$ with t vertices and label the vertices consecutively by $[0]$, $[1]$, ..., $[t-1]$ in a clockwise direction, i.e. vertex set is $\mathbb{Z}_t$. Let $\mathbf{C} = \{\texttt{Colour}\ [i] : 0 \leq i \leq t-1\}$ be the set of "colours". For each vertex $[n] \in \mathbb{Z}_t$, we associate the integer $c_{[n]}$ and the $\texttt{Colour}\ [n] \in \mathbf{C}$ such that $\sum_{i=0}^{t-1} c_{[i]} = t-1$.

We begin by providing an algorithm. This algorithm describes a colouring procedure. Henceforth, we call this algorithm the Colouring Algorithm. The steps of the algorithm are indicated by Roman numerals.

I . Set $i = 0$

II . Set $m = c_{[i]}$, $n = i$.

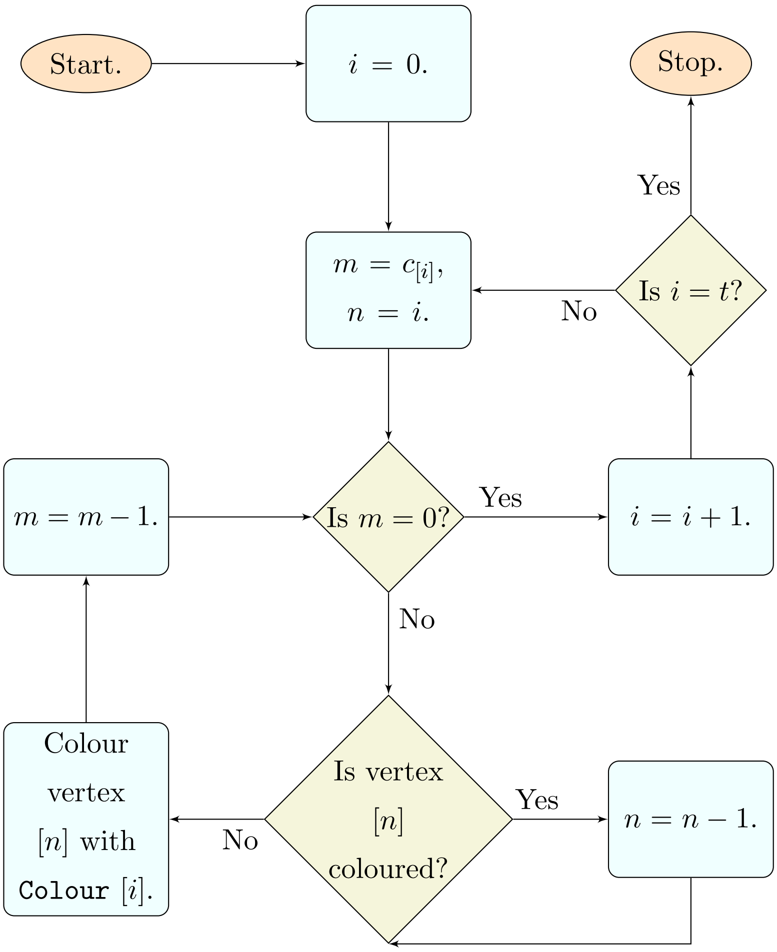
III . If $m \neq 0$, then continue to next step. Otherwise, set $i = i+1$. If $i = t$, then terminate the algorithm; else (i.e. if $i \neq t$), return to Step (II).

IV . If vertex $[n]$ is coloured, then set $n = n-1$ and redo this step.

V . If vertex $[n]$ is not coloured, then colour vertex $[n]$ with $\texttt{Colour}\ [i]$ and set $m = m-1$.

VI . Return to Step (III).

We observe that the Colouring Algorithm can only terminate at Step (III). In the next lemma, we show that this happens in a finite number of steps. The following flow chart illustrates the Colouring Algorithm.



Lemma B.3.1. *The Colouring Algorithm terminates in a finite number of steps.*

`Proof` : By Step (III) the value of i increases if and only if $m = 0$. Further, the previous step of the algorithm that would have been executed must have been Step (II). Therefore at Step (III), either we return to Step (II) with the new i or we terminate the algorithm when i increases to t.

