A STUDY OF BIB DESIGNS WITH REPLICATIONS 11 TO 15

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SUMMARY. The combinations of parameters for BIB designs with replications 11 to 15 have been listed and the actual solutions of the cyclic type have been given in a number of cases. It may be observed that Fisher and Yates (1953) give the solutions to BIB designs up to 10 replications only.

1. Introduction

Sir Ronald Fisher, during his visit to the Indian Statistical Institute in the winter of 1960-61, suggested to the author that a study may be undertaken of the BIB designs with replications 11 to 15. He also gave a list of possible combinations of the parameters v, b, k, r and λ for such designs, except those derivable by doubling a known design with half the number of replications. This list is given in Table 1 together with the reference numbers, which are assigned following the convention of Fisher and Yates (1953). The designs derivable from geometrical configurations, denoted by *o.s.* (orthogonal squares) and *o.c.* (orthogonal cubes) in Fisher and Yates (1953) and Fisher (1945), are given a more explicit representation, indicating the particular geometrical configurations from which the solutions are obtained. The last non-geometrical design listed in Fisher and Yates (1953) has the reference no. 31 and, therefore, the non-geometrical designs in the present list of Table 1 are numbered consecutively from 32, first with increasing values of v within a replication (r), and then with increasing values of r. The actual numbers are written only in cases where the solutions have been found.

It may be seen that in a number of cases denoted by (—), the solution is unknown. In some cases solutions are derived by the methods given by Bose (1939) and Rao (1945, 1946) in so far as they are applicable. Other solutions are obtained by trial and error. The impossibility of certain designs has been established by using the results of Bruck and Ryser (1949), Schützenberger (1949), Chowla and Ryser (1950), Shrikhande (1950), and Hall and Connor (1954). Since their results and the author's (Rao, 1944, 1945, 1946) on the cyclic representations of geometrical designs are published mostly in Journals on Mathematics the relevant theorems are quoted in the next section for ready reference.

Some of the solutions obtained by trial and error and listed in Table 3 can be derived as particular cases of general combinatorial problems. For instance, a *combinatorial assignment problem* of which no. 66 in Table 1 is a special case may be stated as follows. In an establishment, there are v officers, r departments and k types of jobs in each department. Every officer has to be assigned a job in each department such that, in any department there are equal numbers of officers in different jobs and that any two officers have common jobs in exactly λ departments. When $v = s^2$ and s is a prime power a solution exists with $\lambda = 1$. For $s = 6$, no solution is possible with $\lambda = 1$, but a solution can be found with $\lambda = 2$ (derivable from the BIBD design no. 66). It is of some interest to determine for any given number v the minimum λ for which the assignment problem is soluble. All the numbers can then be characterised by the associated minimum λ.

TABLE 1. COMBINATIONS OF THE PARAMETERS FOR BIB DESIGNS WITH REPLICATIONS 11 TO 15

v	b	k	r	λ	ref. no.
12	44	3	11	2	32
12	33	4	11	3	33
12	22	6	11	5	34
23	23	11	11	5	35
45	99	5	11	1	36
45	55	9	11	2	—
56	56	11	11	2	—
100	110	10	11	1	?
111	111	11	11	1	?
13	26	6	12	5	41
19	57	4	12	2	42
21	42	6	12	3	43
22	33	8	12	4	—
25	100	3	12	1	45
33	44	9	12	3	—
34	34	12	12	4	*(Th 1)
37	111	4	12	1	48
45	45	12	12	3	—
55	66	10	12	2	*(Th 2)
61	122	6	12	1	—
67	67	12	12	2	*(Th 4)
121	132	11	12	1	E (2, 11):1
133	133	12	12	1	P (2, 11):1
27	117	3	13	1	E (3, 3):1
27	39	9	13	4	E(3, 3):2
27	27	13	13	6	53
40	130	4	13	1	P (3, 3):1
40	52	10	13	3	—
40	40	13	13	4	P (3, 3):2
53	53	13	13	3	*(Th 3)
66	143	6	13	1	56
66	78	11	13	2	—
79	79	13	13	2	—
144	156	12	13	1	?
157	157	13	13	1	?
15	42	5	14	4	61 (d)
15	35	6	14	5	62
22	77	4	14	2	63
22	44	7	14	4	— (d)
29	58	7	14	3	65
36	84	6	14	2	66 (d)
43	86	7	14	2	67 (d)
78	91	12	14	2	*(Th 2)
85	170	7	14	1	—
92	92	14	14	2	*(Th 1)
169	182	13	14	1	E (2, 13):1
183	183	14	14	1	P (2, 13):1
11	55	3	15	3	71
13	39	5	15	5	72
16	80	3	15	2	73
16	48	5	15	4	74
16	40	6	15	5	75
16	30	8	15	7	E (4, 2):3
21	35	9	15	6	76
26	65	6	15	3	77
28	42	10	15	5	—
31	155	3	15	1	P (4, 2):1
31	93	5	15	2	79
31	31	15	15	7	P (4, 2).3
36	36	15	15	6	—
43	43	15	15	5	—
46	69	10	15	3	—
56	70	12	15	3	—
61	183	5	15	1	84
71	71	15	15	3	—
76	190	6	15	1	—
91	195	7	15	1	87
91	105	13	15	2	*(Th 2)
106	106	15	15	2	*(Th 1)
136	204	10	15	1	—
196	210	14	15	1	*(Th 2)
211	211	15	15	1	*(Th 5)

