Locally Optimal Tests for Multiparameter Hypotheses

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A generalization of a locally most powerful unbiased (LMPU) test, for the single parameter case, to the k-parameter case is proposed. In particular, we define a locally most mean power unbiased (LMMPU) test based on the mean curvature of the power hypersurface. Compared with the type C tests (Neyman and Pearson 1938) and the type D tests, LMMPU tests possess better theoretical properties and enjoy ease of construction of critical regions in practical situations. LMMPU tests are obtained for the important practical case (Perng and Littel 1976) of a two-parameter univariate normal population, for which Isaacson (1951, p. 233) was unable to find a type D test, and for the case of means of a multivariate normal population with independent coordinates.

KEY WORDS: Locally most mean power unbiased tests; Mean curvature; Multiparameter simple hypotheses.

1. INTRODUCTION

For testing multiparameter hypotheses, the likelihood ratio test statistic or the statistic defined by the positive definite quadratic form in the efficient score vector and the inverse of the information matrix may be conveniently used. Their small sample performances relative to reasonable competitors, however, are nearly always unknown.

Alternative procedures construct tests having certain optimal properties, at least locally, by an appeal to the Neyman–Pearson theory of testing and differential geometry (Do Carmo 1976). The type C test (Neyman and Pearson 1938) requires prior preference of parameter directions. The type D and the type E tests (Isaacson 1951; Lehmann 1959) maximize the total curvature of the power hypersurface at the null hypothesis among all regular locally unbiased level α tests.

In this article we propose a locally most mean power unbiased (LMMPU) level α test. It maximizes the mean curvature of the power hypersurface at the null hypothesis among all locally unbiased (LU) level α tests (Sec. 2). Compared with type D, an LMMPU test parallels the former in all of its nice geometrical properties and additionally is leading in the following statistical and practical aspects. (a) An LMMPU test maximizes the average power on a k-dimensional spherical neighborhood of the null hypothesis among the LU tests (Theorem 1). (b) Type D regions have to be guessed, whereas LMMPU critical regions are usually more easily constructed—the Neyman–Pearson lemma yields LMMPU directly (without any appeal to asymptotics) (Theorem 2). (c) The theory for LU optimal tests in practice has been developed almost uniquely for LU tests with regular Hessian matrix for their power. There exist cases, however, in which all LU tests are nonregular. The LMMP criterion is mostly able to discriminate among them, whereas the type D criterion cannot (Result 2). (d) LMMPU tests are not, in general, invariant under parameter transformations. This may be considered a weakness. It is also a strength, since in practice it may be of interest (e.g., see Neyman and Pearson 1938) to emphasize power in certain desired directions away from the null hypothesis, which the standard invariant tests like the type D and E tests do not do. (e) Finally, we note that average power plays a role in one traditional approach to linear model testing (e.g., see Scheffe 1959, p. 48), corresponding to our Result 5.

2. DEFINITIONS AND GEOMETRIC INTERPRETATIONS

Consider a k-parameter family of densities $f(X, \theta)$, where $X = (X_1, \ldots, X_n)' \in \mathcal{X}$, $\theta = (\theta_1, \ldots, \theta_k)' \in \Theta$, $\mathcal{X} \in B(\mathbf{R}^n)$ is the sample space, $\Theta \in \mathbf{R}^k$ is the parameter space, and $B(\mathbf{R}^n)$ denotes the usual Borel algebra of $\mathbf{R}^n$. f may be the sample density for a sample of size N from a p-variate population; then $n = Np$. Consider a test $\phi = I_w$ with critical region $w \in B(\mathcal{X})$ for a simple null hypothesis H_0: $\theta = \theta_0$, θ_0 in the interior of Θ, against the alternative H_1: $\theta \neq \theta_0$. We assume its power function $\beta(\theta)$ is of class C^2 or higher if needed.

Consider in the one-parameter case a locally most powerful unbiased level α test (LMPU). Geometrically, one picks out a test whose power curve $z = \beta(\theta)$ in the (θ, z) plane $\mathbf{R}^2$ takes at the point θ_0 the value α, has horizontal tangent line, and has maximum curvature $\ddot{\beta}(\theta_0)$ among the power curves of all LU tests.

In the k-parameter case, we obtain a power hypersurface $z = \beta(\theta)$ in the (θ, z) space $\mathbf{R}^{k+1}$ and denote it by the same symbol β. The local behavior of β at θ_0 is, as usual, studied by its Taylor expansion at θ_0 and by its behavior in any particular direction in the parameter space at θ_0. Maximizing the power in some direction, however, often pulls down the power in other directions. One needs to consider then some measure of "overall" power.

Definition 1. The level α test ϕ is (strictly) LU if $\beta_\phi(\theta_0)$

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(>) $\geqslant \alpha$ for all θ in some neighborhood of θ_0. Note that a test ϕ with $\beta(\theta)$ in C^2 is strictly LU of level α iff (1) $\beta(\theta_0) = \alpha$, (2) $\dot{\beta}(\theta_0) = 0$, and (3) $\ddot{\beta}(\theta_0)$ is positive definite. A necessary condition for ϕ to be LU is (1), (2) and (3')$\ddot{\beta}(\theta_0)$ is nonnegative definite.

