MATHEMATICS OF OPERATIONS RESEARCH
Vol. 19, No. 3, August 1994
Printed in U.S.A.

THE LINEAR COMPLEMENTARITY PROBLEM WITH EXACT ORDER MATRICES

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Dedicated to Professor K. G. Ramamurthy on the occasion of his 60th birthday.

A real n by n matrix A is called an $N(P)$-matrix of exact order k, if the principal minors of A of order 1 through $(n-k)$ are negative (positive) and $(n-k+1)$ through n are positive (negative). In this paper the properties of exact order 1 and 2 matrices are investigated, using the linear complementarity problem LCP(q, A) for each $q \in R^n$. A complete characterization of the class of exact order 1 based on the number of solutions to the LCP(q, A) for each $q \in R^n$ is presented. In the last section we consider the problem of computing a solution to the LCP(q, A) when A is a matrix of exact order 1 or 2.

0. Notation. Euclidean n-space is denoted by R^n and its nonnegative orthant by R^n_+. By $R^{n \times n}$, we denote the space of all real n by n matrices. We use I to denote the identity matrix of appropriate order. A vector is regarded as a column and superscript t is used to denote transposition. We say that a vector $x \in R^n$ is unisigned, if either $x_i \geqslant 0$ for all $1 \leqslant i \leqslant n$, or $x_i \leqslant 0$ for all $1 \leqslant i \leqslant n$. Let $A \in R^{n \times n}$. For subsets $J, K \subset \{1, 2, \dots, n\}$, we denote by A_{JK} and A^{JK}, the submatrices of A and A^{-1} respectively, with rows and columns corresponding to the index sets J and K. The (i, j)th entry of A and A^{-1} are denoted by a_{ij} and a^{ij} respectively. For $J = \{1, 2, \dots, n\}$, A_{JK} is written for simplicity as $A_{\cdot K}$. The ith column of A is denoted by $A_{\cdot i}$ and the jth row by $A_{j \cdot}$, $A^{\cdot i}$ and $A^{j \cdot}$ similarly denote the ith column and the jth row of A^{-1}, respectively. Finally, for any $J \subseteq \{1, 2, \dots, n\}$, $\bar{J}$ denotes the set $\{1, 2, \dots, n\} \setminus J$.

1. Introduction. For a given n-vector q and $A \in R^{n \times n}$, the linear complementarity problem, denoted by LCP(q, A), is that of finding nonnegative vectors $w \in R^n$, $z \in R^n$ such that

$$w - Az = q, \qquad w^t z = 0. \tag{1.1}$$

It is well known (Mohan and Sridhar 1992) that LCP(q, A) has a unique solution for every $q \ngeq 0$ and has exactly three solutions for every $q > 0$, if and only if A is an *N-matrix of the first category* (i.e., all the principal minors of A are negative and at least one entry of A is positive). N-matrices have been considered in game theory (Parthasarathy and Revindran 1990), in the theory of global univalence of functions (Parthasarathy 1983) and in the literature on the linear complementarity problem (Saigal 1972). In Parthasarathy and Ravindran (1990) prove some characterization theorems for *N-matrices of the second category* (i.e., all principal minors negative, with $A < 0$).

As a sequel to the classes of N-matrices and P-matrices (i.e., matrices whose principal minors are all positive), the classes of almost N and almost P were introduced by Olech, Parthasarathy and Ravindran (1989, 1991). Using game theo-

Received April 3, 1991; revised November 14, 1992.
AMS 1991 subject classification. Primary: 90C33. Secondary: 90D05.
IAOR 1973 subject classification. Main: Programming/Linear. Cross references: Games
OR/MS Index 1978 subject classification. Primary: 622 Programming/Complementarity.
Key words. Linear complementarity problem; exact order matrices of order 1, order 2; completely mixed games, minimax value, Q-matrix.

retic aspects, they studied these classes of matrices extensively, for their significance in the theory of global univalence and the linear complementarity problem.

We now introduce the definition of *exact order* k matrices, which generalizes the above mentioned classes of matrices.

DEFINITION 1.1. A matrix $A \in R^{n \times n}$ is called an *N-matrix* (*P-matrix*) *of exact order* k, $1 \leqslant k \leqslant n$, if every principal submatrix of order $(n - k)$ is an N-matrix (P-matrix) and if every principal minor of order r, $n - k < r \leqslant n$, is positive (negative). A is called a matrix of exact order k, if it is either a P-matrix or an N-matrix of exact order k.

Thus, an $N(P)$-matrix, is an $N(P)$-matrix of exact order 0, and an almost N (almost P)-matrix is an $N(P)$-matrix of exact order 1. The following is an example of an exact order k matrix, for any n and $k, k \leqslant n$.

EXAMPLE 2.1. Let A be an n by n matrix of the form,

$$A = \begin{bmatrix} 1 & x & x & \dots \\ x & 1 & x & \dots \\ x & x & 1 & \dots \end{bmatrix}. \tag{1.2}$$

For $x \neq 1$, $\det(A) = (1 - x)^{n-1}(1 + (n - 1)x)$. When $-1/(n - k) < x < -1/(n - k + 1)$, A is a P-matrix of exact order k. Note also that this is a matrix whose off-diagonal entries are nonpositive. Such a matrix is called a Z-matrix and has been well studied in the literature. See for example Chandrasekaran (1970), Fiedler and Ptak (1962), Mohan (1976), Ramamurthy (1986), and Saigal (1971). Z-matrices arise in a number of applications in operations research and also in the linear complementarity formulations of some engineering applications. See Tamir (1976) and Cryer (1971). Further, as we shall see later in §4, the inverses of certain N-matrices of exact order 2 turn out to be Z-matrices. Our results in this paper will therefore have relevance to these areas of applications. In addition, our result on the global univalence of differentiable functions (to be presented elsewhere) may be relevant to problems in the area of factor price equilibrium in international trade. See Inada (1971).

In the following sections we present some characterization theorems for matrices of exact order 1 and 2, and mention a few results for the general exact order k. We shall be mainly studying only the exact order 1 and 2 matrices. A lot of difficulties crop us as we go to the general exact order k matrices. We shall briefly mention about these. Examples of N-matrices of exact order 2 are given in §4.

§2 consists of the definitions and the results that we require subsequently. In §3 we consider the linear complementarity problem with a matrix of exact order 1, and give a complete characterization of this class based on the number of solutions to the LCP(q, A) for every $q \in R^n$. Section 4 elaborates on matrices of exact order 2. Finally, in §5 some results are presented on the already known algorithms that would process the LCP(q, A), when A is a matrix of exact order k. We also have a result on the global univalence of a function whose Jacobian at any point x in the domain of its definition is an N-matrix of exact order 2. This result which generalizes a similar result of Ravindran (1986) for $n = 3$ and builds upon the proofs of the global univalence theorems of Gale-Nikaidio (1965), Inada (1971) and Olech et al. (1989, 1991) will be presented elsewhere.

2. Preliminaries. We call a diagonal matrix S of order n, a *signature matrix* if $s_{ii} = \pm 1$, for all $1 \leqslant i \leqslant n$. We need the following lemma, proved in Mohan and Sridhar (1992).

LEMMA 2.1. *Let $A \in R^{n \times n}$, $n \geqslant 3$ with its principal minors of order 3 or less being negative. Then there is a signature matrix S such that $SAS < 0$.*

Let $\phi \neq J \subseteq \{1, 2, \ldots, n\}$ be defined as $J = \{i: s_{ii} = 1\}$. Then S induces a partition in A which can be written as (if necessary, after a principal rearrangement of its rows and columns)

$$A = \begin{bmatrix} A_{JJ} & A_{J\bar{J}} \\ A_{\bar{J}J} & A_{\bar{J}\bar{J}} \end{bmatrix} \tag{2.1}$$

with $A_{JJ} < 0$, $A_{\bar{J}\bar{J}} < 0$ and $A_{J\bar{J}}$, $A_{\bar{J}J} > 0$. Thus if $n \geqslant k + 3$ and A is an N-matrix of exact order k, then A has the sign pattern given in (2.1) where we can assume without loss of generality that $J \neq \phi$. Further, $\bar{J} \neq \phi$, unless $A < 0$.

Sign reversal property of matrices plays a key role in the theory of linear complementarity. We say that a matrix A *reverses* the sign of a vector $x \in R^n$, if $x_i(Ax)_i \leqslant 0$, for all $1 \leqslant i \leqslant n$. For P- and N-matrices we have the following theorems already known, on their sign reversal nature.

THEOREM 2.1 (GALE AND NIKAIDO [5]). *Let $A \in R^{n \times n}$. A is a P-matrix if and only if A does not reverse the sign of any nonzero vector $x \in R^n$.*

THEOREM 2.2 (SEE MOHAN AND SRIDHAR 1992, PARTHASARATHY AND RAVINDRAN (1990). *Let $A \in R^{n \times n}$ have the partitioned form as in (2.1), for some $\phi \neq J \subseteq \{1, 2, \ldots, n\}$. A is an N-matrix if and only if A reverses the sign of only those vectors of the form $x_J \geqslant 0$, $x_{\bar{J}} \leqslant 0$ or $x_{\bar{J}} \geqslant 0$, $x_J \leqslant 0$. (If $\bar{J} = \phi$, then $A < 0$ and is an N-matrix if and only if A reverses the sign only of the unsigned vectors.)*

We require the following notions from the theory of linear complementarity.

Let $D(A) = \{q \in R^n$: LCP(q, A) has a solution). We say that A is a Q-matrix if $D(A) = R^n$. It is well-known that the class of P-matrices (Murty 1972, Samelson, et al. 1958) and the class of N-matrices with $A \nless 0$ (Saigal 1972) are Q-matrices.

