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A DYNAMIC ASSIGNMENT PROBLEM*

BY ABHIRUP SARKAR[1]

1. INTRODUCTION

The purpose of this paper is to determine the patterns of efficient specializations in a multi-country, multi-commodity time-phased model of international trade. In a well-known paper, Jones [1961] dealt with a similar problem in a static Ricardian framework. From one standpoint, therefore, this paper can be viewed as an extension of Jones' work to situations where time is explicitly taken into account to affect production and consumption decisions. Now, on the consumption side, once time is introduced, the most important decision is that of intertemporal choice of consumption. This, in turn, is reflected in the savings behaviour of the individuals. On the other hand, goods can be ranked in terms of their capital intensities on the production side. Accordingly, we have shown that if savings ratios differ across countries, the countries with higher savings ratios will have comparative advantage in the production of goods with higher capital intensities. This is different from Jones' paper where, in the spirit of Ricardo, comparative advantage is determined on the basis of differential technology across countries. On the contrary, with comparative advantage being partly determined by factor intensities and partly by savings parameters, our model is more Heckscher-Ohlin in spirit.

It is well-known that the most general dynamic linear production model is that of the von-Neumann variety where, in fullest generality, production and capital can be incorporated. Unfortunately, however, this model is too general to obtain any interesting result in the positive theory of international trade. To obtain specific results, one has to consider special cases. One immediate simplification is to assume that there is no pure joint production. But this is not enough. Now, in the von-Neumann model, inputs are assumed to produce, after one period, final outputs along with one-period older inputs. In this sense, there is a complete reference back to the market at the end of every period. This greatly complicates matters. For, in this case, a machine is characterized not only by its type and vintage but also by its history of use. A possibly tractable special case could then be obtained if one assumes that machines are non-shiftable in the sense that once a piece of machinery is installed in a sector, it stays there until it becomes obsolete. As shown by Mirrlees [1969], with the assumption of non-shiftability of capital goods, all steady-state price ratios are determined when the rate of

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interest is given. It is then possible, by closing the rate of interest by savings behaviour, to predict trade patterns on the basis of differential savings behaviour across countries. Accordingly, we have assumed that production of each commodity is completely vertically integrated: all produced inputs are produced and used up within the process itself and what appears in the market is only a sequence of final outputs. The theoretical building blocks of such a technology is based on Hicks [1973].

In the following text, the basic model is developed in Section 2. Section 3 contains the main results of this paper. In Section 4, we present conclusions.

2. THE BASIC MODEL

We consider a world with n countries and m commodities. Since we shall be concerned only with patterns of complete specializations, we assume that $n \geq m$. Each country is assumed to be in a steady state, growing at a common rate g, which is equal to the growth rate of domestic labor in that country. Also, technology is assumed to be identical in each country.

The production of each good is assumed to be vertically integrated; thus, all produced inputs are produced and used up within the process. The produced inputs include fixed capital in the sense of plant and machinary which last for more than one period of production. Since, by assumption, used equipment stay within the process for their entire life-span, they produce a flow of outputs occuring at different points in time. The technology for the i-th good can, therefore, be represented by

$$\{a_t^i, b_t^i\}_{t=0}^{Ti}, \quad i = 1,\dots, m| \quad a_t^i \geq 0, b_t^i \geq 0$$

where a_t^i is the amount of labor used in an unit process of age t of the i-th good; b_t^i is the flow of output from an unit process of age t of the i-th good; and Ti is the termination date of an unit process of the i-th good. The input-output coefficients, a_t^i, b_t^i, are assumed to be fixed. Also, constant returns to scale are assumed to prevail in the sense that the size of each sector can be increased or decreased without changing the input-output coefficients. Finally, we shall assume that the termination date Ti of a process is fixed, and that it satisfies conditions of economic viability.

With the assumption of a steady state prevailing in each country, prices and wages are constant over time. Let

$$q_t^i = P_i b_t^i - w a_t^i$$

where P_i is the price of the i-th good and w is the wage rate. q_t^i measures the value of net output in the i-th sector obtained from a t-period old process. q_t^i may have any sign: positive, negative or zero. Let

$$k_t^i = \sum_{\tau=t}^{Ti} q_\tau^i R^{t-\tau}$$

where $R = 1+r$ and r is the market rate of interest. Then k_t^i is the capitalized value of a t-period old process in the i-th sector at period t. It is the discounted value of future net outputs when the process is t periods old. We assume that perfect competition prevails so that zero profit conditions imply that $k_0^i=0$. Also, profit maximization implies that $k_t^i \geq 0$ for $0 \leq t \leq Ti$.