Definition 2. The level α test ϕ is LMMPU of level α if it is LU of level α and for any other LU level α test Ψ, there exists an $r_0 > 0$ (depending on Ψ) such that

$$\int_{S_r} \beta_\phi(\theta)\, d\theta > \int_{S_r} \beta_\Psi(\theta)\, d\theta, \qquad r < r_0,$$

where $S_r = \{\theta: |\theta - \theta_0| < r\}$. Note that a test ϕ is regular LMMPU with level α iff $\beta(\theta)$ satisfies (1)–(3) and (4) tr $\ddot{\beta}_\phi(\theta_0) \geqslant$ tr $\ddot{\beta}_\Psi(\theta_0)$ for any other LU level α test Ψ. Replacing (3) by the weaker requirement (3'), $\ddot{\beta}(\theta_0)$ is positive semidefinite gives a second-order LMMPU test.

Some geometric consequences of these definitions are of interest and are presented subsequently, since they make these notions easier to work with. The power of test ϕ in a direction $\delta \in \mathbf{R}^k/\{0\}$ is

$$\beta_\delta(t) = \beta(\theta_0 + t\delta), \qquad t \in \mathbf{R},$$

and its second-order Taylor polynomial is the second-order power in the δ direction,

$$\beta_{\delta,2}(t) = \beta(\theta_0) + \dot{\beta}'(\theta_0)\delta t + \tfrac{1}{2}\delta'\ddot{\beta}(\theta_0)\delta t^2,$$

where $\dot{\beta}'(\theta_0)$ and $\ddot{\beta}(\theta_0)$ are the gradient vector and the Hessian matrix of β at θ_0, respectively. The power curvature $\gamma_\delta(\theta_0)$ in the direction δ at the null hypothesis is the normal curvature of the power hypersurface β at θ_0, that is, the curvature of the curve $\beta_\delta(t)$ at $t = 0$.

Assume a level α test ϕ, that is, (1) $\beta(\theta_0) = \alpha$. Then ϕ is (strictly) LU in a direction $\delta \in \mathbf{R}^k/\{0\}$ if $\beta_\delta(t)$ has a (strict) local minimum at $t = 0$. The test ϕ is (strictly) LU if it is (strictly) LU for all directions $\delta \in \mathbf{R}^k/\{0\}$. In particular, a regular LU test, defined as an LU test with regular Hessian, or defined by (2) and (3) is strictly LU. A regular LU test is most powerful in a direction δ if it has maximum power curvature $\gamma_\delta(\theta_0)$ among all regular LU tests.

The critical values of the normal curvature $\gamma_\delta(\theta_0)$ with respect to δ, which in case $k = 2$ reduce to the extreme value of $\gamma_\delta(\theta_0)$, and the vectors δ for which they are attained, are known as the principal (power) curvatures and corresponding principal (power) directions of the (test ϕ) power hypersurface β at θ_0. They are the eigenvalues λ_i ($1 \leqslant i \leqslant k$) and corresponding eigenvectors of $\ddot{\beta}(\theta_0)$. The total curvature or Gaussian curvature (power) and the mean curvature (power) of β (test ϕ) at θ_0 are, respectively,

$$K = \det \ddot{\beta}(\theta_0) = \prod_{i=1}^k \lambda_i, \qquad H = \operatorname{tr} \ddot{\beta}(\theta_0) = \sum_{i=1}^k \lambda_i.$$

The λ_i, H, K are measures of the way β bends away from its tangent space T_0: $z = \alpha$ at θ_0. A good geometrical interpretation follows from β_2: $z = \beta_2(\theta)$, where now

$$\beta_2(\theta) = \alpha + \tfrac{1}{2}\delta'\ddot{\beta}(\theta)\delta, \qquad \delta = \theta - \theta_0.$$

β and β_2 have at the point θ_0 the same z value, the same tangent space, and the same normal curvature γ_δ for any direction $\delta \in \mathbf{R}^k/\{0\}$. Hence as to tangent space and normal curvatures at θ_0, β_2 is a copy of β. Restrict now to second-order LU tests, that is, besides (1) and (2), all eigenvalues λ_i are nonnegative. The local behavior of β and β_2 at θ_0 is clarified by the intersection E'_ε of β_2 and a hyperplane T_ε: $z = \alpha + \varepsilon$ for small $\varepsilon > 0$; E'_ε is congruent with its projection

$$E_\varepsilon: \tfrac{1}{2}\delta'\ddot{\beta}(\theta_0)\delta = \varepsilon$$

in the parameter space; E_ε is the set of constant second-order power $\alpha + \varepsilon$. It is the $(\alpha + \varepsilon)$ power section, $\varepsilon > 0$, or the parameter set of equidetectability $\alpha + \varepsilon$ of the test ϕ. E_ε gives the set of deviations, defined by the standard Euclidean norm, of θ that will be equally frequently detected by the test. In case $k = 2$ the principal power directions are the directions of the smallest and the largest deviations of θ with a given detectability $\alpha + \varepsilon$; in addition, they are the directions in which equal size deviations of the parameter θ are the most and the least likely to be detected. As an example, requiring equidetectability of equal size positive or negative deviations of the means in both two-parameter normal examples $N(\mu, \sigma^2)$ and $N(\mu, \eta; I)$ of Section 4 translates the parameter directions into the principal power directions.

