Electrical Analog Computing Machine for Solving Linear Equations and Related Problems

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A new iteration method of solving linear equations is described. The convergence of this method depends on a single scalar parameter which can be predetermined. A brief description of an electrical analog machine for solving linear equations with ten variables based on this iteration method is given. In addition to the solution the machine gives the largest eigenvalue of the matrix of the linear system.

INTRODUCTION

BERRY, Wilcox, Rock, and Washburn[1] have described a very interesting analog computing machine for solving linear equations based on a well-known numerical method, *viz.*, the Gauss-Seidel iteration method. Since then the descriptions of many other analog machines based on this method and some other numerical methods have been published. It is noteworthy that analog machines which are based on iteration methods are very sound from the practical viewpoint. The accuracy of an analog machine is limited by the accuracy of the machine components. If large rounding-off and accidental errors are allowed, the error in the final result will be so large as to spoil the prospect of using the machine for numerical work. The beauty of the iteration methods is that they keep down all propagated errors, such that the error in the final result does not exceed the elementary instrumental error of a single analogous multiplication and addition, inherent in the particular instrument and the analog methods employed.

But a serious drawback of the Gauss-Seidel iteration method is that it does not converge always and that its convergence depends on the initial approximation. The machine which is going to be described is based on another iteration method, whose convergence can be proved very easily and which is entirely independent of the initial approximation, but depends on a single scalar parameter that can be predetermined. Further, the eigenvalue (latent root) of the matrix, having the largest absolute value can be determined simultaneously with the solution of a system of linear equations, which is a special feature of the machine. Solution can be obtained even if the rank of the matrix is less than the number of equations. The mathematical details of the method will be published elsewhere. The following gives a brief account of the method:

[1] Berry, Wilcox, Rock, and Washburn, J. Appl. Phys. **17**, 262 (1946).

MATHEMATICS OF THE MACHINE

The system of linear equations,

$$\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n&=g_1,\\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n&=g_2,\\ \cdots\cdots\cdots&\\ \cdots\cdots\cdots&\\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\cdots+a_{nn}x_n&=g_n, \end{aligned}$$

may be expressed in the matrix notation by

$$AX=G, \tag{1}$$

where A is the square matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix},$$

X is the unknown column vector whose components are $(x_1, x_2, x_3, \cdots, x_n)$, and G is the known column vector whose components are $(g_1, g_2, g_3, \cdots, g_n)$. If the determinant of A is not zero, then the solution of the above equations is given symbolically by

$$X=A^{-1}G, \tag{2}$$

where A^{-1} is the reciprocal matrix of A.

Let h be a scalar number, then the Neumann series,

$$I+(I-hA)+(I-hA)^2+\cdots,$$

converges to

$$\left[I-(I-hA)\right]^{-1}=(hA)^{-1}=\frac{1}{h}A^{-1},$$

provided the latent roots of A are all positive and $h<2/\lambda_d$, where λ_d is the largest latent root of the matrix.

Let us assume for the time being that the latent roots of A are all positive, that is to say, that the matrix A is positive-semidefinite, and form the iterative sequence[2]

$$\begin{aligned} X^{(1)}&=hG+(I-hA)X^{(0)}\\ X^{(2)}&=hG+(I-hA)X^{(1)}\\ &\cdots\cdots\\ &\cdots\cdots\\ X^{(m+1)}&=hG+(I-hA)X^{(m)}, \end{aligned} \tag{3}$$

where $X^{(0)}$ is any arbitrary vector. It can be shown that

$$\begin{aligned} X^{(m)}&=A^{-1}G+(1-hA)^m(X^{(0)}-A^{-1}G)\\ &=A^{-1}G+O(1-h\lambda_d)^m(X^{(0)}-A^{-1}G). \end{aligned} \tag{4}$$

Thus

$$\lim_{m\to\infty} X^{(m)}=A^{-1}G \tag{5}$$

whatever the initial approximation $X^{(0)}$ may be, provided, of course, $h<2/\lambda_d$ and the latent roots are all positive.

