Infinite Color Urn Models

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Contents

Chapter 1

Introduction

In recent years, there has been a wide variety of work on *random reinforcement models* of various kinds [26, 55, 47, 41, 5, 34, 48, 54, 14, 22, 25, 23, 44, 20, 17]. Urn models form an important class of random reinforcement models, with numerous applications in engineering and informatics [60, 42, 50] and bioscience [19, 30, 31, 5, 44]. In recent years there have been several works on different kinds of urn models and their generalizations [41, 5, 34, 14, 25, 23, 45, 20, 44, 17]. For *occupancy urn models*, where one considers recursive addition of balls into finite or infinite number of boxes, there are some works which introduce models with infinitely many colors, typically represented by the boxes [29, 37, 39].

As observed in [51], the earliest mentions of urn models are in the post-Renaissance period in the works of Huygen, de Moivre, Laplace and other noted mathematicians and scientists. The rigorous study of urn models began with the seminal work of Pólya [57, 56], where he introduced the model to study the spread of infectious diseases. We will refer to this model as the classical Pólya urn model. Since then, various types of urn schemes with finitely many colors have been widely studied in literature [36, 35, 3, 4, 53, 38, 40, 41, 5, 34, 14, 15, 25, 18, 17]. See [54] for an extensive survey of the known results. However, other than the classical work by Blackwell and MacQueen [13], there has not been much development of infinite color generalization of the Pólya urn scheme. In this thesis, we introduce and analyze a new Pólya type urn scheme with countably infinite number of colors.

1.1 Model description

A generalized Pólya urn model with finitely many colors can be described as follows:

> Consider an urn containing finitely many balls of different colors. At any time $n \geq 1$, a ball is selected uniformly at random from the urn, the color of the selected ball is noted, the selected ball is returned to the urn along with a set of balls of various colors which may depend on the color of the selected ball.

The goal is to study the asymptotic properties of the configuration of the urn. Suppose there are $K \geq 1$, different colors and we denote the configuration of the urn at time n by $U_n = (U_{n,1}, U_{n,2} \ldots, U_{n,K})$, where $U_{n,j}$ denotes the number of balls of color j, $1 \leq j \leq K$. The dynamics of the urn model depend on the *replacement policy*. The replacement policy can be described by a $K \times K$ matrix, say R with non negative entries. The (i, j)-th entry of R is the number of balls of color j which are to be added to the urn if the selected color is i. In literature, R is termed as the *replacement matrix*. Let Z_n denote the random color of the ball selected at the $(n + 1)$-th draw. The dynamics of the model can then be written as

$$U_{n+1} = U_n + R_{Z_n} \tag{1.1.1}$$

where R_{Z_n} is the Z_n-th row of the replacement matrix R.

A replacement matrix is said to be *balanced*, if the row sums are constant. In this case, after every draw a constant number of balls are added to the urn. For such an urn, a standard technique is to divide each entry of the replacement matrix by the constant row sum, thus without loss of generality, one may assume that the row sums are all 1, that is, the replacement matrix is a stochastic matrix. In that case, it is also customary to assume U_0 to be a probability distribution on the set of colors, which is to be interpreted as the probability distribution of the selected color of the first ball drawn from the urn. Note that, in this case the entries of $U_n = (U_{n,1}, U_{n,2} \ldots, U_{n,K})$ are no longer the number of balls of different colors, instead the entries of $U_n/\,(n + 1)$ are the proportion of balls of different colors. We will refer to it as the (random) configuration of the urn. It is useful to note here that the random probability mass function $U_n/(n+1)$ represents the probability distribution of the random color of the $(n+1)$-th selected ball given the n-th configuration of the urn. In other words, if Z_n is the color of the ball selected at the $(n+1)$-th draw, then,

$$\mathbb{P}\left(Z_n = i \,\middle|\, U_0, U_1, \ldots, U_n\right) = \frac{U_{n,i}}{n+1}, \ \ 1 \leq i \leq K. \tag{1.1.2}$$

Since R is a stochastic matrix and U_0 a probability distribution on the set of colors, we can now consider a Markov chain on the set of colors with transition matrix R and initial distribution U_0. We call such a chain, a chain associated with the urn model and vice-versa. In other words, given a balanced urn model we can associate with it a Markov chain on the set of colors and conversely, given a Markov chain there is an associated urn model with colors indexed by the state space.

1.1.1 Urn models with infinitely many colors

The above formulation can now be easily generalized for infinitely many colors. Let the colors be indexed by a finite or countably infinite set S, and the replacement matrix R, be a stochastic matrix suitably indexed by S. Let $U_n := (U_{n,v})_{v \in S} \in [0,\infty)^S$, where $U_{n,v}$ is the weight of the v-th color in the urn after n-th draw. In other words,

$$\mathbb{P}\left((n+1)-\text{th selected ball has color } v \,\middle|\, U_n, U_{n-1}, \cdots, U_0\right) \propto U_{n,v}, \ v \in S. \tag{1.1.3}$$

Starting with U_0 as a probability vector, the dynamics of $(U_n)_{n \geq 0}$ is defined through the following recursion

$$U_{n+1} = U_n + \chi_{n+1} R \tag{1.1.4}$$

where $\chi_{n+1} = (\chi_{n+1,v})_{v \in S}$ is such that $\chi_{n+1,Z_n} = 1$ and $\chi_{n+1,u} = 0$ if $u \neq Z_n$, where Z_n is the random color chosen from the configuration U_n. In other words,

$$U_{n+1} = U_n + R_{Z_n}$$

where R_{Z_n} is the Z_n-th row of the matrix R.

It is important to note here that in general $U_{n,v}$ is not necessarily an integer. When S is infinite, $U_{n,v}$ can be made an integer after suitable multiplication only for certain restrictive cases. One such case is when each row of R is finitely supported with all entries rational. However, as discussed in Section 1.1, $U_n/\left(n+1\right)$ will always denote the proportion of balls of various colors.

When S is infinite we will call such a process an *urn model with infinitely many colors*. The associated Markov chain is on the state space S, with transition probability matrix R and initial distribution U_0. As observed in the finite color case, in general, the random probability mass function $U_n/\left(n+1\right)$ represents the probability distribution of the random color of the $(n+1)$-th selected ball given the n-th configuration of the urn. Recall that Z_n denotes the $(n+1)$-th selected color. Thus for any $v \in S$,

$$\mathbb{P}\left(Z_n = v \,\Big|\, U_n, U_{n-1}, \cdots, U_0\right) = \frac{U_{n,v}}{n+1}, \tag{1.1.5}$$

which implies

$$\mathbb{P}\left(Z_n = v\right) = \frac{\mathbb{E}\left[U_{n,v}\right]}{n+1}. \tag{1.1.6}$$

In other words, the distribution of Z_n is given by the expected proportion of the colors at time n. It is worthwhile to note here, (1.1.3) and (1.1.4) imply that $\left(\frac{U_n}{n+1}\right)_{n\geq 0}$ is a time inhomogeneous Markov chain with state space as the set of all probability measures on S.

It is to be noted here, that we write all vectors as row vectors, unless otherwise stated.

1.2 Motivation

Our main motivations to study such a process have been twofold. It is known in the literature [38, 40, 14, 15, 25], that the asymptotic properties of a finite color urn depend on the qualitative properties of the underlying Markov chain. For example, for an irreducible aperiodic chain with K colors, it is shown in [38, 40] that, as $n \to \infty$,

$$\frac{U_{n,j}}{n+1} \longrightarrow \pi_j \text{ a.s.} \tag{1.2.1}$$

for all $1 \leq j \leq K$, where $\pi = (\pi_j)_{1 \leq j \leq K}$ is the unique stationary distribution of the associated Markov chain. It is also known [40, 41] that if the chain is reducible and j is a transient state then, as $n \to \infty$,

$$\frac{U_{n,j}}{n+1} \longrightarrow 0 \ \text{ a.s.} \tag{1.2.2}$$

Further non-trivial scalings have been derived for the reducible case [40, 41, 14, 15, 25]. So one may conclude that asymptotic properties of an urn model depend on the recurrence/transience of the underlying states. We want to investigate this relation when there are infinitely many colors. In [7], we studied the infinite color model, with colors indexed by $\mathbb{Z}^d$, where R is the transition matrix of a bounded increment random walk on $\mathbb{Z}^d$. The bounded increment random walks on $\mathbb{Z}^d$, is a rich class of examples of Markov chains on infinite states covering both the transient and null recurrent cases. Needless to state, that for the finite color case, the associated Markov chain can posses no null recurrent state. As we shall see later, our study will indicate a significantly different phenomenon for the infinite color urn models associated with the bounded increment random walks on $\mathbb{Z}^d$. In fact, we shall show that the asymptotic configuration is approximately Gaussian, irrespective of whether the underlying walk is transient or recurrent.

Another motivation comes from the work of Blackwell and MacQueen [13], where the authors introduced a possibly infinite color generalization of the Pólya urn scheme. In fact, their generalization even allowed uncountably many colors; the set of colors typically taken as some Polish space. The model then describes a process whose limiting distribution is the *Ferguson distribution* [12, 13], also known as the *Dirichlet process prior* in the Bayesian statistics literature [33]. The replacement mechanism in [13] is a simple diagonal scheme, that is, it reinforces only the chosen color. As in the classical finite color Pólya urn scheme, where R is the identity matrix, this leads to *exchangeable* sequence of colors. We complement the work of [13], by considering replacement mechanisms with non-zero off-diagonal entries. We would like to point out that due to the presence of off-diagonal entries in the replacement matrix, our models do not exhibit exchangeability and hence the techniques used to study our model are entirely different and new. We will present a coupling of the urn model with the associated Markov chain, which will be our most effective method in analyzing the urn models introduced in this work.

1.2.1 A simple but useful example

We present a simple example to motivate the study of urn models with infinitely many colors. Let the colors be indexed by $\mathbb{N} \cup \{0\}$, and $U_0 = \delta_0$. The replacement matrix R is given by

$$R(i,j) = \begin{cases} 1 & \text{if } j = i+1 \text{ for all } i,j \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.3}$$

Note that R has non-zero off diagonal entries. Since R is given by (1.2.3), after every draw a new color is introduced with positive probability. Hence, even though the urn contains only finitely many colors after every draw, we require to index the set of colors by the infinite set $\mathbb{N} \cup \{0\}$, to define the process.

The associated Markov chain $(X_n)_{n\geq 0}$ on the state space $\mathbb{N} \cup \{0\}$, with transition matrix R given by (1.2.3), is a deterministic chain always moving one step to the right. Therefore, we call this Markov chain the *right shift* and the corresponding urn process $(U_n)_{n\geq 0}$ as the *urn model associated with the right shift*. Note that, the right shift is trivially a transient chain.

Theorem 1.2.1. *Consider an urn model $(U_n)_{n\geq 0}$ associated with the right shift, such that the process starts with a single ball of color* 0. *If Z_n denotes the $(n+1)$-th selected color then, as $n \to \infty$,*

$$\frac{Z_n - \log n}{\sqrt{\log n}} \Rightarrow N(0,1). \tag{1.2.4}$$

To prove Theorem 1.2.1 we use the following lemma.

Lemma 1.2.1. *Let $(I_j)_{j\geq 1}$ be a sequence of independent Bernoulli random variables with $\mathbb{E}[I_j] = \frac{1}{j+1}$, $j \geq 1$. If $\tau_n = \sum_{j=1}^{n} I_j$, and $\tau_0 \equiv 0$, then as $n \to \infty$,*

$$\frac{\tau_n - \log n}{\sqrt{\log n}} \Rightarrow N\left(0,1\right). \tag{1.2.5}$$

Proof. $\mathbb{E}[\tau_n] = \sum_{j=1}^{n} \mathbb{E}[I_j] = \sum_{j=1}^{n} \frac{1}{j+1}$. Therefore,

$$\mathbb{E}[\tau_n] \sim \log n, \text{ as } n \to \infty, \tag{1.2.6}$$

where for any two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of positive real numbers, we write $a_n \sim b_n$ to denote $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$.

Similar calculations show that

$$s_n^2 = \mathrm{Var}\left(\tau_n\right) = \textstyle\sum_{j=1}^n \frac{1}{j+1} - \frac{1}{(j+1)^2} \sim \log n, \text{ as } n \to \infty. \tag{1.2.7}$$

Since I_j can possibly take only two values, namely $0,$ and 1, so for any $\epsilon > 0$, we have

$$\frac{1}{s_n^2} \sum_{j=1}^{n} \mathbb{E}\left[\left|I_j - \frac{1}{j+1}\right|^2 1_{\{|I_j - \frac{1}{j+1}| > \epsilon s_n\}}\right] \longrightarrow 0, \text{ as } n \to \infty.$$

Therefore, the Lindeberg-Feller Central Limit theorem (see page 129 of [28]) implies that as $n \to \infty$,

$$\frac{\tau_n - \mathbb{E}\left[\tau_n\right]}{\sqrt{\log n}} \Rightarrow N\left(0, 1\right), \tag{1.2.8}$$

since the variance of τ_n is given by (1.2.7). Observe that,

$$\frac{\tau_n - \log n}{\sqrt{\log n}} = \frac{\tau_n - \mathbb{E}\left[\tau_n\right]}{\sqrt{\log n}} + \frac{\mathbb{E}\left[\tau_n\right] - \log n}{\sqrt{\log n}}.$$

It is easy to see that $\mathbb{E}\left[\tau_n\right] - \log n = \sum_{j=1}^n \frac{1}{j+1} - \log n \longrightarrow \gamma - 1$, as $n \to \infty$, where γ is the Euler's constant (see page 192 of [2]). Therefore, from (1.2.8) and Slutsky's theorem (see page 105 of [28]), we get as $n \to \infty$,

$$\frac{\tau_n - \log n}{\sqrt{\log n}} \Rightarrow N\left(0, 1\right). \tag{1.2.9}$$

$\square$

Proof of Theorem (1.2.1). As observed earlier in (1.1.6), we know that

$$\mathbb{P}\left(Z_n = v\right) = \frac{\mathbb{E}\left[U_{n,v}\right]}{n+1}. \tag{1.2.10}$$

Therefore, the moment generating function $\mathbb{E}\left[e^{\lambda Z_n}\right]$ for Z_n is given by

$$\frac{1}{n+1} \sum_{j \in \mathbb{N} \cup \{0\}} \mathbb{E}\left[U_{n,j}\right] e^{\lambda j}, \text{ for } \lambda \in \mathbb{R}. \tag{1.2.11}$$

For every $\lambda \in \mathbb{R}$, let $x\left(\lambda\right) = \left(e^{\lambda j}\right)_{j \in \mathbb{N} \cup \{0\}}^T$. It is easy to see that e^λ and $x\left(\lambda\right)$ satisfy

$$Rx\left(\lambda\right) = e^\lambda x\left(\lambda\right),$$

where the equality holds coordiante-wise. Define the vector scalar product

$$U_n x(\lambda) = \sum_{j \in \mathbb{N} \cup \{0\}} U_{n,j} e^{\lambda j}. \tag{1.2.12}$$

From (1.1.4) and (1.1.6), we know that

$$\mathbb{E}\left[\chi_n \middle| U_{n-1}\right] = \frac{U_{n-1}}{n}.$$

Therefore, it follows that

$$\mathbb{E}\left[U_n x(\lambda) \middle| U_{n-1}\right] = \left(1 + \frac{e^\lambda}{n}\right) U_{n-1} x(\lambda).$$

This implies,

$$\frac{1}{n+1}\mathbb{E}\left[U_n x(\lambda)\right] = \frac{1}{n+1} \sum_{j \in \mathbb{N} \cup \{0\}} \mathbb{E}\left[U_{n,j}\right] e^{\lambda j} \quad = \quad \frac{1}{n+1}\left(1 + \frac{e^\lambda}{n}\right) \mathbb{E}\left[U_{n-1}\right] x(\lambda).$$

Repeating the same iteration, we obtain

$$\begin{aligned} \frac{1}{n+1}\mathbb{E}\left[U_n x(\lambda)\right] &= \frac{1}{n+1} \prod_{j=1}^{n} \left(1 + \frac{e^\lambda}{j}\right) \\ &= \prod_{j=1}^{n+1} \left(1 - \frac{1}{j+1} + \frac{e^\lambda}{j+1}\right). \end{aligned} \tag{1.2.13}$$

Observe that the right hand side of (1.2.13) gives the moment generating function of τ_n, where τ_n is as in Lemma 1.2.1. Note that τ_n is a non-negative random variable for every $n \geq 0$. Therefore, from Theorem 1 on page 430 of [32] it follows that for all $n \geq 0$,

$$Z_n \stackrel{d}{=} \tau_n, \tag{1.2.14}$$

where for any two random variables X and Y, the notation $X \stackrel{d}{=} Y$ denotes that X and Y have the same distribution. Hence (1.2.5) implies (1.2.4). This completes the proof. □

Later, in Chapters 2 and 5, we will further improve the representation (1.2.14) for urn models with general replacement matrices. This representation is new and will serve as the key tool in deriving the asymptotic properties of the urns with more general replacement matrices.

1.3 Outline and brief sketch of the results

The rest of this work is broadly divided into five chapters. The first three chapters, Chapters 2, 3 and 4 discuss the various asymptotic properties of the urn models associated with bounded increment random walks on $\mathbb{Z}^d$. Chapters 5 and 6 consider urn models with general replacement matrices.

1.3.1 Central limit theorems for the urn models associated with random walks

Based on [7] and [8], in Chapter 2, we study an urn process when $S = \mathbb{Z}^d$, and R is the transition matrix of a bounded increment random walk on $\mathbb{Z}^d$. This is a novel generalization of the Pólya urn scheme, which combines perhaps the two most classical models in probability theory, namely the urn model and the random walk. We prove central limit theorems for the random color of the n-th selected ball and show that, irrespective of the null recurrent or transient behavior of the underlying random walks, the asymptotic distribution is Gaussian after appropriate centering and scaling. In fact, we show that the order of any non-zero centering is always $\mathcal{O}(\log n)$ and the scaling is $\mathcal{O}\left(\sqrt{\log n}\right)$. In this chapter, we also prove Berry-Essen type bounds and show that the rate of convergence of the central limit theorem is of the order $\mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right)$.

1.3.2 Local limit theorems for the urn models associated with random walks

In Chapter 3, we further consider urn models associated with bounded increment random walks on $\mathbb{Z}^d$. In this chapter, we obtain finer asymptotes for the distribution of the randomly selected color. We derive the local limit theorems for the probability mass function of the randomly selected color, [7].

1.3.3 Large deviation principle for the urn models associated with random walks

Based on [8], in Chapter 4, we study further asymptotic properties of urn models associated with bounded increment random walks on $\mathbb{Z}^d$. Here, we show that for the expected configuration a *large deviation principle (LDP)* holds with a *good rate function* and *speed* $\log n$. Moreover, we prove that the rate function is the same as the rate function for the large deviation of the random walk sampled at random stopping times, where the stopping times follow Poisson distribution with mean 1.

1.3.4 Representation theorem

In Chapter 5 we consider general urn models. Here S may be any countable set and the replacement matrix R is any stochastic matrix suitably indexed by S. For this model, we present a *representation theorem* (see Theorem 5.1.1), [6]. This theorem provides a coupling of the marginal distribution of the randomly selected color with the associated Markov chain, sampled at independent, but random times. We show some immediate applications of the representation theorem by rederiving a few known results for finite color urn models.

1.3.5 General replacement matrices

In Chapter 6, based on [6], we consider urn models with infinite but countably many colors and general replacement matrices. In this chapter, we consider several different types of general replacement matrices and apply the representation theorem to deduce the asymptotic properties of the corresponding urn models. In Section 6.1, we consider an urn model with an irreducible, aperiodic R. If R is positive recurrent, with a stationary distribution π, then we show that the distribution of the randomly selected color converges to π. In Sections 6.2 and 6.3, we further generalize the model studied in Chapter 2. Here, we study urn models associated with general random walks, not necessarily with bounded increments, and derive the central limit theorem for the randomly selected color.

1.4 Notations

We mostly follow notations and conventions that are standard in the literature of urn models. For the sake of completeness, we provide a list below.

- As mentioned earlier, for any two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of positive real numbers, we will write $a_n \sim b_n$, if $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$.

- As mentioned earlier, all vectors are written as row vectors, unless otherwise stated. For $x \in \mathbb{R}^d$, we write $x = \left(x^{(1)}, x^{(2)}, \ldots, x^{(d)}\right)$ where $x^{(i)}$ denotes the i^{th} coordinate. The infinite dimensional vectors are written as $y = (y_j)_{j \in \mathcal{J}}$ where y_j is the j^{th} coordinate and $\mathcal{J}$ is the indexing set. To be consistent, column vectors are denoted by x^T, where x is a row vector.

- For any vector x, x^2 will denote a vector with the coordinates squared. That is, if $x = (x_j)_{j \in \mathcal{J}}$ for some indexing set $\mathcal{J}$, then

$$x^2 = \left(x_j^2\right)_{j \in \mathcal{J}}.$$

- The inner product of any two row vectors x and y is denoted by $\langle x, y \rangle$.

- The symbol $\mathbb{I}_d$ will denote the $d \times d$ identity matrix.

- By $N_d(\mu, \Sigma)$ we denote the d-dimensional Gaussian distribution with mean vector $\mu \in \mathbb{R}^d$, and variance-covariance matrix Σ. For $d = 1$, we simply write $N(\mu, \sigma^2)$, where $\sigma^2 > 0$.

- The standard Gaussian measure on $\mathbb{R}^d$ will be denoted by Φ_d. Its density ϕ_d is given by

$$\phi_d(x) := \frac{1}{(2\pi)^{d/2}} e^{-\frac{\|x\|^2}{2}}, x \in \mathbb{R}^d.$$

  For $d = 1$, we will simply write Φ for the standard Gaussian measure on $\mathbb{R}$ and ϕ for its density.

- The symbol $\Rightarrow$ will denote weak convergence of probability measures.

- The symbol $\xrightarrow{p}$ will denote convergence in probability.

- For any two random variables/vectors X and Y, we will write $X \stackrel{d}{=} Y$, to denote that X and Y have the same distribution.

Chapter 2

Central limit theorems for the urn models associated with random walks[1]

The main focus in this chapter is to study the urn models associated with bounded increment random walks on $\mathbb{Z}^d$, $d \geq 1$. Urn models associated with more general random walks are discussed in Section 6.2 of Chapter 6. Here, we derive the central limit theorems for the randomly selected colors, and its rate of convergence. In Section 2.4, we will further generalize the model when the associated random walk takes values in general d-dimensional discrete lattices.

Let $(Y_j)_{j\geq 1}$ be i.i.d. random vectors taking values in $\mathbb{Z}^d$ with probability mass function $p(u) := \mathbb{P}(Y_1 = u), u \in \mathbb{Z}^d$. We assume that the distribution of Y_1 is bounded, that is there exists a non-empty finite subset $B \subset \mathbb{Z}^d$, such that $p(u) = 0$ for all $u \notin B$. We shall always write

$$\begin{aligned} \mu &:= \mathbb{E}[Y_1] \\ \Sigma = ((\sigma_{ij}))_{1\leq i,j\leq d} &:= \mathbb{E}\left[Y_1^T Y_1\right] \\ e(\lambda) &:= \mathbb{E}\left[e^{\langle \lambda, Y_1\rangle}\right], \lambda \in \mathbb{R}^d. \end{aligned} \tag{2.0.1}$$

It is easy to see that Σ is a positive semi definite matrix, that is, for all $a \in \mathbb{R}^d$,

$$a\Sigma a^T = \mathbb{E}\left[\left(aY_1^T\right)^2\right] \geq 0.$$

[1]This chapter is based on the papers entitled "Pólya Urn Schemes with Infinitely Many Colors" [7] and " Rate of Convergence and Large Deviation for the Infinite Color Pólya Urn Schemes" [8]. The paper [8] is published in *Statistics and Probability Letters, Vol. 92, 2014*.

Observe that

$$\Sigma = D + \mu\mu^T,$$

where D is the the variance-covariance matrix of Y_1.

In this chapter, we assume that Σ is positive definite. This will hold, if and only if, the set B contains d linearly independent vectors. Later, in Section 6.2 we will relax the assumption that Σ is positive definite. The matrix $\Sigma^{1/2}$ will denote the unique *positive definite square root* of Σ, that is, $\Sigma^{1/2}$ is a positive definite matrix such that $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$. When the dimension $d = 1$, we will denote the mean and second moment (and not the variance) of Y_1 simply by μ and σ^2 respectively, that is

$$\begin{aligned} \mu &:= \mathbb{E}\left[Y_1\right] \\ \sigma^2 &:= \mathbb{E}\left[Y_1^2\right]. \end{aligned} \tag{2.0.2}$$

In that case we assume $\sigma^2 > 0$.

Let $S_n := Y_0 + Y_1 + \cdots + Y_n, n \geq 0$, be the random walk on $\mathbb{Z}^d$ starting at Y_0 and with increments $(Y_j)_{j\geq 1}$ which are independent. Needless to say, that $(S_n)_{n\geq 0}$ is a Markov chain, with initial distribution given by the distribution of Y_0 and the transition matrix

$$R := ((p\,(v-u)))_{u,v\in\mathbb{Z}^d}. \tag{2.0.3}$$

For the rest of this chapter, we consider the urn process $(U_n)_{n\geq 0}$, with replacement matrix given by (2.0.3). Since the associated Markov chain is a random walk, we will call the urn process as the *urn process associated with a bounded increment random walk.* The urn model associated with the right shift as discussed in Subsection 1.2.1 of Chapter 1, is an example of an urn model associated with a bounded increment random walk, which as discussed earlier, is a deterministic walk always moving one step to the right.

We would like to note here that this model is a further generalization of a subclass of models studied in [20], namely the class of *linearly reinforced models*. In [20], the authors prove that for such models the number of balls of each color grows to infinity. As we will see in the next section, our results will not only show that the number of balls of each color grows to infinity, but will also provide the exact rates of their growths.

2.1 Central limit theorem for the expected configuration

We present in this subsection the central limit theorem for the randomly selected color, [7]. The centering and scaling of the central limit theorem (Theorem 2.1.1) are of the order $\mathcal{O}(\log n)$ and $\mathcal{O}\left(\sqrt{\log n}\right)$ respectively. Such centering and scalings are available because the marginal distribution of the randomly selected color behaves like that of a delayed random walk, where the delay is of the order $\mathcal{O}(\log n)$, see Theorem 2.1.2.

For simplicity, in Sections 2.1 and 2.2 we will assume that the initial configuration of the urn consists of a single ball of color 0, that is, $U_0 = \delta_0$. We will see at the end of Section 2.2 (Remark 2.2.1) that this assumption can be easily removed.

Theorem 2.1.1. *Let $\overline{\Lambda}_n$ be the probability measure on $\mathbb{R}^d$ corresponding to the probability vector $\left(\frac{\mathbb{E}[U_{n,v}]}{n+1}\right)_{v\in\mathbb{Z}^d}$, and let*

$$\overline{\Lambda}_n^{cs}(A) := \overline{\Lambda}_n\left(\sqrt{\log n} A \Sigma^{1/2} + \mu \log n\right),$$

where A is a Borel subset of $\mathbb{R}^d$. Then, as $n \to \infty$,

$$\overline{\Lambda}_n^{cs} \Rightarrow \Phi_d. \tag{2.1.1}$$

If Z_n denotes the $(n+1)$-th selected color then, its probability mass function is given by $\left(\frac{\mathbb{E}[U_{n,v}]}{n+1}\right)_{v\in\mathbb{Z}^d}$. Thus $\overline{\Lambda}_n$ is the probability distribution of Z_n, and $\overline{\Lambda}_n^{cs}$ is the distribution of the scaled and centered random vector $\frac{Z_n-\mu\log n}{\sqrt{\log n}}$. So the following result is a restatement of (2.1.1).

Corollary 2.1.1. *Consider the urn model associated with the random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d,\ d \geq 1$, then as $n \to \infty$,*

$$\frac{Z_n - \mu \log n}{\sqrt{\log n}} \Rightarrow N_d(0, \Sigma). \tag{2.1.2}$$

We begin by constructing a martingale which we will be need in the proof of Theorem 2.1.1.

Define $\Pi_n(z) = \prod_{j=1}^{n}\left(1 + \frac{z}{j}\right)$ for $z \in \mathbb{C}$. It is known from Euler product formula for gamma function, which is also referred to as Gauss' formula (see page 178 of [21]), that

$$\lim_{n\to\infty} \frac{\Pi_n(z)}{n^z}\Gamma(z+1) = 1, \tag{2.1.3}$$

where the convergence is uniform on compact subsets of $\mathbb{C} \setminus \{-1, -2, \ldots\}$.

Recall for every $\lambda \in \mathbb{R}^d$, $e(\lambda) := \sum_{v \in B} e^{\langle \lambda, v \rangle} p(v)$ is the moment generating function of Y_1. Define $x(\lambda) := \left(e^{\langle \lambda, v \rangle}\right)^T_{v \in \mathbb{Z}^d}$. It is easy to see that

$$Rx(\lambda) = e(\lambda)\, x(\lambda),$$

where the equality holds coordinate-wise.