Neyman and Pearson (1938) defined a type C test, case $k = 2$, as a test corresponding to the smallest ellipse in a set of regular LU tests having coaxial similar ellipses of equidetectability. Hence a type C test, if it exists, is obtained by maximizing the power in one direction in a set of regular LU tests with fixed principal power directions and fixed ratio of the principal power curvatures.

A type D test (Isaacson 1951) is obtained by maximizing the total power among all regular LU tests.

Remark 1. Under (3') no type D test exists. There are, however, interesting situations (e.g., Result 2), where LMMPU tests exist in such cases and where they may even be strictly LU.

If ϕ is a regular LU test, that is, all $\lambda_i > 0$, or equivalently, E_ε is a hyperellipsoid in $\mathbf{R}^k$ (an ellipse if $k = 2$), then λ_i, K, and H can be expressed in the axes a_i ($1 \leq i \leq k$) of E_ε, as follows:

$$a_i^2 = 2\varepsilon\lambda_i^{-1}, \qquad K = 2\varepsilon \prod_i a_i^{-2} = 2\varepsilon c(\text{vol } E_\varepsilon)^{-2},$$

$$H = 2\varepsilon \sum_i a_i^{-2},$$

where c is a constant and vol is the volume.

Property. Let ϕ be a regular LU level α test with second-order $(\alpha + \varepsilon)$ power section E_ε for any given $\varepsilon > 0$. Then (a) ϕ is type D if it minimizes the volume (the area in case $k = 2$) of the inside of E_ε among all regular LU level α tests and (b) ϕ is LMMPU if it maximizes the sum of the squares of the inverse axes of E_ε among all regular LU level α tests.

In the next theorem, parts (a) and (b) provide the motivations and statistical relevance of the mean power H in the following sense. Given an LU test and any small $r >$

0, the average frequency of detecting any deviation of size $\leqslant r$ from a null hypothesis θ_0 is measured by the mean power $H = \text{tr}\, \ddot{\beta}(\theta_0)$, which depends on θ_0 only. An LMMPU test is obtained by maximizing this average detectability among all LU tests of a chosen level α.

Theorem 1. (a) Suppose that $\beta_\Psi(\theta)$ is in C^2 for all LU tests Ψ. A necessary condition for the level α test ϕ to be LMMPU is that it be LU and maximize tr $\ddot{\beta}_\Psi(\theta_0)$ among all LU level α tests Ψ. If ϕ is essentially the only test with this property, then it is LMMPU. (b) A level α test ϕ is LMMPU iff it is LU and for any LU level α test ϕ_0 either (i) $\exists r_0 > 0 \ni B(r) > B_0(r)$ for all r with $0 < r < r_0$ or (ii) $B(r) \bar{_2} B_0(r)$. The symbol $\bar{p}$ stands for the following: have equal Taylor polynomials of order p.

Proof. (a) To prove this part of the theorem, we need the following result to connect average local power with $\ddot{\beta}(\theta_0)$.

Lemma 1. Suppose that a test ϕ has $\beta_\phi \in C^2$, which satisfies (1) and (2). Then,

$$\lim_{r\to 0} \int_{S_r} [\beta_\phi(\theta_0) - \alpha]\, d\theta / r^2 \text{vol}(S_r) = \text{tr}\, \ddot{\beta}_\phi(\theta_0)/2(k+2).$$

Proof of Lemma 1. By (1) and (2) and since $\beta_\phi \in C^2$, the Taylor series expansion for β_ϕ gives

$$\beta_\phi(\theta) = \alpha + \tfrac{1}{2}(\theta - \theta_0)'\ddot{\beta}_\phi(\theta_0)(\theta - \theta_0) + |\theta - \theta_0|^2 h(\theta),$$

where $\lim_{\theta\to\theta_0} h(\theta) = 0$.

Consider a C^2 function f: $\mathbf{R}^k \to \mathbf{R}$; $x \to f(x)$ such that $\dot{f}(0) = 0$, $f(0) = 0$. It suffices to show that then the average of $f(x)$ on a k-spherical neighborhood S_r: $|x| \leqslant r$, $r > 0$, is

$$F(r) = \frac{1}{\text{vol}\, S_r} \int_{S_r} f = \frac{r^2}{2(k+2)} [H + A(r)], \quad (2.1)$$

where $H = \text{tr}\, \ddot{f}(0)$ is the mean curvature of the hypersurface graph $f = \{(x, f(x)) | x \in \mathbf{R}^k\}$ at the point $x = 0$ and the function $A : \mathbf{R} \to \mathbf{R}$ satisfies $\lim_{r\to 0} A(r) = 0$.

Now the function f as given previously can be written as $f(x) = \frac{1}{2}x'\ddot{f}(0)x + |x|^2 R(x)$ for some function R: $\mathbf{R}^k \to \mathbf{R}$ satisfying $\lim_{x\to 0} R(x) = 0$. Let

$$Q(r) = \frac{1}{2\, \text{vol}\, S_r} \int_{S_r} x'\ddot{f}(0)x\, dx_1 \cdots dx_k.$$

Consider an orthogonal change of coordinates in $\mathbf{R}^k$, $x \to y$, $x = Py$, such that $P'\ddot{f}(0)P = \text{diag}(\lambda_1, \ldots, \lambda_k)$.