If, however, A be any real matrix then the latent roots of AA' are all positive, A' being the transpose of A. Consequently, the above method is applicable to the product matrix AA' and the solution of the equation

$$(AA')Y=G$$

can be obtained by the iterative process described. Now,

$$\begin{aligned} Y&=(AA')^{-1}G\\ &=[(A')^{-1}A^{-1}]G. \end{aligned}$$

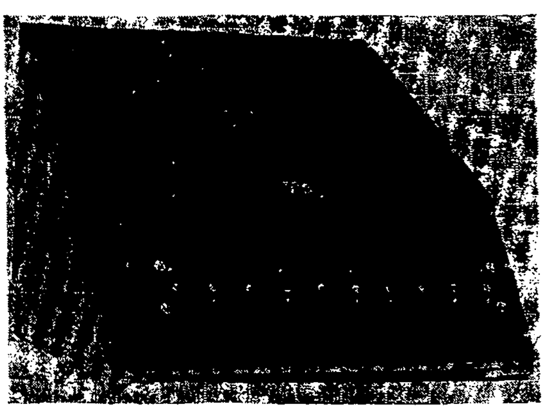

FIG. 1. The photograph of the machine.

[2] Hotelling, Ann. Math. Stats. 14, 23 (1943).

Multiplying this solution by the transposed matrix A' we get

$$A'Y=A^{-1}G=X,$$

which is the required solution.

So, when A is any matrix, the above iteration method is applicable, though some extra computational labor is involved. In the machine the multiplication of A by A' and the multiplication of Y by A' are carried out mechanically.

There is one very interesting feature of the machine. If h, which is analogously represented by a tapped voltage from a potentiometer, were increased to $2/\lambda_d$ (or $2/|\lambda_d|^2$ in the case of an arbitrary matrix) the iteration process would not converge. If a solution has been obtained in the machine with a particular setting of h, if this solution is very slightly disturbed, and if the setting of h is gradually increased to a point where $h=2/\lambda_d$, the iterative operation of the machine will fail to restore the solution and would never tend to reach a final stage. Consequently, this setting of h will give the absolute value of the largest latent root with a sufficient degree of accuracy. Another interesting feature is that a solution can be obtained even though the rank of the matrix A is less than the number of equations, because only matrix multiplication is involved in the iterative method.

DESCRIPTION AND OPERATION OF THE MACHINE

The machine (Fig. 1) is designed for solving a system of linear equations with ten variables. (The basic components of the machine are almost the same as those of the machine designed by Berry *et al.*[1]) Numbers (fractional) are simulated by ac voltages tapped from wire-wound potentiometers. Negative numbers are simulated by ac voltages in opposite phase to the voltages which represent positive numbers. These "positive" and "negative" voltages are supplied by an accurately wound center-tapped transformer suitably loaded. Multiplication of voltages is executed by potentiometers and addition of voltages is done by Kirchhoff network. There are 100 potentiometers to represent the 100 elements, a_{11}, a_{12}, a_{13}, $\cdots$, of the matrix A, and 10

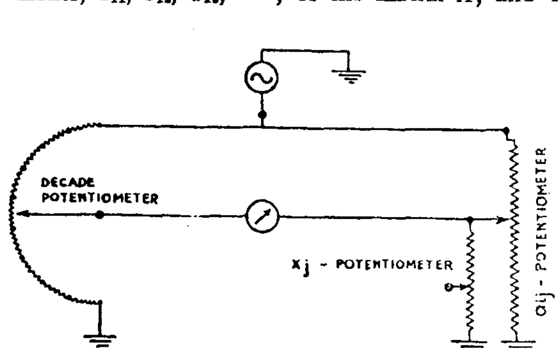

FIG. 2. The bridge circuit for setting the coefficients of the matrix in the corresponding matrix potentiometers.