Consider $[I: -A]$. A pair of column vectors $\{I_{.j}, -A_{.j}\}$ from $[I: -A]$ is called a complementary pair. A matrix $B \in R^{n \times k}$, with $B_{.j}$ the jth column of B being either $I_{.j}$ or $-A_{.j}$ for $1 \leqslant j \leqslant n$, is called a complementary matrix of $[I: -A]$. The nonegative cone generated by B is denoted by pos(B). When B is a square matrix of order n, pos(B) is called a complementary cone of $[I: -A]$. Hence if the LCP(q, A) has a solution, for a $q \in R^n$, then there exists a complementary cone pos(B) of $[I: -A]$ such that $q \in$ pos(B). We use the notation $q \in (\text{pos}(B))^c$ to mean that $q \notin$ pos(B). A $q \in R^n$ is said to be nondegenerate with respect to A, if every solution (w, z) of LCP(q, A), has exactly n nonzero coordinates. We call $\bar{A} = B^{-1}\bar{B}$, the principal pivot transform of A with respect to the nonsingular complementary square matrix B of $[I: -A]$, where $\bar{B}$ is the matrix having columns of $[I: -A]$ that are not in B. For more details on principal pivot transforms, see Mohan and Sridhar (1992), Murty (1972).

In the theory of linear complementarity, it has been of special interest to characterize a class of matrices based on the number of solutions the LCP(q, A) has for each $q \in R^n$. We denote by $m(q)$, the number of solutions the LCP(q, A) has for a given $q \in R^n$. P- and N-matrices have the following characterizations in terms of the number of solutions to the LCP(q, A):

THEOREM 2.3 (SAMELSON, THRALL AND WESLER 1958). *A is a P-matrix if and only if $m(q) = 1$ for all $q \in R^n$.*

THEOREM 2.4 (PARTHASARATHY AND RAVINDRAN 1990). *Let $A < 0$. A is an N-matrix of the second category if and only if $m(q) = 2$, for all $q > 0$.*

THEOREM 2.5 (MOHAN AND SRIDHAR 1992). *Let $A \nleq 0$ have the partitioned form as in* (2.1) *for some $J \subseteq \{1, 2, \dots, n\}$. A is an N-matrix of the first category if and only if $m(q) = 1$ for any $q \ngeq 0$ and $m(q) = 3$ for any $q > 0$.*

Consider a submatrix C of order $n \times (n-1)$, of $[I\colon -A]$ which is a complementary matrix. We call pos(C) an $(n-1)$ face of $[I\colon -A]$ if rank(C) $= n-1$. Let $F = \text{pos}(C)$ be an $(n-1)$ face of $[I\colon -A]$. A complementary cone pos(B) is said to be *incident on* F if the columns of C are also columns of B. Thus, for any $(n-1)$-face F, there are exactly two complementary cones incident on it. If we assume that the matrix A is nondegenerate (i.e., none of the principal minors is zero), then it is valid to say that any $n \times (n-1)$ complementary matrix of $[I\colon -A]$ is an $(n-1)$ face of $[I\colon -A]$. Further the subspace generated by such an $(n-1)$-face will be a hyperplane in R^n. We say that the two complementary cones incident on an $(n-1)$-face F are *properly situated* if they lie on opposite sides of the hyperplane generated by F. F is called a *proper* face, if either F lies on the boundary of $D(A)$ or the two complementary cones incident on it are properly situated. The above notions have been introduced by Saigal (1972). He also proves a lemma which we require in the subsequent sections.

LEMMA 2.2. *Let $A \in R^{n \times n}$. Let F be an $(n-1)$-face of $[I\colon -A]$, with the two complementary cones incident on it being* pos(B) *and* pos(B'). *Let* $\det(B) \neq 0$. *F is proper if and only if*

$$\det(B')/\det(B) \leqslant 0.$$

We frequently make use of the notions of value and optimal strategies in a matrix game and the results pertaining to completely mixed games. For these results the reader may refer to Kaplansky (1945) and Olech, Parthasarathy and Ravindran (1991). In what follows we collect some of these results for ready reference. We denote the minimax value of a matrix game with a pay-off matrix A, by $v(A)$. We assume that the column player is the maximizer and the row player, the minimizer. A *mixed strategy* for a player is a probability vector with which the player chooses his rows or columns. We call a mixed strategy x *completely mixed* if $x > 0$ (that is, every coordinate of x is positive). We say that the *game is completely mixed* if every optimal strategy of either player is completely mixed.

THEOREM 2.6 (KAPLANSKY 1945). *Let A denote the pay-off matrix of order $m \times n$, of a two person zero-sum game. We then have the following*:

(1) *If player* 1 *has a completely mixed optimal strategy then any optimal strategy $q = (q_1, q_2, \dots, q_n)$ for player* 2 *satisfies $\Sigma a_{ij} q_j = v(A)$ for all $1 \leqslant i \leqslant m$.*

(2) *If $m = n$ and the game is not completely mixed then both the players have optimal strategies that are not completely mixed.*

(3) *A game with $v(A) = 0$ is completely mixed if and only if* (a) *$m = n$ and the* rank(A) $= n - 1$ *and* (b) *all the cofactors A_{ij} of A are different from* 0 *and have the same sign.*

(4) *Suppose A is a completely mixed game. Then $v(A) = \det(A)/\Sigma\Sigma A_{ij}$ where A_{ij}'s denote the cofactors of A.*

(5) *Let $V = (V_{ij})$ denote the matrix of order $m \times n$ where V_{Ij} is the value of a game whose pay-off matrix is obtained from A by omitting its ith row and jth column. Then the game with payoff matrix A is not completely mixed if and only if the game with pay-off matrix V has a pure saddle point, that is, if and only if there exists a pair (i_0, j_0) such that*

$$v_{i_0 j} \leqslant v_{i_0 j_0} \leqslant v_{i j_0} \quad \forall i, j$$

and $v_{i_0 j_0} = v(A)$.

PROOF. For a proof of the above theorem see Kaplansky (1945).

REMARK 2.1. If the game with pay-off matrix A is completely mixed then the game with pay-off A^t (transpose of A) is also completely mixed and moreover from Theorem 2.6 it is clear that $v(A) = v(A^t)$.

REMARK 2.2. If A is a nonsingular matrix then $v(A)$ and $v(A^{-1})$ keep the same sign. Furthermore if $A^{-1} > 0$ (or $A^{-1} < 0$), then the game with pay-off matrix A is completely mixed. See Olech, Parthasarathy and Ravindran (1991) for more details.

The concept of degree of a map has been used in linear complementarity theory. For degree theory, see Lloyd (1973). For the use of this concept in linear complementarity, we refer to Cottle, Pang and Stone (1992), Howe and Stone (1983), Morris (1990) and Gowda (1991).

Given the LCP(q, A), define a piecewise linear map P_A as,

$$P_A(I_{\cdot j}) = I_{\cdot j}, \qquad P_A(-I_{\cdot j}) = -A_{\cdot j}, 1 \leqslant j \leqslant n,$$

and

$$P_A(x) = \sum_j x_j^+ P_A(I_{\cdot j}) + \sum_j x_j^- P_A(-I_j),$$

for any $x \in R^n$, where $x_i^+ = \max(0, x_i)$ and $x_i^- = -\min(0, x_i)$. Then it is clear that LCP(q, A) is equivalent to finding $x \in R^n$ such that $P_A(x) = q$.

Let $A \in R^{n \times n}$ be a nondegenerate matrix and $x \in R^n$ be a point in the interior of an orthant of R^n. Define the *index* of P_A by ind $P_A(x) = +1$ or -1 depending on whether det $J_A(x)$ is positive or negative, where $J_A(x)$ stands for the Jacobian of the map P_A at x. As shown in Cottle, Pang and Stone (1992) the quantity

$$\sum_{x \in P_A^{-1}(q)} \operatorname{ind} P_A(x)$$

is the same for all nondegenerate $q \in R^n$, is called the *degree* of P_A, and is denoted by deg P_A. As there exists an $x \in R^n$ corresponding to each solution of the LCP(q, A) such that $P_A(x) = q$, we can rewrite the degree of P_A as

$$\deg P_A = \sum_{J:\, q \in \operatorname{pos}(B_J)} \operatorname{sgn} \det(A_{JJ})$$

where $q \in R^n$ is nondegenerate with respect to A and

$$B_J = \begin{bmatrix} -A_{JJ} & 0 \\ -A_{\bar{J}J} & I_{\bar{J}\bar{J}} \end{bmatrix}$$

is a complementary matrix of $[I\colon -A]$. The assumptions of nondegeneracy of the vector $q \in R^n$ with respect to A and of the matrix A made in defining the degree of the map P_A may be relaxed. See Gowda (1991) for more details.

The following result based on the degree of P_A, is well known in linear complementarity.

THEOREM 2.7. *Let $A \in R^{n \times n}$ be such that LCP$(0, A)$ has a unique solution. If* $\deg P_A \neq 0$, *then $A \in Q$.*

We present below a lemma that is useful in proving results on exact order 2, in §4.

LEMMA 2.3. *Let $A \in R^{n \times n}$ be a nondegenerate matrix and $\bar{A}$ be its principal pivot transform of A. If* deg P_A *is r, then* deg $P_{\bar{A}}$ *is* $\pm r$.

For a proof of this see Theorem 6.6.23 of Cottle, Pang and Stone (1992).

3. Matrices of exact order one. At first, we would categorize the matrices of exact order 1 as follows. (As mentioned earlier, these matrices are known as almost N- and almost P-matrices, in the literature.)

DEFINITION 3.1. An N-matrix $A \in R^{n\times n}$ of exact order 1, for $n \geqslant 4$, is said to be of the first category if both A and A^{-1} contain at least one positive entry each; otherwise, it is called an N-matrix of exact order 1 of the second category.

From the definition, it is clear that if $A \in R^{n\times n}$ is an N-matrix of exact order 1, then so is A^{-1}. Also if $A \in R^{n\times n}$ for $n \geqslant 4$ is of the first category then in the representation of A as in (2.1) $J \neq \phi$ and $\bar{J} \neq \phi$.

DEFINITION 3.2. A P-matrix of exact order 1, $A \in R^{n\times n}$, is said to be of the first category, if A^{-1} has a positive entry; otherwise it is said to be of the second category.

The characterization of P-matrices of exact order 1, in terms of the number of solutions LCP(q, A) has for each $q \in R^n$, become easier, due to the fact that a P-matrix of exact order 1 is the inverse of an N-matrix. As one can easily deduce these results from Theorems 2.4 and 2.5, we would not mention them here.

We therefore restrict ourselves to N-matrices of exact order 1 in this section and obtain characterization results for this class of matrices.