Note that S_r is symmetric about the origin. Let

$$S_r^+ = y = \{(y_1, \ldots, y_k) \in S_r \mid y_i \geqslant 0 \text{ for all } i\}.$$

Then

$$Q(r) = \frac{1}{2k} H \frac{1}{\text{vol}\, S_r} 2^k \int_{S_r^+} \left(\sum_i y_i^2\right) dy_1 \cdots dy_k. \quad (2.2)$$

Now transform to spherical coordinates in k space, as follows:

$$(y_1, \ldots, y_k) \to (\rho, \theta), \qquad \theta = (\theta_1, \ldots, \theta_{k-1})'$$

subject to

$$0 \leqslant \rho < \infty, \qquad -\pi/2 \leqslant \theta_j \leqslant \pi/2, \qquad 1 \leqslant j \leqslant k-1.$$

Then,

$$dy_1 \cdots dy_k = \rho^{k-1} g(\theta)\, d\rho\, d\theta, \qquad d\theta = \prod_{j=1}^{k-1} d\theta_j,$$

where $g(\theta)$ is positive almost everywhere and a function of θ only. One obtains

$$\int_{S_r^+} \left(\sum y_i^2\right) dy_1 \cdots dy_k = \frac{r^{k+2}}{k+2} \int_G g(\theta)\, d\theta$$

and

$$\text{vol}\, S_r = 2^k \frac{r^k}{k} \int_G g(\theta)\, d\theta.$$

Integration over $S_r : |x| \leq r$ $(r > 0)$ and substitution of these two results in (2.2) gives $F(r)$ as in (2.1), where

$$A(r) = \frac{2(k+2)}{r^2 \text{vol}\, S_r} \int |x|^2 R(x).$$

Then $\lim_{r\to 0} A(r) = 0$ will follow from $\lim_{x\to 0} R(x) = 0$. Indeed for any $\varepsilon > 0$ there exists an $r_\varepsilon > 0$ such that $|R(x)| < \varepsilon$ for all x satisfying $0 < |x| < r_\varepsilon$. Hence for any $\varepsilon > 0$ there exists an $r_\varepsilon > 0$ such that

$$\left| \int_{S_r} |x|^2 R(x) \right| \leq \int |x|^2 |R(x)| \leq r^2 \varepsilon \int 1 = \varepsilon r^2 \text{vol}\, S_r$$

for all r satisfying $0 < r < r_\varepsilon$.

(b) One has to prove $H \geq H_0 \leftrightarrow$ (i) or (ii). Part (a) gives

$$B(r) - B_0(r) = \frac{r^2}{2(k+2)} [(H - H_0) + A(r) - A_0(r)],$$

where $\lim_{r\to 0} A(r) = 0 = \lim_{r\to 0} A_0(r)$; so if $H \neq H_0$, there exists an $r_1 > 0$ such that $|A(r) - A_0(r)| < |H - H_0|$ for all r with $0 < r < r_1$. This enhances the theorem in terms of the following properties: (1) $H = H_0 \leftrightarrow$ (ii), (2) $H > H_0 \to$ (i), (3) (i) $\to H \geq H_0$. Indeed, (1) is obvious, (2) also by choosing $r_0 = r_1$, (3) follows from the fact that (i) cannot induce $H < H_0$—given (i) and $H < H_0$ and taking $0 < r < \min(r_0, r_1)$ would result in $B(r) < B_0(r)$, which contradicts (i).

3. CONSTRUCTION OF LMMPU CRITICAL REGION

Theorem 2. Let $f(x, \theta)$, $\theta \in \Theta \subset \mathbf{R}^k$, be a k-parameter family of densities, $\in \mathbf{R}^n$. Let H_0: $\theta = \theta_0$ be a null hypothesis. Assume that the integral and derivative can be interchanged in β. Consider any Borel set w of the form

$$w : \sum_{i=1}^{k} \ddot{f}_{ii}(x, \theta_0) \geq cf(x, \theta_0) + \sum_{i=1}^{k} c_i \dot{f}_i(x, \theta_0), \quad (3.1)$$

where the constants $c, c_1, \ldots, c_k$ satisfy the conditions (1) $\int_w f(x, \theta_0) = \alpha$ and (2) $\int_w \dot{f}_i(x, \theta_0) = 0$ $(1 \leq i \leq k)$. Moreover, if w is essentially the only set with the property (3) $\ddot{\beta}(\theta_0) = (\int_w \ddot{f}_{ij}(x, \theta_0)) \in \mathbf{R}^{k\times k}$ is positive definite, then w is a regular LMMPU level α critical region. If $\ddot{\beta}(\theta_0)$ is only positive semidefinite, one has obtained a second-order LMMPU level α critical region.

Proof. An application of theorem 5 of Lehmann (1959, p. 83), one general nonrandomized form of the Neyman–Pearson fundamental lemma, gives that a region w that maximizes

$$\operatorname{tr} \ddot{\beta}(\theta_0) = \sum_i \int_w \ddot{f}_{ii}(x, \theta_0) = \int_w \sum_i \ddot{f}_{ii}(x, \theta_0),$$

subject to the conditions (1) and (2) of the theorem, is of the form (3.1), provided that such constants $c, c_1, \ldots, c_k$ exist. Definition 2 completes the proof.

Similar results can be obtained with randomized tests if no nonrandomized test satisfies the requirements of Theorem 2.