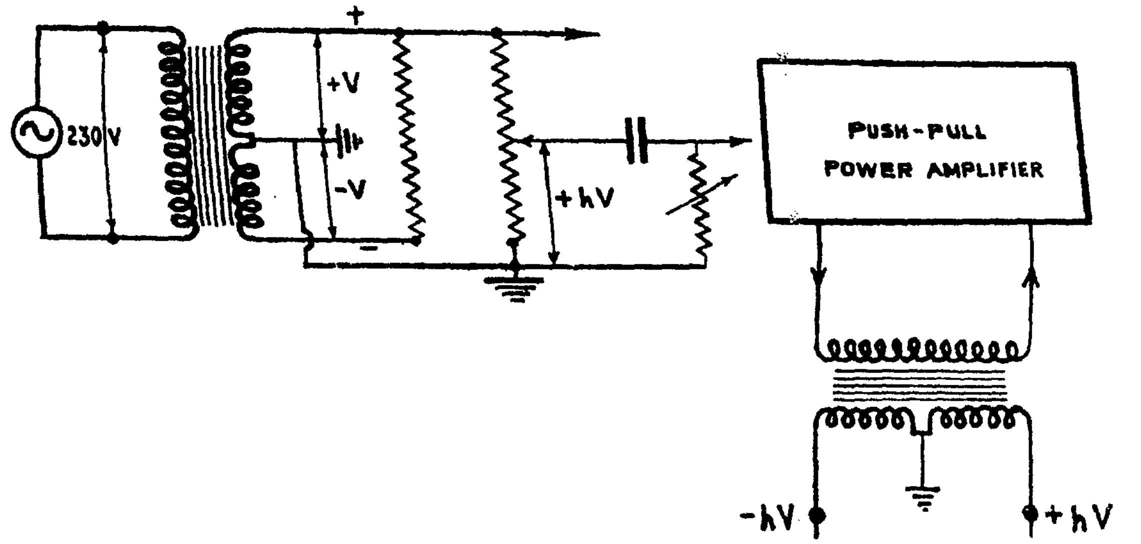

Fig. 3. The circuit for multiplying the coefficients of the matrix and of the column vector by the parameter h.

potentiometers to represent the 10 elements, $g_1, g_2, g_3, \cdots$, of the column vector G. There are also 10 potentiometers each to represent the iterated column vectors $X^{(m)}$ and $X^{(m+1)}$ given in Eq. (3). The first step is setting the coefficients of the matrix A and the column vector G in the corresponding potentiometers. For the sake of clarity, it is desirable to denote these potentiometers by their corresponding symbols in Eq. (3), *viz.*, $a_{11}, a_{12}, a_{13}, \cdots; g_1, g_2, g_3, \cdots$. There are ten rotary selector switches each with ten positions for selecting the particular element of the matrix in a column, and there is another such switch for selecting the elements of the column vector G. Each one of these switches has two poles. One pole is switched to the upper terminal of a potentiometer while the other pole is connected to its tapping terminal, the lower terminals of all the potentiometers being grounded. There is a four-dial decade potentiometer in which a four-digit number representing the coefficients of the matrix A or of the column vector G can be put. For example, if the element of the matrix a_{11} has a numerical value such as 0.8962, the selector switch corresponding to the first column of the matrix is set on the first row position and the number 0.8962 is put on the decade potentiometer. The a_{11}-potentiometer is then adjusted to match the voltage given by the decade potentiometer by means of a Wheatstone bridge circuit. In order to avoid loading errors, each potentiometer belonging to a column of the matrix is loaded with the corresponding X-potentiometer during this setting operation, but the potentiometers belonging to the diagonal elements $a_{11}, a_{22}, a_{33}, \cdots$, are not loaded with their corresponding x_1-, x_2-, x_3-, $\cdots$,-potentiometers, for reasons to become obvious in the next paragraph. The circuit is described in Fig. 2.