By Step (V), every time a vertex is coloured with `Colour` $[i]$, the value of m is decreased by 1. Further, from Step (VI) we see that if $m = 0$, then the value of i is increased by 1. If the new $i \leq t-1$, then m is reset to $c_{[i]}$ for the new i by Step (II). Otherwise, $i = t$ and the algorithm terminates. Hence for every colour `Colour` $[i]$ there can be a maximum of $c_{[i]}$ vertices with `Colour` $[i]$. Hence at any time there can be a maximum of

$$\sum_{i=0}^{t-1} c_{[i]} = t - 1. \tag{B.3.1}$$

coloured vertices. Hence there will always be at least one vertex with no colour. This implies that Step (IV) decreases the value of n by 1, until the vertex $[n]$ has no colour.

This procedure will terminate in finite steps.

Now we see that apart from Step (IV) when we return to a lower numbered step from a higher numbered step, either the value of m is decreased or the value of i is increased. But we know that i increases from 0 to t. As long as i remains fixed m decreases from $c_{[i]}$ to 0. Thus after a finite number of returns $i = t$ and $m = 0$. As all returns are to Step (II) or Step (III), Step (III) is always executed after a return. Hence the algorithm terminates in finite steps. $\square$

Now we discuss the outcomes of the Colouring Algorithm.

Lemma B.3.2. *For each i, with $0 \leq i \leq t-1$, there are exactly $c_{[i]}$ vertices of G which are coloured with* `Colour` *$[i]$.*

`Proof` : Since the algorithm terminates in finite number of steps, all values of i from 0 to $t-1$ must be taken consecutively starting from 0. Further, increment of i only happens if $m = 0$ and after the increment m equals $c_{[i]}$ for the new value of i. Thus for each i, with $0 \leq i \leq t-1$, m must decrease from $c_{[i]}$ to 0, at which instance the value of i is increased. Hence for each i, m is decreased by 1 exactly $c_{[i]}$ times. But as m is decreased exactly after a vertex is coloured with `Colour` $[i]$ and no coloured vertex is recoloured [see Step (IV)], we have exactly $c_{[i]}$ vertices with `Colour` $[i]$. $\square$

Lemma B.3.3. *There exists a unique vertex say $[\mu]$ of G, which is not coloured and it satisfies the following properties.*

(a) $c_{[\mu]} = 0$.

(b) *For each n, with $1 \leq n \leq t-1$, all vertices of G with* `Colour` *$[\mu + n]$ is a subset of $\{[\mu + i] : 1 \leq i \leq n\}$.*

(c) *For each n, with $1 \leq n \leq t-1$, $\sum_{i=1}^{n}(1 - c_{[\mu+i]}) \geq 0$.*

`Proof` : By using Lemma B.3.2, we have there are $c_{[i]}$ vertices with `Colour` $[i]$, where $0 \leq i \leq t-1$. Hence $\sum_{i=0}^{t-1} c_{[i]} = t-1$ vertices are coloured. As a consequence, exactly one vertex of G is not coloured. This proves the existence a unique vertex say $[\mu]$ of G, which is not coloured. At the time when $i = s$, $m = c_{[i]} = c_{[\mu]}$. If $c_{[\mu]} \neq 0$, then according to the algorithm, we move from Step (III) to Step (V) and colour the vertex $[\mu]$ with `Colour` $[\mu]$. This means vertex $[\mu]$ receives `Colour` $[\mu]$, a contradiction arises since the algorithm terminates without colouring that vertex $[\mu]$. Hence (a) follows.

Fix integers i and j, with $1 \leq i \leq t-1$ and $1 \leq j \leq i$. Recall that we started colouring with `Colour` $[\mu + i]$ at $n = [\mu + i]$. Then we kept on decreasing n by 1 and colouring whichever vertex was not coloured (until there were $c_{[\mu+i]}$ vertices with `Colour` $[\mu+i]$) but as vertex $[\mu]$ is not coloured n was never equal to $[\mu]$. Hence n could only have taken the values $[\mu + i], [\mu + i - 1], [\mu + i - 2], ..., [\mu + 1]$. Thus all the vertices with `Colour` $[\mu + i]$ need to be in this set which proves (b).

$c_{[\mu+m]}$ vertices are coloured with the colour `Colour` $[\mu + m]$, where $1 \leq m \leq n$. Hence $\sum_{m=1}^{n} c_{[\mu+m]}$ vertices are coloured with the colours `Colour` $[\mu + 1]$, `Colour` $[\mu + 2]$, ..., `Colour` $[\mu + n]$. Therefore by using part (b) we have,

$$\sum_{m=1}^{n} c_{[\mu+m]} \leq n.$$

This proves (c). $\square$

Recall that, for any finite sequence $(x_0, \ldots, x_{t-1})$ its cyclic shifts are the t sequences $(x_{i+1}, \ldots, x_{i+t})$ where $0 \leq i \leq t - 1$.