Remark 2. (a) Theorem 2 gives a definite method of construction of LMMPU critical region. In addition to Remark 1, this is yet another advantage of the LMMPU test over the type D test, where the critical region has to be guessed.

(b) Clearly, any unique LMMPU or unique type D test is admissible.

(c) Consider a pre-LMMPU test, that is, a test having maximum mean power in the class of all level-α tests satisfying (1) and (2). It needs to be emphasized that a pre-LMMPU critical region w of the aforementioned form becomes useful only when the minimum requirement of local unbiasedness is satisfied. If it turns out that w makes $\ddot{\beta}(\theta_0)$ positive semidefinite and nondefinite, one has to verify the local unbiasedness in all singular directions of $\ddot{\beta}(\theta_0)$.

(d) Since the proof of Theorem 2 is essentially the Neyman–Pearson fundamental lemma, LMMPU and second-order LMMPU regions of the form (3.1) will be called *Neyman–Pearson type regions.* Conditions under which regions maximizing an integral under certain side constraints are necessarily of Neyman–Pearson type have been developed (Chernoff and Scheffe 1952; Dantzig and Wald 1951).

(e) Expressed in the log-likelihood $l(x, \theta) = \log f(x, \theta)$, a Neyman–Pearson type LMMPU critical region can be written as

$$w : \sum_i [\ddot{l}_{ii}(x, \theta_0) + \dot{l}_i^2(x, \theta_0)] \geq c + \sum_i c_i \dot{l}_i(x, \theta_0), \quad (3.2)$$

where $c, c_1, \ldots, c_k \in \mathbf{R}$. This follows easily from (3.1) and the relations

$$\frac{\dot{f}_i}{f} = \dot{l}_i, \qquad \frac{\ddot{f}_{ij}}{f} = \ddot{l}_{ij} + \dot{l}_i \dot{l}_j.$$

(f) Note that the LMMPU property is not, in general, parameter invariant (whereas the type D property is). This follows easily. A parameter transform $g : \theta \to \tilde{\theta} = g(\theta)$ transforms the power function of a test ϕ_w, $\beta \to \zeta : \beta = \zeta \cdot g$. If this transformation has an invertible jacobian J, then $\operatorname{tr} \ddot{\beta}(\theta_0) = \operatorname{tr} J \ddot{\zeta}(\tilde{\theta}_0) J'$; so unless $JJ' = a^2 I$, maximizing $\operatorname{tr} \ddot{\zeta}$ need not be equivalent to maximizing $\operatorname{tr} \ddot{\beta}$. In addition,

$$\frac{1}{2(k+2)} \operatorname{tr} \ddot{\zeta}(\tilde{\theta}_0) = \lim_{r \to 0} \frac{1}{r^2} \left[\left[\int_{E_r} \beta(\theta) \, d\theta / \operatorname{vol}(E_r) - \alpha \right] \right],$$

where $E_r = \{\theta \mid \theta'(JJ')^{-1}\theta \leq r^2\}$. Thus changing parameters and finding the local average power over a sphere is equivalent to finding the local average power in the original parameter over an ellipse.

Generalization of the LMMPU test to the case with nuisance parameters follows easily.

Definition 3. Let $\{f(x, \theta) \mid \theta \in \Theta\}$ be a k-parameter family of densities, with parameter space $\Theta \subset \mathbf{R}^k$. Denote $\theta = (\lambda', \mu')'$, $\lambda = (\theta_1, \ldots, \theta_s)'$, $\mu = (\theta_{s+1}, \ldots, \theta_k)'$, $0 < s < k$. For $\lambda_0 \in \mathbf{R}^s$, let $\Theta_0 = \{(\lambda, \mu) \in \Theta \mid \lambda = \lambda_0\}$. Consider the null hypothesis $H_0 : \theta \in \Theta_0$. Hence $\mu = (\lambda_{s+1}, \ldots, \lambda_k)'$ is a $(k - s)$-dimensional nuisance parameter. A test ϕ with critical region w is a regular LMMPU similar level α test for H_0 if its power β satisfies the following conditions for all $(\lambda_0, \mu) \in \Theta_0$:

1. $\beta(\lambda_0, \mu) = \alpha$.
2. $\dot{\beta}_i(\lambda_0, \mu) = 0$, $1 \leq i \leq s$.
3. $B(\lambda_0, \mu) = (\ddot{\beta}_{ij}(\lambda_0, \mu))$, $1 \leq i, j \leq s$ is positive definite.
4. $\operatorname{tr} B(\lambda_0, \mu) \geq \operatorname{tr} B_{w_0}(\lambda_0, \mu)$ for any test ϕ_{w_0} satisfying conditions 1–3.

In case condition 3 is weakened to positive semidefiniteness one obtains a second-order LMMPU level α test.

Theorem 3 can be proven through slight modifications of the proof of Theorem 2.

Theorem 3. Consider the null hypothesis H_0 in Definition 3. A Borel set $w \in \mathbf{R}^n$ of the form

$$w : \sum_{i=1}^{s} \ddot{f}_{ii}(x; \lambda_0, \mu) \geq c(\mu) f(x; \lambda_0, \mu) + \sum_{i=1}^{s} c_i(\mu) \dot{f}_i(x; \lambda_0, \mu),$$

where $c, c_i : \mathbf{R}^{k-s} \to \mathbf{R}$ are functions of the nuisance parameter μ and w does not depend on μ, will be a regular LMMPU similar level α critical region for H_0, provided the functions $c, c_1, \ldots, c_s$ satisfy conditions 1–3 in Definition 3.