The next thing we need to have is a measure of capital intensity in each sector. Now, by definition,

$$k_0^i = q_0^i + q_1^i R^{-1} + \cdots + q_{t-1}^i R^{-(t-1)} + k_t^i R^{-t} = 0.$$

Therefore,

$$k_t^i = (-q_0^i)R^t + (-q_1^i)R^{(t-1)} + \cdots + (-q_{t-1}^i)R.$$

But q_τ^i is the value of the net product obtained at period τ. Therefore, $(-q_\tau^i)$ is the value of the net input used at period τ and k_t^i measures the capital invested up to period t accumulated by the rate of interest. Then the sum of discounted values of capital stock held every period over the entire process gives us the present value of capital stock held for the process as a whole. Also, to compare factor intensities in different industries we express these present values as proportions of the present values of the wage bills spent over the entire life span of each process. Thus, $\Sigma k_t^i R^{-t}/\Sigma w a_t^i R^{-t}$ is a measure of capital intensity in the i-th sector. Also, so far as the relative ranking of the industries according to factor intensity is concerned, we assume that

Assumption 1.

$$\Sigma k_t^1 R^{-t}/\Sigma a_t^1 R^{-t} < \Sigma k_t^2 R^{-t}/\Sigma a_t^2 R^{-t} < \cdots < \Sigma k_t^m R^{-t}/\Sigma a_t^m R^{-t}$$

for all R where the i-th ratio refers to the relative capital intensity of commodity i. Hence, factor-intensity reversal is ruled out by assumption.

Finally, we make a very simplifying assumption about saving: we assume that in each country saving is a fixed proportion of total profits in that country and countries can be ranked by their saving ratios in the following way:

Assumption 2. $$s_1 < s_2 < \cdots < s_n$$

where s_j refers to the saving ratio in the j-th country.

Steady state in each country implies that capital grows at the rate g so that investment in any period is equal to gK where K is the stock of capital in that period. In equilibrium, this must be equal to savings which gives us $gK=srK$, or, $r=g/s$. Thus, in each country, given the rate of growth of population and the saving ratio, the rate of interest is determined. Since, by assumption, the growth rates are the same across countries, we must have

Assumption 3. $$r_1 > r_2 > \cdots r_n.$$

3. THE ASSIGNMENT PROBLEM

In this section, we shall develop the main results of our paper. Competitive equilibrium implies that for the i-th good and the j-th country

$$P_i \leq w_j \Sigma a_t^i R_j^{-t} / \Sigma b_t^i R_j^{-t}.$$

For notational simplicity, we define

$$\Sigma a_t^i R_j^{-t} / \Sigma b_t^i R_j^{-t} = A_j^i.$$

Following Jones [1961], we define an assignment as a pattern of complete specialization where each country is producing only one commodity. Also, two (or more) assignments are defined to belong to the same class if every commodity has the same number of countries assigned to it in one assignment as in the other. Following Jones' argument, we can show that a necessary and sufficient condition for an assignment to be efficient in its class is that the product of A_j^i (i.e., the j-th country assigned to the i-th good) is minimum. Consider the case where $n = m$. If the j–j assignment (i.e., the j-th country producing the j-th good) is optimal, we must have

$$P_j = w_j A_j^j$$

and

$$P_k \leq w_j A_j^k$$

so that

$$P_j / P_k \geq A_j^j / A_j^k.$$

Multiplying the inequalities for all j and k with j not equal to k, we get the minimum product criterion. Also following the same logic we can get a similar criterion for n not equal to m. Thus if $n = m$ and the j–j assignment is efficient we must have

$$\Pi(A_j^j / A_j^{j'}) < 1$$

where j–j' is any other assignment in the class where the j-th country specializes in the j'-th good.

Given this efficiency criterion, we can determine the efficient production patterns by comparing factor intensities and the saving ratios. Let us first consider the even case where $n = m$. Assuming that all goods are produced, each good should be assigned to only one country. Our problem is to find out the efficient assignment, given that the countries are ranked by their saving ratios and the commodities are ranked by their factor intensities. The proposition we want to prove is as follows:

PROPOSITION. *If countries are ranked by their saving ratios as in Assumption 2, and if the commodities are ranked by their factor intensities as in Assumption 1, then the efficient assignment will be the one where the first country is assigned to the first good, the second country to the second good and so on up to the n-th country assigned to the n-th good.*

In other words, we want to show that the j–j assignment is efficient.