—solution unknown, * solution does not exist, ? presumably non-existing, (d) double of insoluble type.

P and E have been used for PG, and EG, the Projective and Euclidean finite geometrical configurations whose construction is considered in Section 3.

2. Theorems on non-existence of certain BIB designs

Theorem 1 : (Schützenberger, 1949; Chowla and Ryser, 1950; Shrikhande, 1950) : *For the existence of a symmetrical BIBD with parameters, v, b, k, r, λ, a necessary condition is that $(r-\lambda)$ is a perfect square, when v is even.*

Theorem 2 : (Hall and Connor, 1954; Shrikhande, 1960) : *The existence of the BIBD, $v-r$, $b-1$, $r-\lambda$, r, λ, implies that of the BIBD, $v = b$, $k = r$, λ, when $\lambda = 1$ or 2.*

For example, the designs 68 and 70 with the parameters

$$v = 78,\ b = 91,\ k = 12,\ r = 14,\ \lambda = 2$$

$$v = 92,\ b = 92,\ k = 14,\ r = 14,\ \lambda = 2$$

do not exist. By Theorem 1, the latter does not exist as $(k-\lambda) = 12$ is not a perfect square, and Theorem 2 excludes the possibility of the former. Similarly the designs 88 and 89 do not exist. No. 47 does not exist by Theorem 1, but this does not necessarily imply the non-existence of 44.

Theorem 3 : (Chowla and Ryser, 1950): *If $v \equiv 1 \mod 4$ and there exists an odd prime p such that p divides the square free part of $k-\lambda$, and, moreover, if the Legendre symbol $(\lambda | p) = -1$, then the symmetric BIBD does not exist.*

Theorem 4 : (Chowla and Ryser, 1950) : *If $v \equiv 3 \mod 4$ and if there exists an odd prime p such that p divides the square free part of $k-\lambda$, and moreover, if $(-\lambda | p) = -1$, then the symmetric BIBD has no solution.*

For the design 52, ($v = 67 = b$, $r = 12 = k$, $\lambda = 2$), $v \equiv 3 \mod 4$, $k-\lambda = 10$ and $p = 5$ is an odd prime dividing the square free part of 10. We find $-\lambda = -2$ is a quadratic non-residue of 5, so that the Legendre symbol $(-2|5) = -1$. Hence, by Theorem 4, the symmetrical design 52 does not exist. By Theorem 2, 50 does not exist. By an application of Theorem 3, it can be shown in a similar way that 55 does not exist. This does not imply the non-existence of 54, as Theorem 2 is not applicable for $\lambda = 3$.

Theorem 5 : (Bruck and Ryser 1949): *If $n \equiv 1$ or $2 \mod 4$ and p, a prime dividing the square free part of n, is of the form $4k+3$, then $(n-1)$ mutually orthogonal squares do not exist.*

Designs 91 and 92 are not possible for they imply the existence of 13 mutually orthogonal squares of order 14, which is impossible by Theorem 5.

