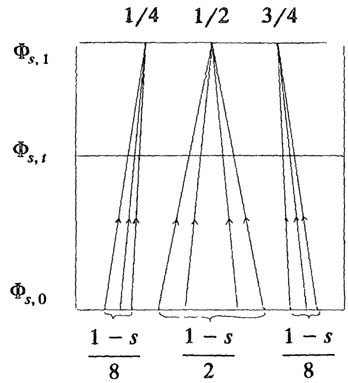

FIG. 4. A homotopy from the identity to the Cantor function.

The triangular regions are chosen with their top vertices at rational numbers of the form $(2k+1)/2^m$. Thus if we delete the interior of all such triangles, the intersection of the resulting set with the line $y = t$ is precisely the Cantor set $C_{t+s(1-t)}$ described above. Define a two-parameter family of maps $F_{s,t}\colon I \times I \to I \times I$ by $F_{s,t}(x, y) = (\Phi_{s,t}(x), \Phi_{s,t}(y))$ for $s \in (0, 1]$ and $t, x, y \in [0, 1]$. This is clearly jointly continuous in all four variables. For fixed s the family $F_{s,t} \circ f_s$ is precisely the family $f_{t+s(1-t)}$. Running this homotopy backwards gives us an unfolding of $I \times I$ into the arc f_s with Lebesgue measure s^2.

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Unions and Common Complements of Subspaces

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The following exercise can be found in most undergraduate text-books on linear algebra: show that if S and T are subspaces of a vector space, $S \cup T$ is a subspace if and only if either $S \subseteq T$ or $T \subseteq S$. In our experience, most of the students are able to show this and a few get curious about what happens for three or more subspaces.

The most natural conjecture for the general case would be that the union of a family $\mathscr{S}$ of subspaces of a vector space V over a field F is itself a subspace of V

if and only if some member of $\mathscr{S}$ contains all the others. But one can immediately see that this is false since any vector space with dimension at least two is the union of its subspaces with codimension 1 and none of these can contain the others. So we raise the question: can we impose some reasonable restrictions on the cardinalities of $\mathscr{S}$ and F and the dimensions of the subspaces in $\mathscr{S}$ to make the conjecture true? The answer is, indeed, yes. For example, the conjecture is known to be true if $\mathscr{S}$ is assumed to be finite and F infinite, see Problem 21, p. 177 of [**2**] or Theorem 1 below. In this note we present other reasonable conditions under which the conjecture is true. We also study the existence of a common complement for a family of subspaces, which turns out to be related to the above problem, and deduce a result of Lord [**4**].

To set up the notation let V be a (not necessarily finite dimensional) vector space over a (not necessarily infinite) field F. We will use $|A|$ to denote the cardinality of set A. For a subspace S, $\dim(S)$ denotes the dimension of S while $\operatorname{cod}(S)$ denotes the codimension of S. It may be noted that $\operatorname{cod}(S)$ is the dimension of any complement of S and equals $\dim(V/S)$. $\operatorname{Sp}(X)$ will denote the linear span of the set X of vectors.

To start with we will show that for any vector space V over F with dimension at least 2, we must have $|I| < |F| + 1$ for the conjecture to have any chance of holding. To see this, fix a basis $\{x_i\colon i \in I\}$ of V and two distinct elements j and k of I. Consider the $|F| + 1$ subspaces $W_\alpha = \operatorname{Sp}(\{x_j + \alpha x_k\} \cup \{x_i\colon i \neq j, k\})$ for all $\alpha \in F$ and $W_\infty = \operatorname{Sp}(\{x_i\colon i \neq j\})$. It is easy to check that these are distinct subspaces with codimension 1 (so none of them can contain the others) and that their union is V.

However, $|I| < |F| + 1$ alone is not sufficient: consider the vector space V of all polynomials over $\mathbb{R}$ and the subspaces $S_0, S_1, S_2, \ldots$ where S_i consists of all polynomials with degree at most i—clearly no S_i contains all the others though their union is V. Thus, to prove the conjecture, we have to make some further assumption: for example, the family is finite or the subspaces are of bounded dimension. We now proceed to prove the conjecture under each of these assumptions.