The next circuit is the iteration operation of Eq. (3). The first equation of this vectorial formula is

$$x_1^{(1)}=hg_1+(1-ha_{11})x_1^{(0)}-ha_{12}x_2^{(0)} -ha_{13}x_3^{(0)}-\cdots-ha_{110}x_{10}^{(0)}. \quad (6)$$

The left-hand side of this equation shows that in order to get the first element of the iterated vector $X^{(1)}$ from its initial approximation $X^{(0)}$ it is necessary to (a) multiply the voltage given by the g_1-potentiometer by a fractional number h; (b) multiply the voltages given by the potentiometers representing the first row of the matrix $a_{11}, a_{12}, a_{13}, \cdots a_{110}$ by a negative fraction $-h$; (c) from the voltage $-ha_{11}$ given by the a_{11}-potentiometer to build a voltage proportional to $1-ha_{11}$ by an electronic circuit; (d) to multiply the voltages proportional to $1-ha_{11}, -ha_{12}, -ha_{13}, \cdots, -ha_{1\,10}$, respectively, by the numbers $x_1^{(0)}, x_2^{(0)}, \cdots, x_{10}^{(0)}$; (e) to add all these voltages and store the result in a potentiometer.

The operations (a) and (b) are performed by a circuit described in Fig. 3. The operation (c) is performed by the circuit described in Fig. 4. The operation (d) is effected by a simple potentiometric multiplication and operation (e) is effected by the usual addition circuit based on Kirchhoff's laws, which is shown in Fig. 5. For the second and other elements of the vector $X^{(1)}$ the same operations are only repeated.

The circuit described in Fig. 3 provides "positive" and "negative" voltages proportional to h. This is a negative feedback current amplifier with high input impedance designed in such a way that the output voltage is exactly equal in magnitude and phase to the input voltage. This input voltage, which is proportional to h, is tapped from a potentiometer connected to the "positive" side of the main center-tapped transformer and ground. The value of h can be read from this potentiometer by means of the decade potentiometer described earlier. The circuit shown in Fig. 4 is an

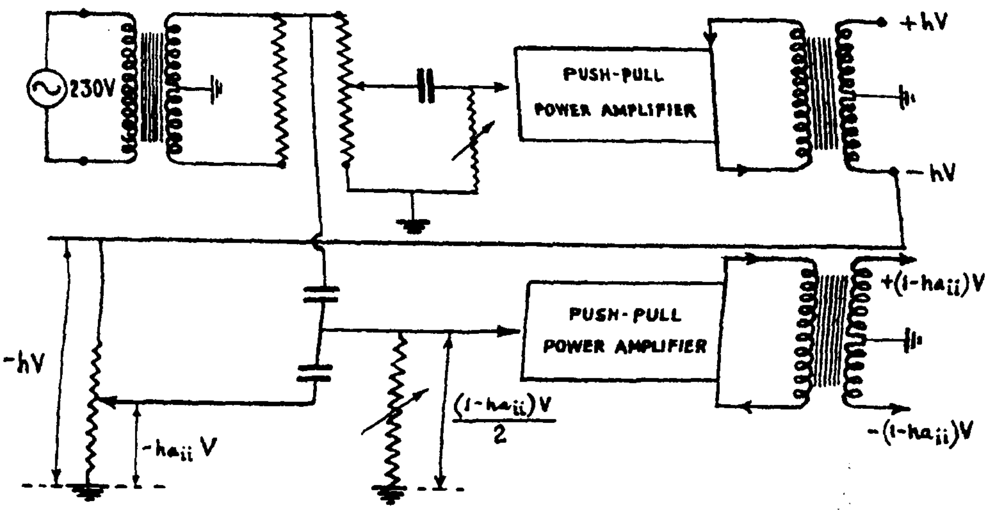

FIG. 4. The circuit for producing low impedance source of voltage proportional to $1-ha_{ii}$.

exactly similar current amplifier, the only difference is that the input voltage is proportional to $1-ha_{ii}$.

After the coefficients of the matrix and the known column vector are taken into the machine as described earlier the computing operation starts. This consists in selecting the potentiometers corresponding to the first row of the matrix with the help of the eleven selector switches. Proper signs of the coefficients a_{11}, a_{12}, $\cdots$ $a_{1\,10}$, g_1 are then put by applying "positive" or "negative" voltages through eleven toggle switches and the corresponding x_1'-potentiometer is adjusted until a null is shown on the null detector. The mathematics of this operation has been stated in Eq. (6). The corresponding circuit is given in Fig. 5.