Let $\mathcal{F}_n = \sigma(U_j \colon 0 \leq j \leq n),\, n \geq 0$, be the natural filtration. Define

$$\overline{M}_n(\lambda) := \frac{U_n x(\lambda)}{\Pi_n(e(\lambda))}.$$

From the fundamental recursion (1.1.4), we get,

$$U_{n+1} x(\lambda) = U_n x(\lambda) + \chi_{n+1} R x(\lambda).$$

Thus,

$$\mathbb{E}\left[U_{n+1} x(\lambda) \middle| \mathcal{F}_n\right] = U_n x(\lambda) + e(\lambda)\, \mathbb{E}\left[\chi_{n+1} x(\lambda) \middle| \mathcal{F}_n\right] = \left(1 + \tfrac{e(\lambda)}{n+1}\right) U_n x(\lambda).$$

Therefore, $\overline{M}_n(\lambda)$ is a non-negative martingale for every $\lambda \in \mathbb{R}^d$. In particular, $\mathbb{E}\left[\overline{M}_n(\lambda)\right] = \overline{M}_0(\lambda)$. We now present a representation of the marginal distribution of Z_n, in terms of the increments $(Y_j)_{j \geq 1}$, where the distribution of Y_j is given by $p(\cdot)$ for every $j \geq 1$.

Theorem 2.1.2. *For each $n \geq 1$,*

$$Z_n \stackrel{d}{=} Z_0 + \sum_{j=1}^{n} I_j Y_j, \tag{2.1.4}$$

where $(I_j)_{j \geq 1}$ are independent random variables such that $I_j \sim Bernoulli\left(\frac{1}{j+1}\right), j \geq 1$ and are independent of $(Y_j)_{j \geq 1}$; and Z_0 is a random vector taking values in $\mathbb{Z}^d$ distributed according to the probability vector U_0 and is independent of $\left((I_j)_{j \geq 1}; (Y_j)_{j \geq 1}\right)$.

The representation in (2.1.4) is interesting and non-trivial, as it necessarily demonstrates that the marginal distribution of the randomly selected color behaves like a delayed random walk.

Proof. As noted before, the probability mass function for Z_n is $\left(\frac{\mathbb{E}[U_{n,v}]}{n+1}\right)_{v \in \mathbb{Z}^d}$. So, for $\lambda \in \mathbb{R}^d$, the moment generating function of Z_n is given by

$$\begin{aligned}
\frac{1}{n+1}\sum_{v\in\mathbb{Z}^d} e^{\langle\lambda,v\rangle}\mathbb{E}\left[U_{n,v}\right] &= \frac{\Pi_n\left(e(\lambda)\right)}{n+1}\mathbb{E}\left[\overline{M}_n(\lambda)\right] \\
&= \frac{\Pi_n\left(e(\lambda)\right)}{n+1}\overline{M}_0(\lambda) && (2.1.5) \\
&= \overline{M}_0(\lambda)\prod_{j=1}^{n}\left(1-\frac{1}{j+1}+\frac{e(\lambda)}{j+1}\right). && (2.1.6)
\end{aligned}$$

The right hand side of (2.1.6) is the moment generating function of $Z_0+\sum_{j=1}^{n} I_j Y_j$. This proves (2.1.4). □

Proof of Theorem 2.1.1. Since the initial configuration of the urn consists of a single ball of color 0, that is, $U_0 = \delta_0$, hence $Z_0 \equiv 0$. It follows from (2.1.4) that

$$Z_n \stackrel{d}{=} \sum_{j=1}^{n} I_j Y_j. \tag{2.1.7}$$

Now, we observe that,

$$\mathbb{E}\left[\sum_{j=1}^{n} I_j Y_j\right] - \mu\log n = \sum_{j=1}^{n}\frac{1}{j+1}\mu - \mu\log n \longrightarrow (\gamma-1)\,\mu, \tag{2.1.8}$$

where γ is the Euler's constant.

Case I: Let $d=1$. Let $s_n^2 = \mathrm{Var}\left(\sum_{j=1}^{n} I_j Y_j\right)$. It is easy to note that

$$s_n^2 = \sum_{j=1}^{n}\frac{1}{j+1}\mathbb{E}\left[Y_1^2\right] - \frac{\mu^2}{(j+1)^2} \sim \sigma^2\log n.$$

The cardinality of B is finite, so for any $\epsilon>0$, we have

$$\frac{1}{s_n^2}\sum_{j=1}^{n}\mathbb{E}\left[\left|I_j Y_j - \frac{\mu}{j+1}\right|^2 1_{\{|I_j Y_j - \frac{\mu}{j+1}|>\epsilon s_n\}}\right] \longrightarrow 0 \text{ as } n\to\infty.$$

Therefore, by the Lindeberg Central Limit theorem (see page 129 of [28]), we conclude that as $n\to\infty$,

$$\frac{Z_n-\mu\log n}{\sigma\sqrt{\log n}} \Rightarrow N(0,1).$$

This completes the proof in this case.

Case II: Now suppose $d\geq 2$. Let $\Sigma_n := \left((\sigma_{k,l}(n))\right)_{d\times d}$ denote the variance-covariance matrix for $\sum_{j=1}^{n} I_j Y_j$. Then by calculations similar to those in one-dimension, it is easy to see that for all $k, l \in \{1, 2, \ldots d\}$,

$$\frac{\sigma_{k,l}(n)}{\sigma_{k,l}\,\log n} \longrightarrow 1 \text{ as } n \to \infty.$$

Therefore, for every $\theta \in \mathbb{R}^d$, by Lindeberg Central Limit Theorem in one dimension,

$$\frac{\langle \theta, \sum_{j=1}^{n} I_j Y_j \rangle - \langle \theta, \mu \log n \rangle}{\sqrt{\log n\, (\theta \Sigma \theta^T)}} \Rightarrow N(0,1) \text{ as } n \to \infty.$$

Now using Cramer-Wold device (see Theorem 29.4 on page 383 of [11]), it follows that as $n \to \infty$,

$$\frac{\sum_{j=1}^{n} I_j Y_j - \mu \log n}{\sqrt{\log n}} \Rightarrow N_d\left(0, \Sigma\right).$$

So we conclude that, as $n \to \infty$,

$$\frac{Z_n - \mu \log n}{\sqrt{\log n}} \Rightarrow N_d\left(0, \Sigma\right).$$

This completes the proof. □

The following corollary is an immediate consequence of Corollary 2.1.1 .

Corollary 2.1.2. *Consider the urn model associated with the simple symmetric random walk on* $\mathbb{Z}^d, d \geq 1$*. Then, as* $n \to \infty$*,*

$$\frac{Z_n}{\sqrt{\log n}} \Rightarrow N_d(0, d^{-1}\mathbb{I}_d),$$

where $\mathbb{I}_d$ *is the* $d \times d$ *identity matrix.*

The above result essentially shows that irrespective of the recurrent or transient behavior of the under lying random walk, the associated urn models have similar asymptotic behavior. In particular, the limiting distribution is always Gaussian with universal centering and scaling orders, namely, $\mathcal{O}\left(\log n\right)$ and $\mathcal{O}\left(\sqrt{\log n}\right)$ respectively.

2.2 Weak convergence of the random configuration

In this section we will present an asymptotic result for the random configuration of the urn. Let $\mathcal{M}_1$ be the space of probability measures on $\mathbb{R}^d,\ d \geq 1$, endowed with the topology of weak convergence. Let Λ_n be the random probability measure on $\mathbb{Z}^d \subset \mathbb{R}^d$ corresponding to the random probability vector $\frac{U_n}{n+1}$. It is easy to see that Λ_n is measurable.

Theorem 2.2.1. *Consider the random measure*

$$\Lambda_n^{cs}\left(A\right) = \Lambda_n\left(\sqrt{\log n} A \Sigma^{1/2} + \mu \log n\right),$$

for any Borel subset A of $\mathbb{R}^d$. Then, as $n \to \infty$,

$$\Lambda_n^{cs} \xrightarrow{p} \Phi_d \text{ on } \mathcal{M}_1. \tag{2.2.1}$$

We note that Theorem 2.2.1 is a stronger version of Theorem 2.1.1, as the later follows from the former by taking expectation.

We first present the results required to prove Theorem 2.2.1. We have already introduced the martingales $\left(\overline{M}_n\left(\cdot\right)\right)_{n\geq 0}$ in Section 2.1. The next theorem states that on a non-trivial closed subset of $\mathbb{R}^d$ with 0 in its interior, the martingales $\left(\overline{M}_n\left(\lambda\right)\right)_{n\geq 0}$ are uniformly (in λ) L_2-bounded.

Theorem 2.2.2. *There exists $\delta > 0$, such that*

$$\sup_{\lambda \in [-\delta,\delta]^d} \sup_{n\geq 1} \mathbb{E}\left[\overline{M}_n^2\left(\lambda\right)\right] < \infty. \tag{2.2.2}$$

Proof. From (1.1.4), we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(U_{n+1}x\left(\lambda\right)\right)^2 \Big| \mathcal{F}_n\right] &= \left(U_n x\left(\lambda\right)\right)^2 + 2e\left(\lambda\right) U_n x\left(\lambda\right) \mathbb{E}\left[\chi_{n+1}x\left(\lambda\right) \Big| \mathcal{F}_n\right] \\
&\quad + e^2\left(\lambda\right) \mathbb{E}\left[\left(\chi_{n+1}x\left(\lambda\right)\right)^2 \Big| \mathcal{F}_n\right].
\end{aligned}$$

It is easy to see that,

$$\mathbb{E}\left[\chi_{n+1}x\left(\lambda\right) \Big| \mathcal{F}_n\right] = \frac{1}{n+1} U_n x\left(\lambda\right),$$

and

$$\mathbb{E}\left[\left(\chi_{n+1}x\left(\lambda\right)\right)^2 \Big| \mathcal{F}_n\right] = \frac{1}{n+1} U_n x\left(2\lambda\right).$$

Therefore, we get the recursion

$$\mathbb{E}\left[\left(U_{n+1}x\left(\lambda\right)\right)^2\right] = \left(1+\frac{2e\left(\lambda\right)}{n+1}\right)\mathbb{E}\left[\left(U_n x\left(\lambda\right)\right)^2\right] + \frac{e^2\left(\lambda\right)}{n+1}\mathbb{E}\left[U_n x\left(2\lambda\right)\right]. \tag{2.2.3}$$

Let us write

$$\Pi^2_{n+1}\left(e\left(\lambda\right)\right) := \prod_{j=1}^{n+1}\left(1+\frac{e\left(\lambda\right)}{j}\right)^2.$$

Dividing both sides of (2.2.3) by $\Pi^2_{n+1}\left(e\left(\lambda\right)\right)$,

$$\mathbb{E}\left[\overline{M}^2_{n+1}\left(\lambda\right)\right] = \frac{\left(1+\frac{2e(\lambda)}{n+1}\right)}{\left(1+\frac{e(\lambda)}{n+1}\right)^2}\mathbb{E}\left[\overline{M}^2_n\left(\lambda\right)\right] + \frac{e^2\left(\lambda\right)}{n+1}\times\frac{\mathbb{E}\left[U_n x\left(2\lambda\right)\right]}{\Pi^2_{n+1}\left(e\left(\lambda\right)\right)}. \tag{2.2.4}$$

The sequence $\left(\overline{M}_n\left(2\lambda\right)\right)_{n\geq 0}$ being a martingale, we obtain $\mathbb{E}\left[U_n x\left(2\lambda\right)\right] = \Pi_n\left(e\left(2\lambda\right)\right)\overline{M}_0\left(2\lambda\right)$. Therefore, from (2.2.4), we get

$$\begin{aligned}\mathbb{E}\left[\overline{M}^2_n\left(\lambda\right)\right] &= \frac{\Pi_n\left(2e\left(\lambda\right)\right)}{\Pi_n\left(e\left(\lambda\right)\right)^2}\overline{M}^2_0\left(\lambda\right) \\ &\quad + \sum_{k=1}^{n}\frac{e^2\left(\lambda\right)}{k}\left\{\prod_{j>k}^{n}\frac{\left(1+\frac{2e(\lambda)}{j}\right)}{\left(1+\frac{e(\lambda)}{j}\right)^2}\right\}\frac{\Pi_{k-1}\left(e\left(2\lambda\right)\right)}{\Pi^2_k\left(e\left(\lambda\right)\right)}\overline{M}_0\left(2\lambda\right). \end{aligned} \tag{2.2.5}$$

Recall that $e\left(\lambda\right) = \sum_{v\in B} e^{\langle\lambda,v\rangle}p(v)$. We observe that, as $e\left(\lambda\right) > 0$, so $\frac{1+\frac{2e(\lambda)}{j}}{\left(1+\frac{e(\lambda)}{j}\right)^2} < 1$ and hence $\frac{\Pi_n(2e(\lambda))}{\Pi^2_n(e(\lambda))} < 1$. Thus,

$$\mathbb{E}\left[\overline{M}^2_n\left(\lambda\right)\right] \leq \overline{M}^2_0\left(\lambda\right) + e^2\left(\lambda\right)\overline{M}_0\left(2\lambda\right)\sum_{k=1}^{n}\frac{1}{k}\frac{\Pi_{k-1}\left(e\left(2\lambda\right)\right)}{\Pi^2_k\left(e\left(\lambda\right)\right)}. \tag{2.2.6}$$

Using (2.1.3), we know that

$$\Pi^2_n\left(e\left(\lambda\right)\right) \sim \frac{n^{2e(\lambda)}}{\Gamma^2\left(e\left(\lambda\right)+1\right)}. \tag{2.2.7}$$

Note that $e\left(0\right) = 1$, and $e\left(\lambda\right)$ is continuous as a function of λ. So given $\eta > 0$, there exists $0 < K_1, K_2 < \infty$, such that for all $\lambda \in [-\eta,\eta]^d$, $K_1 \leq e\left(\lambda\right) \leq K_2$. Since the convergence in (2.1.3) is uniform on compact subsets of $[0,\infty)$, given $\epsilon > 0$, there exists $N_1 > 0$, such that for all $n \geq N_1$ and $\lambda \in [-\eta,\eta]^d$,

$$\left(1-\epsilon\right)\frac{\Gamma^2\left(e\left(\lambda\right)+1\right)}{\Gamma\left(e\left(2\lambda\right)+1\right)}\sum_{k\geq N_1}^{n}\frac{1}{k^{1+2e(\lambda)-e(2\lambda)}}$$

$$\leq \sum_{k \geq N_1}^{n} \frac{1}{k} \frac{\Pi_{k-1}\left(e\left(2\lambda\right)\right)}{\Pi_k^2\left(e\left(\lambda\right)\right)}$$
$$\leq (1+\epsilon) \frac{\Gamma^2\left(e\left(\lambda\right)+1\right)}{\Gamma\left(e\left(2\lambda\right)+1\right)} \sum_{k \geq N_1}^{n} \frac{1}{k^{1+2e(\lambda)-e(2\lambda)}}.$$

Since the set B is finite, $\lambda \mapsto e\left(\lambda\right)$, and $\lambda \mapsto 2e\left(\lambda\right) - e\left(2\lambda\right)$ are clearly continuous functions, that take value 1 at $\lambda = 0$. Hence there exists a $\delta_0 > 0$, such that $\min_{\lambda \in [-\delta_0, \delta_0]^d} 2e\left(\lambda\right) - e\left(2\lambda\right) > 0$. Choose $\delta = \min\{\eta, \delta_0\}$ and $\lambda_0 \in [-\delta, \delta]^d$ so that $\min_{\lambda \in [-\delta, \delta]^d} 2e\left(\lambda\right) - e\left(2\lambda\right) = 2e\left(\lambda_0\right) - e\left(2\lambda_0\right) > 0$. Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{k^{1+2e(\lambda)-e(2\lambda)}} \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+2e(\lambda_0)-e(2\lambda_0)}}.$$

Now find $N_2 > 0$, such that $\forall \lambda \in [-\delta, \delta]^d$,

$$\sum_{k > N_2}^{\infty} \frac{1}{k^{1+2e(\lambda)-e(2\lambda)}} \leq \sum_{k > N_2}^{\infty} \frac{1}{k^{1+2e(\lambda_0)-e(2\lambda_0)}} < \epsilon.$$

The functions $\frac{\Gamma^2(e(\lambda)+1)}{\Gamma(e(2\lambda)+1)}$, $e^2\left(\lambda\right)$ and $\overline{M}_0\left(2\lambda\right)$ are continuous in λ. Therefore, these are bounded on $[-\delta, \delta]^d$. Choose $N = \max\{N_1, N_2\}$. From (2.2.6), we obtain for all $n \geq N$,

$$\mathbb{E}\left[\overline{M}_n^2\left(\lambda\right)\right] \leq \overline{M}_0^2\left(\lambda\right) + C_1 \sum_{k=1}^{N} \frac{1}{k} \frac{\Pi_{k-1}\left(e\left(2\lambda\right)\right)}{\Pi_k^2\left(e\left(\lambda\right)\right)} + C_2\epsilon \tag{2.2.8}$$

for appropriate positive constants C_1, C_2.

The functions $\sum_{k=1}^{N} \frac{1}{k} \frac{\Pi_{k-1}(e(2\lambda))}{\Pi_k^2(e(\lambda))}$ and $\overline{M}_0^2\left(\lambda\right)$ are continuous in λ, and hence bounded on $[-\delta, \delta]^d$. Therefore, from (2.2.8) we obtain that there exists $C > 0$, such that for all $\lambda \in [-\delta, \delta]^d$ and for all $n \geq 1$,

$$\mathbb{E}\left[\overline{M}_n^2\left(\lambda\right)\right] \leq C.$$

This proves (2.2.2). □

Lemma 2.2.1. *Let δ be as in Theorem 2.2.2, then for every $\lambda \in [-\delta, \delta]^d$, as $n \to \infty$,*

$$\overline{M}_n\left(\frac{\lambda}{\sqrt{\log n}}\right) \xrightarrow{p} 1. \tag{2.2.9}$$

Proof. Since $U_0 = \delta_0$, from equation (2.2.5), we get

$$\mathbb{E}\left[\overline{M}_n^2(\lambda)\right] = \frac{\Pi_n(2e(\lambda))}{\Pi_n^2(e(\lambda))} + \frac{\Pi_n(2e(\lambda))}{\Pi_n^2(e(\lambda))} \sum_{k=1}^{n} \frac{e^2(\lambda)}{k} \frac{\Pi_{k-1}(e(2\lambda))}{\Pi_k(2e(\lambda))}.$$

Replacing λ by $\lambda_n = \frac{\lambda}{\sqrt{\log n}}$, we obtain,

$$\mathbb{E}\left[\overline{M}_n^2(\lambda_n)\right] = \frac{\Pi_n(2e(\lambda_n))}{\Pi_n^2(e(\lambda_n))} + \frac{\Pi_n(2e(\lambda_n))}{\Pi_n^2(e(\lambda_n))} \sum_{k=1}^{n} \frac{e^2(\lambda_n)}{k} \frac{\Pi_{k-1}(e(2\lambda_n))}{\Pi_k(2e(\lambda_n))}. \tag{2.2.10}$$

We observe that

$$\lim_{n\to\infty} e(\lambda_n) = 1. \tag{2.2.11}$$

Since the convergence in formula (2.1.3) is uniform on compact sets of $[0, \infty)$, using (2.2.11) we observe that for $\lambda \in [-\delta, \delta]^d$, and every fixed k

$$\lim_{n\to\infty} \frac{\Pi_n(2e(\lambda_n))}{\Pi_n^2(e(\lambda_n))} = \frac{\Gamma^2(2)}{\Gamma(3)} = \frac{1}{2}. \tag{2.2.12}$$

Using (2.2.11) and (2.2.12), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Pi_n(2e(\lambda_n))}{\Pi_n^2(e(\lambda_n))} \frac{e^2(\lambda_n)}{k} \frac{\Pi_{k-1}(e(2\lambda_n))}{\Pi_k(2e(\lambda_n))} &= \frac{1}{2}\frac{1}{k}\frac{\Pi_{k-1}(1)}{\Pi_k(2)} \\
&= \frac{1}{(k+2)(k+1)}.
\end{aligned}$$

Now using Theorem 2.2.2 and the dominated convergence theorem, we get

$$\lim_{n\to\infty} \frac{\Pi_n(2e(\lambda_n))}{\Pi_n^2(e(\lambda_n))} \sum_{k=1}^{n} \frac{e^2(\lambda_n)}{k} \frac{\Pi_{k-1}(e(2\lambda_n))}{\Pi_k(2e(\lambda_n))} = \sum_{k=1}^{\infty} \frac{1}{(k+2)(k+1)} = \frac{1}{2}.$$

Therefore, from (2.2.10) we obtain

$$\mathbb{E}\left[\overline{M}_n^2(\lambda_n)\right] \longrightarrow 1 \text{ as } n \to \infty. \tag{2.2.13}$$

Observing that $\mathbb{E}\left[\overline{M}_n(\lambda_n)\right] = 1$, we get

$$\text{Var}\left(\overline{M}_n(\lambda_n)\right) \to 0, \text{ as } n \to \infty. \tag{2.2.14}$$

This implies

$$\overline{M}_n(\lambda_n) \xrightarrow{p} 1 \text{ as } n \to \infty,$$

completing the proof of the lemma. $\square$

We will now present an elementary but technical result (Theorem 2.2.3) which we will use in the proof of Theorem 2.2.1. It is really a generalization of the classical result for Laplace transform, namely, Theorem 22.2 of [11], a slightly weaker version appears as Theorem 5 in [43]. However, in the proof of Theorem 2.2.3 presented below, some of the arguments are similar to that of Theorem 5 in [43]. It is important to note here that though Theorem 5 of [43] is stated for $d = 2$, the author in [43] observes at the beginning of Section 3 in [43] that similar result can be obtained for any dimensions $d \geq 1$.

Theorem 2.2.3. *Let ν_n be a sequence of probability measures on $\left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$ and let $m_n(\cdot)$ be the corresponding moment generating functions. Suppose there exists $\delta > 0$, such that $m_n(\lambda) \longrightarrow e^{\frac{\|\lambda\|^2}{2}}$, as $n \to \infty$, for every $\lambda \in [-\delta, \delta]^d \cap \mathbb{Q}^d$, then, as $n \to \infty$,*

$$\nu_n \Rightarrow \Phi_d. \tag{2.2.15}$$

Proof. Let $\delta' \in \mathbb{Q}$, such that $0 < \delta' < \delta$. Observe that for every $a > 0$,

$$\nu_n \left(\left([-a, a]^d \right)^c \right) \leq \sum_{i=1}^{d} e^{-\delta' a} \left(m_n(-\delta' e_i) + m_n(\delta' e_i) \right), \tag{2.2.16}$$

where $(e_i)_{i=1}^d$ are the d-unit vectors. Now for our assumption, we get, $m_n(\delta' e_i) \to e^{\frac{\delta'^2}{2}}$ and $m_n(-\delta' e_i) \to e^{\frac{\delta'^2}{2}}$ as $n \to \infty$, for every $1 \leq i \leq d$. Thus, we get

$$\sup_{n \geq 1} \nu_n \left(\left([-a, a]^d \right)^c \right) \longrightarrow 0 \text{ as } a \to \infty.$$

So the sequence of probability measures $(\nu_n)_{n \geq 1}$ is tight. Therefore, Helly selection theorem (see Theorem 2 on page 270 of [32]) implies that for every subsequence $(n_k)_{k \geq 1}$ there exists a further subsequence $\left(n_{k_j}\right)_{j \geq 1}$ and a probability measure ν such that as $j \to \infty$,

$$\nu_{n_{k_j}} \Rightarrow \nu. \tag{2.2.17}$$

We will show that

$$m_{n_{k_j}}(\lambda) \longrightarrow m_\infty(\lambda), \;\; \forall \; \lambda \in (-\delta, \delta)^d \cap \mathbb{Q}^d \tag{2.2.18}$$

where m_∞ is the moment generating function of ν. It follows from Theorem 5 of [43] that to prove (2.2.18) it is enough to show that for $1 \leq i \leq d$, and for any $|\lambda_i| < \delta$

$$\lim_{a \to \infty} e^{|\lambda_i| a} \sup_{j \geq 1} \nu_{n_{k_j}, i} \left(([-a, a])^c \right) = 0, \tag{2.2.19}$$

where $\nu_{n_{k_j},i}$ denotes the i-th dimensional marginal for $\nu_{n_{k_j}}$.

For any $|\lambda_i| < \delta$, we can choose $\delta' \in \mathbb{Q}$, such that $0 < |\lambda_i| < \delta' < \delta$. Observe that a calculation similar to (2.2.16) implies that for any $a > 0$, and for the chosen δ'

$$\nu_{n_{k_j},i}\left(([-a,a])^c\right) \leq e^{-\delta' a}\left(\left(m_{n_{k_j}}(-\delta' e_i) + m_{n_{k_j}}(\delta' e_i)\right)\right). \tag{2.2.20}$$

Now using the assumption $m_n(\lambda) \longrightarrow e^{\frac{\|\lambda\|^2}{2}}$, as $n \to \infty$, for every $\lambda \in [-\delta,\delta]^d \cap \mathbb{Q}^d$, we obtain,

$$e^{|\lambda_i|a} \sup_{j\geq 1} \nu_{n_{k_j},i}\left(([-a,a])^c\right) \leq K e^{(|\lambda_i|-\delta')a}, \tag{2.2.21}$$

for an appropriate constant $K > 0$. This proves (2.2.19) and hence, (2.2.18) holds. But from our assumption

$$m_{n_{k_j}}(\lambda) \to e^{\frac{\|\lambda\|^2}{2}}, \;\; \forall\, \lambda \in [-\delta,\delta]^d \cap \mathbb{Q}^d.$$

So, we conclude that

$$m_\infty(\lambda) = e^{\frac{\|\lambda\|^2}{2}}, \;\; \forall\, \lambda \in (-\delta,\delta)^d \cap \mathbb{Q}^d.$$

Since both sides of the above identity are continuous functions on their respective domains, we get that $m_\infty(\lambda) = e^{\frac{\|\lambda\|^2}{2}}$ for every $\lambda \in (-\delta,\delta)^d$. We know from (21.22) and Theorem 30.1 of [11] for $d = 1$, the standard Gaussian is characterized by the values of its moment generating function in an open neighborhood of 0. For $d \geq 2$, we can conclude that the standard Gaussian distribution is characterized by the values of its moment generating function in an open neighborhood of 0, by using Theorem 30.1 of [11] and the Cramer-Wold device. So we conclude that every sub-sequential limit is standard Gaussian. This proves (2.2.15). □

Proof of Theorem 2.2.1. Λ_n is the random probability measure on $\mathbb{Z}^d \subset \mathbb{R}^d$, corresponding to the random probability vector $\frac{U_n}{n+1}$. That is, for any Borel subset A of $\mathbb{R}^d$,

$$\Lambda_n(A) = \frac{1}{n+1}\sum_{v\in A} U_{n,v}.$$

For $\lambda \in \mathbb{R}^d$, the corresponding moment generating function is given by

$$\frac{1}{n+1}\sum_{v\in\mathbb{Z}^d} e^{\langle\lambda,v\rangle} U_{n,v} = \frac{1}{n+1} U_n x(\lambda) = \frac{1}{n+1}\overline{M}_n(\lambda)\,\Pi_n(e(\lambda)). \tag{2.2.22}$$

The moment generating function corresponding to the scaled and centered random measure Λ_n^{cs} is

$$\sum_{v\in\mathbb{Z}^d} e^{\langle\lambda, \frac{v-\mu\log n}{\sqrt{\log n}}\Sigma^{-1/2}\rangle}\,\frac{U_{n,v}}{n+1}$$

$$= \frac{1}{n+1} e^{-\langle \lambda, \mu\sqrt{\log n}\Sigma^{-1/2}\rangle} U_n x \left(\frac{\lambda\Sigma^{-1/2}}{\sqrt{\log n}}\right) \tag{2.2.23}$$

$$= \frac{1}{n+1} e^{-\langle \lambda, \mu\sqrt{\log n}\Sigma^{-1/2}\rangle} \overline{M}_n \left(\frac{\lambda\Sigma^{-1/2}}{\sqrt{\log n}}\right) \Pi_n \left(e\left(\frac{\lambda\Sigma^{-1/2}}{\sqrt{\log n}}\right)\right). \tag{2.2.24}$$

To show (2.2.1), it is enough to show that for every subsequence $(n_k)_{k\geq 1}$, there exists a further subsequence $\left(n_{k_j}\right)_{j\geq 1}$ such that, as $j \to \infty$,

$$\frac{e^{-\langle \lambda, \mu\sqrt{\log n_{k_j}}\rangle}}{n_{k_j}+1} \overline{M}_{n_{k_j}} \left(\frac{\lambda}{\sqrt{\log n_{k_j}}}\right) \Pi_{n_{k_j}} \left(e\left(\frac{\lambda}{\sqrt{\log n_{k_j}}}\right)\right) \longrightarrow e^{\frac{\lambda\Sigma\lambda^T}{2}} \tag{2.2.25}$$

for all $\lambda \in [-\delta, \delta]^d$, a.s., where δ is as in Theorem 2.2.2. From Theorem 2.1.1, we know that

$$\frac{Z_n - \mu \log n}{\sqrt{\log n}} \Rightarrow N_d\left(0, \Sigma\right) \text{ as } n \to \infty.$$

Therefore, using (2.1.6), we obtain as $n \to \infty$,

$$e^{-\langle \lambda, \mu\sqrt{\log n}\rangle} \mathbb{E}\left[e^{\langle \lambda, \frac{Z_n}{\sqrt{\log n}}\rangle}\right] = \frac{1}{n+1} e^{-\langle \lambda, \mu\sqrt{\log n}\rangle} \Pi_n \left(e\left(\frac{\lambda}{\sqrt{\log n}}\right)\right) \longrightarrow e^{\frac{\lambda\Sigma\lambda^T}{2}}.$$

Now using Theorem 2.2.3, it is enough to show (2.2.25) only for $\lambda \in \mathbb{Q}^d \cap [-\delta, \delta]^d$. This is equivalent to proving that for every $\lambda \in \mathbb{Q}^d \cap [-\delta, \delta]^d$, as $j \to \infty$,

$$\overline{M}_{n_{k_j}} \left(\frac{\lambda}{\sqrt{\log n_{k_j}}}\right) \longrightarrow 1 \text{ a.s.}$$

From Lemma 2.2.1 we know that for all $\lambda \in [-\delta, \delta]^d$

$$\overline{M}_n \left(\frac{\lambda}{\sqrt{\log n}}\right) \xrightarrow{p} 1 \text{ as } n \to \infty.$$

Therefore, using the standard diagonalization argument we can say that given a subsequence $(n_k)_{k\geq 1}$, there exists a further subsequence $\left(n_{k_j}\right)_{j\geq 1}$, such that for every $\lambda \in \mathbb{Q}^d \cap [-\delta, \delta]^d$,

$$\overline{M}_{n_{k_j}} \left(\frac{\lambda}{\sqrt{\log n_{k_j}}}\right) \longrightarrow 1 \text{ a.s.}$$

This completes the proof. □

Remark 2.2.1. *It is worth noting here that the proofs of Theorems 2.1.1 and 2.2.1 go through if we assume U_0 to be non random probability vector such that there exists $r > 0$, with*

$\sum_{v\in\mathbb{Z}^d} e^{\langle\lambda,v\rangle}U_{0,v} < \infty$, *whenever* $\|\lambda\| < r$. *Further, if* U_0 *is random and there exists* $r > 0$, *such that for any* $\|\lambda\| < r$,

$$\sum_{v\in\mathbb{Z}^d} e^{\langle\lambda,v\rangle}U_{0,v} < \infty, \text{ a.s.}$$

then (2.1.1) *and* (2.2.1) *holds a.s.with respect to* U_0.