Lemma B.3.4 (Raney). *Let $(r_0, r_1, \ldots, r_{t-1})$ be a finite sequence of integers such that $\sum_{i=0}^{t-1} r_i = 1$. Then, one of the t cyclic shifts of this sequence has all its partial sums strictly positive.*

`Proof` : Put $r_n = 1 - c_{[n]}$ for each n, with $0 \leq n \leq t - 1$. We choose $\mu^{'} \in [\mu]$ such that $0 \leq \mu^{'} \leq t - 1$. Then by Lemma B.3.3, $\mu^{'}$ is the required index. The next part of the result follows from (a) and (c) of Lemma B.3.3. $\square$

B.4 On the number of transversals of $\mathbb{F}(k, 2)$

We start this section with the following definition.

Definition. A family of $k-$sets $\mathcal{F}$ is said to be a *transversally minimal family of $k-$sets* if $\mathrm{tr}(\mathcal{F}) < \infty$ and for each $B \in \mathcal{F}$ we have $\mathrm{tr}(\mathcal{F} \setminus \{B\}) = \mathrm{tr}(\mathcal{F}) - 1$. In addition, if $\mathcal{F}$ is an intersecting family then $\mathcal{F}$ is said to be *transversally minimal intersecting family of $k-$sets with transversal size t* in short TMIF(k, t).

Fix a block $B \in \mathbb{F}(k, t)$. Suppose $B = X_{n_0} \sqcup \left\{x_{p_i}^{n_0+i} : 1 \leq i \leq k - |X_{n_0}|\right\}$, where addition in the superscript is modulo t. Consider the sets X_j, $0 \leq j \leq t - 1$, except those that correspond to j, with $j = n_0 + i$, $0 \leq i \leq k - |X_{n_0}|$ and the addition is modulo t. Let

Y be a set consisting of one element from each such X_j. Hence $|Y| = t - (k - |X_{n_0}| + 1) = t - 1 - (k - |X_{n_0}|)$. We observe that for $1 \leq i \leq k - |X_{n_0}|$,

$$p_i = \sum_{j=1}^{i} \epsilon_j,$$

where for each j, with $1 \leq j \leq i$, $\epsilon_j \in \{0, 1\}$. Now we construct a finite sequence q_1, q_2,. . ., $q_{k-|X_{n_0}|}$ as follows, $q_1 = 1 - \epsilon_1$ and for $i \geq 2$

$$q_i := \sum_{j=1}^{i-1} \epsilon_j + (1 - \epsilon_i).$$

Set $Z = \{x_{q_i}^{n+i} : 1 \leq i \leq k - |X_{n_0}|\}$, where addition in the superscript is modulo t. Then $Y \sqcup Z$ is a $(t-1)$−set disjoint from B. Since by Theorem 5.2.5, $\mathrm{tr}(\mathbb{F}(k, t)) = t$ it shows that, $\mathbb{F}(k, t)$ is an example of TmIF(k, t). We recall that,

$$\mathrm{M}^{\top}(k, t) := \max \left\{ |\mathcal{F}^{\top}| : \mathcal{F} \text{ is an intersecting family with } \mathrm{k}(\mathcal{F}) = k \text{ and } \mathrm{tr}(\mathcal{F}) = t \right\}.$$

If $\mathcal{F}$ is an intersecting family of k−sets with transversal size t, then there exists a TmIF(k, t) $\mathcal{G}$ such that $\mathcal{G} \subseteq \mathcal{F}$. As a consequence $\mathcal{F}^{\top} \subseteqq \mathcal{G}^{\top}$. Therefore, if $\mathcal{F}$ itself is a TmIF(k, t), then it may contain the maximum number of transversals. Our guess is that for large positive integers t and k, with $k \geq t + 1$ $\mathrm{M}^{\top}(k, t) = |\mathbb{F}(k, t)|$ (see (b) of Conjecture 5.3.9).