3. Cyclic designs derivable from geometrical configurations

A finite projective geometry of t dimensions with coordinates as elements of a Galois Field $GF(s)$ is represented by $PG(t, s)$ and the corresponding Euclidean geometry by $EG(t, s)$. Bose (1959) observed that by choosing the points as varieties and all d dimensional flats as blocks from either $PG(t, s)$ or $EG(t, s)$ one can generate a BIBD. Designs so obtained are represented by $PG(t, s)$: d or $EG(t, s)$: d under the reference number in Table 1. The parameters of such designs are :

	$PG(t, s) : d$	$EG(t, s) : d$
v	$(s^{t+1}-1)/(s-1)$	s^t
b	$\phi(t, s, d)$	$s^{t-d}\phi(t-1, s, d-1)$
k	$(s^{d+1}-1)/(s-1)$	s^d
r	$\phi(t-1, s, d-1)$	$\phi(t-1, s, d-1)$
λ	$\phi(t-2, s, d-2)$	$\phi(t-2, s, d-2)$

where
$$\phi(t, s, d) = \frac{(s^{t+1}-1)\ldots(s^{t-d+1}-1)}{(s^{d+1}-1)\ldots(s-1)}$$

Such geometries can be constructed for s equal to a prime or a prime power and, therefore, the designs in Table 1 with reference numbers, of the type $PG(t, s) : d$ and $EG(t, s) : d$ exist.

We shall demonstrate how cyclic solutions can be obtained in a simple manner for all designs derivable from finite geometrical configurations. The main results are contained in Rao (1944, 1945, 1946) where the theorems of Singer (1938) and Bose (1942) have been generalised and applied to the construction of cyclic solutions to BIB designs. The theorems quoted in this section are from Rao (1946), which is devoted to the construction of cyclic solutions to BIB designs derivable from cyclic representations of finite geometrical configurations.

Theorem 6 : *Let $x^{t+1}-a_tx^t-\ldots-a_0$ be a minimum function generating the elements of $GF(s^{t+1})$ and the sequence ξ_d, $[d = 0, 1, \ldots, (s^{t+1}-s)/(s-1)]$, be derived from the recurrence relation*

$$\xi_{d+1} = a_t\xi_d+a_{t-1}\xi_{d-1}+\ldots+a_0\xi_{d-t}$$

with the initial values $\xi_0 = \ldots = \xi_{t-1} = 0$, $\xi_t = 1$. The set of integers d such that $\xi_d=0$ constitutes a difference set mod $v = (s^{t+1}-1)/(s-1)$.

The number of d's such that $\xi_d = 0$ is $k = (s^t-1)/(s-1)$ and among the $k(k-1)$ differences reduced mod v, each integer less than v occurs $\lambda = (s^{t-1}-1)/(s-1)$ times. The difference set provides a cyclic solution to the BIBD, $PG(t, s) : (t-1)$ with the parameters

$$v = \frac{s^{t+1}-1}{s-1} = b, \quad r = \frac{s^t-1}{s-1} = k, \quad \lambda = \frac{s^{t-1}-1}{s-1}$$

The difference sets for $t = 2$ and $s = 3, 4, 5, 7, 8, 9$ are given in Fisher and Yates (1953) and also in Rao (1946). The solutions for the following are obtained using the minimum functions indicated.

ref. no.	parameters	minimum function
$PG(2,11):1$	$v = b = 133,\ r = k = 12,\ \lambda = 1$	x^3-3x+1
$PG(2,13):1$	$v = b = 183,\ r = k = 14,\ \lambda = 1$	x^3+2x+2*
$PG(3,3):2$	$v = b = 40,\ r = k = 13,\ \lambda = 4$	$x^4-2x^3-2x^2-x-1$
$PG(4,2):3$	$v = b = 31,\ r = k = 15,\ \lambda = 7$	x^3-x^2-1

We shall demonstrate the method for $PG(3,3):2$. The initial values are $\xi_0 = \xi_1 = \xi_2 = 0$, $\xi_3 = 1$ and the rest ξ_4 to ξ_{39} are obtained from the recurrence relation

$$\xi_{d+1} = 2\xi_d+2\xi_{d-1}+\xi_{d-2}+\xi_{d-3}.$$

Thus $\xi_4 = 2$, $\xi_5 = 0$, $\xi_6 = 2$, ... and so on. The suffixes corresponding to zero values are

$$(0,\ 1,\ 2,\ 5,\ 12,\ 18,\ 22,\ 24,\ 26,\ 27,\ 29,\ 32,\ 33)$$

which constitute a difference set mod 40. On cyclic development we obtain the BIBD, $PG(3,3):2$, where the varieties are numbered 0 to 39. Similarly we find the cyclic solution to $PG(4,2):3$

$$(0,\ 1,\ 2,\ 3,\ 5,\ 6,\ 8,\ 11,\ 12,\ 18,\ 19,\ 20,\ 23,\ 27,\ 29) \bmod 31.$$

Two difference sets for this design obtained by alternative methods are

$$(1,\ 2,\ 4,\ 5,\ 7,\ 8,\ 9,\ 10,\ 14,\ 16,\ 18,\ 19,\ 20,\ 25,\ 28) \bmod 31$$

due to Bose (1939) and

$$(1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 15,\ 16,\ 17,\ 23,\ 24,\ 27,\ 29,\ 30) \bmod 31$$

due to Marshal Hall (1958). The last two are non-isomorphic and their relation to the first solution has not been investigated.