2.3 Rate of convergence of the central limit theorem: the Berry-Essen bound

In this section, we obtain the rate of convergence for the central limit theorem as discussed in Theorem 2.1.1. We show that the rate of convergence is of the order $\mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right)$, by deducing the classical Berry-Essen type bound for any dimension $d \geq 1$. The results discussed in this section are available in [8].

2.3.1 Berry-Essen Bound for $d = 1$

We first consider the case when the associated random walk is a one dimensional walk and the set of colors are indexed by the set of integers $\mathbb{Z}$.

Theorem 2.3.1. *Suppose* $U_0 = \delta_0$, *then*

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{Z_n-\mu h_n}{\sqrt{n\rho_2}} \leq x\right) - \Phi\left(x\right)\right| \leq 2.75 \times \frac{\rho_3}{\sqrt{n}\rho_2^{3/2}} = \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right), \tag{2.3.1}$$

where $h_n := \sum_{j=1}^{n}\frac{1}{j+1}$, Φ *is the standard normal distribution function and*

$$\rho_2 := \frac{1}{n}\left(\sigma^2 h_n - \mu^2\sum_{j=1}^{n}\frac{1}{(j+1)^2}\right) \tag{2.3.2}$$

and

$$\rho_3 := \frac{1}{n}\left(\sum_{j=1}^{n}\frac{1}{j+1}\mathbb{E}\left[\left|Y_1 - \frac{\mu}{j+1}\right|^3\right] + |\mu|^3\sum_{j=1}^{n}\frac{j}{(j+1)^4}\right). \tag{2.3.3}$$

Proof. We first note that when $U_0 = \delta_0$, then (2.1.4) can be written as

$$Z_n \stackrel{d}{=} \sum_{j=1}^{n} I_j Y_j \tag{2.3.4}$$

where $(Y_j)_{j\geq 1}$ are i.i.d. increments of the random walk $(S_n)_{n\geq 0}$, $(I_j)_{j\geq 1}$ are independent Bernoulli variables such that $I_j \sim$ Bernoulli $\left(\frac{1}{j+1}\right)$ and are independent of $(Y_j)_{j\geq 1}$. Now observe that

$$n\rho_2 = \sum_{j=1}^{n} \mathbb{E}\left[\left(I_jY_j - \mathbb{E}\left[I_jY_j\right]\right)^2\right] \text{ and } n\rho_3 = \sum_{j=1}^{n} \mathbb{E}\left[\left|I_jY_j - \mathbb{E}\left[I_jY_j\right]\right|^3\right].$$

Thus from the *Berry-Essen Theorem* for the independent but non-identical increments (see Theorem 12.4 of [10]), we get

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\sum_{j=1}^{n} I_jY_j - \mu h_n}{\sqrt{n\rho_2}} \leq x \right) - \Phi\left(x\right) \right| \leq 2.75 \times \frac{\rho_3}{\sqrt{n}\rho_2^{3/2}}. \tag{2.3.5}$$

The identities (2.3.4) and (2.3.5) imply the bound in (2.3.1).

Finally to prove the last part of the equation (2.3.1), we note that from definition $n\rho_2 \sim C_1 \log n$, and $n\rho_3 \sim C_2 \log n$, where $0 < C_1, C_2 < \infty$, are some constants. Thus,

$$\frac{\rho_3}{\sqrt{n}\rho_2^{3/2}} = \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right).$$

This completes the proof of the theorem. □

The next result follows easily from the above theorem by observing the facts $h_n \sim \log n$, and $n\rho_2 \sim C_1 \log n$, where $C_1 > 0$ is a constant.

Theorem 2.3.2. *Suppose $U_{0,k} = 0$, for all but finitely many $k \in \mathbb{Z}$, then there exists a constant $C > 0$, such that*

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{Z_n - \mu \log n}{\sigma\sqrt{\log n}} \leq x \right) - \Phi\left(x\right) \right| \leq C \times \frac{\rho_3}{\sqrt{n}\rho_2^{3/2}} = \mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right), \tag{2.3.6}$$

Φ is the standard normal distribution function and ρ_2 and ρ_3 are as defined in (2.3.2) *and* (2.3.3) *respectively.*

It is worth noting that unlike in Theorem 2.3.1 the constant C, which appears in (2.3.6) above, is not a universal constant, it may depend on the increment distribution, as well as on U_0.

Proof. Observe that

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left(\frac{Z_n - \mu \log n}{\sigma\sqrt{\log n}} \leq x \right) - \Phi\left(x\right) \right| \leq \sup_{x\in\mathbb{R}} J_n(x) + \sup_{x\in\mathbb{R}} K_n(x),$$

where

$$J_n(x) = \left| \mathbb{P}\left(\frac{Z_n - \mu h_n}{\sqrt{n\rho_2}} \leq x_n \right) - \Phi(x_n) \right|,$$

and $x_n = \mu \frac{(\log n - h_n)}{\sqrt{n\rho_2}} + x\frac{\sigma\sqrt{\log n}}{\sqrt{n\rho_2}}$ and

$$K_n(x) = \left| \Phi\left(\mu \frac{(\log n - h_n)}{\sqrt{n\rho_2}} + x\frac{\sigma\sqrt{\log n}}{\sqrt{n\rho_2}} \right) - \Phi(x) \right|.$$

From Theorem 2.3.1, we observe that

$$\sup_{x\in\mathbb{R}} J_n(x) \leq 2.75 \times \frac{\rho_3}{\sqrt{n}\rho_2^{3/2}} = \mathcal{O}\left(\frac{1}{\sqrt{\log n}} \right). \tag{2.3.7}$$

For a suitable choice of $C_1 > 0$, we have

$$\begin{aligned} K_n(x) &= \left| \frac{1}{\sqrt{2\pi}} \int\limits_{x}^{\mu \frac{(\log n - h_n)}{\sqrt{n\rho_2}} + x\frac{\sigma\sqrt{\log n}}{\sqrt{n\rho_2}}} e^{\frac{-t^2}{2}} \, \mathrm{d}t \right| \\ &\leq C_1 e^{-\frac{x^2}{2}} \left| \mu \frac{(\log n - h_n)}{\sqrt{n\rho_2}} + x\frac{\sigma\sqrt{\log n}}{\sqrt{n\rho_2}} - x \right| \\ &\leq C_1 \left| \mu \frac{(\log n - h_n)}{\sqrt{n\rho_2}} \right| + C_1 e^{-\frac{x^2}{2}} |x| \left| \sigma \frac{\sqrt{\log n}}{\sqrt{n\rho_2}} - 1 \right| \end{aligned}$$

Observe that $h_n = \log n + \gamma + \epsilon_n$, where $\epsilon_n \longrightarrow 0$, as $n \to \infty$, and γ is the Euler constant. Also $\sqrt{n\rho_2} \sim \sqrt{\log n}$. Therefore, there exists a constant $C_2 > 0$, such that for all $n \in \mathbb{N}$,

$$C_1 \left| \mu \frac{(\log n - h_n)}{\sqrt{n\rho_2}} \right| \leq C_2 \frac{\rho_3}{\sqrt{n}\rho_2^{3/2}} = \mathcal{O}\left(\frac{1}{\sqrt{\log n}} \right). \tag{2.3.8}$$

Note that the function $e^{-\frac{x^2}{2}} |x|$ attains its maximum at $x = 1$. Therefore,

$$C_1 e^{-\frac{x^2}{2}} |x| \left| \sigma \frac{\sqrt{\log n}}{\sqrt{n\rho_2}} - 1 \right| \leq C_1 e^{-\frac{1}{2}} \left| \sigma \frac{\sqrt{\log n}}{\sqrt{n\rho_2}} - 1 \right|.$$

Since $|\sqrt{x} - 1| \leq \sqrt{|x - 1|}$ for all $x \in \mathbb{R}$, we obtain

$$C_1 e^{-\frac{1}{2}} \left| \sigma \frac{\sqrt{\log n}}{\sqrt{n\rho_2}} - 1 \right| \leq C_3 \sqrt{\left| \frac{\sigma^2 \log n - n\rho_2}{n\rho_2} \right|} \tag{2.3.9}$$

for an appropriate constant $C_3 > 0$. Observe that for some constant $C_4 > 0$,

$$\begin{aligned}
\frac{n\rho_2 - \sigma^2 \log n}{\sqrt{n\rho_2}} &= \frac{\sigma^2 \left(\sum_{j=1}^{n} \frac{1}{j+1} - \log n \right) - \mu^2 \sum_{j=1}^{n} \frac{1}{(j+1)^2}}{\sqrt{n\rho_2}} \\
&\leq C_4 \frac{\rho_3}{\sqrt{n}\rho_2^{3/2}} = \mathcal{O}\left(\frac{1}{\sqrt{\log n}} \right). \qquad (2.3.10)
\end{aligned}$$

Therefore, combining (2.3.8), (2.3.9) and (2.3.10) we can choose an appropriate constant $C > 0$, such that (2.3.6) holds. □

2.3.2 Berry-Essen bound for $d \geq 2$

Now, we consider the case when the associated random walk is two or higher dimensional and the colors are indexed by $\mathbb{Z}^d$. Before we present our main result, we introduce a few notations.

For a matrix $A = ((a_{ij}))_{1 \leq i,j \leq d}$ we denote by $A(i,j)$, the $(d-1) \times (d-1)$ sub-matrix of A, obtained by deleting the i-th row and j-th column. Let

$$\rho_2^{(d)} := \frac{1}{n} \sum_{j=1}^{n} \frac{1}{(j+1)} \frac{\det\left(\Sigma - \frac{1}{j+1} M \right)}{\det\left(\Sigma(1,1) - \frac{1}{j+1} M(1,1) \right)}, \qquad (2.3.11)$$

where $M := \left(\left(\mu^{(i)} \mu^{(j)} \right) \right)_{1 \leq i,j \leq d}$ and

$$\rho_3^{(d)} := \frac{1}{nd} \sum_{i=1}^{d} \gamma_n^3(i) \left(\sum_{j=1}^{n} \beta_j(i) \right), \qquad (2.3.12)$$

where

$$\gamma_n^2(i) := \max_{1 \leq j \leq n} \frac{\det\left(\Sigma(i,i) - \frac{1}{(j+1)} M(i,i) \right)}{\det\left(\Sigma(1,1) - \frac{1}{j+1} M(1,1) \right)}$$

and

$$\beta_j(i) = \frac{1}{j+1} \mathbb{E}\left[\left| Y_1^{(i)} - \frac{\mu^{(i)}}{j+1} \right|^3 \right] + \frac{j}{(j+1)^4} \left| \mu^{(i)} \right|^3.$$

For any two vectors x and $y \in \mathbb{R}^d$, we will write $x \leq y$, if the inequality holds coordinate wise.

Theorem 2.3.3. *Suppose $U_0 = \delta_0$, then there exists a universal constant $C(d) > 0$, which may* *depend on the dimension d, such that,*

$$\sup_{x\in\mathbb{R}^d}\left|\mathbb{P}\left(\left(Z_n-\mu h_n\right)\Sigma_n^{-1/2}\le x\right)-\Phi_d\left(x\right)\right|\le C\left(d\right)\frac{\rho_3^{(d)}}{\sqrt{n}\left(\rho_2^{(d)}\right)^{3/2}}=\mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right), \tag{2.3.13}$$

where $\Sigma_n := \sum_{j=1}^{n}\frac{1}{j+1}\left(\Sigma-\frac{1}{j+1}M\right)$ and Φ_d is the distribution function of a standard d-dimensional normal random vector.

Proof. As in the one dimensional case, we start by observing that when $U_0=\delta_0$, then (2.1.4) can be written as

$$Z_n \overset{d}{=} \sum_{j=1}^{n} I_j Y_j \tag{2.3.14}$$

where $\left(Y_j\right)_{j\ge1}$ are i.i.d. increments of the random walk $\left(S_n\right)_{n\ge0}$, $\left(I_j\right)_{j\ge1}$ are independent Bernoulli variables such that $I_j\sim$ Bernoulli $\left(\frac{1}{j+1}\right)$ and are independent of $\left(Y_j\right)_{j\ge1}$.

Now the proof of the inequality in (2.3.13) follows from equation (D) of [9] which deals with d-dimensional version of the classical Berry-Essen inequality for independent but non-identical summands, which in our case are the random variables $\left(I_jY_j\right)_{j\ge1}$. It is sufficient to observe that

$$\beta_j(i)=\mathbb{E}\left[\left|I_jY_1^{(i)}-\mathbb{E}\left[I_jY_j^{(i)}\right]\right|^3\right],$$

and

$$\Sigma_n=\sum_{j=1}^{n}\mathbb{E}\left[\left(I_jY_j-\mathbb{E}\left[I_jY_j\right]\right)^T\left(I_jY_j-\mathbb{E}\left[I_jY_j\right]\right)\right].$$

Finally, to prove the last part of the equation (2.3.13) as in the one dimensional case, we note that from definition $n\rho_2^{(d)}\sim C_1'\log n$ and $n\rho_3^{(d)}\sim C_2'\log n$, where $0<C_1',C_2'<\infty$, are some constants. Thus,

$$\frac{\rho_3^{(d)}}{\sqrt{n}\left(\rho_2^{(d)}\right)^{3/2}}=\mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right).$$

This completes the proof of the theorem. □

Remark 2.3.1. *If we define $\Sigma\left(1,1\right)=1$, and $M\left(1,1\right)=0$, when $d=1$, then Theorem 2.3.1 follows from the above theorem except in Theorem 2.3.1 the constant is more explicit.*

Just as in the one dimensional case, the following result follows easily from the above theorem by observing $h_n\sim\log n$.

Theorem 2.3.4. *Suppose $U_0=\left(U_{0,v}\right)_{v\in\mathbb{Z}^d}$, is such that $U_{0,v}=0$ for all but finitely many $v\in\mathbb{Z}^d$, then there exists a constant $C>0$, which may depend on the increment distribution,* *such that*

$$\sup_{x\in\mathbb{R}^d}\left|\mathbb{P}\left(\left(\frac{Z_n-\mu\log n}{\sqrt{\log n}}\right)\Sigma^{-1/2}\le x\right)-\Phi_d\left(x\right)\right|\le C\times\frac{\rho_3^{(d)}}{\sqrt{n}\left(\rho_2^{(d)}\right)^{3/2}}=\mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right), \tag{2.3.15}$$

where Φ_d is the distribution function of a standard d-dimensional normal random vector.

Proof. Observe that

$$\sup_{x\in\mathbb{R}^d}\left|\mathbb{P}\left(\left(\frac{Z_n-\mu\log n}{\sqrt{\log n}}\right)\Sigma^{-1/2}\le x\right)-\Phi_d\left(x\right)\right|\le\sup_{x\in\mathbb{R}^d}J_n(x)+\sup_{x\in\mathbb{R}^d}K_n(x),$$

where

$$J_n(x)=\left|\mathbb{P}\left(\left(Z_n-\mu h_n\right)\Sigma_n^{-1/2}\le x_n\right)-\Phi_d\left(x_n\right)\right|, \tag{2.3.16}$$

where $x_n=\mu\left(\log n-h_n\right)\Sigma_n^{-1/2}+x\sqrt{\log n}\Sigma^{1/2}\Sigma_n^{-1/2}$ and

$$K_n(x)=\left|\Phi_d\left(x_n\right)-\Phi_d\left(x\right)\right|. \tag{2.3.17}$$

It follows from Theorem 2.3.3, that

$$\sup_{x\in\mathbb{R}^d}J_n(x)\le C\left(d\right)\frac{\rho_3^{(d)}}{\sqrt{n}\left(\rho_2^{(d)}\right)^{3/2}}=\mathcal{O}\left(\frac{1}{\sqrt{\log n}}\right).$$

Further, writing $x_n:=\left(x_n^{(1)},x_n^{(2)},\ldots,x_n^{(d)}\right)$, we get

$$K_n(x)\le\sum_{i=1}^{d}\frac{1}{\sqrt{2\pi}}\left|\int_{x^{(i)}}^{x_n^{(i)}}e^{\frac{-t^2}{2}}\,\mathrm{d}t\right|.$$

Note that $\Sigma_n=h_n\Sigma-\left(\sum_{j=1}^{n}\frac{1}{(j+1)^2}\right)M$, so $h_n^{-1}\Sigma_n\longrightarrow\Sigma$. The rest of the argument is exactly similar to that of the one dimensional case. This completes the proof. □

2.4 Urns with Colors Indexed by other lattices on $\mathbb{R}^d$

So far in this chapter, we have discussed urn models with colors indexed by $\mathbb{Z}^d$. We can further generalize the urn models with colors indexed by certain countable lattices in $\mathbb{R}^d$. Such a model will be associated with the corresponding random walk on the lattice. To state the results rigorously we consider the following notations.

Let $(Y_i)_{i\geq 1}$ be a sequence of random d-dimensional i.i.d. vectors with non empty support set $B \subset \mathbb{R}^d$, and probability mass function p. We assume that B is finite. Consider the countable subset

$$S^d := \left\{ \sum_{i=1}^{k} n_i b_i \colon n_1, n_2, \ldots, n_k \in \mathbb{N}, b_1, b_2, \ldots, b_k \in B \right\}$$

of $\mathbb{R}^d$, which will index the set of colors.

As earlier, we consider $S_n := Y_0 + Y_1 + \cdots + Y_n, n \geq 0$, the random walk starting at Y_0 and taking values now in S^d. The transition matrix for this walk is given by

$$R := \left(\left(p\left(u - v\right)\right)\right)_{u,v \in S^d}.$$

In this section, we consider an urn model $(U_n)_{n\geq 0}$ with colors indexed by S^d and replacement matrix R. We will call this process $(U_n)_{n\geq 0}$, an *infinite color urn model associated with the random walk* $(S_n)_{n\geq 0}$ *on* S^d. Naturally, when $S^d = \mathbb{Z}^d$, this is exactly the process which is discussed earlier.

We will use same notations as earlier for the mean, non-centered second moment matrix and the moment generating function for the increment Y_1 (see (2.0.1) for the definitions).

We will still denote by Z_n, the $(n+1)$-th selected color and the expected configuration of the urn at time n will be given by the distribution of Z_n, but now on S^d.

We first note that Theorem 2.1.2 is still valid with exactly the same proof. This enable us to generalize Theorem 2.1.1 and Theorem 2.2.1 as follows.

Theorem 2.4.1. *Let* $\overline{\Lambda}_n$ *be the probability measure on* $\mathbb{R}^d$*, corresponding to the probability vector* $\frac{1}{n+1}\left(\mathbb{E}[U_{n,v}]\right)_{v\in S^d}$ *and let*

$$\overline{\Lambda}_n^{cs}(A) := \overline{\Lambda}_n\left(\sqrt{\log n} A \Sigma^{1/2} + \mu \log n\right),$$

where A *is a Borel subset of* $\mathbb{R}^d$*. Then, as* $n \to \infty$*,*

$$\overline{\Lambda}_n^{cs} \Rightarrow \Phi_d. \tag{2.4.1}$$

Theorem 2.4.2. *Let* $\Lambda_n \in \mathcal{M}_1$ *be the random probability measure corresponding to the random* *probability vector $\frac{U_n}{n+1}$. Let*

$$\Lambda_n^{cs}(A) = \Lambda_n\left(\sqrt{\log n} A \Sigma^{1/2} + \mu \log n\right).$$

where A is a Borel subset of $\mathbb{R}^d$. Then, as $n \to \infty$,

$$\Lambda_n^{cs} \xrightarrow{p} \Phi_d \text{ in } \mathcal{M}_1. \tag{2.4.2}$$

The proofs of these theorems are similar to their counter part for the walks on $\mathbb{Z}^d$, and hence are omitted.

We now consider a specific example, namely, the triangular lattice in two dimensions (see Figure 2.1). For this the support set for the i.i.d. increment vectors is given by

$$B = \left\{(1,0), (-1,0), \omega, -\omega, \omega^2, -\omega^2\right\},$$

where ω, ω^2 are the complex cube roots of unity. The law of Y_1 is uniform on B. This gives the random walk on the triangular lattice in two dimensions.

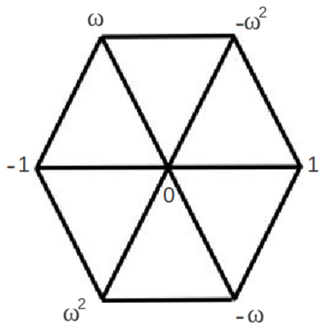

Figure 2.1: Triangular Lattice

Following is an immediate corollary of Theorem 2.4.1.

Corollary 2.4.1. *Consider the urn model associated with the random walk on two dimensional triangular lattice, then, as $n \to \infty$*

$$\frac{Z_n}{\sqrt{\log n}} \Rightarrow N_2\left(0, \frac{1}{2}\mathbb{I}_2\right). \tag{2.4.3}$$

Proof. Since $1 + \omega + \omega^2 = 0$, therefore it is immediate that $\mu = 0$. Also we know that $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, where i is the imaginary square root of -1. Writing $\omega = Re(\omega) + iIm(\omega)$, observe that $\mathbb{E}\left[\left(Y_1^{(1)}\right)^2\right] = \frac{2}{6}\left(1 + \left(Re\left(\omega\right)\right)^2 + \left(Re\left(\omega^2\right)\right)^2\right)$.

Since $Re\left(\omega\right) = Re\left(\omega^2\right)$, therefore, $\mathbb{E}\left[\left(Y_1^{(1)}\right)^2\right] = \frac{2}{6}\left(1 + 2\left(Re\left(\omega\right)\right)^2\right) = \frac{1}{2}$. Similarly, $Im\left(\omega\right) = -Im\left(\omega^2\right)$, and hence $\mathbb{E}\left[\left(Y_1^{2}\right)^2\right] = \frac{2}{6}\left(\left(Im\left(\omega\right)\right)^2 + \left(Im\left(\omega^2\right)\right)^2\right) = \frac{1}{2}$. Finally, $\mathbb{E}\left[X_1^{(1)} X_1^{(2)}\right] = -\frac{2}{6} Im\left(1 + \omega + \omega^2\right) = 0$. So $\Sigma = \frac{1}{2}\mathbb{I}_2$. The rest is just an application of (2.4.1). $\square$

Chapter 3

Local limit theorems for the urn models associated with random walks [1]

In this chapter, we obtain finer asymptotic properties for the distribution of the randomly selected color and derive the local limit theorems. The local limit theorems derived here use the representation (2.1.4). The proofs of the local limit theorems follow techniques similar to that used in the classical case with i.i.d. increments. However, in our case the increments are independent, but not identically distributed.

3.1 Local limit theorems for the expected configuration

Throughout this section, we consider an urn model associated with the bounded increment random walk on $\mathbb{Z}^d$, $d \geq 1$. In Theorem 2.1.1, it is shown that the sequence of random variables/vectors $(Z_n)_{n\geq 0}$ satisfy the central limit theorem. In this section, we show that $(Z_n)_{n\geq 0}$ also satisfies the local limit theorems. We first prove the local limit theorems for one dimension, and then prove the same for dimensions higher than or equal to 2. As introduced in Chapter 2, let $S_n = Y_0 + \sum_{j=1}^{n} Y_j$, denote a random walk on $\mathbb{Z}^d$, with bounded increments $(Y_j)_{j\geq 1}$. For the urn model $(U_n)_{n\geq 0}$, associated with the random walk $(S_n)_{n\geq 0}$, the replacement matrix R is given by (2.0.3). This section is based on [7].

[1] This chapter is partially based on the paper entitled "Pólya Urn Schemes with Infinitely Many Colors", [7].

3.1.1 Local limit theorems for one dimension

In this subsection, we present the local limit theorems for urns with colors indexed by $\mathbb{Z}$. As in Chapter 2, μ, σ and $e(\cdot)$, will denote the mean, non-centered second moment and the moment generating function of Y_1 respectively. Note that, for Y_1 a lattice random variable, we can write

$$\mathbb{P}\left(Y_1 \in a + h\mathbb{Z}\right) = 1, \tag{3.1.1}$$

where $a \in \mathbb{R}$ and $h > 0$, is maximum value such that (3.1.1) holds. h is called the span for Y_1 (see Section 3.5 of [28] for details on lattice random variables). We define

$$\mathcal{L}_n^{(1)} := \left\{ x \colon x = \frac{n}{\sigma\sqrt{\log n}} a - \frac{\mu}{\sigma}\sqrt{\log n} + \frac{h}{\sigma\sqrt{\log n}} z,\ z \in \mathbb{Z} \right\}. \tag{3.1.2}$$

Theorem 3.1.1. *Consider the urn model associated with a bounded increment random walk on $\mathbb{Z}$. Assume that $\mathbb{P}\left(Y_1 = 0\right) > 0$. Then, as $n \to \infty$*

$$\sup_{x \in \mathcal{L}_n^{(1)}} \left| \sigma \frac{\sqrt{\log n}}{h} \mathbb{P}\left(\frac{Z_n - \mu \log n}{\sigma\sqrt{\log n}} = x \right) - \phi(x) \right| \longrightarrow 0. \tag{3.1.3}$$

Proof. From Theorem 2.1.2, we know that $Z_n \stackrel{d}{=} Z_0 + \sum_{j=1}^{n} I_j Y_j$. Y_j is a lattice random variable, therefore, $I_j Y_j$ is also a lattice random variable. Now by our assumption, $\mathbb{P}\left(Y_1 = 0\right) > 0$, we have $0 \in B$, where B is the support of Y_1. Therefore, $I_j Y_j$ and Y_j are supported on the same lattice.