We shall prove the above proposition by contradiction. Suppose the j–j assignment is not efficient. Then there exists some other assignment, say the j–j' assignment, which is efficient. Hence, we must have

$$\Pi A_j^j > \Pi A_j^{j'}.$$

We shall prove that the above inequality is impossible. Now, starting from the j–j assignment it is possible to go to the j–j' assignment by making a series of binary switches between countries and commodities. A binary switch refers to an operation where we choose any two countries, say the k-th and the l-th and assign the k-th country the good the l-th country was producing and assign the l-th country the good the k-th country was producing.

First we shall construct a series of binary switches by which we can move from the j–j assignment to the j–j' assignment. Note that in the j–j' assignment the j-th country produces the j'-th good. Let $\bar{j}$ be the commodity to which country j is assigned to at a particular assignment. For the j–j assignment, for example, $\bar{j}=j$ for all j. Define

$$S_1 = \{j \mid j' > \bar{j}\}, \quad S_2 = \{j \mid j' < \bar{j}\}, \quad S_3 = \{j \mid j' = \bar{j}\}.$$

It is obvious that $S_1 \cup S_2 \cup S_3 = \{1, 2, \ldots, n\}$ and the intersection of the sets is a null set. Also note that there exists at least one j belonging to S_1 (assuming that S_1 is non-empty) such that the country corresponding to j' belongs to S_2. This implies two things. First, S_1 is nonempty iff S_2 is nonempty; and second, there exists at least one j belonging to S_1 (assuming S_1 is nonempty) such that there is l belonging to S_2 and $\bar{l} > \bar{j}$. Finally, note that if both S_1 and S_2 are empty then the j–j and the j–j' assignments are identical.

Start with the j–j assignment. Define, for any particular assignment, the distance between countries j_1 and j_2 as $|\bar{j}_2 - \bar{j}_1|$. Choose j_1 belonging to S_1 and j_2 belonging to S_2 such that $\bar{j}_2 > \bar{j}_1$ and $|\bar{j}_2 - \bar{j}_1|$ is minimum. Such a minimum always exists because j–j and j–j' are different assignments so that S_1, S_2 are nonempty and also by the above argument, there exists j_1 belonging to S_1 and j_2 belonging to S_2 such that $\bar{j}_2 > \bar{j}_1$.

Switch the j_1-th country to the $\bar{j}_2$-th good and the j_2-th country to the $\bar{j}_1$-th good. Hence, we get a new assignment. Calculate j_1, j_2 for the new assignment and switch the countries and commodities accordingly. Repeat the same procedure again and again.

We have to prove that by repeating the above procedure we shall reach the j–j' assignment after a finite number of steps.

Suppose, by making the series of binary switches we never reach the j–j' assignment. Since the above procedure can stop only when S_1 and S_2 are empty, i.e., when the j–j' assignment is reached, by hypothesis, the process goes on forever. Now, for any j_1 and j_2 chosen by the above procedure, if $\bar{j}_3$ lies between $\bar{j}_1$ and $\bar{j}_2$, then j_3 must belong to S_3; if not, then either j_3 belongs to S_1 or j_3 belongs to S_2.

If j_3 belongs to S_1 then $|\bar{j}_2-\bar{j}_3|<|\bar{j}_2-\bar{j}_1|$; and, if j_3 belongs to S_2, then $|j_3-j_2|<|j_2-j_1|$. Hence, both cases contradicting the minimality of $|\bar{j}_2-\bar{j}_1|$. Therefore, for any j that moves in a particular switch, it moves by either one place in the direction of j' or if it jumps, it jumps in the direction of j' over indices that are already in their places. In particular, any j, while moving towards j', cannot go beyond j'. Hence, the maximum number of times any j can move is given by $|j-j'|$. This is true for any j that does not belong to S_3 and, moves in some switch. Therefore, the maximum number of times binary switches can take place is bounded above. This means the process cannot go on forever; the j–j' assignment is reached after a finite number of switches.

Let us consider the k-th and $(k+1)$-th assignments in the series of binary switches. Suppose between the k-th and the $(k+1)$-th assignments countries j and k have switched. Suppose further that in the k-th assignment j is in good x and k is in good y. Then in the $(k+1)$-th assignment, j is in good y and k is in good x. We must have, by the nature of the switch described above, $k>j$ and $y>x$. Now for any country

$$A^x/A^y = (\Sigma a_t^x R^{-t}/\Sigma b_t^x R^{-t})/(\Sigma a_t^y R^{-t}/\Sigma b_t^y R^{-t}).$$