Theorem 7 : *Let* $x^t-a_{t-1}x^{t-1}-\ldots-a_0$ *be a minimum function generating* $GF(s^t)$, *and the sequence* $\xi_d, [d = 0, \ldots, s^t-1]$, *be derived from the recurrence relation* :

$$\xi_{d+1} = a_{t-1}\xi_d+\ldots+a_0\xi_{d-t+1}$$

with the initial values $\xi_0 = \xi_1 = \ldots = \xi_{t-2} = 0$, $\xi_{t-1} = 1$. *The set of integers* d *such that* $\xi_d = \alpha$ *(fixed)* $\neq 0$ *provides a difference set such that among the differences mod* (s^t-1), *all integers less than* (s^t-1) *and not divisible by* $\theta = (s^t-1)/(s-1)$ *occur an equal number,* (s^{t-2}), *of times and those divisible by* θ, *zero times.*

*Minimum functions of certain orders are given in Charmichael (1937). Some algebraic methods of deriving minimum functions of the second order and, in a few cases, of the third order have been given by Bose, Chowla and Rao (1944, 1945a, 1945b). The third order minimum functions for $s = 13$ have not been reported anywhere in the literature. At the request of my colleague Dr. I. M. Chakravorty, Dr. Jack Alanen found some solutions with the help of a Burroughs—220 computer. I wish to thank Dr. Alanen for supplying a few minimum functions of the third order for $s = 13$, of which I am quoting one. A note by Dr. Alanen is appearing in this issue of *Sankhyā*.

Such a difference set provides a compact representation of the resolvable BIBD, $EG(t, s) : (t-1)$, with the parameters

$$v = s^t,\ b = (s^{t+1}-s)/(s-1),\ k = s^{t-1},\ r = (s^t-1)/(s-1),\ \lambda = (s^{t-1}-1)/(s-1).$$

The s^t varieties are represented by the residues $0, 1, \ldots, s^t-1$ and an element ∞ which is invariant under addition with the residues. Let $d_1, \ldots, d_k$ be the difference set obtained as in Theorem 7. Consider the $(s-1)$ sets derived by adding $0, \theta, \ldots, (s-2)\theta$ to each element of the initial set, i.e., in the cyclic development of $(d_1, \ldots, d_k)$ we take the sets obtained after $\theta, 2\theta, \ldots$ steps. This operation may be denoted by

$$[(d_1, \ldots, d_k)S(\theta)] \bmod (s^t-1)$$

To this add another set consisting of the remaining $(k-1)$ residues of (s^t-1) and an invariant element ∞. This operation is denoted by

$$[(d_1, \ldots, d_k)S(\theta)+R] \bmod (s^t-1)$$

which gives one complete replication of the resolvable BIBD. The other replications are obtained by adding $1, 2, \ldots (\theta-1)$, to each member of the first replication. The procedure for developing the whole solution of the resolvable design may be represented as

$$PC(\theta)[(d_1, \ldots, d_k)S(\theta)+R] \bmod (s^t-1)$$

where PC indicates a partial cycle, up to θ only.

As an example let us consider $EG(4, 2)$: 3 and the minimum function x^4-x^3-1. We find

$$\xi_0 = \xi_1 = \xi_2 = \xi_7 = \xi_9 = \xi_{12} = \xi_{13} = 0$$
$$\xi_3 = \xi_4 = \xi_5 = \xi_6 = \xi_8 = \xi_{10} = \xi_{11} = \xi_{14} = 1$$

and the set of d, such that $\xi_d = 1$,

$$3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 11,\ 14$$

is a difference set mod 15 with the property stated in Theorem 7. One replication of the resolvable BIBD, $v = 16$, $b = 30$, $k = 8$, $r = 15$, $\lambda = 7$, is

$$[(3,\ 4,\ 5,\ 6, 8,\ 10,\ 11,\ 14)S(15)+R] \bmod 15$$

which on development yields one replication

$$(3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 11,\ 14)$$
$$(\infty,\ 1,\ 2,\ 7,\ 9,\ 12,\ 13,\ 15)$$

and the other 14 replications are obtained by adding 1, 2, ..., 14 to the elements of the first replication and reducing to mod 15. Similarly solutions are obtained for the following and given in Table 2.

