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SIZE DISTRIBUTION OF SUSPENDED PARTICLES — UNIMODALITY, SYMMETRY AND LOGNORMALITY

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SUMMARY. The shape of the grain size distribution of suspended material is related to the parameters of flow and the grain size distribution in the bed. It is noted that under certain conditions lognormality may be achieved in suspension even with a hyperbolic distribution in the bed.

KEY WORDS. lognormality, hyperbolic distribution, suspension, diffusion.

1. INTRODUCTION

Grain-size frequencies of naturally occurring sediment populations often follow lognormal (Krumbein, 1938) or hyperbolic distributions (Barndorff-Nielsen, 1977; Bagnold and Barndorff-Nielsen, 1979). Controlled flume experiments indicated that under suitable conditions lognormality can be attained through a process of grain sorting during suspension transportation in water flows, even when the source (bed) materials are not lognormal. These experiments also showed that the suspension load's grain-size distribution is related to flow velocity, height of suspension, and nature of the bed material (Sengupta, 1975, 1979). Lognormality was explained as a transitional phenomenon attained through a process of size sorting within a critical range of velocity and height above a sand bed of a given composition.

The purpose of the present paper is to develop a theoretical framework with a view to relating the suspension load's grain-size distribution, particularly the occurrence of unimodality, symmetry

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and lognormality, to flow parameters and grain-size distribution of the bed material. Grain-size data of two bed materials used for the earlier experiments (bed nos. 2 and 3 of Sengupta, 1979) have been utilized for the present discussion.

The theoretical model that we use here is a simpler version of an earlier model developed from the diffusion equation (Ghosh *et al.*, 1979). The model is briefly explained in Section 2 and compared with some observed data. The statistical consequences relating to shape of the suspension distribution are studied in Section 3, both for the observed data of Section 2 and for general lognormal and hyperbolic beds. Among other things it will be clear from the following discussion that if the bed is lognormal or hyperbolic with a mode lying between 1 and 5ϕ, (where $\phi = -\log_2 D$, D is grain diameter in mm) then under certain flow conditions the suspension distribution follows lognormality.

2. THE MATHEMATICAL MODEL

Consider a steady uniform flow of depth d and longitudinal velocity $u(y)$ at height y above the bed surface. The weight frequency $w_b(\phi)$ is the weight of grains in the sand bed in the range $\phi - .5$ to $\phi + .5$. (We shall also regard it as the weight frequency density at ϕ.) Let $w_b'(\phi) = w_b(\phi)/(\sum w_b(\phi))$. The bed roughness k_s is that value of ϕ for which $\sum_{\phi \geq k_s} w_b'(\phi) = .65$. The average concentration $S_y(\phi)$ at height y is the weight frequency in the range $\phi - .5$ to $\phi + .5$ per unit volume. The average concentration $S_y'(\phi) = S_y(\phi)/(\sum_\phi S_y(\phi))$. The average concentration will be assumed to depend only on the space coordinate y.

One needs a model for predicting $S_y'(\phi)$ for $w_b'(\phi)$. In an earlier paper (Ghosh *et al.*, 1979), we advocated a diffusion model for achieving this, and compared numerically our results with other existing approaches due to Einstein (1950) and Gessler (1965). We develop below a simplified version of this. A key step in our earlier approach was the observation that if we fit Hunt's (1954) velocity profile to the observed $u(y)$ and extrapolate it up to k_s, then the fitted value is zero at k_s. To retain this feature but simplify the model in Ghosh *et al.* (1979) we tried the following logarithmic profile

$$u(y) = (u_*/\chi)\log(y/k_s) \tag{1}$$

where u_* is the shear velocity, χ is the von-Kármán constant (0.4) and the constants have been adjusted to make $u(k_s) = 0$. It was found that the fit with observed velocity was excellent. Again for simplicity, we use the single diffusion equation for sediment

$$\frac{\partial S_y(\phi,t)}{\partial t} = \frac{\partial}{\partial y}\,[c(\phi)S_y(\phi,t)] + \frac{\partial}{\partial y}\,[\varepsilon(y)\frac{\partial S_y(\phi,t)}{\partial y}] \tag{2}$$

where $c(\phi)$ is the settling velocity of the grain (Terminal fall velocity of quartz spheres in water) and