In this section we show that

$$\mathrm{M}^{\top}(k, 2) = k^2 - k + 1 \text{ for } k \geq 3.$$

This result is not new. It appeared in [8, Proposition 1, Page 143]. However the solution technique used here is independent and completely new. Our answer is a little stronger (see Theorem B.4.9). Our aim is to give a uniform solution to establish,

$$\mathrm{M}^{\top}(k, 3) = (k-1)^3 + 3(k-1) = |\mathbb{F}^{\top}(k, 3)| \text{ for } k \geq 4.$$

This problem has a solution for $k = 4, 5$ in [4, 12] and for $k \geq 9$ in [8, Theorem 1]. In [10], Frankl et al. studied about $\mathrm{M}^{\top}(k, t)$, where k and t are positive integers with $k \geq t$.

We recall the construction of $\mathbb{F}(k, 2)$.

Construction. Let $k \geq 3$ be an integer. Let $X_0 = \{x_i^0 : 0 \leq i \leq k-1\}$ and $X_1 = \{x_i^1 : 0 \leq i \leq k-2\}$, be 2 pairwise disjoint sets. The family of k−sets $\mathbb{F}(k, 2)$ consists of three k−sets X_0, $X_1 \sqcup \{x_0^0\}$ and $X_1 \sqcup \{x_1^0\}$.

The complete list of transversals of $\mathbb{F}(k, 2)$ are $\{x, y\}$ and $\{x_0^0, x_1^0\}$, where $x \in X_0$ and

$y \in X_1$. Therefore there are $k^2 - k + 1$ transversals of $\mathbb{F}(k, 2)$. The purpose of this section is to prove Theorem B.4.9. In this section, we assume $k \geq 3$ is a positive integer. Let $\mathcal{F}$ be a TMIF$(k, 2)$. Fix B_1, $B_2 \in \mathcal{F}$ and let $|B_1 \setminus B_2| = n$, where $1 \leq n \leq k - 1$. Let $\mathcal{G} := \mathcal{F} \setminus \{B_1, B_2\}$. It is a non empty family with $\mathrm{tr}(\mathcal{G}) = 1$. We define $X := \{x : \{x\} \in \mathcal{G}^\top\}$. The proof of Theorem B.4.9 is based on the parameter $|X|$.

Lemma B.4.1. *For each $B \in \mathcal{G}$, $X \subseteq B$.*

`Proof` : Suppose there exists $B \in \mathcal{G}$ such that $X \nsubseteq B$. This means there exists $x \in X$ such that $x \notin B$. It contradicts that the transversal of $\mathcal{G}$ namely $\{x\}$ intersects B. Hence the result follows. □

Lemma B.4.2. $2 \leq |X| \leq k$.

`Proof` : Since $\mathcal{G}$ is a non empty family of $k-$sets therefore the upper bound follows from Lemma B.4.1.

X is disjoint from $B_1 \cap B_2$. (If not, then $x \in X \cap (B_1 \cap B_2)$ and $\{x\}$ is a blocking set of $\mathcal{F}$. It contradicts that $\mathrm{tr}(\mathcal{F}) = 2$.) Since for $i = 1$ and $i = 2$ we have $\mathrm{tr}(\mathcal{F} \setminus \{B_i\}) = 1$, so $(\mathcal{F} \setminus \{B_i\})^\top \subset \mathcal{G}^\top$. Therefore X intersects both $B_1 \setminus B_2$ and $B_2 \setminus B_1$. The lower bound follows since $B_1 \setminus B_2$ is disjoint from $B_2 \setminus B_1$. □

We prove in Lemma B.4.2 that X is disjoint from $B_1 \cap B_2$ and X intersects both $B_1 \setminus B_2$ and $B_2 \setminus B_1$. Let G_1 denote the collection of transversals of $\mathcal{F}$ of the form $\{x, y\}$, where $x \in X$ and $y \in B_1 \cap B_2$.

Let $\{x, y\} \in \mathcal{F}^\top$, where $x \in X$ and $y \notin B_1 \cap B_2$. Then $y \in (B_1 \cup B_2) \setminus (B_1 \cap B_2)$. So if such $y \in B_1 \setminus B_2$ (respectively, $y \in B_2 \setminus B_1$) then $x \in X \cap (B_2 \setminus B_1)$ (respectively, $x \in X \cap (B_1 \setminus B_2)$). Let G_2 denote the collection of transversals of $\mathcal{F}$ of the form $\{x, y\}$, where either $x \in X \cap (B_1 \setminus B_2)$ and $y \in B_2 \setminus B_1$ or $x \in X \cap (B_2 \setminus B_1)$ and $y \in B_1 \setminus B_2$.