ref. no.	parameters of resolvable BIBD					minimum function
	v	b	k	r	λ	
$EG(2,11) : 1$	121	132	11	12	1	x^2-4x+2, $GF(11)$
$EG(3,2) : 2$	27	30	9	13	4	x^3-x-2, $GF(3)$
$EG(2,13) : 1$	169	182	13	14	1	x^2-x-1, $GF(13)$
$EG(4,2) : 3$	16	30	8	15	7	x^4-x^3-1, $GF(2)$

Theorem 8: *If s is a prime or a prime power and $\theta = (s^{t+1}-1)/(s^2-1)$ is not integral, it is possible to find $y = (s^t-1)/(s^2-1)$ sets*

$$(d_{0j}, d_{1j}, \ldots, d_{mj}), \quad j = 1, \ldots, y$$

such that all the differences mod $v = (s^{t+1}-1)/(s-1)$ contain integers less than v once and once only.

For the actual method of construction references may be made to Rao (1945, 1946). The solution obtained for $PG(4, 2) : 1$, with parameters, 31, 155, 3, 15, 1 is given in Table 2.

Theorem 9 : *If s is a prime or prime power, there exist $\eta = (s^{t-1}-1)/(s-1)$ sets*

$$(d_{1i}, \ldots, d_{si}), \quad i = 1, \ldots, \eta$$

such that all the differences mod $v(=s^t-1)$ contain integers less than v and not divisible by $\theta = v/(s-1)$ once and those divisible by θ, zero times.

If we add to the difference set of Theorem 9 the set $(\infty, 0, \theta, \ldots)$ with a partial cycle θ, we obtain a compact representation of the BIBD, $EG(t, s) : 1$, with the parameters

$$v = s^t, \; b = s^{t-1}(s^t-1)/(s-1), \quad k = s, \quad r = (s^t-1)/(s-1), \; \lambda = 1.$$

The solution for $EG(3,3) : 1$ is given in Table 2.

Theorems providing difference sets for the designs $PG(t, s) : d$ and $EG(t, s) : d$ for values of d other than those covered by Theorems 6, 7, 8, 9 are given in Rao (1945, 1946). They have not been quoted here as they do not provide solutions to any of the designs listed in Table 1.

TABLE 2. CYCLIC SOLUTIONS TO BIB DESIGNS DERIVABLE FROM FINITE GEOMETRICAL CONFIGURATIONS

ref. no.	solution
	Non-resolvable designs
PG (2,11) : 1	(1, 2, 4, 13, 21, 35, 39, 82, 89, 95, 105, 110) mod 133
PG (2,13) : 1	(0, 1, 3, 24, 41, 52, 57, 66, 70, [illegible] 102, 149, 164, 176) mod 183
PG (3,3) : 1	(0, 1, 26, 32), (0, 7, 19, 30), (0, 3, 16, 38) mod 40, *PC* (10) (0, 10, 20, 30) mod 40
PG (3,3) : 2	(0, 1, 2, 5, 12, 18, 22, 24, 26, 27, 29, 32, 33) mod 40
PG (4,2) : 3 (3 solutions)	(*i*) (0, 1, 2, 3, 5, 6, 8, 11, 12, 18, 19, 20, 23, 27, 29) mod 31
	(*ii*) (1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28) mod 31
	(*iii*) (1, 2, 3, 4, 6, 8, 12, 15, 16, 17, 23, 24, 27, 29, 30) mod 31
PG (4,2) : 1	(0, 1, 18), (0, 2, 5), (0, 4, 10), (0, 8, 20), (0, 9, 16) mod 31
EG (3,3) : 1	(0, 1, 22), (0, 2, 8), (0, 3, 14), (0, 7, 17) mod 26, *PC* (13) (∞, 0, 13) mod 26
	Resolvable designs
EG (2,11) : 1	*PC* (12)[(0, 9, 27, 29, 46, 50, 76, 104, 107, 114, 115) S (12) + *R*] mod 120
EG (3,3) : 2	*PC* (13)[(0, 1, 2, 8, 11, 18, 20, 22, 23) + R] mod 26
EG (2,13) : 1	*PC* (14)[(0, 9, 23, 35, 72, 92, 97, 110, 136, 151, 157, 158, 160)S(14) + *R*] mod 168
EG (4,2) : 3	*PC* (14)[(3, 4, 5, 6, 8, 10, 11, 14) + *R*] mod 15.