$$\varepsilon(y) = u_*^2(1-y/d)/\frac{du}{dy}\ .$$

Under equilibrium conditions, the equation (2) leads to, *vide* Rouse (1938),

$$\varepsilon(y)\ \frac{dS_y(\phi,t)}{dy} + c(\phi)S_y(\phi,t) = 0. \tag{3}$$

Integrating the equation (3) from k_s to y, we get

$$S_y = S_{k_s}\left(\frac{d-y}{y}\cdot\frac{k_s}{d-k_s}\right)^{\frac{c(\phi)}{\chi u_*}} \tag{4}$$

and hence

$$S'_y = S'_{k_s}(\phi)\ e^{-\psi c(\phi) + g(\psi)} \tag{5}$$

$$\text{where}\quad \psi = \frac{1}{\chi u_*}\ \log\left(\frac{y}{d-y}\cdot\frac{d-k_s}{k_s}\right) \tag{6}$$

is a parameter summarizing the effect of flow and

$$e^{-g(\psi)} = \sum_{\phi} S'_{k_s}(\phi)\ e^{-\psi c(\phi)}. \tag{7}$$

Assuming $S'_{k_s} \simeq w'_b$, we can calculate $S'_y(\phi)$ from equation (5). Calculated values of $S'_y(\phi)$ as well as observed values of $S'_y(\phi)$ for beds 2 and 3 are shown in Figures 1 and 2, respectively.

Formula (5) is frequently used for obtaining $S'_y(\phi)$ from $S_{y_1}(\phi)$, where $0 < y_1 < y$, are both within the so-called suspension zone. This formula is not used for obtaining $S'_y(\phi)$ from w'_b for two reasons: (a) expression (5) breaks down if $y_1 = 0$ and (b) $y_1 = 0$ is outside the suspension region. Our main observation in this context is that (i) effective height of the bed (bed roughness, k_s) is not zero and (ii) at least for the grain sizes $0 \leq \phi \leq 5$, use of a simple diffusion model does not lead to greater error than the more complicated diffusion models discussed earlier (Ghosh *et al.*, 1979).

It is worth pointing out that equation (2) has an elegant probabilistic interpretation. Consider a particle of size ϕ whose displacement is Markovian with drift $b(y) = \varepsilon'(y) - c(\phi)$ and variance per unit time $a(y) = 2\varepsilon(y)$. Then Kolmogorov's forward differential equation becomes

$$\frac{\partial S_y(\phi,t)}{\partial t} = \tfrac{1}{2}\frac{\partial^2}{\partial y^2}[a(y)S_y(\phi,t)] - \frac{\partial}{\partial y}[b(y)S_y(\phi,t)] \qquad (8)$$

where $S_y(\phi,t)$ is the concentration at time t (before the steady state is reached); clearly equation (8) is identical with (2). Following Dynkin and Yushkevich (1969, Chapter 4), the upper boundary $y = d$ can be taken as repelling and the lower boundary reflecting. If we take the lower boundary to be reflecting, then the stationary distribution is given by

$$p(y,\phi) = \frac{K}{\varepsilon(y)} \exp\{\int_{k_s}^{y} \frac{b(y)}{2\varepsilon(y)}\, dy\} \qquad (9)$$

where K is chosen such that $\int_{k_s}^{y} p(y,\phi)dy = 1$.

If we assume that the supply $S''_y(\phi)$ of material of size ϕ from the bed is such that S_y is continuous at $y = k_s$, then $S''_{k_s} = S_{k_s}\varepsilon(k_s)/K$ and $S_y(\phi) = S''_{k_s}(\phi)p_y(\phi)$ is the solution (4).

Note that more generally as long as $b(y)$ and $a(y)$ are such that $y = d$ is repelling and $y = k_s$ is reflecting, our