REMARK 3.1. In the statements of the theorems and lemmas that follow we assume that the matrix A is a square matrix of order $n \geqslant 4$. There are two reasons for this. (i) In the proofs of the statements we use the sign pattern given by Lemma 2.1 which requires that all the principal minors of order up to 3 be negative. This means that for an N-matrix A of exact order 1, in order to invoke this lemma, A has to be of order at least 4. (ii) Some of the theorems that follow may not hold for 3×3 matrices. For similar reasons, we assume in §4 that the order of the matrix A considered there is at least 5.

To start with we prove a lemma which plays a crucial role in the results that follow.

LEMMA 3.1. *Let $A \in R^{n\times n}$ be an N-matrix of exact order 1 with $n \geqslant 4$. If LCP(q, A) for $q \in R^n$ has two solutions (w^1, z^1) and (w^2, z^2) with $w_i^1 = w_i^2 = 0$, for some $i = 1, 2, \dots, n$ then $q \in$ pos$(-A)$.*

PROOF. Without loss of generality, assume that $i = 1$, i.e., $w_1^1 = w_1^2 = 0$. We consider two cases:

Case (i): $A^{-1} < 0$. Since LCP(q, A) has a solution (w^1, z^1), considering the system

$$(3.1) \qquad z - A^{-1}w = -A^{-1}q,$$

$$z \geqslant 0, \qquad w \geqslant 0, \qquad z^t w = 0,$$

we see that the LCP$(-A^{-1}q, A^{-1})$, has a solution (z^1, w^1). As $A^{-1} < 0$ we note that $-A^{-1}q \geqslant 0$ and hence $q \in$ pos$(-A)$.

Case (ii): A^{-1} has a positive entry. We notice that (z^1, w^1) and (z^2, w^2) are two solutions to (3.1) with $w_1^1 = w_1^2 = 0$. Let $\bar{A}$ be the principal submatrix of A^{-1} got by omitting its first row and first column. Extracting the system

$$(3.2) \qquad \bar{z} - \bar{A}\bar{w} = \bar{q}$$

obtained by dropping the first entry of $(-A^{-1}q)$ in (3.1), we note that all principal minors of $\bar{A}$ are negative and the reduced LCP$(\bar{q}, \bar{A})$ has two solutions. Let them be denoted as $(\bar{w}^1, \bar{z}^1)$ and $(\bar{w}^2, \bar{z}^2)$. As $\bar{A}$ is an N-matrix, this implies that $\bar{q} \geqslant 0$, i.e., $(-A^{-1}q)_i \geqslant 0$, for $i = 2, \dots, n$. See Mohan and Sridhar (1992), Parthasarathy and Ravindran (1990).

If $(-A^{-1}q)_1 \geqslant 0$, then $q \in \operatorname{pos}(-A)$ and Lemma 3.1 follows. On the contrary let $(-A^{-1}q)_1 < 0$. From (3.1) we note that

$$z_1^1 - \sum_{j=1}^{n} a^{1j} w_j^1 = z_1^1 - \sum_{j=2}^{n} a^{1j} w_j^1 = (-A^{-1}q)_1 < 0. \tag{3.3}$$

Similarly,

$$z_1^2 - \sum_{j=2}^{n} a^{1j} w_j^2 = (-A^{-1}q)_1 < 0.$$

It follows from (3.3) that there exist indices j and k, $1 \leqslant j,\ k \leqslant n$ such that,

$$w_j^1 > 0, \qquad a^{1j} > 0 \tag{3.4}$$

and

$$w_k^2 > 0, \qquad a^{1k} > 0.$$

If in (3.4), $j = k$, then by complementarity, $z_j^1 = z_j^2 = 0$ and this violates Lemma 3.1 of Kojima and Saigal (1979) when applied to LCP$(\bar{q}, \bar{A})$. Hence $j \neq k$ and the following must hold:

$$w_j^1 > 0,\ z_j^1 = 0; \qquad w_j^2 = 0, \qquad z_j^2 > 0 \tag{3.5}$$

$$w_k^1 > 0,\ z_k^1 = 0; \qquad w_k^2 = 0,\ z_k^2 > 0.$$

Note that $\bar{A}$ reverses the sign of $(\bar{w}^1 - \bar{w}^2)$ which is not unisigned, in view of (3.5). Hence by Theorem 2 of Olech, Parthasarathy and Ravindran (1991) $\bar{A}$ is not an N-matrix of the second category, and $\bar{A}$ should have at least one positive entry. By Lemma 2.1, there is a $\phi \neq J \subset \{1, \ldots, n\}$ such that $(\bar{w}^1 - \bar{w}^2)_J \leqslant 0$ and $(\bar{w}^1 - \bar{w}^2)_{\bar{J}} \geqslant 0$. Further, $j \in J$ and $k \in \bar{J}$, by (3.5). Hence we have

$$a^{ij} a^{ik} < 0 \quad \text{for all} \quad 2 \leqslant i \leqslant n.$$

The partitioned form of A^{-1} is as given in (2.1),

$$A^{-1} = \begin{bmatrix} A^{LL} & A^{L\bar{L}} \\ A^{\bar{L}L} & A^{\bar{L}\bar{L}} \end{bmatrix}$$

where either $L = J \cup \{1\}$, $\bar{L} = \bar{J}$ or $L = J$, $\bar{L} = \bar{J} \cup \{1\}$. In either case using Lemma 3.2 of Mohan and Sridhar (1992) we get $a^{1j} a^{1k} < 0$, a contradiction to (3.4). Hence $q \in \operatorname{pos}(-A)$ and the proof is complete. □

The following theorems present the number of solutions LCP(q, A) has, for each $q \in R^n$, when A is an N-matrix of exact order 1.

THEOREM 3.1. *Let $A < 0$, $A \in R^{n \times n}$ be an N-matrix of exact order* 1 *of the second category. Then*
(i) $m(q) = 4$ *for* $q \in \operatorname{int}[\operatorname{pos}(-A)]$;
(ii) $m(q) = 2$ *for* $q > 0$, $q \notin \operatorname{pos}(-A)$.

PROOF. Since $A < 0$, we have $D(A) = R_+^n$ and it is sufficient to show that if $m(q) > 2$, then $q \in \operatorname{pos}(-A)$, and $m(q) = 4$. Observe that LCP(q, A) has at least two solutions for $q > 0$, due to a result on parity of solutions by Murty (1972). Suppose for some $q > 0$, there exist two solutions for LCP(q, A), other than the trivial solution, $w = q$, $z = 0$. Let (w^1, z^1) and (w^2, z^2) be two solutions such that $z^1 \neq z^2 \neq 0$. Note that A reverses the sign of the nonzero vector $(z^1 - z^2)$. Now by Theorem 3.1 of Olech, Parthasarathy and Ravindran (1991) either $(z^1 - z^2)$ is unisigned or there is a signature matrix $S_0 \neq \pm I$, such that $S_0(z^1 - z^2)$ is unisigned and $S_0 A^{-1} S_0 < 0$.

Suppose $(z^1 - z^2)$ is unisigned. Without loss of generality, we can assume that $(z^1 - z^2) \geqslant 0$. Since $z^2 \neq 0$, there exists an index i, $1 \leqslant i \leqslant n$, such that $z_i^2 > 0$ and $z_i^1 > 0$. Hence $w_i^1 = w_i^2 = 0$, and by applying Lemma 3.1, we have $q \in \operatorname{pos}(-A)$. If $S_0(z^1 - z^2)$ were unisigned, we have $S_0 A^{-1} S_0 < 0$ and let the partition of A^{-1} induced by S_0 be

$$A^{-1} = \begin{bmatrix} A^{JJ} & A^{J\bar{J}} \\ A^{\bar{J}J} & A^{\bar{J}\bar{J}} \end{bmatrix}.$$

Further, we have $(z^1 - z^2)_J \geqslant 0$ and $(z^1 - z^2)_{\bar{J}} \leqslant 0$. If there is an index i such that $z_i^1 > 0$ and $z_i^2 > 0$, $1 \leqslant i \leqslant n$, we are done as in the previous paragraph. Otherwise, we have

$$z_J^1 \geqslant 0; \qquad z_J^2 = 0,$$

$$z_{\bar{J}}^1 = 0; \qquad z_{\bar{J}}^2 \geqslant 0.$$

In this case we notice that

$$-A^{\bar{J}\bar{J}} w_{\bar{J}}^1 = (-A^{-1}q)_{\bar{J}} \geqslant 0, \qquad -A^{JJ} w_J^2 = (-A^{-1}q)_J \geqslant 0,$$

and hence $-A^{-1}q \geqslant 0$ or in other words, $q \in \operatorname{pos}(-A)$. Hence if LCP$(q, A)$ for $q > 0$ has more than two solutions, then $q \in \operatorname{pos}(-A)$.

To complete the proof, we note from Lemma 3.2 of Olech, Parthasarathy and Ravindran (1991) that LCP(q, A^{-1}) has exactly 4 solutions if $q > 0$. As there is a one-to-one correspondence between the complementary cones of $[I\colon -A]$ and that of $[I\colon -A^{-1}]$ (see Saigal 1972), it follows that $m(q) = 4$, if $q \in \operatorname{int}[\operatorname{pos}(-A)]$. The proof follows. □

As has been pointed out by one of the referees, if we use degree theory, this theorem follows immediately. This is so because as $A < 0$, $\deg P_A = 0$ and the only complementary bases of positive index are I and $-A$.

REMARK 3.2. From Theorem 3.1 and Lemma 3.2 of Olech, Parthasarathy and Ravindran (1991) it follows that if $A \ngtr 0$, $A \in R^{n\times n}$ is an N-matrix of exact order 1 of the second category, then $m(q) = 4$ for $q > 0$ and $m(q) = 2$, for $q \in \operatorname{int}[D(A) \cap \operatorname{pos}(I)^C]$.

REMARK 3.3. For $A \nleqslant 0$, $A \in R^{n\times n}$, $n \geqslant 4$, an N-matrix of exact order 1 of the second category, one can easily check that LCP(q, A) has exactly 1 or 2 solutions, for $q \geqslant 0$, with $q_i = 0$, for at least one i, $1 \leqslant i \leqslant n$, depending on the sign pattern of A. The proof is similar to the one given in Theorem 3.1 of Mohan and Sridhar (1992).