For further details, the reader is referred to SenGupta and Vermeire (1982).

4. EXAMPLES

Result 1 (Exponential Family). For a regular exponential family with log-likelihood $l(x, \theta) = \log g(x) + \theta' x - \psi(\theta)$, where $x = (x_1, \ldots, x_k)'$, $\theta = (\theta_1, \ldots, \theta_k)'$ is the parameter, a Neyman–Pearson-type LMMPU critical region for $H_0 : \theta = \theta_0$ will be of the form $w : \sum_i (x_i - a_i)^2 \geq a$, where the constants $a, a_1, \ldots, a_k$ should be determined from the LMMPU level α conditions.

Proof. From $\dot{l}_i = x_i - \dot{\psi}_i$, $\ddot{l}_{ij} = \ddot{\psi}_{ij}$, and (3.2) the critical region w is of the form

$$w : \sum_i [(x_i - \dot{\psi}_{i,0})^2 - \ddot{\psi}_{ii,0}] \geq c + \sum_i c_i (x_i - \dot{\psi}_{i,0}),$$

where $c, c_1, \ldots, c_k$ are constants. Observing that $\dot{\psi}_{i,0}$ and $\ddot{\psi}_{ii,0}$ are constants, one obtains the claimed result.

Example 1: Application to Univariate Normal Population

With Unknown Mean and Variance. For the interesting and important practical (Perng and Littell 1976) problem of testing that the observed random sample comes from a specified normal population, Isaacson (1951, p. 233) was unable to establish his "conjectured" region to be a region of type D. We present an LMMPU level α test for this problem. The cases $n = 1$ and $n > 1$ turn out to have different properties, mainly because their sufficient statistics for the two-dimensional parameter (μ, σ^2) are one- and two-dimensional, respectively.

By Remark 2 (f), the LMMPU tests for $(\mu, \sigma^2) = (0, 1)$ and $(\mu, \eta = 1/\sigma^2) = (0, 1)$ are the same. The computations will be done for the parameters (μ, η).

Result 2: Case of One Observation. Suppose that $X \sim N(\mu, \sigma^2)$ with both mean and variance unknown. In testing the null hypothesis $H_0 : (\mu, \sigma^2) = (0, 1)$ against $H_1 : (\mu, \sigma^2) \neq (0, 1)$ on the basis of one observation, at a level α $(0 < \alpha < 1)$ we observe that

1. Any LU test has power curvature zero in the μ direction and thus total power zero. Hence there does not exist a regular LU test.
2. Any Neyman–Pearson type LMMPU critical region can be written as

$$w = (-\infty, -b] \cup [-a, a] \cup [b, \infty), \qquad 0 < a < b,$$

where the constants a and b should satisfy the two conditions (a) $a = z_{1/2-\alpha_2}$, $b = z_{\alpha_1}$, $\alpha_1 + \alpha_2 = \alpha/2$, and (b) $af(a) = bf(b)$, where f is the standard normal pdf and z_α is its upper $\alpha\%$ point, $0 \leq \alpha < 1$. This test is strictly locally unbiased.

3. For any $\alpha < .68$ the system (a), (b) has a unique solution (a, b), $0 < a < b$. Moreover, $0 < a < 1$, $b > 1$.

The case of one observation, though mathematically quite interesting, lacks practical significance. The proof requires substantial computations and manipulations. The interested readers are referred to SenGupta and Vermeire (1982).

Remark 3. (a) Since all LU tests have power curvature zero in the μ direction, the LMMP criterion selects among them a test that is LMP in the σ^2 direction. (b) For a given α, the solution (a, b) of the system (a), (b) should be found numerically, for example, by the Gauss–Seidel method.

Result 3: Case of n Observations, n > 1. Let $x = (x_1, \ldots, x_n)' \in \mathbf{R}^n$ be a sample of size n from a univariate normal population $X \sim N(\mu, \sigma^2 = 1/\eta)$ with unknown mean and variance. Let

$$\bar{x} = \frac{1}{n}\sum_i x_i, \quad s^2 = \frac{1}{n}\sum_i (x_i - \bar{x})^2;$$

$$u = \sqrt{n}\bar{x}, \quad v = ns^2.$$

For testing the null hypothesis H_0: $(\mu, \eta) = (0, 1)$ against the alternative H_1: $(\mu, \eta) \neq (0, 1)$ a Borel set $w \subset \mathbf{R}^n$ is a Neyman–Pearson type (second-order) LMMPU level α critical region iff it is of the form $w : (v + u^2 - A)^2 + 4nu^2 - K^2 \geq 0$. Letting $w_0 = \{(u, v) \in \overline{w} \mid u \geq 0\}$, p_0 equal the standard normal density, and q_0 equal the chi-squared density with $(n - 1)$ df, the constants A, $K \in \mathbf{R}$ should satisfy the following:

$$(1') \quad 2\int_{w_0} p_0(u)q_0(v)\, du\, dv = 1 - \alpha,$$

$$(2') \quad 2\int_{w_0} (u^2 + v)p_0(u)q_0(v)\, du\, dv = n(1 - \alpha),$$

$$(3') \quad 2\int_{w_0} u^2 p_0(u)q_0(v)\, du\, dv \leq 1 - \alpha,$$

$$(4') \quad 2\int_{w_0} (u^2 + v)^2 p_0(u)q_0(v)\, du\, dv \leq n(n + 2)(1 - \alpha).$$