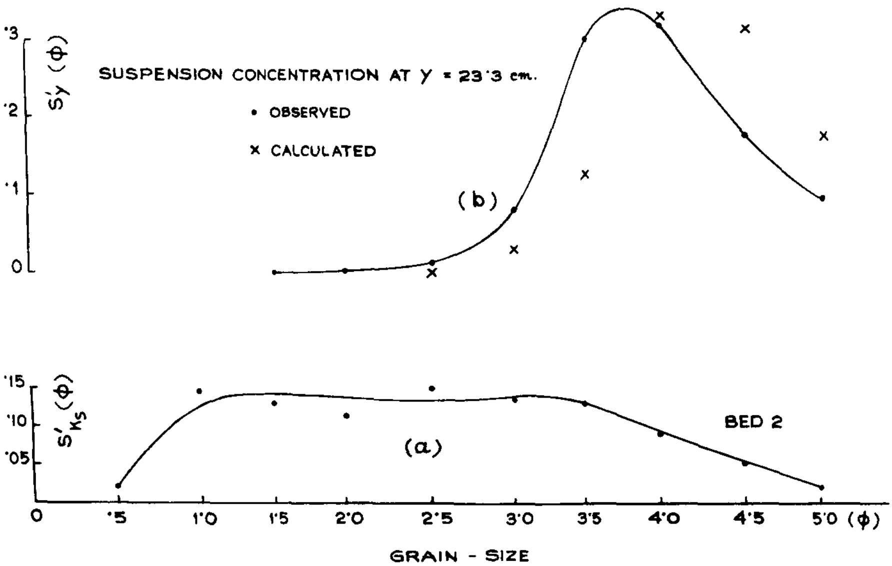

FIG. 1: Bed 2, grain-size distribution (relative concentration) (a) in the bed S'_{k_s} (ϕ)*; (b) in suspension* $S'_y(\phi)$ *at* $y = 23.3$ *cm.*

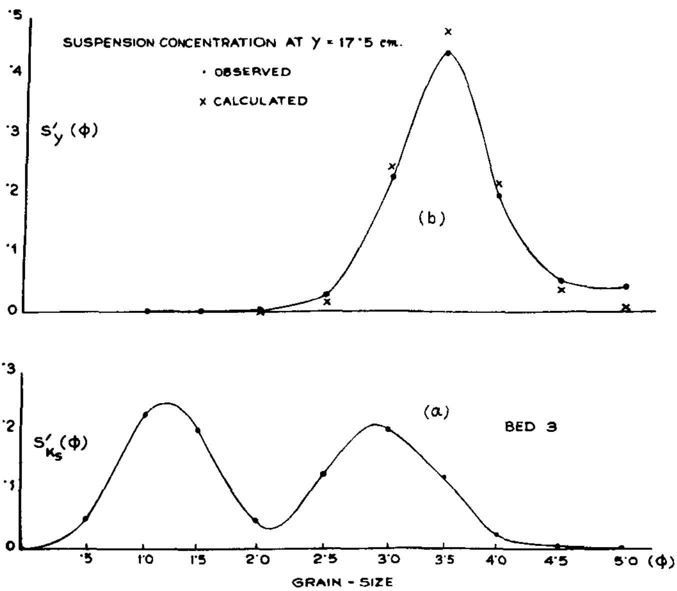

FIG. 2: Bed 3, grain-size distribution (relative concentration) (a) in the bed S'_{k_s} (ϕ)*;(b) in suspension* S'_{k_s} (ϕ) *at* $y = 17.5$ *cm.*

method works and yields a solution

$$S'_y(\phi) = S'_{k_s}(\phi)\ \exp\{A(y)c(\phi)\}/\sum_{\phi} S'_{k_s}(\phi)\ \exp\{A(y)c(\phi)\}. \quad (10)$$

For fixed y, $A(y)$ is a constant and can be determined by the method of least squares. This was done for both beds 2 and 3. For bed 3 there was practically no improvement over (5); for bed 2 the improvement was much greater.

3. SHAPES OF SUSPENSION DISTRIBUTIONS FOR LOGNORMAL AND HYPERBOLIC BEDS

In this section we shall think of $S'_y(\phi)$ as a density. Among other things equation (7) should be interpreted as

$$e^{-g(\psi)} = \int_{-\infty}^{\infty} S'_{k_s}(\phi)\ e^{-\psi c(\phi)} d\phi. \quad (11)$$

In view of equation (5) it is more convenient to write $S'_y(\phi)$ as $S'_\psi(\phi)$. We shall assume throughout that expression (5) is true but in most of the discussion the actual form (6) of ψ will not play any role; thus our discussion would cover the more general form (10).

Our main problem is to study the change in shape of the suspension distribution S'_ψ with change in ψ for a given bed distribution $S'_{k_s}(\phi)$. More specifically, given $S'_{k_s}(\phi)$, we wish to determine whether there is one or more values of ψ for which S'_ψ is unimodal and either approximately normal or at least a symmetric distribution. In the later case one would also be interested in the kurtosis of S'_ψ.

In the context the following mathematical problem is of interest. Can one choose $S'_{k_s}(\phi)$ such that equation (5) does not alter the shape and only the location is changed? In other words we want a solution of

$$S'_\psi(\phi) = \text{const.}\ S'_{k_s}(\phi - \alpha(\psi)). \quad (12)$$

Here $\alpha(\psi)$ would be the difference in the mean or median of $S'_{k_s}(\phi)$ and $S'_\psi(\phi)$. The following proposition contains a complete answer:

Proposition. If $S'_{k_s}(\phi)$ and $S'_\psi(\phi)$ satisfy equation (12) and $\alpha(\psi)$ is continuously differentiable for $0 < \psi < \infty$, then $c(\phi)$ is linear in ϕ and $S'_{k_s}(\phi)$ is normal.