THEOREM 3.2. *$A \in R^{n\times n}$ be an N-matrix of exact order 1 of the first category with $n \geqslant 4$. Then*

(i) $m(q) = 3$ *if* $q \in \operatorname{int}[\operatorname{pos}(-A)]$ *or* $\operatorname{int}[\operatorname{pos}(I)]$;

(ii) $m(q) = 1$ *if* $q \in [\operatorname{pos}(I) \cup \operatorname{pos}(-A)]^c$.

PROOF. It is clear from Lemma 3.3 of Olech, Parthasarthy and Ravindran (1991) that $m(q) = 3$, if $q > 0$. As A^{-1} is also an N-matrix of exact order 1 of the first category, it follows that if $q \in \text{int}[\text{pos}(-A)]$, then $m(q) = 3$.

Since we know that A is a Q-matrix using Lemma 3.3 of Olech, Parthasarathy and Ravindran (1991), we need to prove that if $m(q) > 1$, then either $q \in \text{pos}(-A)$ or $q \in \text{pos}(I)$.

Suppose that LCP(q, A) has two distinct solutions (w^1, z^1) and (w^2, z^2) with $z^1 \neq z^2$. Then A reverses the sign of the vector $x = (z^1 - z^2)$. This implies that there exist signature matrices S_0, S_1, such that either S_0x is unisigned or S_1x is unisigned, where $S_0, S_1 \neq \pm I$ and $S_0AS_0 < 0$, $S_1A^{-1}S_1 < 0$.

Suppose S_0x is unisigned. Since $S_0AS_0 < 0$, the signature matrix S_0 induces a partition as in Lemma 2.1 for some $\phi \neq J \subset \{1, 2, \ldots, n\}$ with $\bar{J} \neq \phi$ and we have $(z^1 - z^2)_J \geqslant 0$ $(z^1 - z^2)_{\bar{J}} \leqslant 0$. If there is an index i such that $z_i^1 \geqslant z_i^2 > 0$ or $z_i^2 \geqslant z_i^1 > 0$ for $1 \leqslant i \leqslant n$, then as in the proof of Theorem 3.1 it follows that $q \in \text{pos}(-A)$. Otherwise, we have

$$z_J^1 \geqslant 0; \qquad z_J^2 = 0,$$

$$z_{\bar{J}}^1 = 0; \qquad z_{\bar{J}}^2 \geqslant 0.$$

From the sign pattern of the partitioned form as in Lemma 2.1, it follows that $q_J \geqslant 0$ and $q_{\bar{J}} \geqslant 0$. Hence $q \in \text{pos}(I)$. We can proceed similarly in the case of S_1x being unisigned, where the signature matrix S_i is such that $S_1A^{-1}S_1 < 0$, using A^{-1}. Then we would arrive at the fact that $q \in \text{pos}(-A)$. Hence, the theorem follows. □

REMARK 3.4. The above theorem does not hold for $n = 3$. The following example due to Olech, Parthasarathy and Ravindran (1991) illustrates this.

EXAMPLE 3.1. Let

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & -3 & -2 \\ -4 & -5 & -3 \end{bmatrix}.$$

It is easy to see that this is an N-matrix of exact order 1 of the first category. But $A \notin Q$. If we take $q^t = (1, 1, -10)$, the LCP(q, A) does not have a solution.

One of the referees has pointed out to us that the following well-known example of a 3×3 matrix due to Murty (1972) is also an N-matrix of exact order 1 of the first category and is a Q-matrix with an even number of solutions to each nondegenerate q, and thus a counterexample to the above theorem for $n = 3$.

EXAMPLE 3.2. Let

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

The following theorems establish the converse of Theorems 3.1 and 3.2, respectively.

THEOREM 3.3. *Let $A < 0$ be a square matrix of order $n \geqslant 4$. Suppose $m(q)$ satisfies the following conditions*:

(i) $m(q) < \infty$ *for all* $q \in R^n$;

(ii) $m(q) > 2$ *if* $q \in \text{int}(\text{pos}(-A))$],

(iii) $m(q) \leqslant 2$, *if* $q \notin \text{pos}(-A)$.

Then A is an N-matrix of exact order 1 of the second category.

PROOF. Since $m(q) < \infty$, for all $q \in R^n$, none of the principal minors of A is zero. We shall show that if F is an $(n-1)$-face of $[I\colon -A]$, which is not a face of pos(I) or pos$(-A)$, then F is proper. Let F be an $(n-1)$-face generated by k columns of I and $(n-k)$ columns of $-A$, $1 < k < (n-1)$, such that the two complementary cones pos(B) and pos(B^1) incident on it are not properly situated.

Without loss of generality, let us assume that

$$F = \mathrm{pos}(I_1, \ldots, I_{\cdot k}, \quad -A_{k+2}, \ldots, -A_{\cdot n})$$

with $\{I_{\cdot k+1}, -A_{k+1}\}$ as the left out complementary pair. Since $A < 0$, it is easy to see that the vector

$$q = \sum_{r=1}^{k} I_{\cdot r} + \delta \sum_{s=k+2}^{n} (-A_{\cdot s}) \in F$$

is not contained in pos$(-A)$ for δ sufficiently small and $q > 0$. As F is not proper, there exists an $\epsilon > 0$ such that

$$q + \epsilon(I_{\cdot k+1}) \notin \mathrm{pos}(-A) \quad \text{and}$$

$$q + \epsilon(I_{\cdot k+1}) \in \mathrm{pos}(B) \cap \mathrm{pos}(B^1).$$

We then find the LCP$(q + \epsilon(I_{\cdot k+1}), A)$ has at least 3 solutions, contrary to our hypothesis. Hence our claim that F, which is not a face of pos(I) or pos$(-A)$, is proper, follows.

Now by Lemma 2.5 of Mohan and Sridhar (1992) it follows that all the proper principal minors of A have the same sign. Since $A < 0$, all the proper principal minors are negative. If $\det(A) < 0$, A will be an N-matrix. This however contradicts Theorem 2.4.

Therefore, $\det(A) > 0$ and A is an N-matrix of exact order 1. Since $A < 0$, it follows that A is an N-matrix of exact order 1 of the second category. □

THEOREM 3.4. *Suppose $A \in R^{n \times n}$, with $n \geqslant 4$ and suppose we have the following*:
 (i) $m(q) < \infty$ *for all* $q \in R^n$;
 (ii) $m(q) = 2$ *or* 0 *if* $q \notin R^n_+$;
 (iii) $m(q) > 2$ *for all* $q > 0$.
Also suppose that pos$(I) \subseteq$ int[pos$(-A)$]. *Then A is an N-matrix of exact order* 1 *of the second category.*

PROOF. Since $m(q) < \infty$, for all $q \in R^n$, none of the principal minors of A is zero. Notice that as pos$(I) \subseteq$ int[pos$(-A)$], pos$(-A^{-1}) \subseteq$ int[pos(I)]. Therefore $A^{-1} < 0$ and A^{-1} satisfies all the conditions of Theorem 3.3 and the conclusion follows. □

THEOREM 3.5. *Let $A \in R^{n \times n}$ be such that each of the columns of A as well as A^{-1} contains a positive entry and suppose that A satisfies the following*:
 (i) $m(q) < \infty$ *for all* $q \in R^n$;
 (ii) $m(q) = 1$, *if* $q \notin$ [pos$(I) \cup$ pos$(-A)$];
 (iii) $m(q) > 1$ *for* $q \in$ int[pos(I)] *or* $q \in$ int[pos$(-A)$].
Then A is an N-matrix of exact order 1 *of the first category.*

PROOF. For $n = 1$ or $n = 2$, it is easy to verify that there are no matrices satisfying all the hypotheses of the theorem and hence it holds vacuously. In what follows we shall assume that $n \geqslant 3$.

Since $m(q) < \infty$ for all $q \in R^n$, it follows that none of the principal minors of A is zero. We now claim that if F is an $(n-1)$-face of $[I: -A]$, which is not a face of pos(I) or pos($-A$), then F is proper. Suppose this is not true. Then there exists a k, $1 < k < (n-1)$, and an $(n-1)$-face F, generated by k columns of I and $(n-k-1)$ columns of $-A$, such that the two complementary cones that are incident on F lie on the same side of it. Without loss of generality let us assume that

$$F = \text{pos}(I_{\cdot 1}, \ldots, I_{\cdot k}, -A_{\cdot k+2}, \ldots, -A_{\cdot n}).$$

We now construct a $q \in R^n$, contained in the relative interior of F which is neither in pos(I) nor in pos($-A$). First note that, for any j, $1 \leqslant j \leqslant n$, $I_{\cdot j} \notin \text{pos}(-A)$, for otherwise $-A^{\cdot j} \in \text{pos}(I)$, contradicting our hypothesis about A^{-1}. By hypothesis, $A_{\cdot k+2}$ contains a positive entry, i.e., $\exists$ an index r, such that $a_{r(k+2)} > 0$. Consider for $\lambda > 0$, $s \neq r$, $\tilde{q} = -A_{\cdot k+2} + \lambda I_{\cdot s} \ngeq 0$, where $1 \leqslant s \leqslant k$, and $s \neq r$. Hence $\tilde{q} \notin \text{pos}(I)$.

Also, as $I_{\cdot s} \notin \text{pos}(-A)$, there exists a $\lambda_0 > 0$ such that

$$-A_{\cdot k+2} + \lambda_0 I_{\cdot s} \notin \text{pos}(-A).$$

Let $q = -A_{\cdot k+2} + \lambda_0 I_{\cdot s} + \delta\sum_{j=1}^{k} I_{\cdot j} + \delta\sum_{j=k+3}^{n}(-A_{\cdot j}) \in F$. For sufficiently small δ, q is neither contained in pos(I) nor in pos($-A$). Now, we note that there exists an $\epsilon > 0$ such that LCP($q + \epsilon(-A_{\cdot k+1}), A$) has at least two solutions and $q + \epsilon(-A_{\cdot k+1}) \notin \text{pos}(I)$ or pos($-A$), which contradicts our hypothesis.

The rest of the proof follows in similar lines as that of Theorem 3.3. □

4. Matrices of exact order two. In this section, we study the properties of matrices of exact order 2. Characterizing these matrices, regarding their Q-nature, forms the main result of this section.

We start with some examples.