Observe that Z_n is a lattice random variable. Therefore, applying Fourier inversion formula, for all $x \in \mathcal{L}_n^{(1)}$, we obtain

$$\mathbb{P}\left(\frac{Z_n - \mu \log n}{\sigma\sqrt{\log n}} = x \right) = \frac{h}{2\pi\sigma\sqrt{\log n}} \int_{-\frac{\pi\sigma}{h}\sqrt{\log n}}^{\frac{\pi\sigma}{h}\sqrt{\log n}} e^{-itx} \psi_n(t)\, \mathrm{d}t \tag{3.1.4}$$

where $\psi_n(t) = \mathbb{E}\left[e^{it\frac{Z_n - \mu \log n}{\sigma\sqrt{\log n}}} \right]$. Also, by Fourier inversion formula,

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{\frac{-t^2}{2}}\, \mathrm{d}t, \text{ for all } x \in \mathbb{R}. \tag{3.1.5}$$

Given $\epsilon > 0$, there exists N, large enough, such that, for all $n \geq N$,

$$\int_{\frac{\sigma\pi}{h}\sqrt{\log n}}^{\infty} \phi(t)\,\mathrm{d}t < \epsilon.$$

Therefore, for all $n \geq N$,

$$\begin{aligned}
&\left|\frac{\sigma\sqrt{\log n}}{h}\mathbb{P}\left(\frac{Z_n - \mu\log n}{\sigma\sqrt{\log n}} = x\right) - \phi(x)\right| \\
\leq\ & \frac{1}{2\pi}\int_{-\frac{\sigma\pi}{h}\sqrt{\log n}}^{\frac{\sigma\pi}{h}\sqrt{\log n}} \left|\psi_n(t) - e^{\frac{-t^2}{2}}\right|\mathrm{d}t + \frac{1}{\pi}\int_{\frac{\sigma\pi}{h}\sqrt{\log n}}^{\infty}\phi(t)\,\mathrm{d}t \\
<\ & \frac{1}{2\pi}\int_{-\frac{\sigma\pi}{h}\sqrt{\log n}}^{\frac{\sigma\pi}{h}\sqrt{\log n}} \left|\psi_n(t) - e^{\frac{-t^2}{2}}\right|\mathrm{d}t + \frac{\epsilon}{\pi}.
\end{aligned}$$

This implies that it is enough to prove that as $n \to \infty$

$$\int_{-\frac{\sigma\pi}{h}\sqrt{\log n}}^{\frac{\sigma\pi}{h}\sqrt{\log n}} \left|\psi_n(t) - e^{\frac{-t^2}{2}}\right|\mathrm{d}t \longrightarrow 0. \tag{3.1.6}$$

Given $M > 0$, we can write for all n

$$\begin{aligned}
\int_{-\frac{\pi\sigma}{h}\sqrt{\log n}}^{\frac{\sigma\pi}{h}\sqrt{\log n}} \left|\psi_n(t) - e^{\frac{-t^2}{2}}\right|\mathrm{d}t \ \leq\ & \int_{-M}^{M}\left|\psi_n(t) - e^{\frac{-t^2}{2}}\right|\mathrm{d}t + 2\int_{M}^{\frac{\sigma\pi}{h}\sqrt{\log n}}\left|\psi_n(t)\right|\mathrm{d}t \\
& +2\int_{M}^{\frac{\sigma\pi}{h}\sqrt{\log n}} e^{\frac{-t^2}{2}}\,\mathrm{d}t.
\end{aligned} \tag{3.1.7}$$

We know from Theorem 2.1.1, that as $n \to \infty$, $\frac{Z_n - \mu\log n}{\sigma\sqrt{\log n}} \Rightarrow N(0,1)$. Hence, for all $t \in \mathbb{R}$, $\psi_n(t) \longrightarrow e^{\frac{-t^2}{2}}$. Therefore, for any fixed $M > 0$, by bounded convergence theorem, we get as $n \to \infty$,

$$\int_{-M}^{M}\left|\psi_n(t) - e^{\frac{-t^2}{2}}\right|\mathrm{d}t \longrightarrow 0.$$

Let

$$\mathcal{I}(n,M) = \int_{M}^{\frac{\sigma\pi}{h}\sqrt{\log n}} \Big|\psi_n(t)\Big|\,\mathrm{d}t. \tag{3.1.8}$$

We will show that for any $\epsilon > 0$, we can choose $M > 0$, such that for all n large enough

$$\mathcal{I}(n,M) < \epsilon. \tag{3.1.9}$$

Since $Z_n \stackrel{d}{=} Z_0 + \sum_{j=1}^{n} I_j Y_j$, therefore,

$$\psi_n(t) = e^{-it\frac{\mu \log n}{\sigma}}\,\mathbb{E}\left[e^{it\frac{Z_0}{\sigma\sqrt{\log n}}}\right]\mathbb{E}\left[e^{it\frac{\sum_{j=1}^{n} I_j Y_j}{\sigma\sqrt{\log n}}}\right]. \tag{3.1.10}$$

Let us denote by

$$g_n(t) := \mathbb{E}\left[e^{it\frac{\sum_{j=1}^{n} I_j Y_j}{\sigma\sqrt{\log n}}}\right]. \tag{3.1.11}$$

It is easy to see from (3.1.10) that for all $t \in \mathbb{R}$,

$$\Big|\psi_n(t)\Big| \le \Big|g_n(t)\Big|. \tag{3.1.12}$$

Therefore, from (3.1.8) we obtain

$$\mathcal{I}(n,M) \le \int_{M}^{\frac{\sigma\pi}{h}\sqrt{\log n}} \Big|g_n(t)\Big|\,\mathrm{d}t.$$

Applying a change of variables $\frac{t}{\sqrt{\log n}} = w$, we obtain,

$$\mathcal{I}(n,M) \le \sqrt{\log n} \int_{M/\sqrt{\log n}}^{\frac{\pi\sigma}{h}} \Big|g_n\left(w\sqrt{\log n}\right)\Big|\,\mathrm{d}w. \tag{3.1.13}$$

Observe that,

$$\begin{aligned}
\mathbb{E}\left[e^{it\sum_{j=1}^{n} I_j Y_j}\right] &= \prod_{j=1}^{n}\left(1 - \frac{1}{j+1} + \frac{e(it)}{j+1}\right) \\
&= \frac{1}{n+1}\Pi_n\left(e(it)\right)
\end{aligned}$$

where $e\left(it\right)=\mathbb{E}\left[e^{itY_1}\right]$. Therefore,

$$g_n\left(t\right)=\frac{1}{n+1}\Pi_n\left(e\left(\frac{it}{\sigma\sqrt{\log n}}\right)\right). \tag{3.1.14}$$

Now, there exists $\delta>0$, such that for all $t\in(0,\delta)$ (see page 133 of [28])

$$\left|e\left(it\right)\right|\leq 1-\frac{t^2}{4}. \tag{3.1.15}$$

Therefore, using the inequality $1-x\leq e^{-x}$, we obtain $1-\frac{1}{j+1}+\frac{|e(it)|}{j+1}\leq e^{-\frac{1}{j+1}\frac{t^2}{4}}$. Hence, for all $t\in(0,\delta\sigma)$

$$\frac{1}{n+1}\left|\Pi_n\left(e\left(\frac{it}{\sigma}\right)\right)\right|\leq e^{-\frac{t^2}{4\sigma^2}\sum\limits_{j=1}^{n}\frac{1}{j+1}}. \tag{3.1.16}$$

Let us write

$$\sqrt{\log n}\int\limits_{M/\sqrt{\log n}}^{\frac{\pi\sigma}{h}}\left|g_n\left(w\sqrt{\log n}\right)\right|\mathrm{d}w=\mathcal{I}_1(n,M)+\mathcal{I}_2(n),$$

where

$$\mathcal{I}_1(n,M):=\sqrt{\log n}\int\limits_{M/\sqrt{\log n}}^{\sigma\delta}\left|g_n\left(w\sqrt{\log n}\right)\right|\mathrm{d}w$$

and

$$\mathcal{I}_2(n):=\sqrt{\log n}\int\limits_{\sigma\delta}^{\frac{\pi\sigma}{h}}\left|g_n\left(w\sqrt{\log n}\right)\right|\mathrm{d}w.$$

From (3.1.14), we obtain

$$g_n\left(w\sqrt{\log n}\right)=\frac{1}{n+1}\Pi_n\left(e\left(\frac{iw}{\sigma}\right)\right).$$

From (3.1.16), for a suitable constant $C_1>0$, we obtain

$$\mathcal{I}_1(n,M)\leq C_1\sqrt{\log n}\int\limits_{M/\sqrt{\log n}}^{\delta\sigma}e^{-\frac{w^2}{4\sigma^2}\log n}\,\mathrm{d}w.$$

Applying a change of variables $w\sqrt{\log n} = t$, we obtain

$$\mathcal{I}_1(n, M) \leq C_1 \int_M^{\delta\sigma\sqrt{\log n}} e^{-\frac{t^2}{4\sigma^2}}\, \mathrm{d}t. \tag{3.1.17}$$

Observe that for $\epsilon > 0$, we can choose $M > 0$, such that

$$C_1 \int_M^{\infty} e^{-\frac{t^2}{4\sigma^2}}\, \mathrm{d}t < \epsilon. \tag{3.1.18}$$

This implies that

$$C_1 \int_M^{\delta\sigma\sqrt{\log n}} e^{-\frac{t^2}{4\sigma^2}}\, \mathrm{d}t \leq C_1 \int_M^{\infty} e^{-\frac{t^2}{4\sigma^2}}\, \mathrm{d}t < \epsilon. \tag{3.1.19}$$

This implies that for the chosen M,

$$\mathcal{I}_1(n, M) < \epsilon. \tag{3.1.20}$$

Observe, that the span of Y_1 is h. Therefore, for all $t \in \left[\delta, \frac{2\pi}{h}\right)$, $|e\,(it)| < 1$. The characteristic function being continuous in t, there exists $0 < \eta < 1$, such that $|e\left(\frac{it}{\sigma}\right)| \leq \eta$, for all $t \in \left[\delta\sigma, \frac{\pi\sigma}{h}\right]$. Therefore,

$$1 - \frac{1}{j+1} + \frac{|e\left(\frac{it}{\sigma}\right)|}{j+1} \leq 1 - \frac{1}{j+1} + \frac{\eta}{j+1} \leq e^{-\frac{1-\eta}{j+1}}.$$

It follows that,

$$\frac{1}{n+1}\Big|\Pi_n\left(e\left(\frac{it}{\sigma}\right)\right)\Big| \leq e^{-\sum\limits_{j=1}^{n}\frac{1-\eta}{j+1}} \leq C_2 e^{-(1-\eta)\log n}$$

where C_2 is some positive constant. So, as $n \to \infty$,

$$\mathcal{I}_2(n) \leq C_2 \sigma e^{-(1-\eta)\log n}\,(\pi - \delta)\sqrt{\log n} \longrightarrow 0. \tag{3.1.21}$$

Given $\epsilon > 0$, choose M large enough, such that (3.1.18) holds and

$$\int_M^{\infty} e^{\frac{-t^2}{2}}\, \mathrm{d}t < \epsilon.$$

Therefore,

$$\int_{M}^{\frac{\pi\sigma\sqrt{\log n}}{h}} e^{-\frac{t^2}{2}}\,\mathrm{d}t \leq \int_{M}^{\infty} e^{\frac{-t^2}{2}}\,\mathrm{d}t < \epsilon. \tag{3.1.22}$$

Since (3.1.20) holds, and $\mathcal{I}_2(n) \longrightarrow 0$ as $n \to \infty$, therefore from (3.1.7) we obtain

$$\overline{\lim_{n\to\infty}} \int_{-\frac{\pi\sigma}{h}\sqrt{\log n}}^{\frac{\sigma\pi}{h}\sqrt{\log n}} \left|\psi_n(t) - e^{\frac{-t^2}{2}}\right| \mathrm{d}t < 4\epsilon. \tag{3.1.23}$$

□

Theorem 3.1.1 covers the case when $\mathbb{P}(Y_1 = 0) > 0$. Suppose now, $\mathbb{P}(Y_1 = 0) = 0$ and let $\tilde{h}$ be the span for Y_1. We can now write $\mathbb{P}(I_1Y_1 \in a + h\mathbb{Z}) = 1$, where $a \in \mathbb{R}$ and $h > 0$ is the span for I_1Y_1. It is easy to note that $h \leq \tilde{h}$. The following result gives a local limit theorem for the case when $\tilde{h} < 2h$. An example of such a walk is when $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = 2) = 1/2$. Then, $\tilde{h} = 1$. The support for I_1Y_1 is $\{0, 1, 2\}$ and $h = 1$. This example illustrates the case when $\tilde{h} < 2h$ holds.

Theorem 3.1.2. *Assume that $\tilde{h} < 2h$, then, as $n \to \infty$*

$$\sup_{x\in\mathcal{L}_n^{(1)}} \left|\sigma\frac{\sqrt{\log n}}{h}\mathbb{P}\left(\frac{Z_n - \mu\log n}{\sigma\sqrt{\log n}} = x\right) - \phi(x)\right| \longrightarrow 0, \tag{3.1.24}$$

where $\mathcal{L}_n^{(1)} = \left\{x \colon x = \frac{n}{\sigma\sqrt{\log n}}a - \frac{\mu}{\sigma}\sqrt{\log n} + \frac{h}{\sigma\sqrt{\log n}}z \; z \in \mathbb{Z}\right\}$.

Proof. The proof is similar to the proof of Theorem 3.1.1. So we omit certain details. Since for $j \in \mathbb{N}$, the span of I_jY_j is h, for all $x \in \mathcal{L}_n^{(1)}$, we obtain by Fourier inversion formula,

$$\mathbb{P}\left(\frac{Z_n - \mu\log n}{\sigma\sqrt{\log n}} = x\right) = \frac{h}{2\pi\sigma\sqrt{\log n}} \int_{-\frac{\pi\sigma\sqrt{\log n}}{h}}^{\frac{\pi\sigma\sqrt{\log n}}{h}} e^{-itx}\psi_n(t)\,\mathrm{d}t$$

where $\psi_n(t) = \mathbb{E}\left[e^{it\frac{Z_n-\mu\log n}{\sigma\sqrt{\log n}}}\right]$.

Also, by Fourier inversion formula, for all $x \in \mathbb{R}$,

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{\frac{-t^2}{2}}\,\mathrm{d}t.$$

The bounds for $\left|\frac{\sigma\sqrt{\log n}}{h}\mathbb{P}\left(\frac{Z_n-\mu\log n}{\sigma\sqrt{\log n}}=x\right)-\phi(x)\right|$ are similar to those in the proof of Theorem 3.1.1 except for that of $\mathcal{I}_2(n)$, where

$$\mathcal{I}_2(n)=\sqrt{\log n}\int_{\sigma\delta}^{\frac{\sigma\pi}{h}}\left|g_n\left(w\sqrt{\log n}\right)\right|\,\mathrm{d}w$$

and δ is chosen as in (3.1.15) and $g_n(\cdot)$ is as in (3.1.11). We have to show

$$\mathcal{I}_2(n)\longrightarrow 0 \text{ as } n\to\infty.$$

The span of Y_1 being $\tilde{h}$, for all $t\in\left[\delta,\frac{2\pi}{\tilde{h}}\right)$, $|e\left(it\right)|<1$. We have assumed that $\tilde{h}<2h$. The characteristic function being continuous in t, there exists $0<\eta<1$, such that, $|e\left(\frac{it}{\sigma}\right)|\leq\eta$ for all $t\in\left[\delta\sigma,\frac{\pi\sigma}{h}\right]\subset\left[\delta\sigma,\frac{2\pi\sigma}{\tilde{h}}\right)$. So, as $n\to\infty$,

$$\mathcal{I}_2(n)\leq C_2\sigma e^{-(1-\eta)\log n}\left(\pi-\delta\right)\sqrt{\log n}\longrightarrow 0. \tag{3.1.25}$$

□

Remark 3.1.1. *Observe that in the proof of Theorem 3.1.2, the assumption $\tilde{h}<2h$ is required only to obtain the bound in* (3.1.25)*. This assumption implies that $\left[\delta\sigma,\frac{\pi\sigma}{h}\right]\subset\left[\delta\sigma,\frac{2\pi\sigma}{\tilde{h}}\right)$, which guarantees the existence of $0<\eta<1$, such that $|e\left(\frac{it}{\sigma}\right)|\leq\eta$ for all $t\in\left[\delta\sigma,\frac{\pi\sigma}{h}\right]$.*

The next theorem is stated for the special case when the urn is associated with simple symmetric random walk on $\mathbb{Z}$, which is not covered by Theorem 3.1.1 or its generalization given by Theorem 3.1.2.

Theorem 3.1.3. *Assume that $\mathbb{P}\left(Y_1=1\right)=\mathbb{P}\left(Y_1=-1\right)=\frac{1}{2}$. Then, as $n\to\infty$*

$$\sup_{x\in\mathcal{L}_n^{(1)}}\left|\sqrt{\log n}\,\mathbb{P}\left(\frac{Z_n}{\sqrt{\log n}}=x\right)-\phi(x)\right|\longrightarrow 0 \tag{3.1.26}$$

where $\mathcal{L}_n^{(1)}$ is given by (3.1.2) *with $\mu=0=a$ and $\sigma=1=h$.*

The following result is immediate from Theorem 3.1.3.

Corollary 3.1.1. *Assume that $\mathbb{P}\left(X_1=1\right)=\mathbb{P}\left(X_1=-1\right)=\frac{1}{2}$. Then, as $n\to\infty$*

$$\mathbb{P}\left(Z_n=0\right)\sim\frac{1}{\sqrt{2\pi\log n}}. \tag{3.1.27}$$

Proof of Theorem 3.1.3. In this case $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$. Thus the span of Y_1 is 2. The random variables $I_1 Y_1$ is supported on the set $\{0, 1, -1\}$ and it has span 1. We have $\mu = 0$ and $\sigma = 1$, so from (3.1.2), we get $\mathcal{L}_n^{(1)} = \frac{1}{\sqrt{\log n}}\mathbb{Z}$.

For all $x \in \mathcal{L}_n^{(1)}$, we obtain by Fourier Inversion formula,

$$\mathbb{P}\left(\frac{Z_n}{\sqrt{\log n}} = x\right) = \frac{1}{2\pi\sqrt{\log n}} \int\limits_{-\pi\sqrt{\log n}}^{\pi\sqrt{\log n}} e^{-itx}\psi_n(t)\,\mathrm{d}t$$

where $\psi_n(t) = \mathbb{E}\left[e^{it\frac{Z_n}{\sqrt{\log n}}}\right]$. Furthermore, by Fourier inversion formula, for all $x \in \mathbb{R}$,

$$\phi(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-itx}e^{\frac{-t^2}{2}}\,\mathrm{d}t.$$

The proof of this theorem is also very similar to that of Theorem 3.1.1. The bounds for $\left|\sqrt{\log n}\,\mathbb{P}\left(\frac{Z_n}{\sqrt{\log n}} = x\right) - \phi(x)\right|$ are similar to those in the proof of Theorem 3.1.1 except for that of $\mathcal{I}_2(n)$, where

$$\mathcal{I}_2(n) = \sqrt{\log n}\int\limits_{\delta}^{\pi}\left|\psi_n\left(w\sqrt{\log n}\right)\right|\,\mathrm{d}w,$$

and δ is chosen as in (3.1.15). To show that $\mathcal{I}_2(n) \longrightarrow 0$ as $n \to \infty$, we observe that

$$\begin{aligned}\mathbb{E}\left[e^{itZ_n}\right] &= \prod_{j=1}^{n}\left(1 - \frac{1}{j+1} + \frac{\cos t}{j+1}\right)\\ &= \frac{1}{n+1}\Pi_n(\cos t)\end{aligned}$$

since $\mathbb{E}\left[e^{itY_1}\right] = \cos t$. Therefore,

$$\psi_n(w\sqrt{\log n}) = \mathbb{E}\left[e^{iwZ_n}\right] = \frac{1}{n+1}\Pi_n(\cos w).$$

We note that $\cos w$ is decreasing in $\left[\frac{\pi}{2}, \pi\right]$ and for all $w \in \left[\frac{\pi}{2}, \pi\right]$, $-1 \leq \cos w \leq 0$. Therefore, there exists $\eta > 0$ (small enough), such that, $[\pi - \eta, \pi) \subset \left(\frac{\pi}{2}, \pi\right]$ and for all $w \in [\pi - \eta, \pi)$, we have $-1 < \cos(\pi - \eta) < 0$, and

$$\left|\psi_n(w\sqrt{\log n})\right| \leq \frac{1}{n+1}\Pi_n(\cos(\pi - \eta)).$$

Since $-1 < \cos(\pi - \eta) < 0$, so for all $j \geq 1$, $\left(1 + \frac{\cos(\pi-\eta)}{j}\right) < 1$. Therefore,

$$\Pi_n\left(\cos(\pi - \eta)\right) \leq 1. \tag{3.1.28}$$

Let us write

$$\mathcal{I}_2(n) = \mathcal{J}_1(n) + \mathcal{J}_2(n),$$

where

$$\mathcal{J}_1(n) = \sqrt{\log n} \int_{\delta}^{\pi-\eta} \left|\psi_n\left(w\sqrt{\log n}\right)\right| \, \mathrm{d}w, \tag{3.1.29}$$

and

$$\mathcal{J}_2(n) = \sqrt{\log n} \int_{\pi-\eta}^{\pi} \left|\psi_n\left(w\sqrt{\log n}\right)\right| \, \mathrm{d}w.$$

It is easy to see from (3.1.28) that

$$\mathcal{J}_2(n) \leq \frac{\eta}{n+1}\sqrt{\log n} \longrightarrow 0 \text{ as } n \to \infty.$$

For all $t \in [\delta, \pi - \eta]$, $0 \leq |\cos t| < 1$, so there exists $0 < \alpha < 1$, such that, $0 \leq |\cos t| \leq \alpha$ for all $t \in [\delta, \pi - \eta]$. Recall that

$$\psi_n(w\sqrt{\log n}) = \prod_{j=1}^{n}\left(1 - \frac{1}{j+1} + \frac{\cos w}{j+1}\right).$$

Using the inequality $1 - x \leq e^{-x}$, it follows that for all $t \in [\delta, \pi - \eta]$

$$1 - \frac{1}{j+1} + \frac{|\cos t|}{j+1} \leq 1 - \frac{1}{j+1} + \frac{\alpha}{j+1} \leq e^{-\frac{1-\alpha}{j+1}},$$

and hence,

$$\frac{1}{n+1}|\Pi_n\left(\cos t\right)| \leq e^{-\sum\limits_{j=1}^{n}\dfrac{1-\alpha}{j+1}} \leq Ce^{-(1-\alpha)\log n}$$

where C is some positive constant. Therefore, from (3.1.29) we obtain as $n \to \infty$,

$$\mathcal{J}_1(n) \leq Ce^{-(1-\alpha)\log n}\left(\pi - \eta - \delta\right)\sqrt{\log n} \longrightarrow 0.$$

$\square$

3.1.2 Local limit theorems for higher dimensions

Now, we consider the case $d \geq 2$. As in (2.0.1), μ, Σ and $e(\cdot)$, will denote the mean, non-centered second moment matrix and the moment generating function of Y_1 respectively. For Y_1 a lattice random vector taking values in $\mathbb{Z}^d$, let $\mathcal{L}$ be its *minimal lattice*, that is, $\mathbb{P}(Y_1 \in x + \mathcal{L}) = 1$ for every $x \in \mathbb{Z}^d$, such that, $\mathbb{P}(Y_1 = x) > 0$. We refer to the pages 226 – 227 of [10] for a formal definition of the minimal lattice of a d-dimensional lattice random variable. If $\mathcal{L}'$ is any closed subgroup of $\mathbb{R}^d$, such that $\mathbb{P}(Y_1 \in y + \mathcal{L}') = 1$ for some $y \in \mathbb{Z}^d$, then from the definition of minimal lattice it follows that $\mathcal{L} \subseteq \mathcal{L}'$. We assume Y_1 to be non-degenerate. This implies that the rank of $\mathcal{L}$ is d. Let $l = |\det(\mathcal{L})|$ (see the pages 228 – 229 of [10] for more details). Now, x_0 be such that, $\mathbb{P}(Y_1 \in x_0 + \mathcal{L}) = 1$ and we define

$$\mathcal{L}_n^{(d)} := \left\{ x \colon x = \frac{n}{\sqrt{\log n}} x_0 \Sigma^{-1/2} - \sqrt{\log n}\, \mu\, \Sigma^{-1/2} + \frac{1}{\sqrt{\log n}} z \Sigma^{-1/2} \,:\, z \in \mathcal{L} \right\}. \tag{3.1.30}$$

Theorem 3.1.4. *Assume that* $\mathbb{P}(Y_1 = 0) > 0$. *Then, as* $n \to \infty$

$$\sup_{x \in \mathcal{L}_n^{(d)}} \left| \frac{\det(\Sigma^{1/2}) \left(\sqrt{\log n}\right)^d}{l} \mathbb{P}\left(\frac{Z_n - \mu \log n}{\sqrt{\log n}} \Sigma^{-1/2} = x \right) - \phi_d(x) \right| \longrightarrow 0. \tag{3.1.31}$$

Proof. From Theorem 2.1.2, we obtain $Z_n \stackrel{d}{=} Z_0 + \sum_{j=1}^n I_j Y_j$. The random vector Y_j is a lattice random vector. Therefore, $I_j Y_j$ is also a lattice random vector. By our assumption, $\mathbb{P}(Y_1 = 0) > 0$, so $0 \in B$, therefore, Y_j and $I_j Y_j$ are supported on the same lattice.

Observe that Z_n is a lattice random vector, for every $n \in \mathbb{N}$. For $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}$, we define

$$xA := \{xy \colon y \in A\},$$

and

$$A\Sigma^{1/2} := \{y\Sigma^{1/2} \colon y \in A\}.$$

By Fourier inversion formula (see 21.28 on page 230 of [10]), we get for $x \in \mathcal{L}_n^{(d)}$,

$$\mathbb{P}\left(\frac{Z_n - \mu \log n}{\sqrt{\log n}} \Sigma^{-1/2} = x \right) = \frac{l}{(2\pi\sqrt{\log n})^d \det(\Sigma^{1/2})} \int\limits_{(\sqrt{\log n}\mathcal{F}^* \Sigma^{1/2})} \psi_n(t) e^{-i\langle t, x \rangle} \, \mathrm{d}t$$

where $\psi_n(t) = \mathbb{E}\left[e^{i\langle t, \frac{Z_n - \mu \log n}{\sqrt{\log n}} \Sigma^{-1/2} \rangle} \right]$, $l = |\det(\mathcal{L})|$ and $\mathcal{F}^*$ is the fundamental domain for Y_1

as defined in equation (21.22) on page 229 of [10]. Also, by Fourier inversion formula

$$\phi_d(x) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{-i\langle t,x\rangle} e^{-\frac{\|t\|^2}{2}}\,\mathrm{d}t.$$

Given $\epsilon > 0$, there exists $N > 0$ such that $n \geq N$,

$$\begin{aligned}
&\Big|\frac{\det(\Sigma^{1/2})\left(\sqrt{\log n}\right)^d}{l}\mathbb{P}\left(\frac{Z_n - \mu\log n}{\sqrt{\log n}}\Sigma^{-1/2} = x\right) - \phi_d(x)\Big| \\
\leq\ & \frac{1}{(2\pi)^d} \int\limits_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})} \left|\psi_n(t) - e^{-\frac{\|t\|^2}{2}}\right| \mathrm{d}t + \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d\setminus\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2}} e^{-\frac{\|t\|^2}{2}}\,\mathrm{d}t \\
\leq\ & \frac{1}{(2\pi)^d} \int\limits_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})} \left|\psi_n(t) - e^{-\frac{\|t\|^2}{2}}\right| \mathrm{d}t + \epsilon.
\end{aligned}$$

Therefore, it is enough to prove that as $n \to \infty$

$$\int\limits_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})} \left|\psi_n(t) - e^{-\frac{\|t\|^2}{2}}\right| \mathrm{d}t \longrightarrow 0. \tag{3.1.32}$$

Given any compact set $A \subset \mathbb{R}^d$, we have

$$\begin{aligned}
\int\limits_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})} \left|\psi_n(t) - e^{-\frac{\|t\|^2}{2}}\right| \mathrm{d}t \ \leq\ & \int\limits_A \left|\psi_n(t) - e^{-\frac{\|t\|^2}{2}}\right| \mathrm{d}t + \int\limits_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})\setminus A} \left|\psi_n(t)\right| \mathrm{d}t \\
& + \int\limits_{\mathbb{R}^d\setminus A} e^{-\frac{\|t\|^2}{2}}\,\mathrm{d}t.
\end{aligned} \tag{3.1.33}$$

By Theorem 2.1.1, we know that $\frac{Z_n - \mu\log n}{\sqrt{\log n}}\Sigma^{-1/2} \Rightarrow N_d(0, \mathbb{I}_d)$ as $n \to \infty$. Therefore, for any compact set $A \subset \mathbb{R}^d$, by bounded convergence theorem,

$$\int\limits_A \left|\psi_n(t) - e^{-\frac{\|t\|^2}{2}}\right| \mathrm{d}t \longrightarrow 0 \text{ as } n \to \infty.$$

Let us write

$$\mathcal{I}(n, A) = \int\limits_{(\sqrt{\log n}\mathcal{F}^*\Sigma^{1/2})\setminus A} \left|\psi_n(t)\right| \mathrm{d}t. \tag{3.1.34}$$

We will show that for any $\epsilon > 0$, we can choose a compact subset A of $\mathbb{R}^d$, such that for all n

large enough

$$\mathcal{I}(n, A) < \epsilon.$$

Since $Z_n \overset{d}{=} Z_0 + \sum_{j=1}^{n} I_j Y_j$, calculations similar to (3.1.10) show that for every $t \in \mathbb{R}^d$, and each n

$$\Big|\psi_n(t)\Big| \leq \Big|g_n(t)\Big|,$$

where

$$g_n\left(t\right) := \mathbb{E}\left[e^{\langle it, \frac{\sum_{j=1}^{n} I_j Y_j}{\sqrt{\log n}} \Sigma^{-1/2}\rangle}\right]. \tag{3.1.35}$$

Therefore, from (3.1.34) we obtain

$$\mathcal{I}(n, A) \leq \int\limits_{(\sqrt{\log n}\mathcal{F}^* \Sigma^{1/2}) \setminus A} \Big|g_n(t)\Big| \,\mathrm{d}t. \tag{3.1.36}$$

Observe that

$$\begin{aligned}
\mathbb{E}\left[e^{i\langle t, \sum_{j=1}^{n} I_j Y_j\rangle}\right] &= \prod_{j=1}^{n}\left(1 - \frac{1}{j+1} + \frac{e\left(it\right)}{j+1}\right) \\
&= \frac{1}{n+1}\Pi_n\left(e\left(it\right)\right)
\end{aligned}$$

where $e\left(it\right) = \mathbb{E}\left[e^{i\langle t, Y_1\rangle}\right]$. So, from (3.1.35)

$$g_n(t) = \frac{1}{n+1}\Pi_n\left(e\left(\frac{1}{\sqrt{\log n}} it\Sigma^{-1/2}\right)\right).$$

Applying a change of variables $t = \frac{1}{\sqrt{\log n}} w$, to (3.1.34), we obtain

$$\mathcal{I}(n, A) \leq (\sqrt{\log n})^d \int\limits_{\mathcal{F}^* \Sigma^{1/2} \setminus \frac{1}{\sqrt{\log n}} A} \Big|g_n\left(\sqrt{\log n} w\right)\Big| \,\mathrm{d}w. \tag{3.1.37}$$

We can choose $\delta > 0$, such that for all $w \in B(0, \delta) \setminus \{0\}$, there exists $b > 0$, such that

$$|e(iw)| \leq 1 - \frac{b\|w\|^2}{2}, \tag{3.1.38}$$

(see Lemma 2.3.2(a) of [46] for a proof). Therefore, using the inequality $1 - x \leq e^{-x}$,

$$|g_n(\sqrt{\log n} w)| \quad = \quad \frac{1}{n+1}|\Pi_n\left(e(iw\Sigma^{-1/2})\right)|$$

$$
\begin{aligned}
&\leq \prod_{j=1}^{n+1}\left(1-\frac{1}{j+1}+\frac{\left|e\left(iw\Sigma^{-1/2}\right)\right|}{j+1}\right) \\
&\leq e^{-\sum_{j=1}^{n}\frac{b}{j+1}\frac{\|w\|^2}{2}} \leq C_1 e^{-b\frac{w\Sigma w^T}{2}\log n} \qquad (3.1.39)
\end{aligned}
$$

for some positive constant C_1. We write

$$
(\sqrt{\log n})^d \int\limits_{\mathcal{F}^*\Sigma^{1/2}\setminus\frac{1}{\sqrt{\log n}}A} \left|g_n\left(\sqrt{\log n}w\right)\right| \mathrm{d}w = \mathcal{I}_1(n,A)+\mathcal{I}_2(n)
$$

where

$$
\mathcal{I}_1(n,A) := (\sqrt{\log n})^d \int\limits_{\left(B(0,\delta)\Sigma^{1/2}\setminus\frac{1}{\sqrt{\log n}}A\right)\cap\mathcal{F}^*\Sigma^{1/2}} \left|g_n\left(\sqrt{\log n}w\right)\right| \mathrm{d}w
$$

and

$$
\mathcal{I}_2(n) = (\sqrt{\log n})^d \int\limits_{\mathcal{F}^*\Sigma^{1/2}\setminus B(0,\delta)\Sigma^{1/2}} |g_n\left(\sqrt{\log n}w\right)| dw.
$$

From (3.1.39), we obtain

$$
\mathcal{I}_1(n) \leq C_1(\sqrt{\log n})^d \int\limits_{B(0,\delta)\Sigma^{1/2}\setminus\frac{A}{\sqrt{\log n}}} e^{-b\frac{w\Sigma w^T}{2}\log n}\,\mathrm{d}w, \qquad (3.1.40)
$$

for an appropriate positive constant C_1. Applying a change of variables $w\left(\sqrt{\log n}\right) = t$, we obtain

$$
\mathcal{I}_1(n,A) \leq C_1 \int\limits_{B(0,\delta\sqrt{\log n})\Sigma^{1/2}\setminus A} e^{-b\frac{w\Sigma w^T}{2}}\,\mathrm{d}w. \qquad (3.1.41)
$$

Given $\epsilon > 0$, choose a compact subset A of $\mathbb{R}^d$, such that

$$
C_1 \int\limits_{A^c} e^{-b\frac{w\Sigma w^T}{2}}\,\mathrm{d}w < \epsilon. \qquad (3.1.42)
$$

Therefore, for this choice of A,

$$
\mathcal{I}_1(n,A) < \epsilon. \qquad (3.1.43)
$$

Since the lattices for Y_1 and I_1Y_1 are same, for all $w \in \mathcal{F}^* \setminus B(0,\delta)$, we get $|e(iw)| < 1$, so there exists an $0 < \eta < 1$, such that, $|e(iw\Sigma^{-1/2})| \leq \eta$, for all $w \in \mathcal{F}^*\Sigma^{1/2} \setminus B(0,\delta)\Sigma^{1/2}$

Therefore, using the inequality $1 - x \leq e^{-x}$, we obtain

$$|g_n(\sqrt{\log n}w)| \leq e^{-\sum_{j=i}^{n} \frac{1}{j+1}(1-\eta)} \leq C_2 e^{-(1-\eta)\log n} \tag{3.1.44}$$

for some positive constant C_2. Therefore, using equation (21.25) on page 230 of [10], we obtain

$$\mathcal{I}_2(n) \leq C_2'(\sqrt{\log n})^d e^{-(1-\eta)\log n} \longrightarrow 0 \text{ as } n \to \infty \tag{3.1.45}$$

where C_2' is an appropriate positive constant.

