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SELECTORS FOR BOREL SETS WITH LARGE SECTIONS

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ABSTRACT. We prove a result asserting the existence of a Borel selector for a Borel set in the product of two Polish spaces. This subsumes a number of results about Borel selectors for Borel sets having large sections.

1. Introduction. The principal selection theorems for Borel sets with large sections can be stated as follows. If X is analytic, Y a Polish space, B a Borel subset of $X \times Y$ such that every vertical section B_x is of positive measure or nonmeager, then B admits a Borel measurable selector, i.e., there exists a Borel measurable function f: $X \to Y$ such that $(x, f(x)) \in B$ for each $x \in X$. The measure result was proved by Blackwell and Ryll-Nardzewski [**1**], while the category result has been observed by several authors (see, for example, [**5** and **6**]). A selection theorem of Burgess [**2**], when specialized to the situation above, extends the category result and subsumes a result of Srivastava [**8**] concerning Borel selectors for the case when the sections B_x are G_δ sets (see also [**4**] in this connection). A precise version of Burgess' result will be given in §4.

The aim of the present article is to show that the results described above are really instances of a single theorem. Moreover, the proof we give here is elementary in nature, using nothing more than the well-known characterization of a Borel subset of a Polish space as a one-one, continuous, bimeasurable image of a closed subset of the space of irrationals (see, for instance, [**3**]). In particular, the separation principle for analytic sets is avoided. Lastly, the proof works for X, an arbitrary separable metric space (Srivatsa [**7**] observed that the results mentioned in the previous paragraph hold for separable metric X, not just for analytic X).

The situation obtaining for Borel sets with large sections should be contrasted with that for Borel sets with small sections, i.e., when all vertical sections are countable or compact or σ-compact. It is known that such Borel sets admit Borel selectors when the horizontal axis X is analytic [**5**]. The proofs of these results, however, involve in an essential manner the separation principle for analytic sets (indeed, the separation principle for a sequence of analytic sets). Furthermore, the results are not true for arbitrary separable metric X. (I am indebted to J. P. Burgess for the last remark.)

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The main result of this paper is as follows.

THEOREM. *Let X be a separable metric space and Y a Polish space. For each $x \in X$, let $\mathcal{I}_x$ be a σ-ideal of subsets of the Borel σ-field of Y such that for each Borel set C in $X \times Y$, the set $\{x \in X: C_x \notin \mathcal{I}_x\}$ is Borel in X, where $C_x = \{y \in Y: (x, y) \in C\}$.*

Let B be a Borel set in $X \times Y$ such that $B_x \notin \mathcal{I}_x$ for each $x \in X$. Then there is a Borel measurable function $f: X \to Y$ such that $(x, f(x)) \in B$ for each $x \in X$.

The proof of the theorem will be given in §3. §2 explains the notation to be used. §4 will discuss special cases of our result.

2. Notation. We denote the set of natural numbers by ω. The letters k, m, n, with or without primes, will stand for natural numbers. The set of finite sequences of natural numbers is denoted by $\omega^{<\omega}$. Elements of $\omega^{<\omega}$ will be denoted by s, t, with or without primes. The letter e will denote the empty sequence in $\omega^{<\omega}$. If $s \in \omega^{<\omega}$, $\text{lh}(s)$ denotes the length of s. If $s \in \omega^{<\omega}$ and $m \in \omega$, sm denotes the catenation of the sequence s followed by the sequence $\langle m \rangle$.

The set of infinite sequences of natural numbers will be denoted by ω^ω. Elements of ω^ω will be denoted by α, β. For $\alpha \in \omega^\omega$ and $n \in \omega$, $\alpha(n)$ stands for the nth coordinate of α and $\alpha \upharpoonright n$ will denote the finite sequence $\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$. For $s \in \omega^{<\omega}$ and $\alpha \in \omega^\omega$, write $s \subseteq \alpha$ if $\alpha \upharpoonright \text{lh}(s) = s$. We denote the set $\{\alpha \in \omega^\omega: s \subseteq \alpha\}$ by $N(s)$ for $s \in \omega^{<\omega}$. The sets $N(s)$, $s \in \omega^{<\omega}$, form a base for a topology on ω^ω. Endowed with this topology, ω^ω becomes a Polish space which is homeomorphic to the space of irrationals.