Suppose there exists a blocking set of $\mathcal{G}$ which is disjoint from X. Let C be such a blocking set. Then $C \in \mathcal{F}^\top$ if C intersects B_1 and B_2 and $|C| = 2$. Let G_3 denote the collection of transversals of $\mathcal{F}$ of the form $\{x, y\}$, where $\{x, y\}$ is disjoint from X and it intersects B_1 or B_2.

Lemma B.4.3. $\mathcal{F}^\top = \mathsf{G}_1 \cup \mathsf{G}_2 \cup \mathsf{G}_3$.

`Proof` : We already have $\mathsf{G}_1 \cup \mathsf{G}_2 \subset \mathcal{F}^\top$. If G_3 is empty then $\mathcal{F}^\top = \mathsf{G}_1 \cup \mathsf{G}_2$ and we are done. Let $\mathcal{F}^\top \setminus (\mathsf{G}_1 \cup \mathsf{G}_2)$ be non empty and T be such a transversal. We show that $T \in \mathsf{G}_3$. Let $T = \{x, y\}$. If possible suppose $T \cap X \neq \emptyset$. Without loss of generality let

$x \in X$. Then either $y \in B_1 \cap B_2$ or $y \notin B_1 \cap B_2$. But in both the cases, $T \in \mathsf{G}_1 \cup \mathsf{G}_2$, a contradiction. Hence T is disjoint from X but $T \in \mathcal{F}^{\top}$. So T is a a blocking set of $\mathcal{G}$ and intersects both B_1 and B_2. Hence $T \in \mathsf{G}_3$. □

Lemma B.4.4. *If $2 \leq |X| \leq k-2$, then $\mathcal{F}$ has at most $k^2 - 2k + 3$ transversals.*

`Proof` : Suppose $|X| = m$. We need to estimate G_1, G_2 and G_3 of Lemma B.4.3. There are $m(k-n)$ transversals in G_1. We use the inclusion-exclusion principle to estimate the number of transversals in G_2. There are at most $mn - 1$ transversals in G_2. Consider the family of sets which consists of the blocks of the form $B \setminus X$, where $B \in \mathcal{G}$. Call it $\mathcal{G}_{\bar{X}}$. By Lemma B.4.1 $\mathcal{G}_{\bar{X}}$ is a family of $(k-m)-$sets. We observe that each $T \in \mathsf{G}_3$ is a minimal blocking set of $\mathcal{G}_{\bar{X}}$. Hence there are at most $(k-m)^2$ transversals in G_3. Therefore there are at most $m(k-n) + mn - 1 + (k-m)^2$ transversals. The result follows since $\max\{k^2 - mk + m^2 - 1 : 2 \leq m \leq k-2\} = k^2 - 2k + 3$. □

Lemma B.4.5. *If $|X| = k$, then $\mathcal{F}$ has at most $k^2 - k + 1$ transversals.*

`Proof` : Since $|X| = k$ then by using Lemma B.4.1 we have $\mathcal{G} = \{X\}$ and (say) $\mathcal{A} := \{B_1, B_2, X\} \subset \mathcal{F}$. But $\mathrm{tr}(\mathcal{A}) = 2$ and $\mathcal{F}$ is a TMIF$(k, 2)$, hence $\mathcal{F} = \mathcal{A}$. To count the number of transversals we need to estimate G_1, G_2 and G_3 of Lemma B.4.3. But G_3 is empty. There are $k(k-n)$ transversals in G_1 and there are at most n^2 transversals in G_2. Therefore there are at most $k(k-n) + n^2$ transversals. The result follows since $\max\{k(k-n) + n^2 : 1 \leq n \leq k-1\} = k^2 - k + 1$. □

Lemma B.4.6. *If $|X| = k-1$, then for $k \geq 4$, $\mathcal{F}$ has at most $k^2 - k - 1$ transversals. Also for $k = 3$, $\mathcal{F}$ has at most 6 transversals.*