Given $\epsilon > 0$, choose A, such that, (3.1.42) holds and

$$\int_{A^c} e^{-\frac{\|t\|^2}{2}} \, dt < \epsilon. \tag{3.1.46}$$

From (3.1.33), (3.1.43), (3.1.45) and (3.1.46), we get

$$\overline{\lim_{n\to\infty}} \int_{(\sqrt{\log n}\mathcal{F}^* \Sigma^{1/2})} \left| \psi_n(t) - e^{-\frac{\|t\|^2}{2}} \right| dt < 2\epsilon. \tag{3.1.47}$$

$\square$

Observe that as in the one dimensional case, Theorem 3.1.4 covers only the case when $\mathbb{P}(Y_1 = 0) > 0$. The next theorem is stated for the special case when the urn is associated with simple symmetric random walk on $\mathbb{Z}^d$, $d \geq 2$, which is not covered by the Theorem 3.1.4.

Theorem 3.1.5. *Assume that $\mathbb{P}(Y_1 = \pm e_i) = \frac{1}{2d}$, for $1 \leq i \leq d$, where e_i is the i-th unit vector in direction i. Then, as $n \to \infty$,*

$$\sup_{x \in \mathcal{L}_n^{(d)}} \left| (d)^{\frac{d}{2}} \left(\sqrt{\log n}\right)^d \mathbb{P}\left(\frac{\sqrt{d}}{\sqrt{\log n}} Z_n = x \right) - \phi_d(x) \right| \longrightarrow 0, \tag{3.1.48}$$

where $\mathcal{L}_n^{(d)}$ is as defined in (3.1.30) *with $\mu = 0 = x_0$, $\Sigma = \mathbb{I}_d$ and $\mathcal{L} = \sqrt{d}\,\mathbb{Z}^d$.*

Similar to the one dimensional case, the next result is immediate from the above theorem.

Corollary 3.1.2. *Assume that $\mathbb{P}(X_1 = \pm e_i) = \frac{1}{2d}$, for $1 \leq i \leq d$, where e_i is the i-th unit vector in direction i. Then, as $n \to \infty$,*

$$\mathbb{P}(Z_n = 0) \sim \frac{1}{\left(\sqrt{2\pi d \log n}\right)^d}. \tag{3.1.49}$$

Proof of Theorem 3.1.5. In this case, $\mathbb{P}(Y_1 = \pm e_i) = \frac{1}{2d}$, for $1 \leq i \leq d$, where e_i is the i-th unit vector in direction i, thus $\mu = 0$ and $\Sigma = \frac{1}{d}\mathbb{I}_d$.

For notational simplicity, we prove (3.1.48) for $d = 2$. The proof for general d can be written similarly.

Now for each $j \in \mathbb{N}$, $I_j Y_j$ is a lattice random vector with the minimal lattice $\mathbb{Z}^2$. It is easy to see that $2\pi\mathbb{Z} \times 2\pi\mathbb{Z}$ is the set of all periods for $I_j Y_j$, and its fundamental domain is given by $(-\pi, \pi)^2$. To prove (3.1.48), it is enough to show

$$\sup_{x \in \frac{1}{\sqrt{2}}\mathcal{L}_n^{(2)}} \left| (\log n)\, \mathbb{P}\left(\frac{Z_n}{\sqrt{\log n}} = x \right) - \phi_{2,\frac{1}{2}\mathbb{I}_2}(x) \right| \longrightarrow 0 \text{ as } n \to \infty,$$

where $\phi_{2,\frac{1}{2}\mathbb{I}_2}(x) = \frac{1}{\pi} e^{-\|x\|^2}$ is the bivariate normal density with mean vector 0 and variance-covariance matrix $\frac{1}{2}\mathbb{I}_2$ and $\frac{1}{\sqrt{2}}\mathcal{L}_n^{(2)} = \frac{1}{\sqrt{\log n}}\mathbb{Z}^2$. By Fourier inversion formula (see 21.28 on page 230 of [10]), we get for $x \in \frac{1}{\sqrt{2}}\mathcal{L}_n^{(2)}$

$$\mathbb{P}\left(\frac{Z_n}{\sqrt{\log n}} = x \right) = \frac{1}{(2\pi)^2 \log n} \int\limits_{\left(-\sqrt{\log n}\pi, \sqrt{\log n}\pi\right)^2} \psi_n(t) e^{-i\langle t, x\rangle}\, \mathrm{d}t$$

Also, by Fourier inversion formula,

$$\phi_{2,\frac{1}{2}\mathbb{I}_2}(x) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2} e^{-i\langle t,x\rangle} e^{-\frac{\|t\|^2}{4}}\, \mathrm{d}t.$$

Let us write $H_n = \left(-\sqrt{\log n}\pi, \sqrt{\log n}\pi\right)^2$. Given $\epsilon > 0$, there exists $N > 0$, such that, for $n \geq N$,

$$\begin{aligned} \left| \log n\, \mathbb{P}\left(\frac{Z_n}{\sqrt{\log n}} = x \right) - \phi_{2,\frac{1}{2}\mathbb{I}_2}(x) \right| &\leq \frac{1}{(2\pi)^2} \int\limits_{H_n} \left| \psi_n(t) - e^{-\frac{\|t\|^2}{4}} \right| \mathrm{d}t + \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \setminus H_n} e^{-\frac{\|t\|^2}{4}}\, \mathrm{d}t \\ &< \frac{1}{(2\pi)^2} \int\limits_{H_n} \left| \psi_n(t) - e^{-\frac{\|t\|^2}{4}} \right| \mathrm{d}t + \epsilon. \end{aligned}$$

Given any compact set $A \subset \mathbb{R}^2$, for all n large enough, we have

$$\int\limits_{H_n} \left| \psi_n(t) - e^{-\frac{\|t\|^2}{4}} \right| \mathrm{d}t \leq \int\limits_{A} \left| \psi_n(t) - e^{-\frac{\|t\|^2}{4}} \right| \mathrm{d}t + \int\limits_{H_n \setminus A} \left| \psi_n(t) \right| \mathrm{d}t + \int\limits_{A^c} e^{-\frac{\|t\|^2}{4}}\, \mathrm{d}t.$$

By Theorem 2.1.1, we know that $\frac{Z_n}{\sqrt{\log n}} \Rightarrow N_2(0, \frac{1}{2}\mathbb{I}_2)$, as $n \to \infty$. Therefore, for any compact set $A \subset \mathbb{R}^2$, by bounded convergence theorem,

$$\int\limits_{A} \left| \psi_n(t) - e^{-\frac{\|t\|^2}{4}} \right| \mathrm{d}t \longrightarrow 0 \text{ as } n \to \infty.$$

Let us write

$$\mathcal{I}(n, A) = \int\limits_{H_n \setminus A} \left|\psi_n(t)\right| \mathrm{d}t.$$

We will show that for any $\epsilon > 0$, we can choose a compact subset A of $\mathbb{R}^2$, such that, for all n large enough

$$\int\limits_{A^c} e^{-\frac{\|t\|^2}{4}} \, \mathrm{d}t < \epsilon,$$

and

$$\mathcal{I}(n, A) < \epsilon.$$

Applying a change of variables $t = \frac{1}{\sqrt{\log n}} w$, we obtain

$$\mathcal{I}(n, A) = \log n \int\limits_{(-\pi,\pi)^2 \setminus \frac{1}{\sqrt{\log n}} A} \left|\psi_n\left(\sqrt{\log n} w\right)\right| \mathrm{d}w.$$

We can write

$$\mathcal{I}(n, A) = \mathcal{I}_1(n, A) + \mathcal{I}_2(n)$$

where

$$\mathcal{I}_1(n, A) = \log n \int\limits_{\left(B(0,\delta) \setminus \frac{1}{\sqrt{\log n}} A\right) \cap (-\pi,\pi)^2} \left|\psi_n\left(\sqrt{\log n} w\right)\right| \mathrm{d}w$$

and

$$\mathcal{I}_2(n) = \log n \int\limits_{(-\pi,\pi)^2 \setminus B(0,\delta)} \left|\psi_n\left(\sqrt{\log n} w\right)\right| dw.$$

where δ is as in (3.1.38). Using arguments similar to (3.1.43), we can show that for any $\epsilon > 0$, we can choose a compact subset A of $\mathbb{R}^2$, such that for all n large enough

$$\mathcal{I}_1(n, A) < \epsilon.$$

Therefore, it is enough to show that $\mathcal{I}_2(n) \longrightarrow 0$, as $n \to \infty$. To do so, we first observe that for $t = \left(t^{(1)}, t^{(2)}\right) \in \mathbb{R}^2$, the characteristic function for Y_1 is given by

$e\,(it) = \frac{1}{2}\left(\cos t^{(1)} + \cos t^{(2)}\right)$. So, if $t \in [-\pi,\pi]^2$, be such that, $|e\,(it)| = 1$, then $t \in \{(\pi,\pi),(-\pi,\pi),(\pi,-\pi),(-\pi,-\pi)\}$. The function $\cos\theta$ is continuous and decreasing as a function of θ for $\theta \in \left[\frac{\pi}{2},\pi\right]$. Choose $\eta > 0$, such that for $t \in A_1 = (-\pi,\pi)^2 \cap B^c(0,\delta) \cap D^c$, we have $|e\,(it)| < 1$, where $D = [\pi-\eta,\pi)^2 \cup [-\pi+\eta,-\pi) \times [\pi-\eta,\pi) \cup [-\pi+\eta,-\pi)^2 \cup [\pi-\eta,\pi) \times [-\pi+\eta,-\pi)$. Let us write

$$\mathcal{I}_2(n) = \mathcal{J}_1(n) + \mathcal{J}_2(n),$$

where

$$\mathcal{J}_1(n) = \log n \int\limits_{A_1} |\psi_n\left(\sqrt{\log n}w\right)| dw$$

and

$$\mathcal{J}_2(n) = \log n \int\limits_{D} |\psi_n\left(\sqrt{\log n}w\right)| dw.$$

It is easy to note that,

$$\mathcal{J}_1(n) \leq \log n \int\limits_{\overline{A}_1} |\psi_n\left(\sqrt{\log n}w\right)| dw,$$

where $\overline{A}_1$ denotes the closure of A_1. For $w \in \overline{A}_1$, there exists some $0 < \alpha < 1$, such that, $|e\,(it)| \leq \alpha$. Therefore, using bounds similar to that in (3.1.44), we can show that

$$\mathcal{J}_1(n) \longrightarrow 0 \text{ as } n \to \infty.$$

We observe that

$$\mathcal{J}_2(n) \leq 4\log n \int\limits_{[\pi-\eta,\pi]^2} |\psi_n\left(\sqrt{\log n}w\right)| dw.$$

Hence, it is enough to show that

$$\log n \int\limits_{[\pi-\eta,\pi]^2} |\psi_n\left(\sqrt{\log n}w\right)| dw \longrightarrow 0 \text{ as } n \to \infty.$$

For $w \in [\pi-\eta,\pi]^2$, we have $0 < |\left(1 + \frac{e(iw)}{j}\right)| \leq \left(1 + \frac{\cos(\pi-\eta)}{j}\right) \leq 1$. Therefore,

$$|\psi_n\left(\sqrt{\log n}w\right)| = \frac{1}{n+1}\prod_{j=1}^{n}\Big|\left(1 + \frac{e(iw)}{j}\right)\Big| \leq \frac{1}{n+1}.$$

So,

$$\log n \int\limits_{[\pi-\eta,\pi]^2} |\psi_n\left(\sqrt{\log n}w\right)|dw \leq \frac{\eta^2}{n+1}\log n \longrightarrow 0 \text{ as } n \to \infty.$$

$\square$

Chapter 4

Large deviation principle for the urn models associated with random walks[1]

Urn models associated with bounded increment random walks were introduced in Chapter 2. Let $S_n = Y_0 + \sum_{j=1}^{n} Y_j$, denote a random walk on $\mathbb{Z}^d$, with bounded increments $(Y_j)_{j\geq 1}$. For the urn model $(U_n)_{n\geq 0}$, associated with the random walk $(S_n)_{n\geq 0}$, we consider the replacement matrix R given by (2.0.3). As in (2.0.1), μ, Σ and $e(\cdot)$, will denote the mean, non-centered second moment matrix and the moment generating function of Y_1 respectively. For this model, as shown in Theorem 2.1.1, if Z_n is the randomly selected color at the $(n+1)$-th draw, then

$$\frac{Z_n - \mu \log n}{\sqrt{\log n}} \Rightarrow N_d(0, \Sigma) \text{ as } n \to \infty. \tag{4.0.1}$$

It is easy to see that (4.0.1) implies

$$\frac{Z_n}{\log n} \xrightarrow{p} \mu \text{ as } n \to \infty. \tag{4.0.2}$$

In the following section, we show that the sequence of probability measures $\left(\mathbb{P}\left(\frac{Z_n}{\log n} \in \cdot\right)\right)_{n\geq 2}$ satisfy a *large deviation principle (LDP)* with a *good rate function* and *speed* $\log n$. A characterization of the rate function is also provided.

The large deviation principle discussed here uses the representation (2.1.4). Since the increments $(I_j Y_j)_{j\geq 1}$ are independent, but not identically distributed, we require techniques

[1]This chapter is based on the paper " Rate of convergence and large deviation for the infinite color Pólya urn schemes, *Statist. Probab. Lett.,* 92:232-240, 2014, [8].

different from the classical case. For this we use the Gärtner-Ellis Theorem (see Remark (a) on page 45 of [27] or page 66 of [16]).

4.1 Large deviation principles for the randomly selected color

The following standard notation is used in this section. For any subset $A \subseteq \mathbb{R}^d$, we write A° to denote the *interior* of A and $\bar{A}$ to denote the *closure* of A, under the usual Euclidean topology. We present a few definitions (for more details see [27]), which are standard in the literature.

Definition 4.1.1. *A sequence of probability measures $(\nu_n)_{n\geq 2}$ is said to satisfy a LDP with a rate function I, and speed v_n, if for all Borel subset A of $\mathbb{R}^d$,*

$$-\inf_{x\in A^\circ} I(x) \leq \varliminf_{n\to\infty} \frac{\log \nu_n(A)}{v_n} \leq \varlimsup_{n\to\infty} \frac{\log \nu_n(A)}{v_n} \leq -\inf_{x\in \bar{A}} I(x).$$

Definition 4.1.2. *A rate function I is a lower semicontinuous function $I : \mathbb{R}^d \to [0,\infty]$. A rate function is said to be* good*, if all the level sets $\{x : I(x) \leq a\}$ are compact subsets of $\mathbb{R}^d$.*

In the next theorem, we discuss the asymptotic behavior of the tail probabilities of $\frac{Z_n}{\log n}$.

Theorem 4.1.1. *The sequence of probability measures $\left(\mathbb{P}\left(\frac{Z_n}{\log n} \in \cdot\right)\right)_{n\geq 2}$ satisfy a LDP with rate function $I(\cdot)$ and speed $\log n$, where $I(\cdot)$ is the Fenchel-Legendre dual of $e(\cdot) - 1$, that is, for $x \in \mathbb{R}^d$,*

$$I(x) = \sup_{\lambda\in\mathbb{R}^d} \{\langle x, \lambda\rangle - e(\lambda) + 1\}. \tag{4.1.1}$$

Moreover, $I(\cdot)$ is a good rate function *which is also convex.*

Proof. Let us define

$$\boldsymbol{\alpha}_n(\lambda) := \frac{1}{\log n} \log \mathbb{E}\left[e^{\langle\lambda, Z_n\rangle}\right]. \tag{4.1.2}$$

From (2.1.4), we know that

$$Z_n \stackrel{d}{=} Z_0 + \sum_{j=1}^{n} I_j Y_j.$$

Therefore, it follows from calculations similar to (2.1.5)

$$\mathbb{E}\left[e^{\langle\lambda, Z_n\rangle}\right] = \frac{1}{n+1}\mathbb{E}\left[e^{\langle\lambda, Z_0\rangle}\right]\Pi_n(e(\lambda))$$

where $\Pi_n(z) = \prod_{j=1}^{n}\left(1 + \frac{z}{j}\right)$, $z \in \mathbb{C}$. From (2.1.3), we get

$$\boldsymbol{\alpha}_n(\lambda) \longrightarrow e(\lambda) - 1 < \infty, \ \ \forall\ \lambda \in \mathbb{R}^d. \tag{4.1.3}$$

Thus the LDP now follows from the Gärtner-Ellis Theorem (see Remark (a) on page 45 of [27] or page 66 of [16]).

We next note that $I(\cdot)$ is a convex function because it is the Fenchel-Legendre dual of $e(\lambda) - 1$, which is finite for all $\lambda \in \mathbb{R}^d$.

Finally, we will show that $I(\cdot)$ is good rate function, that is, the level sets $A(a) = \{x \colon I(x) \leq a\}$ are compact for all $a > 0$. The function I is a rate function. Hence by definition, it is lower semicontinuous. So, it is enough to prove that, $A(a)$ is bounded for all $a \in \mathbb{R}$. Observe that, for all $x \in \mathbb{R}^d$,

$$I(x) \geq \sup_{\|\lambda\|=1} \{\langle x, \lambda \rangle - e(\lambda) + 1\}.$$

Now the function $\lambda \mapsto e(\lambda)$ is continuous and $\{\lambda \colon \|\lambda\| = 1\}$ is a compact set. So $\exists\ \lambda_0 \in \{\lambda \colon \|\lambda\| = 1\}$, such that $\sup_{\|\lambda\|=1} e(\lambda) = e(\lambda_0)$. Therefore, for $\|x\| \neq 0$, choosing $\lambda = \frac{x}{\|x\|}$, we have $I(x) \geq \|x\| - e(\lambda_0) + 1$. So, if $x \in A(a)$, then,

$$\|x\| \leq (a + e(\lambda_0) - 1).$$

This proves that the level sets are bounded, which completes the proof. □

Our next result is an easy consequence of (4.1.1) which can be used to compute explicit formula for the rate function I.

Theorem 4.1.2. *The rate function I is same as the rate function for the large deviation of the empirical means of i.i.d. random vectors with distribution corresponding to the distribution of the following random vector*

$$W = \sum_{i=1}^{N} Y_i, \tag{4.1.4}$$

where $N \sim$ Poisson (1) and is independent of $(Y_j)_{j\geq 1}$, which are the i.i.d. increments of the associated random walk.

Proof. We first observe that $\log \mathbb{E}\left[e^{\langle \lambda, W \rangle}\right] = e(\lambda) - 1$. The rest then follows from (4.1.1) and Cramér's Theorem (see Theorem 2.2.30 of [27]). □

For $d = 1$, one can get more information about the rate function I, in particular the following result is a consequence of Theorem 4.1.2 and Lemma 2.2.5 of [27].

Proposition 4.1.1. *Suppose $d = 1$, then $I(x)$ is non-decreasing when $x \geq \mu$ and non-increasing* *when $x \leq \mu$. Moreover,*

$$I(x) = \begin{cases} \sup_{\lambda \geq 0}\{x\lambda - e(\lambda) + 1\} & \text{if } x \geq \mu, \\ \sup_{\lambda \leq 0}\{x\lambda - e(\lambda) + 1\} & \text{if } x \leq \mu. \end{cases} \tag{4.1.5}$$

In particular, $I(\mu) = \inf_{x \in \mathbb{R}} I(x)$.

The following is an immediate corollary of the above result and Theorem 4.1.1.

Corollary 4.1.1. *Let $d = 1$, then for any $\epsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{\log n} \log \mathbb{P}\left(\frac{Z_n}{\log n} \geq \mu + \epsilon\right) = -I\left(\mu + \epsilon\right), \tag{4.1.6}$$

and

$$\lim_{n \to \infty} \frac{1}{\log n} \log \mathbb{P}\left(\frac{Z_n}{\log n} \leq \mu - \epsilon\right) = -I\left(\mu - \epsilon\right). \tag{4.1.7}$$

We end the chapter with explicit computations of the rate functions for two examples of infinite color urn models associated with random walks on one dimensional integer lattice.

Example 4.1.1. *Our first example is the case when the random walk is the right shift as introduced in Subsection 1.2.1, which moves deterministically one step to the right at a time. In other words, $Y_1 = 1$, with probability one. In this case $\mu = 1$ and $\sigma^2 = 1$. Also the moment generating function of Y_1, is given by $e\left(\lambda\right) := e^{\lambda}$, $\lambda \in \mathbb{R}$. By Theorem 4.1.2, the rate function for the associated infinite color urn model is same as the rate function for a Poisson random variable with mean 1 (see page 96 of [28]), that is,*

$$I(x) = \begin{cases} \infty & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ x \log x - x + 1 & \text{if } x > 0. \end{cases} \tag{4.1.8}$$

Example 4.1.2. *Our next example is the case when the random walk is the* simple symmetric random walk *on the one dimensional integer lattice. For this case, we note that $\mu = 0$, $\sigma^2 = 1$ and the moment generating function Y_1 is $e\left(\lambda\right) = \cosh \lambda$, $\lambda \in \mathbb{R}$. Therefore, from* (4.1.5)*, we have*

$$I(x) = \begin{cases} \sup_{\lambda \geq 0}\{x\lambda - \cosh \lambda + 1\} & \text{if } x \geq 0, \\ \sup_{\lambda \leq 0}\{x\lambda - \cosh \lambda + 1\} & \text{if } x \leq 0. \end{cases}$$

Fixing $x \in \mathbb{R}$, define $f_x\left(\lambda\right) := x\lambda - \cosh \lambda + 1$. Differentiating $f_x\left(\lambda\right)$ with respect to λ, we *obtain*

$$\frac{\mathrm{d}f_x(\lambda)}{\mathrm{d}\lambda} = x - \sinh\lambda, \tag{4.1.9}$$

and

$$\frac{\mathrm{d}^2 f_x(\lambda)}{\mathrm{d}\lambda^2} = -\cosh\lambda < 0 \textit{ for all } \lambda \in \mathbb{R}. \tag{4.1.10}$$

$\lambda = \sinh^{-1} x$*, solves the equation* $\frac{\mathrm{d}f_x(\lambda)}{\mathrm{d}\lambda} = 0$*, for every fixed* $x \in \mathbb{R}$*. Therefore, the function* $f_x(\lambda)$ *attains its maximum at* $\lambda = \sinh^{-1} x$*. Hence, the rate function for the associated infinite color urn model turns out to be*

$$I(x) = x\sinh^{-1} x - \sqrt{1+x^2} + 1. \tag{4.1.11}$$

Chapter 5

Representation theorem [1]

In this chapter, we present a coupling of the urn model with the associated Markov chain, which will improve the representation given in (2.1.4). This method is novel and useful in deriving several results for the expected configuration of the urn. There are a few standard methods for analyzing finite color urn models which are mainly based on martingale techniques [38, 55, 14, 15, 25], stochastic approximations [44] and embedding into continuous time pure birth processes [3, 40, 41, 5]. Typically, the analysis of a finite color urn is heavily dependent on the *Jordan decomposition* [24] of matrices and the *Perron-Frobenius theory* [58] of matrices with positive entries, [3, 38, 40, 41, 5, 14, 25]. The absence of such theories for infinite dimensional matrices makes the analysis of urns with infinitely many colors quite difficult and challenging. The improved representation derived here will help us bypass this difficulty.

5.1 Representation theorem

Theorem 5.1.1. *Consider an urn model with colors indexed by a set S, replacement matrix R and initial configuration U_0. Let $(X_n)_{n\geq 0}$ be the associated Markov chain. Then, there exists an increasing non negative sequence of stopping times $(\tau_n)_{n\geq 0}$ with $\tau_0 = 0$, which are independent of the Markov chain $(X_n)_{n\geq 0}$, such that, if Z_n denotes the color of the $(n+1)$-th selected ball, then, for any $n \geq 0$,*

$$Z_n \stackrel{d}{=} X_{\tau_n}. \tag{5.1.1}$$

[1] This chapter is based on the paper entitled "A New Approach to Pólya Urn Schemes and Its Infinite Color Generalization", [6].

Moreover, as $n \to \infty$*,*

$$\frac{\tau_n}{\log n} \longrightarrow 1 \text{ a.s.} \tag{5.1.2}$$

and

$$\frac{\tau_n - \log n}{\sqrt{\log n}} \Rightarrow N(0,1). \tag{5.1.3}$$

Remark 5.1.1. *Theorem 5.1.1 will be referred to as the* representation theorem. *It is worthwhile to note here that* (5.1.1) *gives only a marginal representation of* Z_n*, for each* $n \geq 0$*. The following need not be true:*

$$(Z_n)_{n\geq 0} \stackrel{d}{=} (X_{\tau_n})_{n\geq 0}. \tag{5.1.4}$$

This is because $(X_{\tau_n})_{n\geq 0}$ *is a Markov chain, but* $(Z_n)_{n\geq 0}$ *is not necessarily Markov.*

Observe that the probability mass function for Z_n is $\left(\frac{\mathbb{E}[U_{n,v}]}{n+1}\right)_{v\in S}$. Therefore (5.1.1) will be useful in deriving results about the expected configuration of the urn. However, the representation theorem may not be useful in deriving asymptotic properties of the random configuration of the urn. It is worthwhile to note here that Theorem 5.1.1 holds for any S, be it finite or infinite. Consequently, (5.1.1) will be used in the next section to rederive several known results for finite colors. In the next chapter, we will apply Theorem 5.1.1 to derive new results for the infinite color case.

Proof of Theorem 5.1.1. It is clear that the probability mass function for Z_n is given by $\left(\frac{\mathbb{E}[U_{n,v}]}{n+1}\right)_{v\in S}$. From (1.1.4), we have

$$U_n = U_{n-1} + \chi_n R.$$

Denote by $R(u,v)$ the v-th entry of the u-th row of the matrix R, for all $u, v \in S$. So,

$$\begin{aligned} \frac{U_n}{n+1} &= \frac{n}{n+1}\frac{U_{n-1}}{n} + \frac{1}{n+1}\chi_n R \\ &= \frac{n}{n+1}\frac{U_{n-1}}{n} + \frac{1}{n+1}R(Z_{n-1}, \cdot). \end{aligned} \tag{5.1.5}$$

We will prove this theorem by induction on $n \in \mathbb{N}$. Let $(X_n)_{n\geq 0}$ be the Markov chain associated with the urn process. Then the initial distribution of $(X_n)_{n\geq 0}$ is given by U_0. Let us denote by $\mathbb{P}_{U_0}$ the law of the Markov chain $(X_n)_{n\geq 0}$. We know that the distribution of Z_0 is given by U_0. Therefore, (5.1.1) is trivially true for $n = 0$.