4. Cyclic solutions to non-geometrical designs

Bose (1939) discussed some methods of obtaining cyclic solutions and gave the actual solutions to several series of designs designated by $T_1, T_2, F_1, F_2, G_1, G_2, S_n$. Table 3 gives the cyclic solutions obtained by these methods, in so far as they are applicable in which cases the series to which each solution belongs has been indicated, and other solutions obtained by trial and error (indicated by the symbol —).

There are several types of cyclic solutions as may be seen from Table 3. In a simple cyclic solution such as 35, the varieties are represented by integers $0, 1, ..., v-1$. From the initial set or sets, the whole design is generated by adding integers $1, 2, ..., v-1$ and reducing to mod v. That is, an integer i in an initial set is changed to $i+1, ..., v-1, 0, ... i-1$ in the derived sets. In some cases such as 32, the varieties are represented by integers and an invariant element designated as ∞, (Fisher and Yates, 1953 use I for this purpose). This element remains the same in all the derived sets.

There are dicyclic solutions such as 43, where the varieties are represented by (x, y), $x = 0, ..., p-1$ and $y = 0, 1, ..., q-1$. The cyclic development with respect to one of the coordinates is carried out first keeping the other fixed. From the sets so generated others are derived by a cyclic development of the other coordinate, fixed in the first operation. When the initial sets are given in such a way that no cyclic development is necessary with respect to one of the coordinates for obtaining the complete design a dash is indicated in the symbol, mod (p, q) as in 73 (*ii*). An additional complication in dicyclic solutions is the introduction of invariant elements as in 48 and 62. Some designs such as 36 have tricyclic solutions.

Notes on the designs in Table 3 : 34 and 41 can also be obtained by block section from 35 and 53 respectively. It is not known whether from 76, one can build up the symmetrical design 80 with parameters 36, 36, 15, 15, 6, whose existence implies that of 76. Design 48, obtained by trial, is the most difficult one, and no known technique including the recent methods given by Bose and Shrikhande (1960) could yield a solution. A non-isomorphic solution to 32 is given by Skolem (1958).

Some of the designs in Table 3 are of the resolvable type. But the solutions given are not resolvable except in cases where the index (r) is shown with the reference number. Two solutions are listed for 32, of which only (i) is resolvable. In the case of a resolvable solution, either one complete replication, as in 32 and 33, or the method of generating one or more complete replications, as in 66, is shown within square brackets. All the replications are obtained by the indicated cyclic development of the initial replications. In the case of 66,