Proof. Using Result 1, simplifications yield $w : (v + u^2 - A)^2 + 4nu^2 - Bu - K^2 \geq 0$, where the constants A, B, and K satisfy (1)–(3) of Theorem 2. The power is

$$\beta(\mu, \eta) = P_{\mu,\eta}[(u, v) \in w] = 1 - \int_{\overline{w}} p(u)q(v),$$

where $p(u)$ and $q(v)$ are the respective densities of the independent components u and v.

But then condition (1) of Theorem 2 implies that

$$(1'') \quad \int_{\overline{w}} p_0 q_0 = 1 - \alpha,$$

whereas condition (2) implies that

$$(2'') \quad \int_{\overline{w}} (v + u^2 - n)p_0 q_0 = 0$$

and further, B has to be 0. Finally, since symmetry gives $\ddot{\beta}_{\mu\eta} = 0$, (3) of Theorem 2 requires

$$(3'') \quad \int_{\overline{w}} (u^2 - 1)p_0 q_0 \leq 0$$

and

$$(4'') \quad \int_{\overline{w}} [(v + u^2 - n)^2 - 2n]p_0 q_0 \leq 0.$$

Since the region $\overline{w}$ and all of the integrands are symmetric in u, it suffices to consider w_0. Then the system (1″)–(4″) reduces to the system (1′)–(4′) as claimed in Result 3.

Remark 4. The existence of a solution (A, K) for the system (1′)–(4′) should now be investigated. For $n = 1$ $(u = x, v = 0)$ we proved that the equalities (1′) and (2′) induce the inequalities (3′) and (4′) and have a solution for the usual values of α. Hence there is reasonable hope to achieve the same result in case $n > 1$. Our attempts to prove analytically, however, that (1′) and (2′) establish (3′) and (4′) failed, except for the case $A \leq \min(K, K^2/4n)$. Therefore, we solve the problem numerically. Writing the region as

$$w_0\colon 0 \leq u \leq K/(2\sqrt{n}), \qquad C(u) \leq v \leq D(u),$$

where

$$C(u) = A - u^2 - \sqrt{K^2 - 4nu^2},$$

$$D(u) = A - u^2 + \sqrt{K^2 - 4nu^2},$$

shows that all integrals are of the form

$$\int_{w_0} u^r v^m p_0(u) q_0(v)\, du\, dv = \int_0^{K/(2\sqrt{n})} u^r p_0(u) \int_{C(u)}^{D(u)} v^m q_0(v)\, dv\, du.$$

For given α, n we solve (1′) and (2′) for A, K by a numerical method, for example, the Gauss–Seidel method, and then check if (3′) and (4′) are satisfied. The integrals were computed by the Harwell computer package. Table 1 presents a few results in which, for each of the cases, the test is regular LU. Detailed tables will be published elsewhere. For the last two cases Figure 1 compares the acceptance regions, plotted in the $(u, \sqrt{v})$ plane.

Remark 5. In the foregoing example as soon as an LU critical region w is symmetric in $\bar{x}$ about the null hypothesis, one obtains $\ddot{\beta}_0$ is a diagonal matrix. Hence for $N(\mu, \sigma^2)$, requiring equidetectability for equal amounts of positive or negative deviations of the mean translates the parameter directions into the principal power directions.

Example 2: A Multivariate Normal Population With Known Covariance Matrix and Unknown Means. Consider a k-variate normal population with unknown mean vector and known covariance matrix. Since a suitable orthogonal transformation reduces the covariance matrix to a diagonal matrix, we restrict to the latter case. In addition, we write the theory first for covariance matrix I and $k = 2$; generalization to all values $k > 2$ is straightforward and is given afterwards.

Result 4. Let (X, Y) be a bivariate normal population, where X, Y are independent with means μ, η and variances equal to 1. Tests for the null hypothesis H_0: $(\mu, \eta) = (0, 0)$ at a level α $(0 < \alpha < 1)$ against H_1: $(\mu, \eta) \neq (0, 0)$, on the basis of a sample $x = (x_1, \ldots, x_n)'$ from X and a sample $y = (y_1, \ldots, y_m)'$ from Y, have the following properties. Let

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \bar{y} = \frac{1}{m}\sum_{j=1}^{m} y_j.$$

(1) See Isaacson (1951). The likelihood ratio critical region, $w : n\bar{x}^2 + m\bar{y}^2 \geq a^2$, where $a^2 = \chi^2_{2,\alpha}$ is also a type D critical region.

(2) Any second-order LMMPU critical region of Neyman–Pearson type will be of the form w: $(\sum x_i)^2 + (\sum y_j)^2 \geq a^2$, where a satisfies

$$\int_0^{a/\sqrt{m}} e^{-v^2}/2 \int_0^{A(v)} e^{-u^2}/2\, du\, dv = (\pi/2)(1 - \alpha).$$

w is a regular LMMPU level α critical region.