Let $(I_j)_{j\geq 1}$ be a sequence of independent Bernoulli random variables, with $\mathbb{E}[I_j] = \frac{1}{j+1}$, such that $(I_j)_{j\geq 1}$ is independent of $(X_n)_{n\geq 0}$. Define $\tau_n = \sum_{j=1}^n I_j$, and $\tau_0 \equiv 0$. Observe that, the law of τ_n is same as in Lemma 1.2.1. It is easy to see that,

$$\begin{aligned}\mathbb{P}_{U_0}\left(X_{\tau_1}=v\right) &= \mathbb{P}_{U_0}\left(\left(1-I_1\right)X_0+I_1X_1=v\right)\\ &= \frac{1}{2}\mathbb{P}_{U_0}\left(X_0=v\right)+\frac{1}{2}\mathbb{P}_{U_0}\left(X_1=v\right)\end{aligned} \tag{5.1.6}$$

The distribution of Z_0 is given by $\left(U_{0,v}\right)_{v\in S}$, therefore

$$\mathbb{E}\left[R\left(Z_0,v\right)\right]=\mathbb{P}_{U_0}\left(X_1=v\right). \tag{5.1.7}$$

Using equations (5.1.5), (5.1.6) and (5.1.7), we have proved (5.1.1) for $n=1$. Let us assume that the result is true for n, that is,

$$Z_n \stackrel{d}{=} X_{\tau_n}. \tag{5.1.8}$$

We want to show

$$Z_{n+1} \stackrel{d}{=} X_{\tau_{n+1}}.$$

We know that the distribution of Z_{n+1} is given by $\frac{1}{n+2}\left(\mathbb{E}\left[U_{n+1,v}\right]\right)_{v\in S}$. For $v\in S$, we have

$$\begin{aligned}\mathbb{P}_{U_0}\left(X_{\tau_{n+1}}=v\right) &= \mathbb{P}_{U_0}\left(\left(1-I_{n+1}\right)X_{\tau_n}+I_{n+1}X_{\tau_n+1}=v\right)\\ &= \frac{n+1}{n+2}\mathbb{P}_{U_0}\left(X_{\tau_n}=v\right)+\frac{1}{n+2}\mathbb{P}_{U_0}\left(X_{\tau_n+1}=v\right).\end{aligned}$$

By assumption (5.1.8), we have

$$R\left(Z_n,v\right)=\mathbb{P}\left(X_{\tau_n+1}=v|X_{\tau_n}\right).$$

Therefore,

$$\mathbb{E}\left[R\left(Z_n,v\right)\right]=\mathbb{P}_{U_0}\left(X_{\tau_n+1}=v\right).$$

This proves (5.1.1).

Now we will show (5.1.2) and (5.1.3). As observed in (1.2.6),

$$\mathbb{E}\left[\tau_n\right]\sim\log n \text{ as } n\to\infty. \tag{5.1.9}$$

From (1.2.7), we obtain

$$\operatorname{Var}\left(\tau_n\right)=\textstyle\sum_{j=1}^{n}\frac{1}{j+1}-\frac{1}{(j+1)^2}\sim\log n \text{ as } n\to\infty. \tag{5.1.10}$$

Observe that,

$$\sum_{n\geq1}\frac{\operatorname{Var}(I_n)}{\left(\log n\right)^2}\leq\sum_{n\geq1}\frac{1}{\left(n+1\right)\left(\log n\right)^2}<\infty.$$

Therefore, by the Strong Law of Large Numbers for independent random variables (see Theorem 2 on page 364 of [59]), we obtain as $n \to \infty$,

$$\frac{\tau_n - \mathbb{E}\left[\tau_n\right]}{\log n} \longrightarrow 0 \text{ a.s.} \tag{5.1.11}$$

From (1.2.6), we have $\frac{\mathbb{E}[\tau_n]}{\log n} \longrightarrow 1$ as $n \to \infty$. This together with (5.1.11) implies that as $n \to \infty$,

$$\frac{\tau_n}{\log n} \longrightarrow 1 \text{ a.s.}$$

This proves (5.1.2). The conclusion in (5.1.3) follows from Lemma 1.2.1. □

The following proposition is immediate from the proof of Theorem 5.1.1.

Proposition 5.1.1. *Under the assumptions made in Theorem 5.1.1, it is possible to choose* $\left(\tau_n\right)_{n \geq 1}$*, such that, for all* $n \geq 1$*,*

$$\tau_n = \sum_{j=1}^{n} I_j. \tag{5.1.12}$$

where $\left(I_j\right)_{j \geq 1}$ *is a sequence of independent Bernoulli random variables, with* $\mathbb{E}\left[I_j\right] = \frac{1}{j+1}$*, and* $\left(I_j\right)_{j \geq 1}$ *is independent of* $\left(X_n\right)_{n \geq 0}$*.*

5.2 Color count statistics

For every $v \in S$, and $n \geq 0$, let

$$N_{n,v} := \sum_{m=0}^{n} \mathbb{1}_{\{Z_m = v\}}$$

denote the color count statistics for v. Note that for every $n \geq 0$,

$$\sum_{v \in S} N_{n,v} = n + 1.$$

Here, we present some results for the color count statistics, for both finite and infinite color urn models.

Lemma 5.2.1. *Consider an urn model with colors indexed by a set* S*, replacement matrix* R *and initial configuration* U_0*. If* Z_n *denotes the color of the* $(n+1)$*-th selected ball, then for* *every $v \in S$,*

$$\lim_{n\to\infty}\left(\frac{U_{n,v}}{n+1}-\frac{1}{n+1}\sum_{u\in S}N_{n,u}R(u,v)\right)=0, \text{ a.s.} \tag{5.2.1}$$

where $R(u,v)$ denotes the v-th element of the u-th row of R.

Proof. For $n \geq 0$, the following is obvious from (1.1.4)

$$U_{n+1}=U_n+\chi_{n+1}R=U_0+\sum_{k=1}^{n+1}\chi_k R.$$

Writing $\chi_n = (\chi_{n,u})_{u\in S}$, we observe that $\chi_{n,u} = \mathbb{1}_{\{Z_{n-1}=u\}}$ for every $u \in S$. Therefore, it follows that

$$U_{n,v}=U_{0,v}+\sum_{u\in S}\sum_{m=0}^{n}\mathbb{1}_{\{Z_m=u\}}R(u,v)=U_{0,v}+\sum_{u\in S}N_{n,u}R(u,v). \tag{5.2.2}$$

This implies that, as $n \to \infty$,

$$\left(\frac{U_{n,v}}{n+1}-\frac{1}{n+1}\sum_{u\in S}N_{n,u}R(u,v)\right)=\frac{U_{0,v}}{n+1}\longrightarrow 0 \text{ a.s.} \tag{5.2.3}$$

$\square$

Remark 5.2.1. *It is worthwhile to note that the identity* (5.2.2) *imply that* (5.2.1) *holds for every realization.*

Corollary 5.2.1. *Consider an urn model with colors indexed by a set S, replacement matrix R and initial configuration U_0. Let $\pi = (\pi_u)_{u\in S}$ be a solution to $\pi R = \pi$, such that $\pi_u \geq 0$, for each $u \in S$. If for all $v \in S$,*

$$\frac{N_{n,v}}{n+1}\longrightarrow \pi_v \text{ a.s. as } n\to\infty, \tag{5.2.4}$$

then, for every $v \in S$,

$$\frac{U_{n,v}}{n+1}\longrightarrow \pi_v \text{ a.s. as } n\to\infty. \tag{5.2.5}$$

Remark 5.2.2. *For infinitely many colors, it is well known from the standard theory of Markov chains (see page 130 of [11]) that there exists a unique solution $\pi = (\pi_u)_{u\in S}$, with $\pi_u > 0$, for each $u \in S$ and $\sum_{u\in S}\pi_u = 1$ to*

$$\pi R = \pi,$$

if and only if, R is irreducible and positive recurrent. However, if R is irreducible and null recurrent, then $\pi_u = 0$, for all $u \in S$.

Proof of Corollary 5.2.1. Assume (5.2.4) holds, then to prove (5.2.5), we observe that from (5.2.1), it is enough to prove that, for every $v \in S$,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{u\in S} N_{n,u} R(u,v) = \pi_v \text{ a.s.} \tag{5.2.6}$$

From (5.2.4), we know that for every $u \in S$, as $n \to \infty$,

$$\frac{N_{n,u}}{n+1} \longrightarrow \pi_u \text{ a.s.}$$

Therefore, by dominated convergence theorem, it follows that, as $n \to \infty$,

$$\frac{1}{n+1} \sum_{u\in S} N_{n,u} R(u,v) \longrightarrow \pi_v \text{ a.s.} \tag{5.2.7}$$

for every $v \in S$. □

Corollary 5.2.2. *Consider an infinite color urn model with colors indexed by a set S, replacement matrix R and initial configuration U_0. Let R be irreducible. Then, the following are equivalent:*

(i) For every $v \in S$, $\lim_{n\to\infty} \frac{U_{n,v}}{n+1} = 0$ a.s.;

(ii) For every $v \in S$, $\lim_{n\to\infty} \frac{N_{n,v}}{n+1} = 0$ a.s.

Proof. First we prove that (i) implies (ii). Fix $v \in S$. Since R is irreducible, given $v \in S$, there exists $u \in S$, such that $R(v,u) > 0$. Since we have assumed (i) holds, we have

$$\lim_{n\to\infty} \frac{U_{n,u}}{n+1} = 0.$$

Therefore, (5.2.1) implies that for this chosen $u \in S$,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{w\in S} N_{n,w} R(w,u) = 0 \text{ a.s.} \tag{5.2.8}$$

Since, for every $w \in S$, $N_{n,w} R(w,u) \geq 0$, therefore, (5.2.8) implies that,

$$\lim_{n\to\infty} \frac{1}{n+1} N_{n,w} R(w,u) = 0. \tag{5.2.9}$$

Since $R(v,u) > 0$, (5.2.9) implies that,

$$\lim_{n\to\infty} \frac{N_{n,v}}{n+1} = 0 \text{ a.s.} \tag{5.2.10}$$

This proves that (i) implies (ii).

The proof that (ii) implies (i) is similar to the proof of Corollary 5.2.1. □

Remark 5.2.3. *It is worthwhile to note here that if R is reducible, then it may happen that*

$$\lim_{n\to\infty} \frac{U_{n,v}}{n+1} = 0 \text{ a.s. for some but not all } v. \tag{5.2.11}$$

Consider the following example where the $S = \mathbb{N}$ and $R = ((R(i,j)))_{i,j\in\mathbb{N}}$, such that

$$R(i,j) = \begin{cases} 1 & \text{if } i = 1, j \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

In this case, it follows trivially that

$$\lim_{n\to\infty} \frac{U_{n,1}}{n+1} = 1, \text{ a.s.}$$

and for all $j \geq 2$,

$$\lim_{n\to\infty} \frac{U_{n,j}}{n+1} = 0, \text{ a.s.}$$

5.3 Applications of the representation theorem for finite color urn models

We will now present some applications of the representation theorem for finite color urn models. These results are already available in literature, we will give alternative proofs as applications of the representation theorem. More applications leading to new results for infinite color urn models are available in Chapter 6.

5.3.1 Irreducible and aperiodic replacement matrices

Theorem 5.3.1. *Consider an urn model with colors indexed by $\{1, 2, \ldots, K\}$. Let R be irreducible, and aperiodic, with a unique stationary distribution $\pi = (\pi_j)_{1\leq j\leq K}$. Then, as $n \to \infty$,*

$$\mathbb{P}(Z_n = j) \longrightarrow \pi_j, \text{ for every } j \in \{1, 2, \ldots, K\}. \tag{5.3.1}$$

The proof (5.3.1) is available in [3, 38, 40, 5] using techniques different from the representation (5.1.1). Here, we present the proof as an application of (5.1.1).

Proof. Since $(X_n)_{n \geq 0}$ is irreducible and aperiodic, therefore using the standard limit theorems for Markov chains, see [11], we have as $n \to \infty$,

$$\mathbb{P}(X_n = j) \longrightarrow \pi_j, \text{ for every } j \in \{1, 2, \ldots, K\}. \tag{5.3.2}$$

From (5.1.1), we know that $Z_n \stackrel{d}{=} X_{\tau_n}$, where $(\tau_n)_{n \geq 1}$ is independent of $(X_n)_{n \geq 0}$. From (5.1.2), we know that $\tau_n \longrightarrow \infty$, as $n \to \infty$. This together with (5.3.2) implies that, as $n \to \infty$,

$$\mathbb{P}(Z_n = j) = \mathbb{P}(X_{\tau_n} = j) \longrightarrow \pi_j, \tag{5.3.3}$$

$j \in \{1, 2, \ldots, K\}$. This completes the proof. □

Corollary 5.3.1. *Consider an urn model with colors indexed by* $\{1, 2, \ldots, K\}$. *Let* R *be irreducible, and aperiodic, with a unique stationary distribution* $\pi = (\pi_j)_{1 \leq j \leq K}$. *Then, as* $n \to \infty$,

$$\frac{\mathbb{E}[N_{n,j}]}{n+1} \longrightarrow \pi_j, \textit{ for all } 1 \leq j \leq K. \tag{5.3.4}$$

Proof.

$$\mathbb{E}[N_{n,j}] = \sum_{m=0}^{n} \mathbb{P}(Z_m = j).$$

Therefore, from (5.3.1), it follows that for every $1 \leq j \leq K$,

$$\frac{\mathbb{E}[N_{n,j}]}{n+1} = \frac{1}{n+1} \sum_{m=0}^{n} \mathbb{P}(Z_m = j) \longrightarrow \pi_j \text{ as } n \to \infty. \tag{5.3.5}$$

□

If R be irreducible, and aperiodic, with unique stationary distribution $\pi = (\pi_j)_{1 \leq j \leq K}$, then, it is well known in literature [3, 38, 40, 5], as $n \to \infty$,

$$\frac{U_{n,j}}{n+1} \longrightarrow \pi_j \text{ a.s. for all } j \in \{1, 2, \ldots, K\}. \tag{5.3.6}$$

The proof of (5.3.6) is available in [38, 5], which uses the Perron-Frobenius theory and the Jordan decomposition of matrices. The same is available in [40, 3], using the Athreya-Karlin embedding of the urn processes into continuous time multi-type Markov branching processes. The key to all these techniques is the existence of a dominant eigenvalue, as obtained from Perron-Frobenius theory of matrices with positive entries [58] and the Jordan decomposition of matrices [24]. Here we provide a simplification of the proof given in [5].

If R is irreducible, then by the Perron-Frobenius theory, 1 is a simple eigenvalue of R, and if λ is any other eigenvalue of R, then $Re\,(\lambda) < 1$. As R is irreducible, there exists a non-singular matrix T, such that R admits the following Jordan decomposition

$$T^{-1}RT = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & J_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & J_s \end{pmatrix} \tag{5.3.7}$$

with

$$J_t = \begin{pmatrix} \lambda_t & 1 & 0 & \dots & 0 \\ 0 & \lambda_t & 1 & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_t & 1 \\ 0 & 0 & 0 & \dots & \lambda_t \end{pmatrix} \tag{5.3.8}$$

where $1, \lambda_1, \lambda_2, \dots, \lambda_s$ are the eigenvalues of R, with $Re\,(\lambda_t) < 1$, for all $1 \leq t \leq s$. Let

$$\alpha := \max\,\{Re\,(\lambda_t) : 1 \leq t \leq s\},$$

and

$$\beta := \max\,\{\beta_t \colon 1 \leq t \leq s\},$$

where β_t denotes the order of the matrix J_t. Observe that $\alpha < 1$. It is shown in [5] that for all $n \geq 0$, and for all $1 \leq j \leq K$,

$$\mathrm{Var}\,(U_{n,j}) \leq C V_n^2, \tag{5.3.9}$$

for a suitable constant $C > 0$, where

$$V_n = \begin{cases} \sqrt{n} & \text{if } \alpha < \frac{1}{2}, \\ \sqrt{n}\log^{\beta-\frac{1}{2}} n & \text{if } \alpha = \frac{1}{2}, \\ n^{\alpha}\log^{\beta-1} n & \text{if } \alpha > \frac{1}{2}. \end{cases} \tag{5.3.10}$$

Using (5.3.9) and (5.3.10), the authors in [5] showed that, for every $j \in \{1, 2, \ldots, K\}$,

$$\frac{U_{n,j} - \mathbb{E}\left[U_{n,j}\right]}{n+1} \longrightarrow 0 \text{ a.s. as } n \to \infty.$$

From (5.3.1) it is immediate that, for all $j \in \{1, 2, \ldots, K\}$,

$$\frac{\mathbb{E}\left[U_{n,j}\right]}{n+1} \longrightarrow \pi_j \text{ as } n \to \infty. \tag{5.3.11}$$

This implies (5.3.6).

5.3.2 Reducible replacement matrices

Consider an urn model with finitely many colors, indexed by the set $\{1, 2, \ldots, K\}$. Suppose the replacement matrix is reducible without isolated blocks, and can be written in an upper triangular form, given by

$$R = \begin{pmatrix} r_{11} & r_{12} & \ldots & r_{1k} \\ \vdots & \ddots & & \vdots \\ 0 & & \ldots & Q_{qq} \end{pmatrix} \tag{5.3.12}$$

where Q_{qq} is a $q \times q$ irreducible, aperiodic sub-matrix with a stationary distribution π_q. The block Q_{qq} consists of the colors, indexed by $S_q \subset \{1, 2, \ldots, K\}$. Observe that if $(X_n)_{n \geq 0}$ is the associated Markov chain with transition probability matrix R, then S_q denotes the collection of all its recurrent states. Furthermore, if $X_m \in S_q$ for some $m \geq 0$, then $\mathbb{P}\left(X_n \in S_q\right) = 1$ for all $n \geq m$. It has been proved in Proposition 4.3 of [38], that $\lim_{n\to\infty} \frac{U_n}{n+1} = (0, 0, \ldots, \pi_q)$ a.s. Here, we prove the expected version of this result using Theorem 5.1.1.

Corollary 5.3.2. *Consider an urn model with replacement matrix given by* (5.3.12) *and stationary distribution* $(0, 0, \ldots, \pi_q)$*, where* $\pi_q = \left(\pi_{q,j}\right)_{j \in S_q}$*. Then, as* $n \to \infty$*,*

$$\mathbb{P}\left(Z_n = j\right) \longrightarrow 0, \textit{for } j \notin S_q \tag{5.3.13}$$

and

$$\mathbb{P}\left(Z_n = j\right) \longrightarrow \pi_{q,j}, \textit{for } j \in S_q. \tag{5.3.14}$$

Proof. Since R is upper triangular matrix, given by (5.3.12), it follows that,

$$\mathbb{P}_{U_0}\left(X_n \in S_q \text{ for some } n \geq 0\right) = 1, \tag{5.3.15}$$

where $X_0 \sim U_0$. This implies that for every $j \notin S_q$,

$$\mathbb{P}_{U_0}\left(X_n = j\right) \longrightarrow 0 \text{ as } n \to \infty. \tag{5.3.16}$$

From (5.1.1), we know that $Z_n \stackrel{d}{=} X_{\tau_n}$, where $\left(\tau_n\right)_{n\geq 1}$ is independent of $\left(X_n\right)_{n\geq 0}$. From (5.1.2), we know that $\tau_n \longrightarrow \infty$, as $n \to \infty$. Hence, (5.3.16) implies (5.3.13). Since, Q_{qq} is irreducible and aperiodic, (5.3.14) follows immediately from (5.3.1). □

We will now present a particular example to illustrate an application of Theorem 5.1.1.

Example 5.3.1. *Consider an urn model with colors indexed by* $\{0, 1\}$*, with one "dominant" color. Let the replacement matrix be given by*

$$R = \begin{pmatrix} s & 1-s \\ 0 & 1 \end{pmatrix} \tag{5.3.17}$$

where $0 < s < 1$.

The associated Markov chain is on the state space $S = \{0, 1\}$. The color corresponding to the absorbing state is the dominant color. The rate of growth for the non-dominant color has been calculated [41] and [55, 14], and it is shown that

$$\frac{U_{n,0}}{n^s} \longrightarrow W \text{ a.s.} \tag{5.3.18}$$

where W is some random variable. In [41], the author embedded the urn process into a continuous time multi-type Markov branching process to obtain the distribution of W. In [14], the authors have proved that W is non-degenerate using L_2 bounded martingales. We will show that as $n \to \infty$,

$$\frac{\mathbb{E}\left[U_{n,0}\right]}{n^s} \longrightarrow \frac{1}{\Gamma\left(s+1\right)}. \tag{5.3.19}$$

Though (5.3.19) is weaker than (5.3.18), our approach is simple and avoids martingale techniques as well as the embedding into branching processes. We essentially use the representation (5.1.1).

To prove (5.3.19), note that from (5.1.1), we know that

$$\mathbb{P}\left(Z_n = 0\right) = \mathbb{P}\left(X_{\tau_n} = 0\right).$$

Let us define $T := \inf\{n \geq 1 \colon X_n = 1\}$. Then

$$\mathbb{P}\left(X_{\tau_n} = 0 \middle|\, \tau_n\right) = \mathbb{P}\left(T \geq \tau_n | \tau_n\right) = s^{\tau_n}.$$

Therefore, $\mathbb{P}\left(X_{\tau_n} = 0\right) = \mathbb{E}\left[s^{\tau_n}\right]$. It follows from (5.1.12) that, $\tau_n = \sum_{j=1}^{n} I_j$, where $\left(I_j\right)_{j\geq 1}$ is a sequence of independent Bernoulli random variables, with $\mathbb{E}\left[I_j\right] = \frac{1}{j+1}$, independent of $\left(X_n\right)_{n\geq 0}$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[s^{\tau_n}\right] &= \prod_{j=1}^{n}\left(1 - \frac{1}{j+1} + \frac{s}{j+1}\right) \\
&= \frac{1}{n+1}\prod_{j=1}^{n}\left(1 + \frac{s}{j}\right) \\
&= \frac{1}{n+1}\Pi_n\left(s\right), \qquad (5.3.20)
\end{aligned}$$

where $\Pi_n\left(s\right) = \prod_{j=1}^{n}\left(1 + \frac{s}{j}\right)$. We know that $\mathbb{P}\left(Z_n = 0\right) = \frac{\mathbb{E}[U_{n,0}]}{n+1}$. Therefore, from (5.3.20), we have $\mathbb{E}\left[U_{n,0}\right] = \Pi_n\left(s\right)$. From (2.1.3), it follows that $\lim_{n\to\infty}\frac{\Pi_n(s)}{n^s} = \frac{1}{\Gamma(s+1)}$. This implies that, as $n \to \infty$,

$$\frac{\mathbb{E}\left[U_{n,0}\right]}{n^s} = \frac{\Pi_n\left(s\right)}{n^s} \longrightarrow \frac{1}{\Gamma\left(s+1\right)}.$$

Remark 5.3.1. *Let us consider the Pólya urn model with the set of colors indexed by $\{1, 2, \ldots K\}$, $K \geq 2$. Here, $R = \mathbb{I}_K$. The associated Markov chain is reducible into K classes, that is, every state is an absorbing state. From the representation theorem, it follows that*

$$\frac{\mathbb{E}\left[U_{n,j}\right]}{n+1} = \mathbb{P}\left(Z_n = j\right) = \mathbb{P}\left(X_{\tau_n} = j\right) = U_{0,j}. \qquad (5.3.21)$$

Hence, (5.3.21) *necessarily illustrates that the representation theorem does not provide any new information regarding the Pólya urn model from that already available in literature (see [57, 56, 51]).*

Chapter 6

General replacement matrices [1]

In the previous chapters, Chapters 2, 3 and 4, we have obtained various asymptotic results for urn models associated with bounded increment random walks. This chapter will focus on urn models with countably infinite set of colors and general replacement matrices. We will use the representation (5.1.1), to derive asymptotic properties for these general urn models.

6.1 Infinite color urn models with irreducible and aperiodic replacement matrices

In this section, we assume that R is irreducible and aperiodic, unless stated otherwise. For our model, the weak convergence for the randomly selected color can be obtained from Theorem 5.1.1.

Theorem 6.1.1. *Consider an urn model with colors indexed by a set S, replacement matrix R and initial configuration U_0. Let R be irreducible and aperiodic. Then, as $n \to \infty$,*

$$\mathbb{P}\left(Z_n = v\right) \longrightarrow \pi_v, \text{ for every } v \in S, \tag{6.1.1}$$

where $\left(\pi_v\right)_{v\in S}$ is the unique stationary distribution if R is positive recurrent, otherwise $\pi_v = 0$, for all $v \in S$.

Proof. Since R is irreducible and aperiodic, therefore from the standard limit theorems for Markov chains, see [11], it follows that, as $n \to \infty$,

$$\mathbb{P}\left(X_n = v\right) \longrightarrow \pi_v, \text{ for every } v \in S, \tag{6.1.2}$$

[1] This chapter is based on the paper entitled "A New Approach to Pólya Urn Schemes and Its Infinite Color Generalization", [6].

where $(\pi_v)_{v \in S}$ is the unique stationary distribution if R is positive recurrent, otherwise $\pi_v = 0$, for all $v \in S$. The rest of the proof is similar to the proof of (5.3.1). $\square$

The following corollary is immediate from Theorem 6.1.1 .

Corollary 6.1.1. *Consider an urn model with colors indexed by a set S, replacement matrix R and initial configuration U_0. Let R be irreducible and aperiodic. Suppose there exists a non random sequence $(l_v)_{v \in S}$, such that $0 \le l_v \le 1$, and*

$$\lim_{n\to\infty} \frac{U_{n,v}}{n+1} = l_v \text{ a.s.} \tag{6.1.3}$$

Then, either $l_v > 0$, for every $v \in S$ and $\sum_{v \in S} l_v = 1$, or $l_v = 0$, for every $v \in S$.

Proof. Suppose (6.1.3) holds. Observe that for every $v \in S$, $0 \le \frac{U_{n,v}}{n+1} \le 1$. Therefore, by the bounded convergence theorem we know that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[U_{n,v}\right]}{n+1} = l_v.$$

Note that $\mathbb{P}\left(Z_n = v\right) = \frac{\mathbb{E}[U_{n,v}]}{n+1}$. Therefore, (6.1.1) implies that $\pi_v = l_v$ for all $v \in S$. Hence, it follows that either $l_v > 0$ for every $v \in S$, and $\sum_{v \in S} l_v = 1$, or $l_v = 0$, for every $v \in S$. $\square$

Recall that the distribution of Z_n is given by $\frac{\mathbb{E}[U_n]}{n+1}$. Therefore, when R is irreducible, aperiodic with stationary distribution π, (6.1.1) implies that

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[U_{n,v}\right]}{n+1} = \pi_v, \text{ for every } v \in S. \tag{6.1.4}$$

However, (6.1.4) is a weaker result compared to the finite color case, where we know (see (5.3.6)) that

$$\lim_{n\to\infty} \frac{U_{n,v}}{n+1} = \pi_v, \text{ for every } v \in S. \tag{6.1.5}$$

As observed in Section 5.3, the proof of (6.1.5) requires techniques from matrix algebra, that are mostly unavailable in infinite dimensions. Similar difficulty is observed in discrete time multi-type Markov branching processes with countably many types, [52]. For infinite S, a possible path to prove

$$\lim_{n\to\infty} \frac{U_{n,v}}{n+1} = \pi_v, \text{ for every } v \in S, \tag{6.1.6}$$

is discussed below.