EXAMPLE 4.1. Consider the matrix

$$A = \begin{bmatrix} -.9 & -2 & -2 & 2 & -2 \\ -1 & -.9 & -3 & 3 & -1 \\ -1 & -3 & -.9 & 3 & -1 \\ 1 & 3 & 3 & -.9 & 1 \\ -2 & -2 & -2 & 2 & -.9 \end{bmatrix}.$$

One can directly verify that every principal minor of order 1, 2 or 3 of A is negative and principal minors of order 4 and the determinant of A are positive. Hence A is an N-matrix of exact order 2.

EXAMPLE 4.2.

$$B = \begin{bmatrix} 1 & 2 & 0 & 1.453378 & 0 \\ 1 & 4.989 & 1 & 1.368317 & 1 \\ 0 & 2 & 1.2 & 1.168878 & 0 \\ 1 & 2.6 & 1 & 2.842317 & 1.41 \\ 0 & 2 & 0 & 1.168878 & 1.2 \end{bmatrix}.$$

As before, by looking at the principal minors of B, one can see that B is a P-matrix of exact order 2.

The following notation is followed in this section only. Given a square matrix $A \in R^{n \times n}$, $B_i \in R^{(n-1)\times(n-1)}$, $1 \leqslant i \leqslant n$, will denote the principal submatrix of A, got by deleting the ith row and the ith column of A. By v_{ij}, $1 \leqslant i, j \leqslant n$, we mean the value of the game with the pay-off matrix as the submatrix of A obtained by deleting its ith row and jth column. In the statements of many theorems and lemmas that follow in this section we may require that A be of order at least 5 for the reason stated in Remark 3.1. This is specified in the statements of the relevant theorems and lemmas.

We notice that if A is of exact order k, then B_i, $1 \leqslant i \leqslant n$, are matrices of exact order $(k-1)$.

Depending on the categories of the exact order 1 matrices in A, we classify a matrix of exact order 2, into three categories.

DEFINITION 4.1. We say that a matrix A ($A \nless 0$) of exact order 2, is of *the first category* if there exists at most one index k, $1 \leqslant k \leqslant n$ such that the $(n-1) \times (n-1)$ exact order 1 principal submatrix B_k is nonpositive and every $(n-1) \times (n-1)$ principal submatrix B_i which is $\nless 0$, $1 \leqslant i \leqslant n$, is exact order 1 of the first category. We say that it is of the *second category*, if all B_i's are of the second category. It is of *the third category*, if there are indices, $i, j \in \{1, \ldots, n\}$, such that B_i is of the first category and $B_j \nless 0$ is of the second category. (It must be noted here that if there are more than one index i for which $B_i < 0$, then $A < 0$ and by definition A is of the second category.)

The matrix given in Example 4.1 is an N-matrix of the exact order 2 of the third category. In Example 4.2 we have a P-matrix of exact order 2 which is of the first category.

We observe the following about the principal minors of A^{-1} if A is of exact order 2.

LEMMA 4.1. *If A is an $N(P)$-matrix of exact order 2, then A^{-1} has diagonal entries positive, all proper principal minors of order $\geqslant 2$ negative and the* $\det(A) > 0$ (< 0).

In general we observe the following relationship between the classes of P of exact order r and N of exact order r. We omit the proof as it is simple.

LEMMA 4.2. *Let $A \in R^{n \times n}$, $n > r$, for some positive integer r.*

(i) *Let A be an N-matrix of exact order r and D be a principal submatrix of A^{-1} of order k, $n > k \geqslant r+1$. Then D^{-1} is a P-matrix of exact order r.*

(ii) *Let A be a P-matrix of exact order r and D be a principal submatrix of A^{-1} of order k, $n > k \geqslant r+1$. Then D^{-1} is a P-matrix of exact order r.*

We now prove some game theoretic results for these classes of matrices.

LEMMA 4.3. *Let $A \in R^{n \times n}$ be a matrix of exact order 2, where $n \geqslant 5$. Then $v(A) \neq 0$.*

PROOF. Suppose $v(A) = 0$. Then there exists a probability vector y, such that $y^t A \leqslant 0$. If $y > 0$ from Theorem 2.6 it follows that there is a probability vector z such that $Az = 0$, which contradicts the hypothesis about A. Therefore assume that $y = (y_1, y_2, \ldots, y_{n-1}, 0)^t$. If $B_n < 0$ (which may occur, only if A is an N-matrix of exact order 2) from the fact that $y^t A \leqslant 0$ and the sign pattern of A, it follows that $A < 0$, contradicting our assumption that $v(A) = 0$.

Now, it follows from Lemma 3.1 of Olech, Parthasarathy and Ravindran (1991) (which holds true for P-matrices of exact order 1, also) that $\bar{y} = (y_1, y_2, \ldots, y_{n-1})^t > 0$. Thus $(Az)^t = (0, \ldots, 0, \alpha)$ for any optimal strategy z of the maximizer, where α is a positive scalar.

This implies

$$z = A^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha \end{bmatrix} = \alpha A^{\cdot n}$$

where $A^{\cdot n}$ denotes the nth column of A^{-1}. By Lemma 4.1, since every 2 by 2 principal minor is negative, it follows that $A^{\cdot n} > 0$ as $\alpha \geqslant 0$. Thus $v(A^{-1}) > 0$. From Remark 2.2, it follows that $v(A) > 0$, contradicting our assumption. This completes the proof. □

Lemma 4.3 can be restated as a theorem of the alternatives as follows.

THEOREM 4.1. *Let $A \in R^{n \times n}$ be a matrix of exact order 2, with $n \geqslant 5$. Then exactly one of the following holds.*

(i) *There exists a $y > 0$ such that $y^t A < 0$.*

(ii) *There exists an $x > 0$ such that $Ax > 0$.*

It is well-known that $v(A)$ and $v(A^t)$ keep the same sign whenever A is a matrix of exact order 1 or 0. This result, for matrices of exact order 2, is proved next, in Theorem 4.2. We first prove two lemmas.

LEMMA 4.4. *Let A be a matrix of exact order 2. If all B_i, $1 \leqslant i \leqslant n$, are of the same category, then $v(A)$ and $v(A^t)$ have the same sign.*

PROOF. If the matrix game A is completely mixed, then it is known from our Remark 2.1 (see Kaplansky 1945) that $v(A) = v(A^t)$. Otherwise, from Theorem 2.6 (Kaplansky 1945) it follows that there exist indices $i_0, j_0, \in \{1, \ldots, n\}$, such that

$$v_{i_0 j} \leqslant v(A) \leqslant v_{i j_0} \quad \text{for all} \quad 1 \leqslant i, j \leqslant n. \tag{4.1}$$

Similarly, there exist $i_1, j_1 \in \{1, \ldots, n\}$, such that

$$v^t_{i_1 j} \leqslant v(A)^t \leqslant v^t_{i j_1} \quad \text{for all} \quad 1 \leqslant i, j \leqslant n, \tag{4.2}$$

where by v^t_{ij} we mean the value of the subgame whose pay-off matrix is obtained from A^t by deleting its ith row and the jth column.

Suppose $v(A) > 0$ and $v(A^t) < 0$. From (4.1), (4.2) and Lemma 3.1 of Olech, Parthasarathy and Ravindran (1991) (which holds for a P-matrix of exact order 1 also), we conclude that B_{j_0} is of the first category and B_{i_1} is of the second category contrary to our hypothesis. This concludes the proof. □

Now we prove our desired theorem.

THEOREM 4.2. *Let A be a matrix of exact order 2. Then $v(A)$ and $v(A^t)$ have the same sign.*

PROOF. If A is either of the first category, or of the second category, the theorem follows from Lemma 4.3.

So, let A be an exact order 2 matrix of the third category. Then there exists an $i \in \{1, \ldots, n\}$ such that $B_i \not< 0$ is of the second category. Assume, without loss of generality, that $B_1 \not< 0$. Let us also assume that $v(A) < 0$.

By Lemma 4.1, A^{-1} has no zero entry, with the diagonal entries being positive. Suppose the first row of A^{-1}, i.e., $A^{1\cdot} > 0$. Then as the 2×2 principal minors of A^{-1} are negative, it follows that $A^{\cdot 1} > 0$, and hence $v(A^{-1}) > 0$.

This contradicts our assumption that $v(A) < 0$ by Remark 2.2. Hence $A^{1\cdot}$ contains a negative entry. Thus, there is a $k \in \{2, \dots, n\}$ such that $a^{1k} < 0$.

Define the vector $w \in R^{n-1}$ by taking

$$w_i = 0, \quad \text{if } i \neq k;$$

$$w_k = -1.$$

Let $y = B_1^{-1}w$. As $B_1^{-1} < 0$, $y > 0$.

By taking $u = \begin{bmatrix} 0 \\ y \end{bmatrix}$ as an n-vector, we have $Au = \begin{bmatrix} x \\ w \end{bmatrix}$ where x is a real number. Hence $A^{-1}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$. Now x can be determined from the equation

$$a^{11}x + a^{1k}(-1) = 0.$$

Thus $x < 0$. We have $Au = \begin{bmatrix} x \\ w \end{bmatrix} \leqslant 0$, or $u^tA^t \leqslant 0$.

This with Lemma 4.3 implies that we have $v(A^t) < 0$. Similarly, one can prove the theorem when $v(A) > 0$.

This completes the proof of the theorem. □

We now present some results on the linear complementarity problem with matrices of exact order 2. To start with we prove a general theorem.

THEOREM 4.3. *Let $A \in R^{n \times n}$, with $n \geqslant 2$. Let $v(B_i) < 0$, for $1 \leqslant i \leqslant n$. Then the following are equivalent.*

(i) $v(A) > 0$;

(ii) $A \in Q$;

(iii) *A is nonsingular and $A^{-1} > 0$.*

PROOF. (i) ⇒ (ii): We shall show that for a $q \in R^n$, LCP(q, A) has a unique solution ($w = 0$; $z > 0$), which is also nondegenerate. Let $v = v(A) > 0$, and $v_i = v(B_i)$, for $1 \leqslant i \leqslant n$.

Consider the vector $q = -ve$, where e is the n-vector of 1's. Since $v > 0$, and $v_i < 0$, for all $1 \leqslant i \leqslant n$, it follows that the game is completely mixed. See Theorem 2.6. Hence, there exists a $z > 0$, $z \in R^n$ such that $Az = ve$, or $(0, z)$ solves LCP(q, A).