This may be proved using Theorems 4 and 4a of Dynkin (1951). We have also a direct proof which is omitted because of lack of space.

Since in our case $c(\phi)$ is not linear this must be regarded as a negative result implying the non-existence of $S'_{k_s}(\phi)$ satisfying equation (12).

As the following discussion shows, a more promising line of enquiry is to relax equation (12) by introducing a scale parameter $B(\psi)$ as well as a location parameter $\alpha(\psi)$, requiring a relation like (12) to hold only approximately and that too only for the range of ϕ of interest (from the point of view of studying suspension), namely $1 \leqslant \phi \leqslant 5$, i.e., we want to investigate the possibility of

$$S'_\psi(\phi) \simeq \text{const. } S'_{k_s}\{[\phi - \alpha(\psi)]/B(\psi)\}, \quad 1 \leqslant \phi \leqslant 5. \tag{13}$$

Let us begin by studying the question of unimodality of $S'_\psi(\phi)$. It is noted that $\log c(\phi)$ is approximately linear in ϕ (Table 1), i.e., $\log c(\phi) \simeq a + b\phi$, $1 \leqslant \phi \leqslant 5$, where $a = 3.5758$, $b = -1.1840$.

TABLE 1

ϕ	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Observed $\log c(\phi)$	1.617	1.169	.688	.140	-.329	-1.139	-1.796	-2.526
Computed $\log c(\phi)$	1.800	1.208	.616	.024	-.568	-1.160	-1.752	-2.344

Let $C_\psi(\phi) = \log S'_\psi(\phi)$ and $C_{k_s}(\phi) = \log S'_{k_s}(\phi)$; using (11), equation (5) can be written as

$$C_\psi(\phi) = C_{k_s}(\phi) - \psi c(\phi) + g(\psi). \tag{14}$$

Now $C_\psi(\phi)$ may be expected to be unimodal if we can find a unique solution to the equation for the mode ϕ. Then from equation (14)

$$C'_\psi(\hat{\phi}) = C'_{k_s}(\hat{\phi}) - \psi e^{a+b\hat{\phi}} \cdot b = 0, \qquad (15)$$

where primes denote derivatives with respect to ϕ. Since $1 \leq \phi \leq 5$, we are interested in solutions lying in $2 \leq \phi \leq 4$. Here (I) since $b < 0$, equation (15) can have a solution only when $C'_{k_s}(\hat{\phi}) < 0$, and (II) the solution of (15) is unique if $C''_\psi(\hat{\phi}) < 0$ for $2 \leq \hat{\phi} \leq 4$. Clearly a sufficient condition for (II) is $C''_{k_s}(\phi) \leq 0$ for $2 \leq \phi \leq 4$.

This condition holds if $S'_{k_s}(\phi)$ is normal or hyperbolic (Barndorff-Nielsen, 1977) with mode to the left of 2. For such $S'_{k_s}(\phi)$ the set of values of ψ for which equation (15) has a solution will determine the region of unimodality. Bed 3 in the range $2 \leq \phi \leq 5$ is approximately hyperbolic with mode near 3.0, and will be discussed later.

For bed 2, *vide* Figure 3, (I) implies that equation (15) can have a solution only to the right of $\phi = 3.5$; there a straight line describes $C_{k_s}(\phi)$ adequately so that (II) holds, i.e., if (15) has a solution, the solution is unique. Thus the conclusion of above paragraph holds for this bed. Now we will study the normality of suspension distribution above the bed 2 when equation (15) has a solution and (II) holds at $\phi = \hat{\phi}$. It can be checked from Table 1 that

$$c(\phi) \simeq c(\hat{\phi}) + \tfrac{1}{2} c''(\hat{\phi}) \cdot (\phi - \hat{\phi})^2, \qquad |\phi - \hat{\phi}| \leq 1 \qquad (16)$$

and if we assume

$$C_{k_s}(\phi) \simeq C_{k_s}(\hat{\phi}) + \tfrac{1}{2} C''_{k_s}(\hat{\phi}) \cdot (\phi - \hat{\phi})^2, \qquad |\phi - \hat{\phi}| \leq 1, \qquad (17)$$

then one gets from equation (14)

$$C_\psi(\phi) \simeq C_\psi(\hat{\phi}) + \tfrac{1}{2} C''_\psi(\hat{\phi}) \cdot (\phi - \hat{\phi})^2, \qquad |\phi - \hat{\phi}| \leq 1. \qquad (18)$$