THEOREM 1. *Let $S_1, S_2, \ldots, S_k$ be finitely many subspaces of V with $k < |F| + 1$. Then $S_1 \cup S_2 \cup \cdots \cup S_k$ is a subspace if and only if some S_i contains the others.*

Proof. The "if" part of the theorem is obvious. We prove the other part by induction on k. The case $k = 1$ is trivial. We now assume the result for $k - 1$ and prove it for k. So let $S_1 \cup \cdots \cup S_k$ be a subspace and $k < |F| + 1$. If $S_1 \supseteq S_2 \cup \cdots \cup S_k$, we are through. So let $S_1 \not\supseteq S_2 \cup \cdots \cup S_k$. We shall show that $S_1 \subseteq S_2 \cup \cdots \cup S_k$. To this end, fix a $y \in (S_2 \cup \cdots \cup S_k) - S_1$ and let $x \in S_1$. Then for every scalar α, $\alpha x + y \notin S_1$. Since $S_1 \cup \cdots \cup S_k$ is a subspace, $\alpha x + y \in S_2 \cup \cdots \cup S_k$. Since $\alpha x + y \neq \beta x + y$ whenever $\alpha \neq \beta$ and $|F| \geqslant k$, it follows that there exist j with $2 \leqslant j \leqslant k$ and $\alpha \neq \beta$ in F such that $\alpha x + y$ and $\beta x + y$ are in S_j. Then $x \in S_j$. Thus $S_1 \subseteq S_2 \cup \cdots \cup S_k$. Now the result follows from the induction hypothesis. ∎

We now consider the case of a finite-dimensional vector space before going on to subspaces with bounded dimension in an infinite-dimensional vector space. We state only the non-trivial part in the remaining theorems.

THEOREM 2. *Let $\{S_i: i \in I\}$ be a family of subspaces of a finite-dimensional vector space V with $|I| < |F| + 1$. If $S := \bigcup_{i \in I} S_i$ is a subspace then $S = S_i$ for some $i \in I$.*

Proof. We prove the theorem by induction on n, the dimension of S. For $n = 1$ the result is easy to prove. Next assume the result for $n - 1$ and let $n \geqslant 2$.

Suppose that no S_i equals S. If H is *any* subspace of S with dimension $n - 1$,

$$H = H \cap S = H \cap \left(\bigcup_{i \in I} S_i \right) = \bigcup_{i \in I} (H \cap S_i),$$

so by the induction hypothesis, $H = H \cap S_j$ for some $j \in I$. Then $H \subseteq S_j$ and S_j is a proper subspace of S. Since $\dim(H) = \dim(S) - 1$ it follows that $H = S_j$. Thus *every* subspace of S with dimension $n - 1$ is one of the S_i's. Since, as we saw in the first example before Theorem 1, there are at least $|F| + 1$ such subspaces of S, we get $|I| \geqslant |F| + 1$, a contradiction which proves the theorem. ∎

For a denumerable family Theorem 2 was proved earlier by Byrd [**1**].

We now deduce the result for a family of subspaces with bounded dimension.

THEOREM 3. *Let $\{S_i: i \in I\}$ be a family of subspaces of V with $|I| < |F| + 1$. Also let m be a positive integer such that $\dim(S_i) \leqslant m$ for all $i \in I$. If $S := \bigcup_{i \in I} S_i$ is a subspace then $S = S_i$ for some $i \in I$.*

Proof. We first show that $\dim(S) \leqslant m$. Suppose not. Then there exists a subspace W of S with dimension $m + 1$. Applying Theorem 2 to the family $\{W \cap S_i: i \in I\}$ of subspaces of W, we get $W = W \cap S_i$ for some $i \in I$, a contradiction since $\dim(W) = m + 1$ and $\dim(S_i) \leqslant m$. Now an application of Theorem 2 to the family $\{S_i: i \in I\}$ of subspaces of S yields the present theorem. ∎

We now use the above results to give conditions under which a family of subspaces of a vector space has a common complement.