Suppose (w^1, z^1) is another solution to LCP(q, A). We then have the following equations:

$$Az + q = 0 \tag{4.3}$$

and

$$Az' + q = w'. \tag{4.4}$$

If $w' = 0$, then $z = z'$, since the game is completely mixed. So assume that $w' \neq 0$. Suppose the first coordinate $w_1' > 0$. Then $z_1' = 0$. We note that

$$B_1\bar{z}' + \bar{q} = \bar{w}' \tag{4.5}$$

where for any $x \in R^n$, $\bar{x}$ denotes an $(n-1)$-vector obtained from x by omitting its first coordinate.

Equation (4.5) implies that $v(B_1) = v_1 > 0$, contradicting our hypothesis. Thus $(0, z)$ is the only solution to the LCP(q, A). Similarly, we can show that $(0, 0)$ is the only solution to LCP$(0, A)$.

It now follows from Corollary 3.2 of Saigal (1972), that A is a Q-matrix.

(ii) $\Rightarrow$ (iii): Take $q = -e_i$, for $i \in \{1, \ldots, n\}$, where e_i is the ith column vector of I. Since $A \in Q$, LCP(q, A) should have a solution.

We can observe, as in the previous case, that LCP(q, A) has a unique solution, $(0, z)$, with $z > 0$.

In other words, $-e_i \in \text{pos}(-A)$, for $1 \leqslant i \leqslant n$.

This implies, $\text{pos}(-I) \subset \text{int}[\text{pos}(-A)]$, and hence $A^{-1} > 0$.

(iii) $\Rightarrow$ (i): This follows from Lemma 2.1 of Olech, Parthasarathy and Ravindran (1991). □

As a consequence of Theorems 4.2 and 4.3, for the first category matrices we have the following:

COROLLARY 4.1. *Let A be a matrix of exact order* 2. *If it is of the first category, then $v(A)$ and $v(A^t)$ are positive.*

REMARK 4.1. If $v(A) > 0$ and A is a second category matrix of exact order 2, it follows from Theorem 4.3 that $A^{-1} > 0$ and consequently LCP(q, A) will have an odd number of solutions for any nondegenerate q. This result follows from a theorem of Murty on the constant parity property. See Murty (1972).

Our next theorem, however, demonstrates that the value of a second category matrix is always negative.

THEOREM 4.4. *Let $A \in R^{n \times n}$ be a matrix of exact order* 2, *with $n \geqslant 5$. If A is of the second category, then $v(A) < 0$.*

PROOF. Let A be matrix of exact order 2. Let us assume that A has at least one positive entry, as otherwise there is nothing to prove.

Suppose on the contrary, $v(A) > 0$. By Theorem 4.3 we note that $A^{-1} > 0$. So, for $q > 0$, LCP(q, A^{-1}) has a unique solution and hence, by Lemma 2.3, $\deg P_{A^{-1}} = 1$.

We first consider the case $B_i \nless 0$, $\forall\, i = 1, \ldots, n$. (If A is a P-matrix of exact order 2 then there is no index i, $1 \leqslant i \leqslant n$, such that $B_i < 0$. If A is an N-matrix of exact order 2 then there is at most one i for which $B_i < 0$.)

Since $B_1 \nless 0$ is of the second category, it follows that $B_1^{-1} < 0$. Consider any $q \in R^{n \times n}$, $q > 0$. Let $q^t = (q_1, \bar{q}^t)$ where $\bar{q} \in R^{n-1}$. Let $\bar{y} = -B_1^{-1}(\bar{q})$. Note that $\bar{y} > 0$. Now consider

$$A\begin{bmatrix} 0 \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \Sigma a_{1j}\bar{y}_j \\ -\bar{q} \end{bmatrix}.$$

We now claim that $\Sigma a_{1j}\bar{y}_j > 0$. For otherwise, $v(A^t) < 0$, which contradicts our hypothesis that $v(A) > 0$, in view of Theorem 4.2. Thus we obtain a solution to LCP(q, A) by taking

$$w_1 = \sum a_{1j}\bar{y}_j + q_1, \qquad z_1 = 0,$$

$$w_i = 0, \qquad z_i = \bar{y}_i, 2 \leqslant i \leqslant n.$$

Thus in a similar manner, with each B_i, $B_i \nless 0$, we obtain a distinct solution to LCP(q, A). In calculating the degree of P_A we consider two cases, based on A being a P- or an N-matrix of exact order 2.

Case (i): A is a P-matrix of exact order 2 of the second category. From the previous paragraph, we notice that for a $q \in R^n$, $q > 0$, LCP(q, A) has $n + 1$ solutions and all of them are nondegenerate. It is clear that LCP(q, A) has no other solution, as every other principal submatrix of A is a P-matrix. Using such a $q > 0$ nondegenerate with respect to A, we calculate the degree of A as

$$\deg P_A = \sum_{J:\, q \in \operatorname{pos}(B_J)} \operatorname{sgn}\det(A_{JJ})$$

$$= \sum_{i=1}^{n} \operatorname{sgn}\det B_i + 1 = -n + 1,$$

which implies that $\deg P_A$ is at most -4 for $n \geqslant 5$. As $\deg P_{A^{-1}} = 1$, this contradicts Lemma 2.3. Hence $v(A) \leqslant 0$; and using Lemma 4.3, we conclude that $v(A) < 0$.

Case (ii): Suppose now that A is an N-matrix of exact order 2. By Lemma 2.1 there is a $\phi \neq J \subset \{1, 2, \ldots, n\}$ such that $\bar{J} \neq \phi$ and A has the partition $A = \begin{bmatrix} A_{JJ} & A_{\bar{J}J} \\ A_{J\bar{J}} & A_{\bar{J}\bar{J}} \end{bmatrix}$ where A_{JJ} and $A_{\bar{J}\bar{J}}$ are negative and the other two blocks are positive.

Clearly, for $q > 0$, LCP(q, A) has $n + 1$ distinct solutions, viz., the trivial solution and as noted earlier in the proof, n other distinct solutions each one produced using a principal submatrix B_i, $B_i \nless 0$, $i = 1, \ldots, n$ of exact order 1 of the second category. We note that (from the known results of N-matrices of exact orders 0 and 1), along with these $n + 1$ solutions, for the LCP(q, A) there are exactly two more solutions, whose complementary matrices have principal subdeterminants negative. Hence by choosing a $q > 0$ nondegenerate with respect to A, we notice that

$$\deg P_A = \sum_{i=1}^{n} \operatorname{sgn}\det B_i + 1 - 2 = n - 1.$$

As before, $\deg P_{A^{-1}} = n - 1$, which is not equal to 1 for $n \geqslant 5$ and this leads to a contradiction. Hence $v(A) < 0$.

Suppose now there exists a $B_i < 0$ for some $i \in \{1, \ldots, n\}$. This occurs only when A is an N-matrix of exact order 2. Let us choose a vector $\bar{q} \in R^{n-1}$, $\bar{q} > 0$ nondegenerate with respect to B_i such that $\bar{q} \notin \operatorname{pos}(-B_i)$. Then from Theorem 3.1, LCP$(\bar{q}, B_i)$ has exactly two solutions which can be extended to the problem LCP(q, A) where q is chosen to be $q_i > 0$ and $q_j = \bar{q}_j$, $\forall j \neq i$. Now, we can proceed as in case (ii), concluding that LCP(q, A) has exactly n more solutions, viz., the trivial solution and a distinct solution produced using each principal submatrix B_j, $j \neq i$, $j = 1, \ldots, n$. Thus, we can see that the degree of P_A is $n - 2$ which leads to a contradiction when $n \geqslant 5$, for $\deg P_{A^{-1}}$ is 1. This completes the proof. □

We now have a result on the Q-nature of the exact order 2 matrices of the first category.

THEOREM 4.5. *Let $A \in R^{n \times n}$ be a matrix of exact order 2 of the first category with $n \geqslant 5$. Then $A \in Q$.*

PROOF. Since A is nondegenerate, $(0, A)$ has a unique solution (Murty 1972). We treat P- and N-matrices of exact order 2 separately below.

Suppose A is a P-matrix of exact order 2 of the first category. Consider a $q \in R^n$, $q > 0$. We claim that LCP(q, A) has a unique solution $w = q$, $z = 0$.

Let there exist a solution (w^1, z^1) for LCP(q, A) with $z^1 \neq 0$. The solution can be written in the matrix form, for some $\phi \neq J \subseteq \{1, \ldots, n\}$ as

$$\begin{bmatrix} -A_{JJ} & 0 \\ -A_{\bar{J}J} & I_{\bar{J}} \end{bmatrix} \begin{bmatrix} z_J^1 \\ z_{\bar{J}}^1 \end{bmatrix} = \begin{bmatrix} q_J \\ q_{\bar{J}} \end{bmatrix}, \tag{4.6}$$

i.e., the system,

$$-A_{JJ}z_J > 0, \qquad z_J > 0 \tag{4.7}$$

has a solution, which implies that $v(A'_{JJ}) < 0$. We know that both $v(A_{JJ})$ and $v(A'_{JJ})$ are positive, for $J \subseteq \{1, \dots, n\}$, for a P-matrix of exact order 2 of the first category, using Corollary 4.1. Hence (4.7) is impossible and LCP(q, A) has a unique solution for $q > 0$. By Corollary 3.2 of Saigal (1972), we have $A \in Q$.

Suppose now, A is an N-matrix of exact order 2 of the first category. Two cases arise.