The reason for confining attention to $|\phi - \hat{\phi}| \leq 1$ in (16) is that most of the mass of S'_y is concentrated here. Clearly equation (17) holds for a lognormal or hyperbolic bed $S'_{k_s}(\phi)$ with mode to the left of $\phi = 2$; it also holds for bed 2. Then $S_\psi(\phi)$ looks approximately like a normal with mean $\hat{\phi}$ and

$$\text{Variance} = -C''_\psi(\hat{\phi}) = -C''_{k_s}(\hat{\phi}) + \psi b^2 e^{a+b\hat{\phi}}. \qquad (19)$$

We shall now study in detail the bed 3. In this case we fit a hyperbolic distribution to $S'_{k_s}(\phi)$ for $2 \leqslant \phi \leqslant 5$ as

$$\log S'_{k_s}(\phi) \simeq \nu - \tfrac{1}{2}(\gamma_1+\gamma_2)\{\delta^2+(\phi-\mu)^2\}^{\frac{1}{2}} + \tfrac{1}{2}(\gamma_1-\gamma_2)(\phi-\mu). \qquad (20)$$

We determine the parameters γ_1, γ_2 and μ graphically using the geometrical interpretation of these parameters given by Bagnold and Barndorff-Nielsen (1979) and δ from their equation (2.2). The constant ν has been adjusted to agree with the observed frequency in the range $2 \leqslant \phi \leqslant 5$. The estimated values are $\gamma_1 = 1.787$, $\gamma_2 = 4.294$, $\mu = 3.228$, $\delta = .5039$ and $\nu = -.225$.

Here also from Figure 3, it is clear by (I) that equation (15) can have a solution only to the right of $\phi = 3.0$ and (II) also holds for $\phi \geqslant 3.23$. Hence, whenever (15) has a solution, it is unique so that the corresponding $S'_\psi(\phi)$ is unimodal.

We shall now examine when $S'_\psi(\phi)$ can be symmetrical as well as unimodal. Expanding $c(\phi)$ around $\hat{\phi}$ up to the quadratic term, we get

$$\log S'_\psi(\phi) \simeq g(\psi) + \nu - \tfrac{1}{2}(\gamma_1+\gamma_2)[\delta^2+(\phi-\mu)^2]^{\frac{1}{2}}$$

$$+ \tfrac{1}{2}(\gamma_1-\gamma_2)(\phi-\mu) - \psi c(\hat{\phi})$$

$$- \psi b(\phi-\hat{\phi})c(\hat{\phi}) - \tfrac{1}{2}\psi b^2(\phi-\hat{\phi})^2 c(\hat{\phi}). \qquad (21)$$

In order to get symmetry $\hat{\phi}$ must be nearly equal to μ and

$$\tfrac{1}{2}(\gamma_1-\gamma_2) - \psi b c(\hat{\phi}) \simeq 0. \qquad (22)$$

(Note that (22) follows from (15) if $\hat{\phi} \simeq \mu$.) Then around $\hat{\phi}$, we have

$$\log S_\psi(\phi) \simeq g(\psi) + \nu - \tfrac{1}{2}(\gamma_1+\gamma_2)[\delta^2+(\phi-\hat{\phi})^2]^{\frac{1}{2}}$$