Therefore, denoting proportionate change by a '^' on a variable (e.g. $\hat{x}=dx/x$)

$$\begin{aligned}\widehat{(A^x/A^y)} &= \hat{R}\{\Sigma t q_t^x R^{-t}/\Sigma a_t^x R^{-t}) - (\Sigma t q_t^y R^{-t}/\Sigma a_t^y R^{-t})\} \\ &= \hat{R}\{(\Sigma k_t^x R^{-t}/\Sigma a_t^x R^{-t}) - (\Sigma k_t^y R^{-t}/\Sigma a_t^y R^{-t})\}\end{aligned}$$

which follows from the fact that $k_t^i=0$ and $k_t^i=\sum_{\tau=t}^{Ti} q_\tau^i R^{t-\tau}$ for $i=x,\ y$. Since $k>j$ and $y>x$, by the ranking of countries and commodities we have

$$r_j = (g/s)_j > (g/s)_k = r_k$$

and

$$\Sigma k_t^x R^{-t}/\Sigma a_t^x R^{-t} < \Sigma k_t^y R^{-t}/\Sigma a^y R^{-t}.$$

From these relationships it follows that

$$A_j^x A_k^y < A_j^y A_k^x.$$

Denoting the product of the A_j^i's in the k-th and the $(k+1)$-th assignment by Π^k and Π^{k+1} respectively, we have

$$\Pi^k < \Pi^{k+1}.$$

But this is true for all k and $k+1$. Hence, we must have

$$\Pi A_j^j < \Pi^1 < \cdots < \Pi^k < \Pi^{k+1} < \cdots < \Pi A_j^{j'}$$

which contradicts the hypothesis with which we started. Hence, ΠA_j^j is the minimum among all assignments in the class and the j–j assignment is efficient.

Let us now consider the uneven case where $n>m$. We define a class of assignments as a collection of patterns of complete specialization where n_1 countries are assigned to good 1, n_2 countries to good 2 and so on, and finally, n_m countries are assigned to good m. Following exactly the above procedure, we can prove that the efficient assignment in such a class will be the one where the first n_1 countries will produce the first good, the next n_2 countries will produce the second good and so on, and finally, the last n_m countries will produce the m-th good.

The above two propositions relate the patterns of efficient assignments to factor intensities as well as the saving ratios of countries. They show how it is possible to predict the efficient production patterns and hence, trade patterns on the basis of factor intensities of the goods and saving behaviour of countries in a multi-country multi-commodity world.

Note that the above two propositions are crucially dependent upon the assumption of no-factor-intensity-reversal. In a two-country, two-commodity world this assumption implies a monotonic relationship between autarky relative prices and the rate of interest on the basis of which one can predict that the country with the higher rate of interest (or the lower saving ratio) will export the labor-intensive good (for a detailed discussion on this point, see Sarkar [1982]. In a multi-country multi-commodity world, this assumption enables us to show that the ratio A^j/A^k with $j>k$, increases monotonically with increases in the rate of interest. This means that countries with higher interest rates will have an advantage in the production of good k compared to countries with lower interest rates. With factor-intensity reversal no such monotonic relationship can be obtained and hence, nothing, in general, can be said about efficient patterns of production on the basis of factor intensities and saving ratios.

Finally, it is to be noted that the classical savings function, though it simplified the analysis, was not a crucial assumption. One could, alternatively, start with an explicit utility maximization problem of the representative individual in each country over an infinite horizon. It is well-known that such a maximization yields the modified golden rule $r=\rho+g$, where ρ is the subjective rate of time preference. Then, assuming that countries have identical growth rates but different rates of time preference, one could, once again, order the countries by their rates of interest. A proposition on efficient production pattern could, therefore, be put forward on the basis of factor intensities and differing rates of time preference across countries.

4. CONCLUSIONS

In this paper, we have determined efficient patterns of production in a multi-country, multi-commodity time-phased model of international trade. It has been shown that countries with higher saving ratios will have a comparative advantage in the production of relatively capital intensive goods. One of the shortcomings of our analysis was that it was confined to steady states. A second shortcoming was that in our model trade in used equipment was not allowed which made it a special case of the more general von-Neumann model. We do hope, however, that this special case was able to bring out some useful economic insights.

University of Florida, U. S. A.

REFERENCES

HICKS, J., *Capital and Time* (Oxford: Clarendon Press, 1973).

JONES, R., "The Structure of Simple General Equilibrium Models," *Journal of Political Economy*, 73 (1965), 557–572.

————, "Comparative Advantage and the Theory of Tariffs: A Multi-Country, Multi-Commodity Model," *Review of Economic Studies*, 28 (1961), 161–175.

MIRRLEES, J., "Dynamic Non-Substitution Theorem," *Review of Economic Studies*, 36 (1969), 67–76.

SARKAR, A., "A Neo-Austrian Model International Trade," mimeographed, University of Rochester (1982).