Case (i): There is an i_0, $1 \leqslant i_0 \leqslant n$, such that $B_{i_0} < 0$ (by definition, there may exist one such index). We may assume, without loss of generality, that $i_0 = 1$. The sign pattern of A can be written as

$$A = \begin{bmatrix} - & + & + & \dots & + \\ + & & & & \\ + & & & B_1 & \\ \vdots & & & & \\ + & & & & \end{bmatrix} = \begin{bmatrix} a_{11} & d^t \\ c & B_1 \end{bmatrix} \quad \text{(say)}$$

with $B_1 < 0$. Consider a $q > 0$, whose partitioned form is $q = (q_1, \tilde{q})^t$, where $\tilde{q} \in R^{n-1}$. Choose $\tilde{q} \in R^{n-1}_+$ such that $\tilde{q} \in \operatorname{int}[\operatorname{pos}(-B_1)]$. Since B_1 is an N-matrix of exact order 1 of the second category, and $\tilde{q} \in \operatorname{int}[\operatorname{pos}(B_1)]$, by Theorem 3.1 it follows that the LCP$(\tilde{q}, B_1)$ has exactly four solutions. Also, if $(\bar{w}, \bar{z})$ solves LCP$(\tilde{q}, B_1)$, then the pair (w, z), $w \in R^n$, $z \in R^n$, defined by

$$w_1 = q_1 + d^t\bar{z}; \qquad z_1 = 0,$$

$$w_j = \bar{w}_{j-1}; \qquad z_j = \bar{z}_{j-1}, \quad 2 \leqslant j \leqslant n.$$

solves LCP(q, A). Thus we obtain 4 solutions to LCP(q, A). We construct another solution as follows: Take

$$z_1^1 = q_1/(-a_{11}); \qquad w_1^1 = 0,$$

$$z_i^1 = 0; \qquad w_i^1 = \tilde{q}_{i-1} + z_1^1 a_{11}.$$

Then (w^1, z^1) solves LCP(q, A) and (w^1, z^1) is different from the four solutions constructed before. Thus we have 5 solutions to LCP(q, A) for a q nondegenerate with respect to A, by our construction. Now for this $q \in R^n_+$, we proceed to prove that LCP(q, A) has no other solution.

Suppose (u, v) is a solution to LCP(q, A) distinct from the 5 listed above. Let

$$L = \{i : v_i > 0\}.$$

Then since (u, v) is different from the aforesaid 5 solutions, it follows that,

$$\text{the index } 1 \in L \text{ and } L \cap \{2, \dots, n\} \neq \phi.$$

Now the equation $u - Av = q$, leads us to $A_{LL}v_L < 0$ where either A_{LL} is an N-matrix of exact order 1 or 0 of the first category, or $L = \{1, \dots, n\}$. But this gives rise to a contradiction to the property of an N-matrix of exact order 1 or 0, or if

$L = \{1, 2, \ldots, n\}$, to our assumption about $v(A)$. This shows that there are exactly 5 solutions to LCP(q, A), and by our choice of $\tilde{q} \in \text{int}[\text{pos}(-B_l)]$, q is nondegenerate with respect to A. This along with $(0, A)$ having a unique solution implies (by Corollary 3.2 of Saigal (1972) that A is a Q-matrix.

Case (ii): $\nexists i_0$, such that $B_{i0} < 0$. A can be written in the partitioned form, for some $\phi \neq J \subset \{1, \ldots, n\}$ as

$$A = \begin{bmatrix} A_{JJ} & A_{J\bar{J}} \\ A_{\bar{J}J} & A_{\bar{J}\bar{J}} \end{bmatrix}$$

where $A_{JJ} < 0$, $A_{\bar{J}\bar{J}} < 0$ and $A_{J\bar{J}}, A_{\bar{J}J} > 0$ with $1 < |J| < n - 1$. Note that A_{JJ} and $A_{\bar{J}\bar{J}}$ are N-matrices. Choose a $q > 0$ in R^n such that q is nondegenerate with respect to A and, further, q_J and $q_{\bar{J}}$ are nondegenerate with respect to A_{JJ} and $A_{\bar{J}\bar{J}}$, respectively. Now from Theorem 2.4 it follows that the subproblems LCP(q_J, A_{JJ}) and LCP$(q_{\bar{J}}, A_{\bar{J}\bar{J}})$ have two solutions each, one of which is the trivial solution. These solutions give rise to exactly 3 solutions to the LCP(q, A) and no more. The result now follows from Corollary 3.2 of Saigal (1972). This completes the proof of the theorem. □

It can be seen that if A is an N-matrix of exact order 1 or 0 of the second category, or a P-matrix of exact order 1 of the second category, then $-A \in Q$. In view of this, one may consider the following question:

Question 4.1. Let A be a matrix of exact order 2 of the second category. Is $-A$, a Q-matrix?

Our Theorem 4.7, that follows will provide an affirmative answer to this question. To do this, we need the following results.

LEMMA 4.5. *Let $A \in R^{n \times n}$ be a matrix of exact order* 2. *Let B_1 and B_2 be matrices of exact order* 1 *of the second category, with $B_i \nless 0$, $i = 1, 2$. Suppose that $v(A) < 0$. Then a^{12}, the $(1, 2)$th entry of A^{-1} is negative.*

PROOF. By hypothesis, $B_1^{-1} < 0$. Let $y = (w_1, 0, \ldots, 0)^t \in R^{n-1}$, where $w_2 < 0$. Then $\exists$ a $\tilde{z} \in R^{n-1}$, $\tilde{z} > 0$ such that $y = B_1\tilde{z}$. Let $z = (0, \tilde{z})^t \in R^n$. We note that

$$Az = \begin{bmatrix} w_1 \\ w_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$z = A^{-1}w. \tag{4.8}$$

From Lemma 4.1, the diagonal entries of A^{-1} are positive, and its 2 by 2 principal minors, negative. Hence $a^{12} \neq 0$. From (4.8) we have the equation

$$a^{11}w_1 + a^{12}w_2 = 0. \tag{4.9}$$

Now suppose $a^{12} > 0$. It then follows from (4.9), that $w_1 > 0$. As B_2 is also of the second category, $B_2 \nless 0$, we have $B_2^{-1} < 0$.

Let $x = (-w_1, 0, \ldots, 0)^t \in R^{n-1}$, where w_1 is determined from (4.9). Then there exists a vector $\tilde{v} \in R^{n-1}$, $\tilde{v} > 0$ such that

$$B_2\tilde{v} = x. \tag{4.10}$$

Let $v \in R^n$ be defined by

$$v_i = \tilde{v}_i, \quad \text{for } i \in \{1\} \cup \{3, 4, \ldots, n\},$$
$$v_2 = 0.$$

Note from (4.10) that

$$Av = \begin{bmatrix} -w_1 \\ w_2^* \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{for some } w_2^*.$$

This implies that

$$A(v + z) = \begin{bmatrix} 0 \\ w_2^{**} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{where} \quad w_2^{**} = w_2^* + w_2.$$

Let $u = z + v$. As $u > 0$, and $v(A) < 0$, we see that $w_2^{**} < 0$. On the other hand, the first coordinate of u, i.e.,

$$u_1 = (A^{-1}Au)_1 = a^{12}w_2^{**} < 0,$$

a contradiction. Hence $a^{12} < 0$, and this concludes the proof. □

We say that A is a Z-matrix, if $a_{ij} \leqslant 0$, $\forall\ i \neq j$. For details about Z-matrices, see Fiedler and Ptak (1962).

THEOREM 4.6. *Let $A \in R^{n \times n}$, $n \geqslant 5$ be a matrix of exact order* 2. $A^{-1} \in Z$ *if and only if $v(A) < 0$ and A is of the second category with each $B_i \nless 0$.*

PROOF. The "if" part follows from Lemma 4.5.

Conversely, let A^{-1} be a Z-matrix. By Lemma 4.1 the principal minors of order $2 \leqslant r < n$, are negative. Hence for a $0 \neq z \geqslant 0$, $z \in R^n$, we have $z^t A^{-1} \leqslant 0$. See Fiedler and Ptak (1962). Hence $v(A^{-1}) \leqslant 0$. By Lemma 4.2, we conclude that $v(A) < 0$. We shall show that B_1 is of the second category. This will complete the proof.

B_1 is either an N-matrix of exact order 1 or a P-matrix of exact order 1 depending on whether A is an N- or P-matrix of exact order 2. In either case, if B_i is of the first category then there exists a vector $\bar{y} \in R^{n-1}$, $\bar{y} > 0$ such that

$$B_1 \bar{y} > 0. \tag{4.11}$$

See Lemma 3.1 of Olech, Parthasarathy and Ravindran (1991).

Let $y \in R^n$ be defined as $y = \begin{bmatrix} 0 \\ \bar{y} \end{bmatrix}$.

Note that $A_{1.}y \leqslant 0$, for otherwise, $Ay > 0$ and hence $v(A) > 0$, contradictory to our earlier conclusion. Now,

$$y_1 = (A^{-1}Ay)_1 = 0. \tag{4.13}$$

However, we also note that $(A^{-1}Ay)_1 = \sum_{j=1}^{n} a^{1j}(Ay)_y < 0$ from (4.11), (4.12) and the fact that $a^{1j} \leqslant 0$, for $j \neq 1$, which contradicts (4.13). This contradiction implies that B_1 is of the second category. This concludes the proof of the theorem. □

We now state a theorem that answers Questions 4.1 partially.

THEOREM 4.7. *Let $n \geqslant 5$ and $A \in R^{n\times n}$ be a matrix of exact order 2 of the second category with $B_1 \nless 0$ for $1 \leqslant i \leqslant n$. Then $-A$ is a Q-matrix.*

PROOF. Let $M = -A^{-1}$. Under the hypothesis of the theorem, A^{-1} is a Z-matrix and M has the sign pattern

$$\text{Sign } M = \begin{bmatrix} - & + & + & + & + & + \\ + & - & + & \cdots & \cdots & + \\ + & + & - & \cdots & \cdots & + \\ \vdots & & & & & \\ + & + & + & & & - \end{bmatrix}.$$

We shall show that the degree of the map associated with the LCP(q, M) is $-n + 1$ and conclude from here that M, and hence $-A$, is a Q-matrix.

Note that LCP$(0, M)$ has a unique solution as no principal minor of it is zero. Hence the degree of the map P_M, associated with the LCP(q, M) is well defined.