`Proof` : Since $|X| = k-1$, $\mathcal{G}$ contains at least two blocks. Suppose B and C are two distinct blocks in $\mathcal{G}$. Since $\mathrm{tr}(\mathcal{F} \setminus \{C\}) = 1$, $B \cap B_1 \cap B_2 \neq \emptyset$. By a similar reasoning $C \cap B_1 \cap B_2 \neq \emptyset$. We recall from Lemma B.4.2 that X is disjoint from $B_1 \cap B_2$. Therefore by using Lemma B.4.1 we can assume $B \cap B_1 \cap B_2 = \{p\}$ and $C \cap B_1 \cap B_2 = \{q\}$ and (say) $\mathcal{A} := \{B_1, B_2, X \sqcup \{p\}, X \sqcup \{q\}\} \subset \mathcal{F}$. But $\mathrm{tr}(\mathcal{A}) = 2$ and $\mathcal{F}$ is a TMIF$(k, 2)$, hence $\mathcal{F} = \mathcal{A}$. To count the number of transversals we need to estimate G_1, G_2 and G_3 of Lemma B.4.3. There are $(k-1)(k-n)$ transversals in G_1 and there is exactly one transversal namely $\{p, q\}$ in G_3. We observe that if $|X \cap (B_1 \cup B_2)| \geq 3$, then by using the inclusion-exclusion principle G_2 has at most $n(k-1) - 2$ transversals otherwise G_2 has exactly $2n - 1$ transversals. In the respective cases $\mathcal{F}$ has at most $(k-1)(k-n) + n(k-1) - 2 + 1$ and at most

$(k-1)(k-n)+2n-1+1$ transversals. The result follows since

$$\begin{aligned}\max\{k(k-1)-1, k^2-(n+1)k+3n : 1 \le n \le k-1\} \\ = \max\{k^2-k-1, k^2-2k+3\} \\ = \begin{cases} k^2-k-1 & \text{if} \quad k \ge 4 \\ 6 & \text{if} \quad k = 3. \end{cases}\end{aligned}$$

□

Theorem B.4.7. *Let $k \ge 3$ be a positive integer. Any* TMIF$(k,2)$ *has at most k^2-k+1 transversals. Moreover, $\mathbb{F}(k,2)$ is the unique* TMIF$(k,2)$ *which has k^2-k+1 transversals.*

`Proof` : The first part is a direct consequence of Lemma B.4.4, Lemma B.4.5 and Lemma B.4.6. It also shows that any TMIF$(k,2)$ which has k^2-k+1 transversals contains 3 blocks and satisfies an *intersecting pattern* 1 and $k-1$. (i.e. for each different B_1, $B_2 \in \mathcal{F}$ the size of $B_1 \cap B_2$ is either 1 or $k-1$.) Hence the uniqueness part follows. □

Theorem B.4.8. *Let $k \ge 4$ be a positive integer. Any* TMIF$(k,2)$*, which is not isomorphic to $\mathbb{F}(k,2)$, has at most k^2-k-1 transversals.*

`Proof` : This is a direct consequence of Theorem B.4.7, Lemma B.4.4 and Lemma B.4.6.

□

Theorem B.4.9. *Let $k \ge 3$ be an integer. Any finite intersecting family of $k-$sets with transversal size 2 has at most k^2-k+1 transversals. Moreover, $\mathbb{F}(k,2)$ is the unique intersecting family of $k-$sets which has k^2-k+1 transversals.*

`Proof` : Let $\mathcal{F}$ be an intersecting family of $k-$sets with k^2-k+1 transversals. Let $\mathcal{A}$ be a TMIF$(k,2)$ and it is a subfamily of $\mathcal{F}$. Since $\mathcal{F}^\top \subset \mathcal{A}^\top$, $\mathcal{A}$ has at least k^2-k+1 transversals. Then by Theorem B.4.7 $\mathcal{A}$ is isomorphic to $\mathbb{F}(k,2)$. Hence by using Theorem 5.2.6, we have $\mathcal{A}$ is a CIF$(k,2)$. If possible, suppose $B \in \mathcal{F} \setminus \mathcal{A}$. Then by the closure property of $\mathcal{A}$ there exists $T \in \mathcal{A}^\top$ disjoint from B. So $\mathcal{F}$ has at most $|\mathcal{A}^\top|-1 = k^2-k$ transversals, a contradiction. Therefore $\mathcal{A} = \mathcal{F}$. □

Acknowledgement. The author would like to thank Mr. Satyaki Mukherjee for bringing the Colouring Algorithm to his attention.

[1] Aart Blokhuis, More on maximal intersecting families of finite sets, *Journal of Combinatorial Theory* Series A **44** (1987), No. 2, 299–303.

[2] Béla Bollobás, On generalized graphs, *Acta Mathematica Academiae Scientiarum Hungaricae* **16** (1965), 447–452.