This operation is continued for all the rows of the matrix and the result of the first iterative step is stored in one of the two sets of ten of x-potentiometers, let us call them x'-potentiometers. In the next iterative step, these x'-potentiometers take the place of the x-potentiometers and are switched on to the matrix potentiometers by a ganged rotary switch. The x-potentiometers are used to store the result of the next iterative step. The first row of the matrix is then selected and the first x-potentiometer is adjusted to a balance and the process is repeated exactly as before when the result of this second iterative step is stored in the x-potentiometers. This operation is repeated again and again until the values in the x- and x'-potentiometers are exactly equal. If the convergence is found slow, the value of the parameter h is changed by adjusting the h-potentiometer to an optimum position. By adjusting suitably, most problems can be solved with less than twenty iterative steps. These values are then read off one by one in the decade potentiometer. To find the absolute value of the largest latent root, the voltage output from the h-potentiometer is increased gradually and the solving operation with the x- and x'- potentiometers is repeated after slightly disturbing them from the original solution position. When h is nearly $2/\lambda_d$ the solving operation will fail to restore the potentiometers to the position of solution.

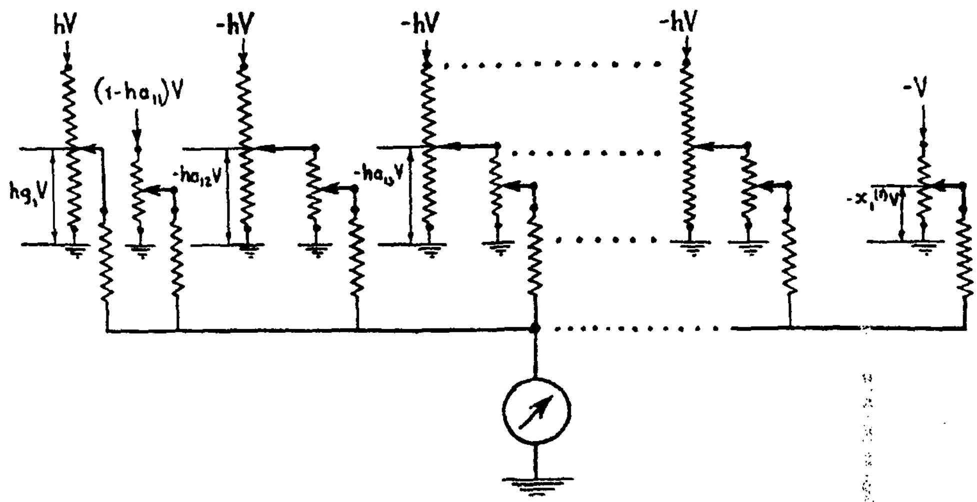

Fig. 5. The circuit corresponding to an iteration step as stated in Eq. (6).

The slight error introduced will not be rectified by the iteration operation but would tend to increase as this operation is continued. At this stage the value of h is read off with the help of the decade potentiometer and the largest latent root is nearly twice the reciprocal of this value of h.

The accuracy of the machine is about 0.5 percent. If precision potentiometers or potentiometers with vernier arrangement were used, the accuracy could be increased considerably. It takes about 90 minutes to solve a system of equations with ten variables, most of the time being consumed for setting the 110 potentiometers.

PROGRAMMING OF PROBLEMS

The inherent defect of representing numbers analogously by potentiometers is that these numbers must be fractional. It is very easy to reduce all the coefficients of the linear equations to proper fractions by dividing them by a suitable number which must be greater than or equal to the magnitude of the largest coefficient occuring in the matrix and the known column vector. Of course, the solution would not be affected in any way. But, to ensure that the elements of the solution vector will also be fractional, some other programming is necessary. If the elements of the known column vector G are divided by a number $c \geqslant n/N(A)$, where n is the number of variables and $N(A)$ is the norm of the matrix A, that is to say

$$N(A) = \left(\sum_{j} \sum_{i} a_{ij}^2 \right)$$

it can be proved with the help of Cauchy-Schwartz inequality that each of the elements of this solution vector, which is $A^{-1}[(1/c)G)]$ is less than one in magnitude. Consequently, the required solution can be obtained simply by multiplying this solution by c.

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