3. Proof of theorem. Fix a complete metric d on Y such that $d - \text{diameter}(Y) < 1$. Choose a system $\{V(s): s < \omega^{<\omega}\}$ of subsets of Y satisfying:

(i) $V(e) = Y$.

(ii) $V(s)$ is nonempty and open in Y.

(iii) $V(s) = \bigcup_{m \geq 0} V(sm)$.

(iv) $\text{cl}(V(sm)) \subseteq V(s)$, where cl denotes closure in Y.

(v) $d - \text{diameter}(V(s)) < 2^{-\text{lh}(s)}$.

Let $\tilde{X}$ be a completion of X, so $\tilde{X}$ is a Polish space. Since B is Borel in $X \times Y$, we can find a Borel set $\tilde{B}$ in $\tilde{X} \times Y$ such that $B = \tilde{B} \cap (X \times Y)$. Fix a closed subset D of ω^ω and a one-one, continuous function g on D onto $\tilde{B}$ such that g takes Borel subsets of D to Borel sets in $\tilde{X} \times Y$.

Next we define a system $\{T(s, t): s, t \in \omega^{<\omega} \,\&\, \text{lh}(s) = \text{lh}(t)\}$ of subsets of X satisfying:

(a) $T(s, t)$ is a Borel set in X,

(b) $T(e, e) = X$,

(c) $T(s, t) \cap T(s', t') = \varnothing$ if $(s, t) \neq (s', t') \,\&\, \text{lh}(s) = \text{lh}(s')$,

(d) $T(s, t) = \bigcup_{m \geq 0} \bigcup_{n \geq 0} T(sm, tn)$,

(e) $T(s, t) \subseteq (g(D \cap N(s)) \cap (X \times Y))^{*V(t)}$,

where $(g(D \cap N(s)) \cap (X \times Y))^{*V(t)} = \{x \in X: (g(D \cap N(s)) \cap (X \times V(t)))_x \notin \mathcal{I}_x\}$. The definition of $T(s, t)$ is by induction on $\text{lh}(s)$. Start the definition by setting $T(e, e) = X$. Assume that $T(s', t')$ has been defined for all s', $t^1 \in \omega^{<\omega}$ such that

$\mathrm{lh}(s') = \mathrm{lh}(t') \leqslant k$. Fix $s, t \in \omega^{<\omega}$ such that $\mathrm{lh}(s) = \mathrm{lh}(t) = k$. Set

$$T'(m, n) = (g(D \cap N(sm)) \cap (X \times Y))^{*V(tn)} \cap T(s, t), \qquad m, n \in \omega.$$

Since $g(D \cap N(sm))$ is Borel in $\tilde{X} \times Y$, the set $g(D \cap N(sm)) \cap (X \times V(tn))$ is Borel in $X \times Y$. So, by hypothesis, $(g(D \cap N(sm)) \cap (X \times Y))^{*V(tn)}$ is Borel in X, hence $T'(m, n)$ is Borel in X. We claim that $T(s, t) = \bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} T'(m, n)$. To see this, let $x \in T(s, t)$. Note that

$$(g(D \cap N(s)))_x \cap V(t) = \bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} [(g(D \cap N(sm)))_x \cap V(tn)].$$

Since $x \in T(s, t)$, the induction hypothesis and clause (e) above imply that

$$(g(D \cap N(s)))_x \cap V(t) \notin \mathcal{J}_x.$$

Since $\mathcal{J}_x$ is a σ-ideal, it follows that there exist $m, n \in \omega$ such that $(g(D \cap N(sm)))_x \cap V(tn) \notin \mathcal{J}_x$, so $x \in T'(m, n)$. Next, disjointify the sets $T'(m, n)$: get Borel sets $T''(m, n)$ in X such that $T''(m, n) \subseteq T'(m, n)$, $T''(m, n) \cap T''(m', n') = \varnothing$ if $(m, n) \neq (m', n')$, and $\bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} T''(m, n) = \bigcup_{m \geqslant 0} \bigcup_{n \geqslant 0} T'(m, n)$. Finally, set $T(sm, tn) = T''(m, n)$. The sets so obtained satisfy clauses (a)–(e).

Define

$$G = \bigcap_{k \geqslant 0} \bigcup [T(s, t) \times \mathrm{cl}(V(t))],$$

where the inner union is over all ordered pairs $(s, t) \in \omega^{<\omega} \times \omega^{<\omega}$ such that $\mathrm{lh}(s) = \mathrm{lh}(t) = k$.