Table 1. Values of (A, K) for Certain (n, α)

n	α	A	K
3	.10	8.2	7.9
10	.10	15.7	13.1
10	.05	17.3	15.1

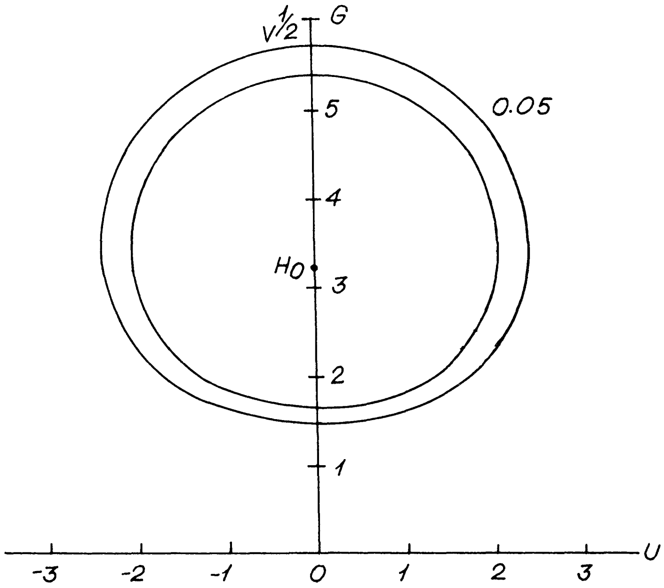

Figure 1. Acceptance Regions of the LMMPU Tests of Result 4.2.2. N = 10.

(3) In case $n = m$ the LMMPU critical region coincides with the type D region from (1).

Remark 6. For a brief comparison between the type D and LMMPU tests at a same level α in the foregoing case we observe the following.

(a) Let w_T and w_M be the critical regions in the (u, v) plane, as follows:

$$w_T : u^2 + v^2 \geq a_T^2, \qquad w_M : nu^2 + mv^2 \geq a_M^2.$$

The constants a_T^2, a_M^2 must be determined such that the volume under the rotation surface p_0 inside the circle S_T or the ellipse S_M,

$$S_T : u^2 + v^2 = a_T^2, \qquad S_M : nu^2 + mv^2 = a_M^2,$$

each equals $1 - \alpha$. If, for example, $n < m$, the LMMPU test is more likely to accept from zero deviating values in the μ direction than the type D test.

We conclude that in a direction with more available information (higher sample size for X or Y) the LMMPU test is more likely to reject H_0 than the type D test, whereas in a direction with less available information the LMMPU test is less likely to reject H_0 than the type D test.

(b) In case $n = m$ any LU test with critical region of the form $(\sum x_i)^2 + (\sum y_j)^2 \geq a^2$ will simultaneously maximize H and K. Indeed, one verifies that the power surface $\beta(\theta)$ for such a test is a surface of revolution with axis $\theta = 0$. Then all eigenvalues of $\ddot{\beta}(\theta_0)$ are equal, say λ, and maximizing $K = \lambda^2$ or $H = 2\lambda$ reduces to maximizing λ.

Result 5 (Generalization of Result 4). Let $(X_1, \ldots, X_k)$ be a k-variate normal population with known covariance matrix $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_k^2)$ and mean vector $\mu = (\mu_1, \ldots, \mu_k)'$, the latter being a k-dimensional parameter, $G = \mathbf{R}^k$, $H_0 : \mu_1 = \cdots = \mu_k = 0$. From each X_i $(1 \leq i \leq k)$ a sample $(x_{i1}, \ldots, x_{in_i})$ of size n_i is drawn. Let $u_i = \sqrt{n_i}\bar{x}_i/\sigma_i$ be the normalized sample mean from X_i under H_0. A regular LMMPU level α critical region for H_0 of the form

$$w\colon \sum_{i=1}^{k} \frac{n_i}{\sigma_i^2} u_i^2 \geq a^2 \tag{4.1}$$

is obtained as soon as the constant $a^2 > 0$ satisfies the level condition. a^2 should be determined from the fact that u_i $(1 \leq i \leq k)$ are iid $N(0, 1)$.

Proof. Writing $\eta_i = 1/\sigma_i^2$ and, after some simplifications by (3.1), one obtains a regular LMMPU level α critical region of the form

$$w\colon \sum_i (n_i \eta_i \bar{x}_i)^2 \geq c + \sum_i c_i \bar{x}_i$$

provided the constants $c, c_1, \ldots, c_k$ satisfy the following conditions on the power β: (a) $\beta_0 = \alpha$, (b) $\dot{\beta}_{\mu_i,0} = 0$ for $1 \leq i \leq k$, (c) $(\ddot{\beta}_{\mu_i\mu_j,0})$ is positive definite. As in the proof of Result 3, the second condition requires $c_i = 0$ $(1 \leq i \leq k)$, which reduces w to the form $w\colon \Sigma_i (n_i \eta_i \bar{x}_i)^2 \geq a^2$ or (4.1).

Note that $\overline{w}$ is an ellipse in terms of u_i $(1 \leq i \leq k)$. Anderson's theorem (see, e.g., Tong 1980) shows that as the mean vector moves away from 0, the probability of the set $\overline{w}$ is strictly decreasing. Thus the test is strictly unbiased and hence the preceding conditions (b) and (c) are satisfied.

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