Note that to prove (6.1.6), from Corollary 5.2.1, it follows that it is enough to prove that for every $v \in S$,

$$\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{Z_k=v\}} \longrightarrow \pi_v \text{ a.s. as } n \to \infty. \tag{6.1.7}$$

Observe that from (6.1.1), we obtain for every $v \in S$,

$$\frac{1}{n+1}\mathbb{E}\left[\sum_{k=0}^{n} \mathbb{1}_{\{Z_k=v\}}\right] = \frac{1}{n+1}\sum_{k=0}^{n} \mathbb{P}\left(Z_k = v\right) \longrightarrow \pi_v \text{ as } n \to \infty. \tag{6.1.8}$$

Further observe that, for $v \in S$,

$$\mathrm{Var}\left(\mathbb{1}_{\{Z_k=v\}}\right) = \mathbb{P}\left(Z_k = v\right)\left(1 - \mathbb{P}\left(Z_k = v\right)\right) \leq 1. \tag{6.1.9}$$

Therefore, from Theorem 6 of [49], it follows that, if we prove

$$\sum_{n\geq 0} \frac{1}{n+1}\mathrm{Var}\left(\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{Z_k=v\}}\right) < \infty, \tag{6.1.10}$$

then, as $n \to \infty$,

$$\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{Z_k=v\}} \longrightarrow \pi_v \text{ a.s.}$$

It is easy to see that,

$$\begin{aligned}\mathrm{Var}\left(\frac{1}{n+1}\sum_{k=0}^{n} \mathbb{1}_{\{Z_k=v\}}\right) &= \frac{1}{(n+1)^2}\sum_{k=0}^{n}\mathrm{Var}\left(\mathbb{1}_{\{Z_k=v\}}\right) \\ &\quad + \frac{2}{(n+1)^2}\sum_{k=0}^{n}\sum_{m=1}^{n-k}\mathrm{Cov}\left(\mathbb{1}_{\{Z_k=v\}}\mathbb{1}_{\{Z_{k+m}=v\}}\right).\end{aligned}$$

Let us denote by

$$J_{n,v}(1) := \frac{1}{(n+1)^2}\sum_{k=0}^{n}\mathrm{Var}\left(\mathbb{1}_{\{Z_k=v\}}\right), \tag{6.1.11}$$

and

$$J_{n,v}(2) := \frac{2}{(n+1)^2}\sum_{k=0}^{n}\sum_{m=1}^{n-k}\mathrm{Cov}\left(\mathbb{1}_{\{Z_k=v\}}\mathbb{1}_{\{Z_{k+m}=v\}}\right). \tag{6.1.12}$$

Therefore, to prove (6.1.10), it is enough to prove that for every $v \in S$,

$$\sum_{n\geq 0}\frac{1}{n+1}J_{n,v}(1) < \infty, \tag{6.1.13}$$

and

$$\sum_{n\geq 0}\frac{1}{n+1}J_{n,v}(2) \text{ converges.} \tag{6.1.14}$$

It is easy to see that,

$$J_{n,v}(1) \;=\; \frac{1}{(n+1)^2}\sum_{k=0}^{n}\mathbb{P}\left(Z_k=v\right)\left(1-\mathbb{P}\left(Z_k=v\right)\right).$$

From (6.1.1) $\frac{1}{(n+1)}\sum_{k=0}^{n}\mathbb{P}\left(Z_k=v\right)\left(1-\mathbb{P}\left(Z_k=v\right)\right)\longrightarrow \pi_v\left(1-\pi_v\right)$ as $n\to\infty$. Therefore, for each $v\in S$, there exists a constant $C_v>0$, such that,

$$\sum_{n\geq 0}\frac{1}{n+1}J_{n,v}(1)\leq C_v\sum_{n\geq 1}\frac{1}{n^2}<\infty. \tag{6.1.15}$$

This proves (6.1.13). Hence, to prove (6.1.10), it is enough to prove (6.1.14) for each $v\in S$. Note that,

$$\text{Cov}\left(\mathbb{1}_{\{Z_k=v\}}\mathbb{1}_{\{Z_{k+m}=v\}}\right)=\mathbb{P}\left(Z_k=v,Z_{k+m}=v\right)-\mathbb{P}\left(Z_k=v\right)\mathbb{P}\left(Z_{k+m}=v\right). \tag{6.1.16}$$

It easy to see that

$$\mathbb{P}\left(Z_k=v,Z_{k+m}=v\right)=\mathbb{P}\left(Z_{k+m}=v\middle|Z_k=v\right)\mathbb{P}\left(Z_k=v\right). \tag{6.1.17}$$

A possible way to estimate $\mathbb{P}\left(Z_{k+m}=v\middle|Z_k=v\right)$ is through the following lemma.

Lemma 6.1.1. *Consider an urn process $\left(U_n\right)_{n\geq 0}$ with replacement matrix R, where R is any stochastic matrix. Then, there exists a Markov chain $\left(\widetilde{X}_n\right)_{n\geq 0}$ on state space S with transition matrix R, such that, for every $u,v\in S$,*

$$\mathbb{P}\left(Z_{k+m}=u\middle|Z_k=v\right)=\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=u\right), \tag{6.1.18}$$

where $\widetilde{\tau}_{m-1}(k+1)\stackrel{d}{=}\sum_{l=k+2}^{k+m}I_l$, for all $m\geq 1$ and $k\geq 0$, and the distribution of $\widetilde{X}_0$ is given by $\frac{\mathbb{E}[U_k]+R_v}{k+2}$, where R_v is the v-th row of R.

Proof. As observed earlier, $\left(\frac{U_n}{n+1}\right)_{n\geq 0}$ is a time inhomogeneous Markov chain with state space as the set of all probability measures on S. Therefore, given $(Z_k=v)$, the law of Z_{k+m} is same as the law of $\widetilde{Z}_{m-1}(k+1)$, for all $m\geq 1$, where $\widetilde{Z}_n(k+1)$ denotes the random color selected at the $(n+1)$-th trial of the urn process $\left(\widetilde{U}_n\right)_{n\geq 0}$ with replacement matrix R, and $\widetilde{U}_0 = \frac{\mathbb{E}[U_k]+R_v}{k+2}$.

Therefore, the distribution of $\widetilde{Z}_0(k+1)$ is given by $\frac{\mathbb{E}[U_k]+R_v}{k+2}$. Observe that the distribution of $\widetilde{Z}_1(k+1)$ is given by $\mathbb{E}\left[\frac{\widetilde{U}_1}{k+3}\right]$, and

$$\mathbb{E}\left[\frac{\widetilde{U}_1}{k+3}\right] = \left(\frac{k+2}{k+3}\right)\frac{\mathbb{E}\left[U_k\right]+R_v}{k+2} + \left(\frac{1}{k+3}\right)\frac{\mathbb{E}\left[U_k\right]+R_v}{k+2}R.$$

Let $\left(\widetilde{X}_n\right)_{n\geq 0}$ be the Markov chain associated with $\left(\widetilde{U}_n\right)_{n\geq 0}$. Then the distribution of $\widetilde{X}_0$ is given by $\widetilde{U}_0 = \frac{\mathbb{E}[U_k]+R_v}{k+2}$. We will prove (6.1.18) by induction on $m\geq 1$. Note that,

$$\mathbb{P}_{\widetilde{U}_0}\left(\widetilde{X}_{\widetilde{\tau}_1(k+1)} = u\right) = \left(\frac{k+2}{k+3}\right)\frac{\mathbb{E}\left[U_{k,u}\right]+R\left(v,u\right)}{k+2} + \left(\frac{1}{k+3}\right)\mathbb{P}_{\widetilde{U}_0}\left(\widetilde{X}_1(k+1)=u\right).$$

This proves (6.1.18) for $m=1$. If we assume (6.1.18) holds for $m\geq 1$, then the proof that (6.1.18) holds for $(m+1)$, follows similar to the proof of (5.1.1). $\square$

We would like to note here, that we have been unable to find a general class of examples of irreducible, and aperiodic replacement matrix R, for which almost sure convergence in (6.1.6) holds. We could prove the almost sure convergence only for the particular example provided below. However, we believe that (6.1.6) holds for any irreducible, and aperiodic replacement matrix R.

Example 6.1.1. *Consider an urn with colors indexed by* $\mathbb{N}\cup\{0\}$*, and the replacement matrix given by* $R := \left(\left(R\left(i,j\right)\right)\right)_{i,j\in\mathbb{N}\cup\{0\}}$*, where*

$$\begin{aligned} &R\left(0,j\right) = \gamma_j > 0, \textit{ for } j\in\mathbb{N}\cup\{0\},\\ &R\left(j,0\right) = 1. \end{aligned} \tag{6.1.19}$$

Since $\gamma_j > 0$, for all $j\in\mathbb{N}\cup\{0\}$, R is irreducible, aperiodic and positive recurrent. Therefore, there exists a stationary distribution $\pi = \left(\pi_j\right)_{j\in\mathbb{N}\cup\{0\}}$, where

$$\pi_0 = \frac{1}{2-\gamma_0}, \text{ and } \pi_j = \frac{\gamma_j}{2-\gamma_0}, \text{ for all } j\geq 1.$$

It follows from (6.1.1), that $\lim_{n\to\infty}\mathbb{P}\left(Z_n=j\right) = \pi_j$, for all $j\in\mathbb{N}\cup\{0\}$. In this particular case, we will prove that for every $i\in\mathbb{N}\cup\{0\}$, as $n\to\infty$,

$$\frac{U_{n,i}}{n+1} \longrightarrow \pi_i \text{ a.s.} \tag{6.1.20}$$

To prove (6.1.20), we first observe that, if we prove

$$\mathbb{P}\left(Z_n = 0 \text{ for some } n \geq 0\right) = 1, \tag{6.1.21}$$

then, without loss of generality, we may assume that $U_0 = \delta_0$.

It is easy to see that $\mathbb{P}\left(Z_n \neq 0 \text{ for all } n \geq 0\right) = \lim_{n\to\infty} \mathbb{P}\left(Z_k \neq 0,\ 1 \leq k \leq n\right)$. Since, R is given by (6.1.19), therefore, $\mathbb{P}\left(Z_k \neq 0,\ 1 \leq k \leq n\right) = \prod_{k=1}^{n}\left(1 - \frac{U_{0,0}+k}{k+1}\right) U_{0,0} \longrightarrow 0$, as $n \to \infty$. This implies that,

$$\mathbb{P}\left(Z_n \neq 0 \text{ for all } n \geq 0\right) = 0.$$

Hence,

$$\mathbb{P}\left(Z_n = 0 \text{ for some } n \geq 0\right) = 1 - \mathbb{P}\left(Z_n \neq 0 \text{ for all } n \geq 0\right) = 1.$$

As observed earlier, we need to show (6.1.14). Note that,

$$\begin{aligned} \operatorname{Cov}\left(\mathbb{1}_{\{Z_k=i\}}\mathbb{1}_{\{Z_{k+m}=i\}}\right) &= \mathbb{P}\left(Z_k = i, Z_{k+m} = i\right) - \mathbb{P}\left(Z_k = i\right)\mathbb{P}\left(Z_{k+m} = i\right) \\ &= \mathbb{P}\left(Z_k = i\right)\left(\mathbb{P}\left(Z_{k+m} = i \middle| Z_k = i\right) - \mathbb{P}\left(Z_{k+m} = i\right)\right) \end{aligned} \tag{6.1.22}$$

Therefore, it follows from (6.1.12), that

$$J_{n,i}(2) = \frac{2}{(n+1)^2}\sum_{k=0}^{n}\mathbb{P}\left(Z_k = i\right)\sum_{m=1}^{n-k}\left(\mathbb{P}\left(Z_{k+m} = i \middle| Z_k = i\right) - \mathbb{P}\left(Z_{k+m} = i\right)\right). \tag{6.1.23}$$

We first prove (6.1.14) for $i = 0$.

$$\mathbb{P}\left(Z_{k+m} = 0\right) = \mathbb{P}\left(Z_{k+m} = 0 \middle| Z_k = 0\right)\mathbb{P}\left(Z_k = 0\right) + \sum_{i\geq 1}\mathbb{P}\left(Z_{k+m} = 0 \middle| Z_k = i\right)\mathbb{P}\left(Z_k = i\right) \tag{6.1.24}$$

Observe that for every $i \geq 1,\ R(i, 0) = 1$. Hence, it follows that for all $i \geq 1$,

$$\mathbb{P}\left(Z_{k+m} = 0 \middle| Z_k = i\right) = \mathbb{P}\left(Z_{k+m} = 0 \middle| Z_k = 1\right).$$

Therefore, from (6.1.24), it follows that,

$$\mathbb{P}\left(Z_{k+m}=0\right)=\mathbb{P}\left(Z_{k+m}=0\middle|Z_k=0\right)\mathbb{P}\left(Z_k=0\right)+\mathbb{P}\left(Z_{k+m}=0\middle|Z_k=1\right)\left(1-\mathbb{P}\left(Z_k=0\right)\right). \tag{6.1.25}$$

Therefore,

$$\begin{aligned}
&\mathbb{P}\left(Z_{k+m}=0\right)-\mathbb{P}\left(Z_{k+m}=0\middle|Z_k=0\right)\\
&=\left(\mathbb{P}\left(Z_{k+m}=0\middle|Z_k=0\right)-\mathbb{P}\left(Z_{k+m}=0\middle|Z_k=1\right)\right)\left(1-\mathbb{P}\left(Z_k=0\right)\right).
\end{aligned} \tag{6.1.26}$$

Now from (6.1.18) it follows that

$$\mathbb{P}\left(Z_{k+m}=0\middle|Z_k=0\right)=\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=0\right),$$

where $\left(\widetilde{X}_n\right)_{n\geq 0}$ is a Markov chain on the state space S, with transition probability matrix R and the distribution of $\widetilde{X}_0$ is given by $\frac{\mathbb{E}[U_k]+(\gamma_j)_{j\geq 0}}{k+2}$. Furthermore, $\widetilde{\tau}_{m-1}(k+1)\overset{d}{=}\sum_{l=k+2}^{k+m} I_l$, for all $m\geq 2$. It is easy to see that,

$$\begin{aligned}
\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=0\right)&=\left(\frac{k+m+\gamma_0}{k+m+1}\right)\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-2}(k+1)}=0\right)\\
&\quad+\frac{1}{k+m+1}\sum_{i\geq 1}\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-2}(k+1)}=i\right)\\
&=\frac{1}{k+m+1}+\left(\frac{k+m-1+\gamma_0}{k+m+1}\right)\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-2}(k+1)}=0\right).
\end{aligned}$$

Repeating the same iteration, we obtain

$$\begin{aligned}
\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=0\right)&=\frac{1}{k+m+1}+\frac{(k+m-1+\gamma_0)}{(k+m+1)(k+m)}\\
&\quad+\frac{(k+m-2+\gamma_0)(k+m-1+\gamma_0)}{(k+m+1)(k+m)(k+m-1)}+\ldots\\
&\quad+A_m(k)\frac{\mathbb{E}\left[U_{k,0}\right]+\gamma_0}{k+2},
\end{aligned} \tag{6.1.27}$$

where

$$A_m(k)=\frac{(k+m-1+\gamma_0)(k+m-2+\gamma_0)\ldots(k+1+\gamma_0)}{(k+m+1)(k+m)\ldots(k+3)}. \tag{6.1.28}$$

Similarly, it is clear that $\mathbb{P}\left(Z_{k+m}=0 \middle| Z_k=1\right) = \mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=0\right)$, where $\left(\widetilde{X}_n\right)_{n\geq 0}$ is a Markov chain on the state space S, with transition probability matrix R and the distribution of $\widetilde{X}_0$ is given by $\frac{\mathbb{E}[U_k]+(1,0,0\ldots)}{k+2}$. Furthermore, $\widetilde{\tau}_{m-1}(k+1) \stackrel{d}{=} \sum_{l=k+2}^{k+m} I_l$, for all $m \geq 2$. Iterating in a fashion similar to (6.1.27), we obtain

$$\begin{aligned}\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=0\right) &= \frac{1}{k+m+1}+\frac{(k+m-1+\gamma_0)}{(k+m+1)\,(k+m)} \\ &+ \frac{(k+m-2+\gamma_0)\,(k+m-1+\gamma_0)}{(k+m+1)\,(k+m)\,(k+m-1)}+\ldots \\ &+ A_m(k)\,\frac{\mathbb{E}\left[U_{k,0}\right]+1}{k+2}.\end{aligned} \tag{6.1.29}$$

where $A_m(k)$ is as in (6.1.28). From (6.1.27) and (6.1.29), it follows that

$$\mathbb{P}\left(Z_{k+m}=0 \middle| Z_k=0\right)-\mathbb{P}\left(Z_{k+m}=0 \middle| Z_k=1\right)=\frac{1}{k+2}A_m(k)\,(\gamma_0-1)\,. \tag{6.1.30}$$

Note that $\frac{1}{k+2}A_m(k) \leq \frac{1}{k+m+1}$. Hence, for all k sufficiently large, we obtain,

$$\begin{aligned}\sum_{m=1}^{n-k}\left|\mathbb{P}\left(Z_{k+m}=0 \middle| Z_k=0\right)-\mathbb{P}\left(Z_{k+m}=0 \middle| Z_k=1\right)\right| &= \sum_{m=1}^{n-k}A_m(k)\frac{1-\gamma_0}{k+2} \\ &\leq (1-\gamma_0)\sum_{m=1}^{n-k}\frac{1}{k+m+1} \\ &\leq C\,(1-\gamma_0)\log(n+1),\end{aligned} \tag{6.1.31}$$

where $C>0$ is an appropriate constant. Therefore, from (6.1.26),

$$\begin{aligned}|J_{n,0}(2)| &\leq \frac{2}{(n+1)^2}\sum_{k=0}^{n}\mathbb{P}\left(Z_k=0\right)\sum_{m=1}^{n-k}\left|\mathbb{P}\left(Z_{k+m}=0 \middle| Z_k=0\right)-\mathbb{P}\left(Z_{k+m}=0\right)\right| \\ &\leq C_1'\frac{\log(n+1)}{(n+1)^2}\sum_{k=1}^{n}\mathbb{P}\left(Z_k=0\right)\left(1-\mathbb{P}\left(Z_k=0\right)\right)\end{aligned} \tag{6.1.32}$$

for a suitable constant $C_1'>0$. It follows from (6.1.1), that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n}\mathbb{P}\left(Z_k=0\right)\left(1-\mathbb{P}\left(Z_k=0\right)\right)=\pi_0\left(1-\pi_0\right).$$

So, for a suitable constant $C_2' > 0$,

$$|J_{n,0}(2)| \leq C_2' \frac{\log(n+1)}{n+1}, \tag{6.1.33}$$

which implies that $\sum_{n\geq 0} \frac{1}{n+1} J_{n,0}(2)$ converges.

We will prove that for every $i \geq 1$, there exists a constant $L_i > 0$, such that,

$$|J_{n,i}(2)| \leq L_i \frac{\log(n+1)}{n+1}. \tag{6.1.34}$$

This will imply that,

$$\Big| \sum_{n\geq 0} \frac{1}{n+1} J_{n,i}(2) \Big| \leq L_i \sum_{n\geq 0} \frac{\log(n+1)}{(n+1)^2} < \infty.$$

It is easy to see that,

$$\mathbb{P}\left(Z_{k+m} = i\right) = \sum_{j\geq 0} \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = j\right) \mathbb{P}\left(Z_k = j\right). \tag{6.1.35}$$

For every $j \geq 1,\ R(j,0) = 1$. Hence, it follows that, for any fixed $i \geq 0$,

$$\mathbb{P}\left(Z_{k+m} = i \middle| Z_k = j\right) = \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 1\right) \text{ for every } j \geq 1.$$

Hence,

$$\begin{aligned}
\mathbb{P}\left(Z_{k+m} = i\right) &= \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 1\right) \sum_{j\geq 1} \mathbb{P}\left(Z_k = j\right) + \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 0\right) \mathbb{P}\left(Z_k = 0\right) \\
&= \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 1\right) \left(1 - \mathbb{P}\left(Z_k = 0\right)\right) + \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 0\right) \mathbb{P}\left(Z_k = 0\right).
\end{aligned} \tag{6.1.36}$$

Using (6.1.36) and the fact that, $\mathbb{P}\left(Z_{k+m} = i \middle| Z_k = i\right) = \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 1\right)$ for every $i \geq 1$, we obtain

$$\begin{aligned}
& \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = i\right) - \mathbb{P}\left(Z_{k+m} = i\right) \\
= & \left[\mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 1\right) - \mathbb{P}\left(Z_{k+m} = i \middle| Z_k = 0\right)\right] \mathbb{P}\left(Z_k = 0\right).
\end{aligned} \tag{6.1.37}$$

Therefore, from (6.1.23) and (6.1.37),

$$J_{n,i}(2) = \frac{2}{(n+1)^2}\sum_{k=0}^{n}\mathbb{P}\left(Z_k=i\right)\mathbb{P}\left(Z_k=0\right)\sum_{m=1}^{n-k}\mathbb{P}\left(Z_{k+m}=i\middle|Z_k=1\right)-\mathbb{P}\left(Z_{k+m}=i\middle|Z_k=0\right). \tag{6.1.38}$$

It follows from (6.1.18) that,

$$\mathbb{P}\left(Z_{k+m}=i\middle|Z_k=1\right)=\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=i\right)$$

where $\left(\widetilde{X}_n\right)_{n\geq 0}$ is a Markov chain on the state space S, with transition probability matrix R and $\widetilde{X}_0 \sim \frac{\mathbb{E}[U_k]+(1,0,0\ldots)}{k+2}$. Furthermore, $\widetilde{\tau}_{m-1}(k+1) \stackrel{d}{=} \sum_{l=k+2}^{k+m} I_l,$ for all $m\geq 2$. Therefore, we have

$$\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-1}(k+1)}=i\right)=\frac{k+m}{k+m+1}\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-2}(k+1)}=i\right)+\frac{\gamma_i}{k+m+1}\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_{m-2}(k+1)}=0\right).$$

Therefore, repeating this iteration, we obtain

$$\mathbb{P}\left(Z_{k+m}=i\middle|Z_k=1\right)=\frac{\mathbb{E}\left[U_{k,i}\right]}{k+m+1}+\frac{\gamma_i}{k+m+1}\sum_{l=0}^{m-1}\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_l(k+1)}=0\right). \tag{6.1.39}$$

Observe that from (6.1.18) for every $l\geq 0$, and

$$\mathbb{P}_{\widetilde{X}_0}\left(\widetilde{X}_{\widetilde{\tau}_l(k+1)}=0\right)=\mathbb{P}\left(Z_{k+l+1}=0\middle|Z_k=1\right).$$

Therefore, from (6.1.39), we obtain

$$\mathbb{P}\left(Z_{k+m}=i\middle|Z_k=1\right)=\frac{\mathbb{E}\left[U_{k,i}\right]}{k+m+1}+\frac{\gamma_i}{k+m+1}\sum_{l=0}^{m-1}\mathbb{P}\left(Z_{k+l+1}=0\middle|Z_k=1\right). \tag{6.1.40}$$

Similarly, we obtain

$$\mathbb{P}\left(Z_{k+m}=i\middle|Z_k=0\right)=\frac{\mathbb{E}\left[U_{k,i}\right]+\gamma_i}{k+m+1}+\frac{\gamma_i}{k+m+1}\sum_{l=0}^{m-1}\mathbb{P}\left(Z_{k+l+1}=0\middle|Z_k=0\right). \tag{6.1.41}$$

From (6.1.30), it follows from

$$\mathbb{P}\left(Z_{k+l+1}=0 \middle| Z_k=0\right)-\mathbb{P}\left(Z_{k+l+1}=0 \middle| Z_k=1\right)=\frac{1}{k+2}A_{l+1}(k)\left(\gamma_0-1\right). \tag{6.1.42}$$

Recall from (6.1.28) that

$$A_m(k)=\frac{\left(k+m-1+\gamma_0\right)\left(k+m-2+\gamma_0\right)\ldots\left(k+1+\gamma_0\right)}{\left(k+m+1\right)\left(k+m\right)\ldots\left(k+3\right)}.$$

Also recall from (2.1.3), that for $\Pi_n\left(z\right)=\prod_{j=1}^{n}\left(1+\frac{z}{j}\right)$,

$$\lim_{n\to\infty}\frac{\Pi_n(z)}{n^z}\Gamma(z+1)=1$$

where the convergence is uniform on compact subsets of $\mathbb{C}\setminus\{-1,-2,\ldots\}$.

Therefore, there exists a constant $C>0$, such that for all k large enough,

$$\begin{aligned}
\frac{1}{k+2}A_{l+1}(k) &= \frac{(k+1)}{(k+l+2)\,(k+l+1)}\left(1+\frac{\gamma_0}{k+1}\right)\left(1+\frac{\gamma_0}{k+2}\right)\left(1+\frac{\gamma_0}{k+l}\right)\\
&= \frac{(k+1)}{(k+l+2)\,(k+l+1)}\,\frac{\Pi_{k+l}\left(\gamma_0\right)}{\Pi_k\left(\gamma_0\right)}\\
&\leq C\frac{(k+1)}{(k+l+2)\,(k+l+1)}\left(\frac{k+l}{k+1}\right)^{\gamma_0}.
\end{aligned} \tag{6.1.43}$$

Hence, for all k large enough and some constant $C'>0$,

$$\left|\mathbb{P}\left(Z_{k+l+1}=0 \middle| Z_k=0\right)-\mathbb{P}\left(Z_{k+l+1}=0 \middle| Z_k=1\right)\right|\leq C'\frac{(k+1)}{(k+l+2)\,(k+l+1)}\left(\frac{k+l}{k+1}\right)^{\gamma_0}.$$

Therefore,

$$\begin{aligned}
\sum_{l=0}^{m-1}\left|\mathbb{P}\left(Z_{k+l+1}=0 \middle| Z_k=0\right)-\mathbb{P}\left(Z_{k+l+1}=0 \middle| Z_k=1\right)\right| &\leq C'\sum_{l=0}^{m-1}\frac{(k+1)^{1-\gamma_0}\,(k+l)^{\gamma_0}}{(k+l+2)\,(k+l+1)}\\
&\leq C_1\frac{(k+1)^{1-\gamma_0}}{(k+m+1)^{1-\gamma_0}}\\
&\leq C_2,
\end{aligned} \tag{6.1.44}$$

for a suitable constants $C_1, C_2 > 0$. From (6.1.40) and (6.1.41), we obtain

$$\Big|\mathbb{P}\left(Z_{k+m} = i \big| Z_k = 1\right) - \mathbb{P}\left(Z_{k+m} = i \big| Z_k = 0\right)\Big| \leq C_3 \frac{1}{k+m+1}$$

for some constant $C_3 > 0$. Therefore,

$$\begin{aligned} \sum_{m=1}^{n-k} \Big|\mathbb{P}\left(Z_{k+m} = i \big| Z_k = 1\right) - \mathbb{P}\left(Z_{k+m} = i \big| Z_k = 0\right)\Big| &\leq C_3 \sum_{m=1}^{n-k} \frac{1}{k+m+1} \\ &\leq C_4 \log\left(n+1\right). \end{aligned}$$

Putting this bound in (6.1.38), we obtain

$$\Big|J_{n,i}(2)\Big| \leq C_4 \frac{\log\left(n+1\right)}{(n+1)^2} \sum_{k=0}^{n} \mathbb{P}\left(Z_k = i\right) \mathbb{P}\left(Z_k = 0\right). \tag{6.1.45}$$

It follows from (6.1.1), that for every $i \geq 1$,

$$\lim_{n\to\infty} \frac{1}{n+1} \textstyle\sum_{k=1}^{n} \mathbb{P}\left(Z_k = i\right) \mathbb{P}\left(Z_k = 0\right) = \pi_i \pi_0.$$

This together with (6.1.45) imply that (6.1.34) holds. This completes the proof of (6.1.20)

6.2 Urn models associated with random walks on $\mathbb{Z}^d$

Urn models associated with random walks was first studied in Chapter 2, where the replacement matrix was the transition matrix for a bounded increment random walk. In this section, we consider urn models associated with random walks on $\mathbb{Z}^d$, $d \geq 1$, where the increments are not necessarily bounded. Let $(Y_j)_{j\geq 1}$ be i.i.d. random vectors taking values in $\mathbb{Z}^d$. Let the law of Y_1 be given by $\mathbb{P}\left(Y_1 = v\right) = p(v)$, for $v \in \mathbb{Z}^d$, where $0 \leq p(v) \leq 1$ and $\sum_{v\in\mathbb{Z}^d} p(v) = 1$. Let $(S_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}^d$, with increments $(Y_j)_{j\geq 1}$, and initial distribution U_0. Therefore, $S_n = S_0 + \sum_{j=1}^{n} Y_j$, where $S_0 \sim U_0$. Consider an urn model $(U_n)_{n\geq 0}$, with colors indexed by $\mathbb{Z}^d$, associated with this random walk $(S_n)_{n\geq 0}$. Therefore, the replacement matrix R, which is the transition matrix for the random walk is given by $R := \left(\left(R\left(u,v\right)\right)\right)_{u,v\in\mathbb{Z}^d}$, where

$$R\left(u,v\right) = p(v-u)$$

for all $u, v \in \mathbb{Z}^d$.

We assume that Y_1 is such that $\mathbb{E}\left[Y_1^T Y_1\right]$ is a well defined matrix, that is, all its entries are finite. This implies that $\mathbb{E}\left[Y_1\right]$ is well defined, with all coordinates finite. We shall always write

$$\begin{aligned} \mu &:= \mathbb{E}\left[Y_1\right], \\ \Sigma &:= \mathbb{E}\left[Y_1^T Y_1\right]. \end{aligned} \tag{6.2.1}$$

Σ is assumed to be positive definite. Also $\Sigma^{1/2}$ will denote the unique *positive definite square root* of Σ, that is, $\Sigma^{1/2}$ is a positive definite matrix such that $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$.

Theorem 6.2.1. *Let $\overline{\Lambda}_n$ be the probability measure on $\mathbb{R}^d$ corresponding to the probability vector $\left(\frac{\mathbb{E}[U_{n,v}]}{n+1}\right)_{v\in\mathbb{Z}^d}$, and let*

$$\overline{\Lambda}_n^{cs}(A) := \overline{\Lambda}_n\left(\sqrt{\log n}A\Sigma^{1/2} + \mu\log n\right),$$

where A is a Borel subset of $\mathbb{R}^d$. Then, as $n \to \infty$,

$$\overline{\Lambda}_n^{cs} \Rightarrow \Phi_d. \tag{6.2.2}$$

Theorem 2.1.1, available in Chapter 2 is immediate from Theorem (6.2.1). Recall that if Z_n denotes the $(n+1)$-th selected color, then, its probability mass function is given by $\left(\frac{\mathbb{E}[U_{n,v}]}{n+1}\right)_{v\in\mathbb{Z}^d}$. Thus $\overline{\Lambda}_n$ is the probability distribution of Z_n, and $\overline{\Lambda}_n^{cs}$ is the distribution of the scaled and centered random vector $\frac{Z_n - \mu\log n}{\sqrt{\log n}}$. So the following result is a restatement of (6.2.2).