We make two observations about G: (1) G_x is a singleton for each $x \in X$, and (2) $G \subseteq B$. To see (1), let $x \in X$. Then there is a unique $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ such that $x \in T(\alpha \restriction k, \beta \restriction k)$ for all $k \geqslant 0$. This follows from clauses (b)–(d) in the definition of the system $\{T(s, t)\}$. So $G_x = \bigcap_{k \geqslant 0} \mathrm{cl}(V(\beta \restriction k))$, which is a singleton set by (iv), (v) and the fact that Y is Polish. For (2), let $(x, y) \in G$. Find $(\alpha, \beta) \in \omega^\omega \times \omega^\omega$ such that $x \in T(\alpha \restriction k, \beta \restriction k)$ for all $k \geqslant 0$. Hence $\{y\} = \bigcap_{k \geqslant 0} \mathrm{cl}(V(\beta \restriction k))$. Since $x \in T(\alpha \restriction k, \beta \restriction k)$ for each k, it follows by (e) that $(g(D \cap N(\alpha \restriction k)))_x \cap V(\beta \restriction k) \neq \varnothing$ for each $k \geqslant 0$. Consequently, $y \in \bigcap_{k \geqslant 0} (\mathrm{cl}(g(D \cap N(\alpha \restriction k))))_x$, so $(x, y) \in \bigcap_{k \geqslant 0} \mathrm{cl}(g(D \cap N(\alpha \restriction k))) = \{g(\alpha)\}$, using continuity of g and closedness of D. Consequently, $(x, y) \in B$.

Let $f\colon X \to Y$ be the function whose graph is G. Since $G \subseteq B$, it follows that $(x, f(x)) \in B$ for every $x \in x$. To see that f is Borel measurable, observe that for any open set W in Y,

$$f^{-1}(W) = \bigcup \{T(s, t)\colon \mathrm{lh}(s) = \mathrm{lh}(t) \;\&\; \mathrm{cl}(V(t)) \subseteq W\},$$

so that $f^{-1}(W)$ is Borel in X. This completes the proof.

4. Special cases. In this final section, we deduce the results mentioned in the Introduction from our result.

Let X be a separable metric space, Y a Polish space and B a Borel set in $X \times Y$.

1°. To consider the measure result first, let $Q(x, E)$, $x \in X$, E a Borel set in Y, be a Borel measurable transition function such that $Q(x, B_x) > 0$ for each $x \in X$. In our theorem, we take $\mathcal{J}_x = \{E \subseteq Y\colon E \text{ is Borel } \& \; Q(x, E) = 0\}$, $x \in X$. It is a standard fact of measure theory that the σ-ideals $\mathcal{J}_x$, $x \in X$, satisfy the condition of our theorem, so that the Blackwell-Ryll-Nardzewski result falls out of ours.

2°. For the category case, assume that B_x is nonmeager for each $x \in X$. We take $\mathcal{J}_x = \{E \subseteq Y$: E is Borel and meager$\}$, $x \in X$. That the $\mathcal{J}_x$, $x \in X$, satisfy the condition of our theorem is a result of P. S. Novikov and, independently, of Vaught [**9**]. The category result is then an immediate consequence of our result.

3°. In Burgess' result, the assumptions on the Borel set B are as follows: (a) the multifunction $x \to B_x$ is Borel measurable, i.e., the set $\{x \in X\colon B_x \cap V \neq \varnothing\}$ is Borel in X for each open set V in Y, and (b) B_x is nonmeager in cl(B_x) for each $x \in X$. We set

$$\mathcal{J}_x = \{E \subseteq \mathrm{cl}(B_x)\colon E \text{ is Borel and meager in } \mathrm{cl}(B_x)\}, \qquad x \in X.$$

A very easy modification of Vaught's proof of the fact mentioned in 2° now shows that the σ-ideals $\mathcal{J}_x$, $x \in X$, satisfy the condition of our theorem. Our theorem then implies that B admits a Borel measurable selector, which is Burgess' result.

Srivatsa [**7**] has an ingenious way of deducing Burgess' result from the category result in 2°. Finally, routine arguments show that the results in 1°–3° hold for an arbitrary measurable space $(X, \mathcal{Q})$, since they hold for a separable metric space (see [**7**]).

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