Observe that if the S_i's have a common complement then they must have the same dimension and the same codimension. We restrict our attention to families of subspaces with common finite dimension or common finite codimension and we start with the latter case which happens to be easier.

THEOREM 4. *Let $\{S_i: i \in I\}$ be a family of subspaces of V with $|I| < |F| + 1$. If the S_i's have a common finite codimension p and either I is finite or V is finite-dimensional, then the S_i's have a common complement.*

Proof. We prove the theorem by induction on p. If $p = 0$, $\{0\}$ is a common complement of the S_i's. Next assume the result for codimension $p - 1$ and let $\operatorname{cod}(S_i) = p$ for all $i \in I$, where $p \geqslant 1$. Then by Theorems 1 and 2, $\bigcup_i S_i \neq V$. Fix an x in $V - \bigcup_i S_i$ and consider $T_i = \operatorname{Sp}(S_i \cup \{x\})$ for all $i \in I$. Clearly $\operatorname{cod}(T_i) = p - 1$ for all i, so by the induction hypothesis, the T_i's have a common complement W. $\operatorname{Sp}(W \cup \{x\})$ is then a common complement of the S_i's. ∎

COROLLARY 5 (Lord [**4**]). *Let V be an n-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $\{S_i: i \in \mathbb{N}\}$ be a countable family of subspaces of V each having the same dimension. Then the S_i's have a common complement.*

The proof of Corollary 5 essentially appears in Byrd [**1**].

THEOREM 6. *Let $\{S_i: i \in I\}$ be a family of subspaces of V with $|I| < |F| + 1$. If the S_i's have a common finite dimension, they have a common complement.*

Proof. If V is finite-dimensional, this follows from Theorem 4. Otherwise, applying Zorn's lemma to the family $\mathscr{T}$ of all subspaces T of V with the property $T \cap S_i = \{0\}$ for all i gives a maximal element T_0 of $\mathscr{T}$. We now prove that T_0 is a common complement of the S_i's. Suppose T_0 is not a complement of S_j. Let φ be the natural homomorphism from V to V/T_0. Since $T \cap S_i = \{0\}$ it follows that $\dim(\varphi(S_i)) = \dim(S_i) = m$ (say) for all $i \in I$. Since T_0 is not a complement of S_j it follows that $\varphi(S_j)$ is a proper subspace of V/T_0, so $\dim(V/T_0) > m$. So by Theorem 3, $\bigcup_i \varphi(S_i) \neq V/T_0$. Hence, there is a 1-dimensional subspace X of V/T_0 such that $X \cap \varphi(S_i) = \{0\}$ for all i. Now $\varphi^{-1}(X)$ is a member of $\mathscr{T}$ properly containing T_0, a contradiction which proves that T_0 is a common complement of the S_i's. ∎

We now show that when $\dim(V) \geqslant 2$, the condition $|I| < |F| + 1$ cannot be dropped in Theorems 4 and 6. Regarding Theorem 4 it is enough to consider (again) $|F| + 1$ subspaces of V with codimension 1 and union V. Regarding Theorem 6 we first choose a subspace Y of V with dimension 2. Then we consider a family $\{S_i: i \in I\}$ of $|F| + 1$ subspaces of Y with common dimension 1 and union Y. Suppose the S_i's have a common complement T in V. Then $S_i + (T \cap Y) = (S_i + T) \cap Y = Y$ for all i, so $T \cap Y$ is a common complement of the S_i's in Y, an impossibility which proves that the S_i's cannot have a common complement in V.

It is easy to see that the condition "either I is finite or V is finite dimensional" in Theorem 4 cannot be dropped. For this, take a basis $B = \{x_j: j \in J\}$ of an infinite-dimensional vector space V and consider $\{\mathrm{Sp}(B - \{x_i\}): i \in I\}$ for any countably infinite subset I of J. Indeed, the same example also shows that "dimension" in Theorem 6 cannot be replaced by "codimension."