$$- \tfrac{1}{2}\psi b^2(\phi-\hat{\phi})^2 c(\hat{\phi}). \qquad (23)$$

Inspection of (23) suggests that the presence of the term

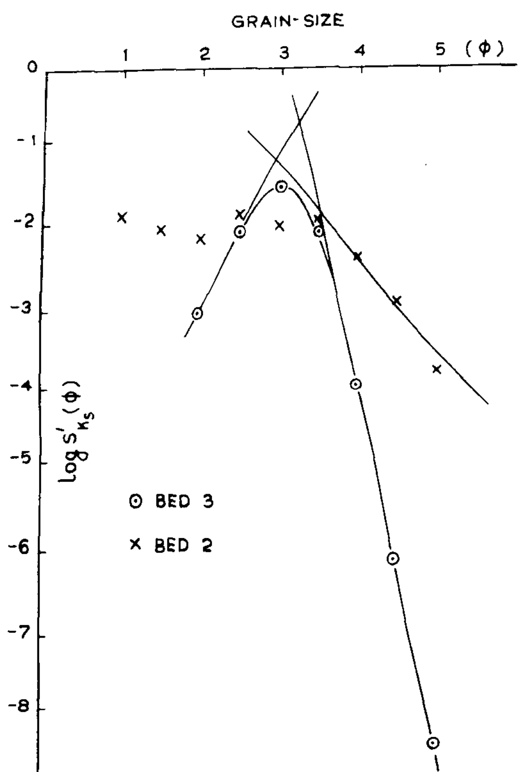

FIG. 3: Grain-size distribution of beds 2 and 3 (log $S'_{k_s}(\phi)$ *against* ϕ).

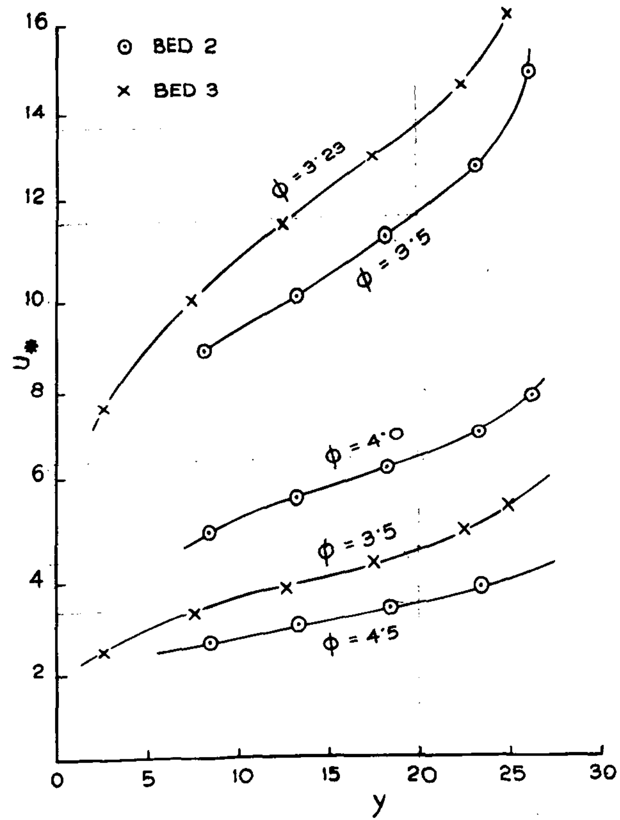

FIG. 4: Graph of height y *against velocity* u_*.

$[\delta^2 + (\phi-\hat{\phi})^2]^{\frac{1}{2}}$ is likely to lead to less peakedness and hence higher values of the coefficient of kurtosis β_2 than the normal. Confirmation of this expectation is provided in the following numerical calculations.

For the values of $y = 17.5$ cm, $u_* = 6.297$ cm/sec, $d = 27.5$ cm and $k_s = .0451$ cm, we get the value $\psi = 2.7676$. The skewness and kurtosis of the corresponding suspension distribution are, respectively, 0.343 and 4.152 (see Figures 12 and 13 of Sengupta, 1979).

The above analysis illustrates how one can study the unimodality, symmetry and normality of S'_ψ for a given hyperbolic bed distribution $S'_{k_s}(\phi)$.

We now study briefly how one can determine, for a given bed distribution, the flow parameters y and u_* leading to unimodality. Here the relation (6) determining ψ will be used. To fix ideas we work with bed 2. Since by (I) and Figure 3, equation (15) can have a solution only for $\hat{\phi} \geq 3.5$ and observed data with $\hat{\phi} > 4.5$ is likely to be scarce, let us work with $3.5 \leq \hat{\phi} \leq 4.5$. For fixed $\hat{\phi}$ in this range we now solve (15) for ψ; for $\hat{\phi} = 3.5$, $\psi = 1.663$ and for $\hat{\phi} = 4.5$, $\psi = 5.441$. The curves obtained by plotting y against u_* for these two fixed values of ψ in (6) are shown in Figure 4. The zone between these curves gives the values of y and u_* which will give unimodality with peak at some $3.5 \leq \hat{\phi} \leq 4.5$. As noted before these unimodal distributions will be approximately lognormal.

A similar analysis was made for bed 3. To achieve symmetry we kept $\hat{\phi}$ in the range 3.23 to 3.5. The resulting curves (y against u_*) are also shown in Figure 4. The combinations (y, u_*) obtained this way agree well with our experimental observations for both beds 2 and 3.

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