$$[(01,\ 06,\ 15,\ 12,\ 23,\ 24) \bmod (5, -)+R)]$$

and

$$[(01,\ 06,\ 35,\ 32,\ 13,\ 14) \bmod (5, -)+R]$$

TABLE 3. CYCLIC SOLUTIONS TO SOME BIB DESIGNS, NOT DERIVABLE FROM FINITE GEOMETRICAL CONFIGURATIONS

ref. no.	v	b	k	r	λ	method	cyclic solution
32′	12	44	3	11	2	—	(i) [(0, 1, 3,), (4, 5, 9), (2, 8, 0), (∞, 7, 10)] mod 11
	(two solutions)					E_1	(ii) (0, 1, 3), (0, 1, 4), (0, 2, 6), (∞, 0, 5) mod 11
33′	12	33	4	11	3	—	[(0, 1, 3, 7), (2, 4, 9, 10), (∞, 5, 6, 8)] mod 11
34	12	22	6	11	5	—	(0, 1, 3, 7, 8, 10), (∞, 0, 5, 6, 8, 10) mod 11
35	23	23	11	11	5	S_1	(1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18) mod 23
36	45	99	5	11	1	G_2	(010, 020, 102, 202, 001), (210, 120, 222, 112, 001) mod (3, 3, 5)
							(000, 001, 002, 003, 004) mod (3, 3, —)
41	13	26	6	12	5	—	(0, 1, 3, 6, 7, 11), (0, 1, 2, 3, 7, 11) mod 13
42	19	57	4	12	2	—	(0, 1, 3, 12), (0, 1, 5, 13), (0, 4, 6, 9) mod 19
43	21	42	6	12	3	—	(00, 05, 14, 11, 22, 23), (00, 01, 03, 10, 11, 13) mod (3, 7)
45	25	100	3	12	1	—	(i) (0, 1, 3), (0, 4, 13), (0, 5, 11), (0, 7, 17) mod 25
	(two solutions)					T_2	(ii) (01, 41, 13), (10, 33, 12), (32, 21, 02), (11, 24, 20) mod (5, 5)
48	37	111	4	12	1	—	(00, 01, 12, 15), (01, 03, 08, 10), (0∞, 07, 15, 21) mod (3, 11)
	[varieties are (x, y) $x = 0, 1, 2$; $y = 0, \ldots, 10, \infty$ and (∞∞)]						(∞∞, 00, 10, 20) mod (—, 11); (0∞, 1∞, 2∞, ∞∞)
53	27	27	13	13	6	S_1	[(001, 100, 120, 111, 202, 110, 102, 020, 021, 121, 211, 022, 221)] mod (3, 3, 3)
61	15	42	5	14	4	—	[(0, 1, 4, 9, 11), (0, 1, 4, 10, 12), (∞, 0, 1, 2, 7)] mod 14
62	15	35	6	14	5	—	[(∞, 00, 10, 11, 12, 14), (∞, 10, 00, 06, 05, 03) (01, 02, 04, 10, 11, 13), (02, 03, 05, 10, 11, 13) (00, 04, 06, 10, 11, 13)] mod (—, 7)
63	22	77	4	14	2	—	[(00, 03, 09, 010), (00, 10, 12, 17), (00, 10, 19, 110) (00, 02, 15, 18), (00, 03, 14, 17), (00, 04, 13, 19) (00, 06, 12, 16)] mod (—, 11)
66′	36	84	6	14	2	—	[(01, 06, 15, 12, 23, 24) mod (5, —) + R] mod (—, 7)
	[varieties are (x,y), $x = 0, \ldots, 4$, $y = 0, \ldots, 6$, and (∞∞)]						[(01, 06, 35, 32, 13, 14) mod (5, —) + R] mod (—, 7)
67	43	86	7	14	2	—	(∞ 0, 01, 06, 15, 12, 23, 24) mod (5, 7)
	[varieties are (x, y)						(∞ 0, 01, 06, 35, 32, 13, 14) mod (5, 7)
	$x = 0, \ldots, 4, \infty$,						(∞ 0, ∞∞, 00, 10, 20, 30, 40) mod (—, 7)
	$y = 0, \ldots, 6$ and (∞, ∞)]						(∞ 0, ∞∞, 00, 10, 20, 30, 40) mod (—, 7)
							(∞0, ∞1, ∞2, ∞3, ∞4, ∞5, ∞6) mod (—,—)
							(∞0, ∞1, ∞2, ∞3, ∞4, ∞5, ∞6) mod (—,—).
71	11	55	3	15	3	—	[(0, 1, 3), (0, 1, 5), (0, 2, 7), (0, 1, 8), (0, 3, 6)] mod 11
72	13	39	5	15	5	—	[(0, 1, 2, 4, 8), (0, 1, 3, 6, 12), (0, 2, 5, 6, 10)] mod 13
73	16	80	3	15	2	—	(i) (0, 1, 3), (0, 3, 8), (0, 2, 12), (0, 1, 7), (0, 4, 9) mod 16
	(two solutions)					E_1	(ii) [(10, 11, 12), (10, 12, 16), (00, 07, 10), (01, 06, 10) (02, 05, 10), (03, 04, 10), (01, 07, 10), (02, 06, 10) (03, 05, 10), (00, 10, 14)] mod (—, 8)
74	16	48	5	15	4	—	[(0, 1, 2, 4, 7), (0, 1, 8, 5, 10), (0, 1, 3, 7, 11)] mod 16
75	16	40	6	15	5	—	[(0, 1, 3, 5, 9, 12), (0, 1, 2, 3, 6, 12)] mod 16
							PC (0, 8, 1, 9, 2, 10) mod 16
76	21	35	9	15	6	—	[(00, 01, 02, 04, 10, 11, 12, 14, 22), (00, 06, 05, 03, 20 24, 23, 22, 10), (10, 16, 15, 13, 20, 24, 23, 22, 00), (04, 01, 03, 10, 12, 16, 24, 21, 22), (00, 02, 06, 14, 11, 13, 24, 21, 22)] mod (—, 7)
84	61	183	5	15	1	G_2	[(1, 9, 20, 58, 34), (4, 36, 19, 49, 114), (16, 22, 15, 13, 50)] mod 61.

give one replication each. The first expression gives the replication

(01, 06, 15, 12, 23, 24), (33, 36, 45, 42, 03, 04)
(11, 16, 25, 22, 33, 34) (41, 46, 05, 02, 12, 14)
(21, 26, 35, 32, 43, 44), (∞ ∞, 00, 10, 20, 30, 40)

and similarly the second expression gives another replication. On cyclic development with respect to the second coordinate the rest of the replications are generated.

