

B-SPACES ARE STANDARD BOREL

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Separated B -spaces introduced by E. B. Dynkin are just standard Borel and thus here we have an intrinsic definition of standard Borel spaces.

We consider a measurable space (Ω, \mathbf{F}) where \mathbf{F} is countably generated and contains singletons. Following Dynkin [1978], we say that a countable family W of bounded functions—always real valued—on (Ω, \mathbf{F}) is a support family if the following two conditions hold:

(A) If (μ_n) is a sequence of probability measures on \mathbf{F} and if for each $f \in W$, $\lim_n \int f d\mu_n = I(f)$ exists then there is a probability μ on \mathbf{F} such that for each $f \in W$, $I(f) = \int f d\mu$.

(B) If a class H of functions on Ω contains W and is closed under addition, multiplication by constants, and bounded convergence then H contains all bounded measurable functions (f_n converges boundedly to f means f_n converges to f pointwise and the functions f_n are uniformly bounded).

(Ω, \mathbf{F}) is said to be a B -space if it has a support family. From Condition (B) it follows that if (Ω, \mathbf{F}) is a B -space and W is a support family then W generates \mathbf{F} . As noted by Dynkin every standard Borel space is a B -space. We now show that the converse is also true, that is, every B -space is a standard Borel space. We find this observation helpful in understanding the content of Dynkin [1978].

So, let (Ω, \mathbf{F}) have a support family $W = \{f_1, f_2, \dots\}$. Let M be the space of all probabilities on (Ω, \mathbf{F}) . The σ -field \mathbf{M} on M is the usual one generated by all functions of the form $p \mapsto p(A)$ for $A \in \mathbf{F}$. By (B), \mathbf{M} is also the σ -field generated by the functions $p \mapsto \int f_i dp$ ($i \geq 1$). Since \mathbf{F} is countably generated it is easy to see that the collection D of point masses belongs to \mathbf{M} and further $(D, D \cap \mathbf{M})$ is Borel isomorphic to (Ω, \mathbf{F}) . We shall now topologize M as follows: consider the map $e: p \mapsto (\int f_1 dp, \int f_2 dp, \dots)$ from M to $R \times R \times \dots = R^\infty$. By (B) e is a 1-1 map and by (A) the range of e , denoted by $e(M)$, is closed in R^∞ . Thus identifying M with $e(M)$ we make M into a complete separable metric space. Let $\bar{\mathbf{M}}$ be the Borel σ -field of M . Since the Borel σ -field on R^∞ is generated by the coordinate maps, $\bar{\mathbf{M}}$ is generated by the maps $p \mapsto \int f_i dp$ ($i \geq 1$). This implies that $\bar{\mathbf{M}} = \mathbf{M}$ and so $(M, \bar{\mathbf{M}})$ is a standard Borel space. As remarked earlier (Ω, \mathbf{F}) is isomorphic to a Borel subspace of $(M, \bar{\mathbf{M}})$ so that (Ω, \mathbf{F}) is also a standard Borel space. This completes the proof of the observation.

We assumed \mathbf{F} to be countably generated. If (Ω, \mathbf{F}) has a countable support family then Condition (B) implies that \mathbf{F} is countably generated. We assumed \mathbf{F} to contain singletons. This is no restriction because otherwise we can look at the space of atoms.

The referee has remarked that the concept of B -space was introduced in Dynkin [1971] and was used extensively in Dynkin and Yushkevich (1979).

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