Corollary 6.2.1. *Consider the urn model associated with the random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$, $d \geq 1$, then, as $n \to \infty$,*

$$\frac{Z_n - \mu\log n}{\sqrt{\log n}} \Rightarrow N_d(0, \Sigma). \tag{6.2.3}$$

Proof of Theorem 6.2.1. Let D denote the variance-covariance matrix of Y_1. Since Σ is assumed to be positive definite, therefore, D is also positive definite.

Since $(Y_j)_{j\geq 1}$ are i.i.d. with finite second moments, from the Central Limit Theorem for i.i.d. random variables (see page 124 of [28]), it follows that

$$\frac{S_n - n\mu}{\sqrt{n}} \Rightarrow N_d(0, D). \tag{6.2.4}$$

This implies that for all $\theta \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{\langle \theta, S_{\log n}\rangle - \log n \langle \theta, \mu\rangle}{\sqrt{\log n (\theta D \theta^T)}} \Rightarrow N(0,1), \text{ as } n \to \infty. \tag{6.2.5}$$

For every $\theta \in \mathbb{R}^d \setminus \{0\}$, we have,

$$\frac{\langle \theta, S_{\tau_n}\rangle - \log n \langle \theta, \mu\rangle}{\sqrt{\log n (\theta D \theta^T)}} = \frac{\langle \theta, S_{\tau_n}\rangle - \tau_n \langle \theta, \mu\rangle}{\sqrt{\log n \, (\theta D \theta^T)}} - \frac{\langle \theta, S_{\log n}\rangle - \log n \langle \theta, \mu\rangle}{\sqrt{\log n \, (\theta D \theta^T)}} + \frac{\langle \theta, S_{\log n}\rangle - \log n \langle \theta, \mu\rangle}{\sqrt{\log n (\theta D \theta^T)}} + \frac{\langle \theta, \mu\rangle}{\sqrt{\theta D \theta^T}} \frac{\tau_n - \log n}{\sqrt{\log n}}.$$

Let us define

$$I_n(1) := \frac{\langle \theta, S_{\tau_n}\rangle - \tau_n \langle \theta, \mu\rangle}{\sqrt{\log n \, (\theta D \theta^T)}} - \frac{\langle \theta, S_{\log n}\rangle - \log n \langle \theta, \mu\rangle}{\sqrt{\log n \, (\theta D \theta^T)}},$$

$$I_n(2) := \frac{\langle \theta, S_{\log n}\rangle - \log n \langle \theta, \mu\rangle}{\sqrt{\log n (\theta D \theta^T)}},$$

and

$$I_n(3) := \frac{\langle \theta, \mu\rangle}{\sqrt{\theta D \theta^T}} \frac{\tau_n - \log n}{\sqrt{\log n}}.$$

We will prove that, as $n \to \infty$,

$$|I_n(1)| = \left| \frac{\langle \theta, S_{\tau_n}\rangle - \tau_n \langle \theta, \mu\rangle}{\sqrt{\log n \, (\theta D \theta^T)}} - \frac{\langle \theta, S_{\log n}\rangle - \log n \langle \theta, \mu\rangle}{\sqrt{\log n \, (\theta D \theta^T)}} \right| \xrightarrow{p} 0. \tag{6.2.6}$$

For $\delta > 0$,

$$\mathbb{P}(|I_n(1)| > \delta) = \mathbb{P}\left(|I_n(1)| > \delta, \left|\frac{\tau_n}{\log n} - 1\right| > \epsilon\right) + \mathbb{P}\left(|I_n(1)| > \delta, \left|\frac{\tau_n}{\log n} - 1\right| \leq \epsilon\right), \tag{6.2.7}$$

where $\epsilon > 0$. From (5.1.2), $\lim_{n\to\infty} \frac{\tau_n}{\log n} = 1$ a.s. Hence,

$$\mathbb{P}\left(|I_n(1)| > \delta, \left|\frac{\tau_n}{\log n} - 1\right| > \epsilon\right) \leq \mathbb{P}\left(\left|\frac{\tau_n}{\log n} - 1\right| > \epsilon\right) \longrightarrow 0, \text{ as } n \to \infty. \tag{6.2.8}$$

Observe that,

$$I_n(1) = \frac{\langle \theta, S_{\tau_n} - S_{\log n}\rangle - (\tau_n - \log n) \langle \theta, \mu\rangle}{\sqrt{\log n \, (\theta D \theta^T)}}.$$

So,

$$\mathbb{P}\left(|I_n(1)| > \delta, \left|\frac{\tau_n}{\log n} - 1\right| \leq \epsilon\right)$$

$$\leq \mathbb{P}\left(\sup_{(1-\epsilon)\log n\leq m\leq(1+\epsilon)\log n}|\langle\theta,S_m\rangle-m\langle\theta,\mu\rangle|>\delta\sqrt{\log n\,(\theta D\theta^T)}\right). \quad (6.2.9)$$

From Kolmogorov maximal inequality (Theorem 2.5.2 of [28]),

$$\begin{aligned}&\mathbb{P}\left(\sup_{(1-\epsilon)\log n\leq m\leq(1+\epsilon)\log n}|\langle\theta,S_m\rangle-m\langle\theta,\mu\rangle|>\delta\sqrt{\log n\,(\theta D\theta^T)}\right)\\ \leq\ &\frac{1}{\delta^2\log n\,(\theta D\theta^T)}\mathrm{Var}\left(\langle\theta,S_{2\epsilon\log n}\rangle\right)=\frac{2\epsilon}{\delta^2}.\end{aligned} \quad (6.2.10)$$

Therefore, (6.2.8) and (6.2.10) imply (6.2.6).

From (6.2.5), we know that $I_n(2)\Rightarrow N(0,1)$, as $n\to\infty$. From (5.1.3), we have $I_n(3)\Rightarrow N\left(0,1\right)$, as $n\to\infty$. Since S_n and τ_n are independent, it follows that

$$\frac{\langle\theta,S_{\tau_n}\rangle-\log n\langle\theta,\mu\rangle}{\sqrt{\log n(\theta D\theta^T)}}\Rightarrow N\left(0,1+\frac{\langle\theta,\mu\rangle^2}{\theta D\theta^T}\right),\text{ as }n\to\infty.$$

Writing $\mu:=\left(\mu^{(1)},\mu^{(2)},\ldots,\mu^{(d)}\right)$, it is easy to see that $\langle\theta,\mu\rangle^2=\theta M\theta^T$, where $M=\left(\left(\mu^{(i)}\mu^{(j)}\right)\right)_{1\leq i,j\leq d}$. D is positive definite, so for every $\theta\in\mathbb{R}^d\setminus\{0\}$, we have $\theta D\theta^T>0$. Therefore,

$$\left(\theta D\theta^T\right)\left(1+\frac{\langle\theta,\mu\rangle^2}{\theta D\theta^T}\right)=\theta\left(M+D\right)\theta^T,$$

where the matrix addition is taken entry-wise. Observe that $M+D=\Sigma$. Therefore, it follows that as $n\to\infty$,

$$\frac{S_{\tau_n}-\mu\log n}{\sqrt{\log n}}\Rightarrow N_d\left(0,\Sigma\right). \quad (6.2.11)$$

From (5.1.1), we have $Z_n\stackrel{d}{=}S_{\tau_n}$. Therefore, we can conclude that, as $n\to\infty$,

$$\frac{Z_n-\mu\log n}{\sqrt{\log n}}\Rightarrow N_d\left(0,\Sigma\right).$$

□

Remark 6.2.1. *If Σ is singular, then the proof of Theorem 6.2.1 clearly indicates that as $n\to\infty$,*

$$\frac{Z_n-\mu\log n}{\sqrt{\log n}}\Rightarrow N_d\left(0,\Sigma\right), \quad (6.2.12)$$

where $N_d\left(0,\Sigma\right)$ is singular Normal (see pages 30-33 of [1] for definition of singular Normal).

6.3 Urn models associated with bounded increment periodic random walk

Fix $k \in \mathbb{N}$. For $n = mk + r$, where $m \in \mathbb{N} \cup \{0\}$, and $0 \leq r < k$, let

$$S_n = S_{mk} + Y_{m+1}(1) + Y_{m+1}(2) + \ldots + Y_{m+1}(r+1), \tag{6.3.1}$$

where $\{Y_j(i), 1 \leq i \leq k,\ j \geq 1\}$ is a collection of independent d-dimensional random vectors, such that, for each fixed $i \in \{1, 2, \ldots k\}$, $\left(Y_j(i)\right)_{j \geq 1}$ are i.i.d. We further assume that for each fixed $1 \leq i \leq k$, there exists a finite non-empty set $B_i \subset \mathbb{R}^d$, such that $\mathbb{P}\left(Y_1(i) \in B_i\right) = 1$. This implies that the law of $Y_1(i)$ is bounded for each $i \in \{1, 2, \ldots, k\}$. We will assume $B_i \cap B_j = \emptyset$, for any $1 \leq i \neq j \leq k$. This ensures that $\left(S_n\right)_{n \geq 0}$ is a k-periodic random walk with increments $\{Y_j(i), 1 \leq i \leq k,\ j \geq 1\}$. This is because, if V denotes the set of all possible sites that the random walk $\left(S_n\right)_{n \geq 0}$ can visit, then we can partition the set V into k disjoint subsets, say $V_0, V_1, \ldots, V_{k-1}$. This partition has the property that for $n = mk + r$, where $m \in \mathbb{N} \cup \{0\}$, and $0 \leq r < k$, $S_{n-1} \in V_r$ and the next increment of the walk is given by $Y_{m+1}(r+1)$. Furthermore, $S_n \in V_{r+1}$, if $0 \leq r < k-1$, and $S_n \in V_0$, if $r = k-1$. This implies that deterministically the random walk will periodically visit sites in V_r exactly after k many steps for every $0 \leq r < k$. It is in this setup that we say that the random walk is k-periodic. Such a partition of V necessarily guarantees that S_n is a k-periodic Markov chain. A sufficient condition for such a partition of V is $B_i \cap B_j = \emptyset$, for all $1 \leq i \neq j \leq k$. Otherwise, the walk may be aperiodic.

For $1 \leq i \leq k$, we shall write

$$\begin{aligned}
\mu(i) &:= \mathbb{E}\left[Y_1(i)\right], \\
\overline{\mu} &:= \tfrac{1}{k} \sum_{i=1}^{k} \mu(i), \\
\Sigma(i) &:= \mathbb{E}\left[Y_1^T(i) Y_1(i)\right].
\end{aligned} \tag{6.3.2}$$

We further assume that $\Sigma(i)$ is positive definite, for each $1 \leq i \leq k$. Let us denote by $\Sigma^{1/2}(i)$ the unique *positive definite square root* of $\Sigma(i)$. Note that, then $\overline{\Sigma} = \frac{1}{k}\sum_{i=1}^{k} \Sigma(i)$ is also *positive definite*. We denote by $\overline{\Sigma}^{1/2}$, the unique *positive definite square root* of $\overline{\Sigma}$.

In the remainder of this subsection, we will consider an urn model $(U_n)_{n\geq 0}$ associated with the k-periodic random walk $(S_n)_{n\geq 0}$.

Theorem 6.3.1. *Let $\overline{\Lambda}_n$ be the probability measure corresponding to the probability vector $\frac{\mathbb{E}[U_n]}{n+1}$, and let*

$$\overline{\Lambda}_n^{cs}(A) := \overline{\Lambda}_n\left(\sqrt{\log n}A\overline{\Sigma}^{1/2} + \overline{\mu}\log n\right),$$

where A is a Borel subset of $\mathbb{R}^d$. Then, as $n \to \infty$,

$$\overline{\Lambda}_n^{cs} \Rightarrow \Phi_d \tag{6.3.3}$$

$\overline{\Lambda}_n$ is the probability distribution of Z_n, and $\overline{\Lambda}_n^{cs}$ is the distribution of the scaled and centered random vector $\frac{Z_n - \mu\log n}{\sqrt{\log n}}$. So the following corollary is a restatement of (6.3.3) .

Corollary 6.3.1. *Consider the urn model associated with the k-periodic random walk $(S_n)_{n\geq 0}$, then as $n \to \infty$,*

$$\frac{Z_n - \overline{\mu}\log n}{\sqrt{\log n}} \Rightarrow N_d(0, \overline{\Sigma}). \tag{6.3.4}$$

Proof of Theorem 6.3.1. For $n = mk + r$, where $m \in \mathbb{N} \cup \{0\}$, and $0 \leq r < k$,

$$\begin{aligned} \mathbb{E}[S_n] &= m\sum_{i=1}^{k}\mathbb{E}[Y_1(i)] + \sum_{i=1}^{r+1}\mathbb{E}[Y_1(i)] \\ &= mk\overline{\mu} + \sum_{i=1}^{r+1}\mu(i). \end{aligned} \tag{6.3.5}$$

Therefore,

$$\frac{\mathbb{E}[S_n]}{n} \longrightarrow \overline{\mu}, \text{ as } n \to \infty. \tag{6.3.6}$$

Similarly, if D_n and $D(i)$ denote the variance-covariance matrix for S_n, and $Y_1(i)$, for each $1 \leq i \leq k$, respectively, then,

$$\frac{D_n}{n} \longrightarrow \overline{D}, \text{ as } n \to \infty, \tag{6.3.7}$$

where $\overline{D} = \frac{1}{k}\sum_{i=1}^{k} D(i)$, and the matrix convergence is entry-wise. Therefore, from the Lindeberg-Feller Central Limit Theorem (see page 129 of [28]), it follows that as $n \to \infty$,

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}} \Rightarrow N_d\left(0, \overline{D}\right). \tag{6.3.8}$$

Let us write
$$\frac{S_n - n\overline{\mu}}{\sqrt{n}} = \frac{S_n - \mathbb{E}\left[S_n\right]}{\sqrt{n}} - \frac{n\overline{\mu} - \mathbb{E}\left[S_n\right]}{\sqrt{n}}. \tag{6.3.9}$$

Now, it is easy to see from (6.3.5) that, for $n = mk + r$, and some $C > 0$,
$$\left|\frac{n\overline{\mu} - \mathbb{E}\left[S_n\right]}{\sqrt{n}}\right| = \left|\frac{r\overline{\mu} - \sum_{i=1}^{r+1} \mu(i)}{\sqrt{n}}\right| \leq \frac{C}{\sqrt{n}} \longrightarrow 0, \text{ as } n \to \infty. \tag{6.3.10}$$

Therefore, from Slutsky's theorem (see page 105 of [28]), we obtain as $n \to \infty$,
$$\frac{S_n - n\overline{\mu}}{\sqrt{n}} \Rightarrow N_d\left(0, \overline{D}\right). \tag{6.3.11}$$

Furthermore, it can be shown that, as $n \to \infty$,
$$\frac{S_{\tau_n} - \overline{\mu}\log n}{\sqrt{\log n}} \Rightarrow N_d\left(0, \overline{\Sigma}\right). \tag{6.3.12}$$

The proof of (6.3.12) is similar to (6.2.11). The rest of the proof now follows by observing that $Z_n \stackrel{d}{=} S_{\tau_n}$. □

Example 6.3.1. *As an application of Theorem 6.3.1, we present the example of the random walk on hexagonal lattice. Let $\mathbb{H} = (V, E)$ be the hexagonal lattice in $\mathbb{R}^2$ (see Figure 6.1). The vertex set $V = V_1 \cup V_2$, where V_1 and V_2 are disjoint. V_1 and V_2 are defined as follows:*
$$V_{1,1} := \left\{1, \omega, \omega^2\right\}, \text{ where } \omega \text{ is a complex cube root of unity,}$$
and
$$V_{2,1} := \left\{v + 1, v + \omega, v + \omega^2 \colon v \in V_{1,1}\right\}.$$
For any $n \geq 2$,
$$V_{1,n} := \left\{v - 1, v - \omega, v - \omega^2 \colon v \in V_{2,n-1}\right\},$$
and
$$V_{2,n} = \left\{v + 1, v + \omega, v + \omega^2 \colon v \in V_{1,n}\right\}.$$
Finally, $V_1 = \cup_{j\geq 1} V_{1,j}$ and $V_2 = \cup_{j\geq 1} V_{2,j}$. For any pair of vertices $v, w \in V$, we draw an edge between them, if and only if, either of the following two cases occur:

(i) $v \in V_1$ and $w \in V_2$ and $w = v + u$ for some $u \in \{1, \omega, \omega^2\}$, or

(ii) $v \in V_2$ and $w \in V_1$ and $w = v + u$ for some $u \in \{-1, -\omega, -\omega^2\}$.

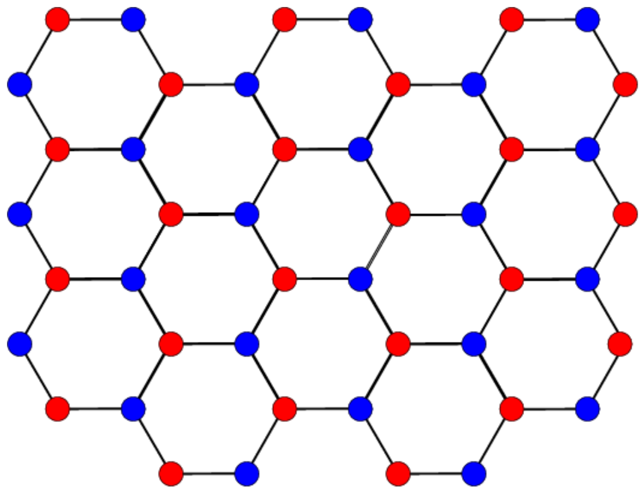

Figure 6.1: Hexagonal Lattice

To define the random walk on $\mathbb{H}$, let us consider $\{Y_j(i)\colon i = 1, 2,\ j \geq 1\}$ to be a sequence of independent random vectors such that $(Y_j(i))_{j\geq 1}$ are i.i.d for every fixed $i = 1, 2$. Let $Y_1(1) \sim \text{Unif}\left\{1, \omega, \omega^2\right\}$, and $Y_1(2) \sim \text{Unif}\left\{-1, -\omega, -\omega^2\right\}$. One can now define a random walk on $\mathbb{H}$, with the increments $\{Y_j(i)\colon i = 1, 2,\ j \geq 1\}$. Needless to say, this random walk has period 2.

Corollary 6.3.2. *Consider the urn process associated with the random walk $(S_n)_{n\geq 0}$ on the hexagonal lattice $\mathbb{H}$. If Z_n is the color of the randomly selected ball at the $(n+1)$-th trial, then, as $n \to \infty$,*

$$\frac{Z_n}{\sqrt{\log n}} \Rightarrow N_2\left(0, \frac{1}{2}\mathbb{I}_2\right),$$

where $\mathbb{I}_2$ is the 2×2 identity matrix.

Proof of Corollary 6.3.2. Since $1+\omega+\omega^2 = 0$, so for the random walk on the hexagonal lattice, $\mu(1) = \mu(2) = 0$. Therefore $\overline{\mu} = 0$. Let

$$\Sigma(1) := \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix}$$

Writing $Y_1(1) := \left(Y_1^{(1)}(1), Y_1^{(2)}(1)\right)$, observe that

$$\sigma_{1,1} = \mathbb{E}\left[\left(Y_1^{(1)}(1)\right)^2\right] \text{ and } \sigma_{2,2} = \mathbb{E}\left[\left(Y_1^{(2)}(1)\right)^2\right].$$

Also,

$$\sigma_{1,2} = \sigma_{2,1} = \mathbb{E}\left[Y_1^{(1)}(1)Y_1^{(2)}(1)\right].$$

Writing $\omega = Re\,(\omega) + i Im\,(\omega)$, it is easy to see that

$$\sigma_{1,1} = \frac{1}{3}\left(1 + (Re\,(\omega))^2 + \left(Re\,(\omega^2)\right)^2\right).$$

Since $Re\,(\omega) = Re\,(\omega^2)$, therefore,

$$\sigma_{1,1} = \frac{1}{3}\left(1 + 2\,(Re\,(\omega))^2\right).$$

Since $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, therefore, this implies $\sigma_{1,1} = \frac{1}{2}$. Similarly, since $Im\,(\omega) = -Im\,(\omega^2)$,

$$\sigma_{2,2} = \frac{1}{3}\left((Im\,(\omega))^2 + \left(Im\,(\omega^2)\right)^2\right) = \frac{2}{3}\,(Im\,(\omega))^2 = \frac{1}{2}.$$

Since, $Re\,(\omega) = Re\,(\omega^2)$, and $Im\,(\omega) = -Im\,(\omega^2)$,

$$\sigma_{1,2} = \sigma_{2,1} = \frac{1}{3}\left(Re\,(\omega)\,Im\,(\omega) + Re\,(\omega^2)\,Im\,(\omega^2)\right) = 0.$$

This proves that $\Sigma(1) = \frac{1}{2}\mathbb{I}_2$. Similar calculations show that $\Sigma(2) = \frac{1}{2}\mathbb{I}_2$. This implies that $\overline{\Sigma} = \frac{1}{2}\Sigma(1) + \frac{1}{2}\Sigma(2) = \frac{1}{2}\mathbb{I}_2$. This completes the proof. □

Bibliography

[1] T. W. Anderson. *An introduction to multivariate statistical analysis*. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition, 2003.

[2] Tom M. Apostol. *Mathematical analysis*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., second edition, 1974.

[3] Krishna B. Athreya and Samuel Karlin. Embedding of urn schemes into continuous time Markov branching processes and related limit theorems. *Ann. Math. Statist.*, 39:1801–1817, 1968.

[4] A. Bagchi and A. K. Pal. Asymptotic normality in the generalized Pólya-Eggenberger urn model, with an application to computer data structures. *SIAM J. Algebraic Discrete Methods*, 6(3):394–405, 1985.

[5] Zhi-Dong Bai and Feifang Hu. Asymptotics in randomized urn models. *Ann. Appl. Probab.*, 15(1B):914–940, 2005.

[6] Antar Bandyopadhyay and Debleena Thacker. A new approach to Pólya urn schemes and its infinite color generalization. Preprint, 2014.

[7] Antar Bandyopadhyay and Debleena Thacker. Pólya urn schemes with infinitely many colors. (http://arxiv.org/pdf/1303.7374v2.pdf), 2014.

[8] Antar Bandyopadhyay and Debleena Thacker. Rate of convergence and large deviation for the infinite color Pólya urn schemes. *Statist. Probab. Lett.*, 92:232–240, 2014.

[9] Harald Bergström. On the central limit theorem in the case of not equally distributed random variables. *Skand. Aktuarietidskr.*, 32:37–62, 1949.

[10] R. N. Bhattacharya and R. Ranga Rao. *Normal approximation and asymptotic expansions*. John Wiley & Sons, New York-London-Sydney, 1976. Wiley Series in Probability and Mathematical Statistics.

[11] Patrick Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, third edition, 1995. A Wiley-Interscience Publication.

[12] David Blackwell. Discreteness of Ferguson selections. *Ann. Statist.*, 1:356–358, 1973.

[13] David Blackwell and James B. MacQueen. Ferguson distributions via Pólya urn schemes. *Ann. Statist.*, 1:353–355, 1973.

[14] Arup Bose, Amites Dasgupta, and Krishanu Maulik. Multicolor urn models with reducible replacement matrices. *Bernoulli*, 15(1):279–295, 2009.

[15] Arup Bose, Amites Dasgupta, and Krishanu Maulik. Strong laws for balanced triangular urns. *J. Appl. Probab.*, 46(2):571–584, 2009.

[16] Arijit Chakrabarty. *When is a Truncated Heavy Tail Heavy?* PhD thesis, Cornell University, 2010.

[17] May-Ru Chen, Shoou-Ren Hsiau, and Ting-Hsin Yang. A new two-urn model. *J. Appl. Probab.*, 51(2):590–597, 2014.

[18] May-Ru Chen and Markus Kuba. On generalized Pólya urn models. *J. Appl. Probab.*, 50(4):1169–1186, 2013.

[19] Joel E. Cohen. *A Model of Simple Competition*, volume 41 of *Annals of the Computation Laboratory*. Harvard University Press, Cambridge, Massachusetts, second edition, 1966.

[20] Andrea Collevecchio, Codina Cotar, and Marco LiCalzi. On a preferential attachment and generalized Pólya's urn model. *Ann. Appl. Probab.*, 23(3):1219–1253, 2013.

[21] John B. Conway. *Functions of one complex variable*, volume 11 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1978.

[22] Codina Cotar and Vlada Limic. Attraction time for strongly reinforced walks. *Ann. Appl. Probab.*, 19(5):1972–2007, 2009.

[23] Edward Crane, Nicholas Georgiou, Stanislav Volkov, Andrew R. Wade, and Robert J. Waters. The simple harmonic urn. *Ann. Probab.*, 39(6):2119–2177, 2011.

[24] Charles W. Curtis. *Linear algebra*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, fourth edition, 1993. An introductory approach.

[25] Amites Dasgupta and Krishanu Maulik. Strong laws for urn models with balanced replacement matrices. *Electron. J. Probab.*, 16:no. 63, 1723–1749, 2011.

[26] Burgess Davis. Reinforced random walk. *Probab. Theory Related Fields*, 84(2):203–229, 1990.

[27] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*. Jones and Bartlett Publishers, Boston, MA, 1993.

[28] Rick Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

[29] Michael Dutko. Central limit theorems for infinite urn models. *Ann. Probab.*, 17(3):1255–1263, 1989.

[30] W. J. Ewens. The sampling theory of selectively neutral alleles. *Theoret. Population Biology*, 3:87–112; erratum, ibid. 3 (1972), 240; erratum, ibid. 3 (1972), 376, 1972.

[31] Warren J. Ewens. *Mathematical population genetics. I*, volume 27 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, second edition, 2004. Theoretical introduction.

[32] William Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[33] Thomas S. Ferguson. A Bayesian analysis of some nonparametric problems. *Ann. Statist.*, 1:209–230, 1973.

[34] Philippe Flajolet, Philippe Dumas, and Vincent Puyhaubert. Some exactly solvable models of urn process theory. In *Fourth Colloquium on Mathematics and Computer Science Algorithms, Trees, Combinatorics and Probabilities*, Discrete Math. Theor. Comput. Sci. Proc., AG, pages 59–118. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2006.

[35] David A. Freedman. Bernard Friedman's urn. *Ann. Math. Statist*, 36:956–970, 1965.

[36] Bernard Friedman. A simple urn model. *Comm. Pure Appl. Math.*, 2:59–70, 1949.

[37] Alexander Gnedin, Ben Hansen, and Jim Pitman. Notes on the occupancy problem with infinitely many boxes: general asymptotics and power laws. *Probab. Surv.*, 4:146–171, 2007.

[38] Raúl Gouet. Strong convergence of proportions in a multicolor Pólya urn. *J. Appl. Probab.*, 34(2):426–435, 1997.

[39] Hsien-Kuei Hwang and Svante Janson. Local limit theorems for finite and infinite urn models. *Ann. Probab.*, 36(3):992–1022, 2008.

[40] Svante Janson. Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Process. Appl.*, 110(2):177–245, 2004.

[41] Svante Janson. Limit theorems for triangular urn schemes. *Probab. Theory Related Fields*, 134(3):417–452, 2006.

[42] Donald E. Knuth. *The art of computer programming. Vol. 3*. Addison-Wesley, Reading, MA, 1998. Sorting and searching, Second edition [of MR0445948].

[43] W. Kozakiewicz. On the convergence of sequences of moment generating functions. *Ann. Math. Statistics*, 18:61–69, 1947.

[44] Sophie Laruelle and Gilles Pagès. Randomized urn models revisited using stochastic approximation. *Ann. Appl. Probab.*, 23(4):1409–1436, 2013.

[45] Mickaël Launay and Vlada Limic. Generalized Interacting Urn Models. available at http://arxiv.org/abs/1201.3495, 2012.

[46] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[47] Vlada Limic. Attracting edge property for a class of reinforced random walks. *Ann. Probab.*, 31(3):1615–1654, 2003.

[48] Vlada Limic and Pierre Tarrès. Attracting edge and strongly edge reinforced walks. *Ann. Probab.*, 35(5):1783–1806, 2007.

[49] Russell Lyons. Strong laws of large numbers for weakly correlated random variables. *Michigan Math. J.*, 35(3):353–359, 1988.

[50] Hosam M. Mahmoud. *Sorting*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000. A distribution theory.

[51] Hosam M. Mahmoud. *Pólya urn models*. Texts in Statistical Science Series. CRC Press, Boca Raton, FL, 2009.

[52] Shu-teh C. Moy. Extensions of a limit theorem of Everett, Ulam and Harris on multitype branching processes to a branching process with countably many types. *Ann. Math. Statist.*, 38:992–999, 1967.

[53] Robin Pemantle. A time-dependent version of Pólya's urn. *J. Theoret. Probab.*, 3(4):627–637, 1990.

[54] Robin Pemantle. A survey of random processes with reinforcement. *Probab. Surv.*, 4:1–79, 2007.

[55] Robin Pemantle and Stanislav Volkov. Vertex-reinforced random walk on $\mathbb{Z}$ has finite range. *Ann. Probab.*, 27(3):1368–1388, 1999.

[56] G. Pólya. Sur quelques points de la théorie des probabilités. *Ann. Inst. H. Poincaré*, 1(2):117–161, 1930.

[57] Georg Pólya. Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. *Math. Ann.*, 84(1-2):149–160, 1921.

[58] Eugene Seneta. *Non-negative matrices and Markov chains*. Springer Series in Statistics. Springer, New York, 2006. Revised reprint of the second (1981) edition [Springer-Verlag, New York; MR0719544].

[59] A. N. Shiryayev. *Probability*, volume 95 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984. Translated from the Russian by R. P. Boas.

[60] Robert T. Smythe and Hosam M. Mahmoud. A survey of recursive trees. *Teor. Ĭmovīr. Mat. Stat.*, (51):1–29, 1994.