We end this note by answering the problems raised in [**3**] and [**4**]. In [**3**] the problem was raised as to what are the results analogous to Corollary 5 which hold for vector spaces over finite fields. The answer is provided by our Theorems 4 and 6.

The problem raised in [**4**] was what is the maximum number of distinct m-dimensional subspaces of an n-dimensional vector space V over a finite field F that have a common complement? The answer is the number of distinct complements of an $(n - m)$-dimensional subspace T since this number is the same for all T. To find this number, fix an ordered basis B of T. Any complement of T can be obtained as $\mathrm{Sp}(C)$ where $B \cup C$ is an extension of B to a basis of V. This extension can be done in

$$\left(|F|^n - |F|^{n-m}\right)\left(|F|^n - |F|^{n-m+1}\right) \cdots \left(|F|^n - |F|^{n-1}\right)$$

ways. But different C's can give rise to the same complement of T. The number of C's giving rise to the same complement is the same as the number of bases of an m-dimensional subspace and this is $(|F|^m - 1)(|F|^m - |F|) \cdots (|F|^m - |F|^{m-1})$. Thus the required number is

$$\frac{\left(|F|^n - |F|^{n-m}\right)\left(|F|^n - |F|^{n-m+1}\right) \cdots \left(|F|^n - |F|^{n-1}\right)}{\left(|F|^m - 1\right)\left(|F|^m - |F|\right) \cdots \left(|F|^m - |F|^{m-1}\right)} = |F|^{m(n-m)}.$$

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The Generation of All Rational Orthogonal Matrices

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In a letter to this MONTHLY [**1**] John Cremona shows how to generate all 3×3 orthogonal matrices with rational coefficients. His method is based on the real algebra of quaternions. By a further application of quaternions one can obtain 4×4 rational orthogonal matrices. (See du Val [**2**].)

In this note we show how to generate all $n \times n$ rational orthogonal matrices and hence all orthonormal bases of the rational vector space $\mathbf{Q}^n$. At the same time we obtain all real orthogonal matrices and all complex unitary matrices. Our method is based on a further piece of mathematics from the last century—Cayley's formula for orthogonal matrices.

Preliminaries. Let A and B be $n \times n$ matrices. We shall say that B *is equivalent to* A, and write $B \sim A$, if and only if there exists a diagonal matrix D with diagonal entries selected from the set $\{-1, 1\}$ such that $B = DA$. Clearly B is equivalent to A if and only if, for each $i = 1, \dots, n$, the ith row of B is $\pm i$th row of A. The relation $\sim$ is an equivalence relation on the set of $n \times n$ matrices, and the equivalence class $\mathscr{E}(A)$ which contains A has at most 2^n members.

LEMMA. *Let A be an $n \times n$ matrix over a field of characteristic $\neq 2$. Then at least one of the matrices in $\mathscr{E}(A)$ does not have eigenvalue* 1.

Proof. By induction on n. For $n = 1$ the result is true, since the 1×1 matrices A and $-A$ cannot both be [1] over a field of characteristic $\neq 2$. Proceeding by way of contradiction let n be the least order for which the lemma is false. Then there exists an $n \times n$ matrix A such that all matrices in $\mathscr{E}(A)$ have eigenvalue 1. Thus for any matrix $DA \in \mathscr{E}(A)$, there exists $\mathbf{x} \neq \mathbf{0}$ such that $DA\mathbf{x} = \mathbf{x}$. Hence $A\mathbf{x} = D\mathbf{x}$, and so the matrix $A - D$ is singular. Consider the determinant function

$$d(t_1, \dots, t_n) = \det(A - \operatorname{diag}(t_1, \dots, t_n)).$$

We have that $d(t_1, \dots, t_n) = 0$ for all 2^n choices of $t_i = \pm 1$, $i = 1, \dots, n$.

Expanding $d(t_1, \dots, t_n)$ according to the first row, we obtain

$$d(t_1, \dots, t_n) = (a_{11} - t_1) d^*(t_2, \dots, t_n) + \text{terms independent of } t_1. \quad (1)$$