Four more solutions, nos. 56, 65, 77 and 79, have been found by trial and error. In the case of no. 77, 26 varieties are represented by (x,y), $x = 0, \ldots, 4$; $y = 0, \ldots, 4$ and ∞. The dicyclic solution in blocks of size 6 is

[(∞, 00, 13, 21, 34, 42), (∞, 00, 12, 24, 31, 43), (∞, 00, 10, 20, 30, 40)] mod (—, 5)

[(01, 04, 12, 13, 21, 24), (01, 04, 22, 23, 32, 33)] mod (5,5)

For no. 79, the parameters are $v = 31$, $b = 93$, $k = 5$, $r = 15$, $\lambda = 2$. A primitive residue of 31 is 3. Consider the set $(3^0, 3^6, 3^{12}, 3^{18}, 3^{24})$ which is same as (1, 2, 4, 8, 16) with an internal multiplier 2. From this two more sets are generated by successively multiplying by 3, (3, 6, 12, 24, 17) and (9, 18, 5, 10, 20). The cyclic solution (in 3 cycles generating 93 blocks) is

[(1, 2, 4, 8, 16), (3, 6, 12, 24, 17), (9, 18, 5, 10, 20)] mod 31

Similarly the solution for no. 65 with parameters $v = 29$, $b = 58$, $k = 7$, $r = 14$, $\lambda = 3$ is

[(1, 7, 16, 20, 23, 24, 25), (3, 21, 19, 2, 11, 14, 17)] mod 29

For no. 56, represent the varieties by (x, y), $x = 0, \ldots, 5$; $y = 0, \ldots, 12$ and ∞. The solution is

[(00, 0 12, 11, 1 11, 24, 28), (01, 0 11, 20, 212, 34, 38)] mod (5, 13)
(∞, 00, 10, 20, 30, 40) mod (—, 13)

A simple method of constructing resolvable BIBD for the series $v = s^2$, $b = s(s+1)$, $k = s$, $\lambda = 1$ has been found when $GF(s)$ exists. One need not go through the construction of Theorem 7, although the result of Theorem 7 is interesting from the number-theoretic view point. Let $\alpha_0, \ldots, \alpha_{s-1}$ represent the elements of $GF(s)$ and $0, 1, \ldots, s-1$, the module $M(s)$ of residue classes mod s and represent the varieties by (x, y) where $x \in M(s)$ and $y \in GF(s)$. Consider the set of pairs (x, y)

$$(0\,\alpha_0, 1\alpha_1, \ldots, (s-1)\ \alpha_{s-1})$$

By cyclic development of y with respect to $GF(s)$ one replication is obtained. The initial set for another replication is obtained by multiplying y in the first set by λ an element of $GF(s)$. Thus.

$$(0\ \lambda\alpha_0, 1\lambda\alpha_1, \ldots, (s-1)\ \lambda\alpha_{s-1})$$

gives on cyclic development of y another replication. Since λ can have s values we obtain s replications. The $(s+1)$-th replication is obtained by the development with respect to x

$$(0\ \alpha_0, 0\ \alpha_1, \ldots, 0\ \alpha_{s-1}) \text{ mod } (s, -).$$

For no. 87 represent the varieties by (x, y), $x = 0, \ldots, 6$; $y = 0, \ldots, 12$ and ∞. The solution is

$$(00, 0\ 12, 14, 18, 51, 5\ 11, 26), (33, 39, 42, 4\ 10, 65, 67, 26) \bmod (7, 13)$$
$$(00, 10, 20, 30, 40, 50, 60) \bmod 13$$

Note added in proof : I am thankful to Prof. S. S. Shrikhande for pointing out that the solutions for nos. 41, 42, 43, 45, 65, 73, 74, 79 obtained by me also follow from the general series given by Sprott (1954).

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