[3] Endre Boros, Zoltán Füredi and Jeff Kahn, Maximal intersecting families and affine regular polygons in $\mathrm{PG}(2, q)$, *Journal of Combinatorial Theory* Series A **52** (1989), No. 1, 1–9.

[4] Shuya Chiba, Michitaka Furuya, Ryota Matsubara and Masanori Takatou, Covers in 4-uniform intersecting families with covering number three, *Tokyo Journal of Mathematics* **35** (2012), No. 1, 241–251.

[5] Stephen J. Dow, David A. Drake, Zoltán Füredi and Jean A. Larson, A lower bound for the cardinality of a maximal family of mutually intersecting sets of equal size, Proceedings of the sixteenth South Eastern international conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1985), Vol. 48, 1985, 47–48.

[6] David A. Drake and Sharad S. Sane, Maximal intersecting families of finite sets and $n-$uniform Hjelmslev planes, *Proceedings of the American Mathematical Society* **86** (1982), No. 2, 358–362.

[7] Paul Erdős and László Lovász, Problems and results on $3-$chromatic hypergraphs and some related questions, *Infinite and finite sets* (Proceedings of a Colloquium held at Keszthely from June 25 to July 1, 1973. Dedicated to Paul Erdős on his 60th birthday), Volume-II, North-Holland, Amsterdam, 1975, pp. 609–627. Colloquia Mathematica Societatis János Bolyai, Volume-10.

[8] Péter Frankl, Katsuhiro Ota and Norihide Tokushige, Uniform intersecting families with covering number four, *Journal of Combinatorial Theory* Series A **71** (1995), No. 1, 127–145.

[9] ——, Covers in uniform intersecting families and a counterexample to a conjecture of Lovász, *Journal of Combinatorial Theory* Series A **74** (1996), No. 1, 33–42.

[10] ——, Uniform intersecting families with covering number restrictions, *Combinatorics, Probability and Computing* **7** (1998), No. 1, 47–56.

[11] Zoltán Füredi, On maximal intersecting families of finite sets, *Journal of Combinatorial Theory* Series A **28** (1980), No. 3, 282–289.

[12] Michitaka Furuya and Masanori Takatou, Covers in 5-uniform intersecting families with covering number three, *Australasian Journal of Combinatorics* **55** (2013), 249–262.

[13] Denis Hanson and Bjarne Toft, On the maximum number of vertices in $n-$uniform cliques, *Ars Combinatoria* **16** (1983), No. A, 205–216.

[14] Michael A. Henning and Anders Yeo, Transversals and matchings in $3-$uniform hypergraphs, *European Journal of Combinatorics* **34** (2013), No. 2, 217–228.

[15] Daniel R. Hughes and Fred C. Piper, Projective planes. Graduate Texts in Mathematics, Vol. 6, *Springer-Verlag, New York-Berlin* (1973).

[16] Jeff Kahn, On a problem of Erdős and Lovász. II. $n(r) = O(r)$, *Journal of the American Mathematical Society* **7** (1994), No. 1, 125–143.

[17] László Lovász, On minimax theorems of combinatorics, *Matematikai Lapok* **26** (1975), No. 3-4, 209–264

[18] David Lubell, A short proof of Sperner's lemma. *Journal of Combinatorial Theory* **1** (1966) 299.

[19] Kaushik Majumder, On the maximum number of points in a maximal intersecting family of finite sets, *Preprint*, arXiv: 1402.7158 (2014).(Online published in *Combinatorica* with DOI: 10.1007/s00493-015-3275-8 on June 24, 2015.)

[20] ——, Closed Intersecting Families of finite sets and their applications, *Preprint*, arXiv:1411.1480 (2014).

[21] ——, Classification of maximal intersecting families of sets with size three, *Preprint*.

[22] Kaushik Majumder and Satyaki Mukherjee, A note on the transversal size of a series of families constructed over Cycle Graph, *Preprint*, arXiv: 1501.02178 (2015).

[23] Nithya Sai Narayana and Sharad Sane, On the maximal clique problem, *Journal of Combinatorics, Information & System Sciences* **34** (2009), Nos. 1–4, 233–240.

[24] George N. Raney, Functional composition patterns and power series reversion, *Transactions of the American Mathematical Society* **94** (1960), 441–451.

[25] Zsolt Tuza, Critical hypergraphs and intersecting set-pair systems, *Journal of Combinatorial Theory* Series B **39** (1985), No. 2, 134–145.