Let $q > 0$. Then LCP(q, M) has the following $(n + 1)$ solutions (w^i, z^i):

$$w_i^i = 0, \qquad z_i^i = q_i/m_{ii}, \qquad 1 \leqslant i \leqslant n$$

$$w_j^i = q_j + m_{ji}z_i^i; \qquad z_j^i = 0, \quad \text{for} \quad j \neq i, i \in \{1,2,\ldots,n\}$$

$$j \in \{1,2,\ldots,n\}$$

and

$$w_j^{n+1} = q; \qquad z_j^{n+1} = 0, \quad \text{for } j \in \{1,\ldots,n\}.$$

where m_{ji} denotes the (j, i)th entry of M. Since $q > 0$, q is nondegenerate with respect to M, from the solutions we have constructed. Thus LCP(q, M) has $(n + 1)$ solutions as mentioned above. We now show that LCP(q, M) has no other solution. Suppose there is another solution $(\bar{w}, \bar{z})$ to LCP(q, M); let

$$L = \{i: \bar{z}_i > 0\}.$$

Then clearly, $|L| \geqslant 2$, as $\bar{z}$ is distinct from each z^i, $1 \leqslant i \leqslant n + 1$. The solution $(\bar{w}, \bar{z})$ can be written as

$$\begin{bmatrix} -M_{LL} & 0 \\ -M_{\bar{L}L} & I_{\bar{L}} \end{bmatrix} \begin{bmatrix} \bar{z}_{\bar{L}} \\ \bar{w}_{\bar{L}} \end{bmatrix} = \begin{bmatrix} q_L \\ q_{\bar{L}} \end{bmatrix} \tag{4.14}$$

which implies that $-M_{LL}\bar{z}_L > 0$ for $\bar{z}_L > 0$, or $-M_{LL}$ is a P-matrix by Theorem 4.3 of Fiedler and Ptak (1962). This contradicts the result about the sign of the principal minors of A^{-1} (Lemma 4.2). Thus it follows that the only solutions to LCP(q, M) are the $(n + 1)$ ones listed above.

We can now calculate the degree of the P_M associated with the M as $-n + 1$. As the degree of the map associated with M is nonzero, it follows from Theorem 2.7 that M is a Q-matrix. This concludes the proof. □

REMARK 4.2. The use of degree theory in the above proof has been suggested to us by an unknown referee. Some details about the concepts of degree theory have been sent to us by Professor M. S. Gowda. Our original proof was long and consisted of using Murty's result on the constant parity property when n is even and of showing that there is a principal pivot transform of M which satisfies the conditions of Todd (1976) for any n. In the latter case, one can then apply Lemke's algorithm to the LCP$(q, \overline{M})$, where $\overline{M}$ is an appropriate principal pivot transform of M, and constructively show that $\overline{M}$, and hence $-A$, is a Q-matrix.

Now we turn our attention to matrices of exact order 2 of the third category. Before we proceed to give a characterization theorem on their Q-nature, we present a few lemmas for the class of matrices of exact order 2.

LEMMA 4.6. *Let $A \in R^{n\times n}$ be a matrix of exact order* 2, *with $n \geqslant 5$ and $v(A) > 0$. Suppose $B_1 \nless 0$ is a matrix of exact order* 1 *of the second category. Then $A^{\cdot 1} > 0$ and $A^{1\cdot} > 0$.*

PROOF. That $a^{ij} \neq 0$, for all $1 \leqslant i, j \leqslant n$, is clear from Lemma 4.1

Suppose $a^{12} < 0$. Taking $w = (w_2, 0, \dots, 0) \in R^{n-1}$ for some $w_2 < 0$, there exists a $\bar{y} > 0$, $\bar{y} \in R^{n-1}$, such that

$$B_1\bar{y} = \begin{bmatrix} w_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let $y \in R^n$ be defined as $y = (0; \bar{y})^t$. Consider $A_{1\cdot}y$. Suppose $A_{1\cdot}y \leqslant 0$. This implies that $y^tA^t \leqslant 0$ which shows that $v(A^t) \leqslant 0$. By our Theorem 4.2 and Lemma 4.3 it follows that $v(A) < 0$, contrary to our hypothesis. Hence $A_{1\cdot}y > 0$. Now

$$(A^{-1}Ay)_1 = 0 \quad \text{or}$$

$$a^{11}(A_{1\cdot}y) + a^{12}w_2 = 0.$$

However, we note that, with $A_{1\cdot}y > 0$, $a^{11} > 0$, $a^{12} < 0$, $w_2 < 0$,

$$a^{11}(A_{1\cdot}y) + a^{12}w_2 > 0,$$

a contradiction. Hence $a^{12} > 0$. This completes the proof. □

LEMMA 4.7. *Let $A \in R^{n\times n}$ be a matrix of exact order* 2, *with $n \geqslant 5$ and $v(A) < 0$. If B_1 is of the first category, then $A^{1\cdot}$ (as well as $A^{\cdot 1}$) has*

$$a^{1j} < 0, \quad a^{j1} < 0, \quad a^{1k} > 0, \quad a^{k1} > 0, \quad \textit{for some } j, k \in \{2, \dots, n\}.$$

PROOF. By our Remark 2.2, Lemma 4.3 and Theorem 4.2 it follows that $v(A^{-1})$, $v(A^t)$ and $v((A^t)^{-1})$ are negative. This shows that the first row and first column of A^{-1} cannot consist only of nonnegative entries. Hence there is a j such that $a^{1j} < 0$. Now from Lemma 4.1 it follows that the diagonal entries of A^{-1} are positive and its 2×2 principal minors are negative. Thus it follows that $j \neq 1$ and $a^{j1} < 0$. To conclude the proof we proceed as follows. Note that B_1 is either a P- or an N-matrix of exact order 1. In either case there is a $\bar{y} \in R^{n-1}$, $\bar{y} > 0$ such that $B_1^{-1}\bar{y} = z > 0$.

See Lemma 3.1 of Olech, Parthasarathy and Ravindran (1991). Take $z = \begin{pmatrix} 0 \\ z \end{pmatrix}$. Now look at $A\bar{z} = \begin{pmatrix} \alpha \\ \bar{y} \end{pmatrix}$. As $\bar{y} > 0$ and $v(A) < 0$, it follows that $\alpha < 0$. Let us look at the equation $A^{-1}\begin{pmatrix} \alpha \\ \bar{y} \end{pmatrix} = \bar{z}$ which leads to the equation

$$a^{11}\alpha + \sum a^{1i}\bar{y}_i = 0.$$

This is possible only if there is a k such that $a^{1k} > 0$, as $a^{11} > 0$, $\alpha < 0$ and $\bar{y}_i > 0$. This concludes the proof. □

The next lemma characterizes the first category matrices through the sign pattern of A^{-1}.

LEMMA 4.8. *Let $A \in R^{n \times n}$ be a matrix of exact order 2, with $n \geqslant 5$ and $v(A) > 0$. If every row (or column) of A^{-1} has a negative entry, then $A \in Q$.*

PROOF. Suppose $a^{12} < 0$. We will prove either that B_1 and B_2 are of the first category, or $B_i < 0$, for some $i \in \{1, 2\}$. Suppose $B_2 \nless 0$ is of the second category. Consider $(w_2, 0, \ldots, 0) \in R^{n-1}$, with $w_2 < 0$. Then there exists a $\bar{y} > 0$, $\bar{y} \in R^{n-1}$,

$$B_1\bar{y} = \begin{bmatrix} w_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By taking $y = (0, \bar{y})^t$ in R^n as before, we have

$$(A^{-1}Ay)_1 = 0$$

which implies

$$a^{11}(A_{1\cdot}y) + a^{12}w_2 = 0.$$

It is clear that $A_{1\cdot}y < 0$ and hence $Ay \leqslant 0$, for $y \geqslant 0$. It follows from here that $v(A^t) \leqslant 0$ and from Theorem 4.2 and Lemma 4.3 it now follows that $v(A) < 0$.

This contradicts our assumption that $v(A) > 0$. Hence all B_i's are of the first category except possibly for one i, for which $B_i < 0$. Thus A is a matrix of exact order 2 of the first category and by Theorem 4.4, $A \in Q$. □

The next theorem gives a characterization of exact order 2 matrices of the third category.

THEOREM 4.8. *Let $A \in R^{n \times n}$, $n \geqslant 5$, be a matrix of exact order 2 of the third category, with $v(A) > 0$. Define*

$$L = \{i : B_i \nless 0, B_i \text{ is of the second category}, 1 \leqslant i \leqslant n\}$$

A is a Q-matrix iff the cardinality of L is 2.

PROOF. First we observe that, since A is a matrix of exact order 2 of the third category, there exists an $i \in \{1, \ldots, n\}$, such that $B_i \nless 0$ and $B_i^{-1} < 0$. Therefore the index set L is nonempty.

(IF): Suppose $|L|$, the cardinality of L is 2. As in the proof of Theorem 4.5, by considering a q nondegenerate with respect to A, $q > 0$ we note that

$$\deg P_A = \begin{cases} 1 & \text{if } A \text{ is a } P\text{-matrix of the exact order 2,} \\ -1 & \text{if } A \text{ is an } N\text{-matrix of the exact order 2.} \end{cases}$$

In either case, $\deg P_A \neq 0$. Hence, using Theorem 2.7, $A \in Q$.

(ONLY IF): Let A be a Q-matrix and $|L| = r$ where r is a positive integer.

Let us assume without loss of generality that $B_1 \nless 0$ is a matrix of the second category in A.

Since $v(A) > 0$, using Lemma 4.6, A^{-1} can be written as

$$A^{-1} = \begin{bmatrix} + & + & \cdots & + \\ + & & & \\ \vdots & & D & \\ + & & & \end{bmatrix}$$

where $D \in R^{(n-1)\times(n-1)}$. Since $A^{\cdot 1} > 0$, for $q = I_{\cdot 1}$, LCP(q, A^{-1}) has a unique solution. This, along with $A \in Q$ implies that $D \in Q$. See 4.9 of Murty (1972).

In a similar manner, for every $i \in L$, we have $A^{\cdot i} > 0$. Hence, by removing the ith row and the ith column for all $i \in L$ from A^{-1}, we would arrive at the fact that the principal submatrix D is a Q-matrix.

If D has any more row (and its corresponding column) of positive entries, we repeat this argument, until we get a principal submatrix $F = A^{JJ}$, for $\phi \neq J \subset \{1, \ldots, n\}$, $|J| = k$, $2 \leqslant k \leqslant n$, such that $F \in Q$ and every row of F has a negative entry. We first note that $k = 2$ is not possible. This is because, if $k = 2$, then F is a Z-matrix with negative determinant and such a matrix cannot be a Q-matrix.

It follows from Lemma 4.2 that F^{-1} is a P-matrix of exact order 2, when $k \geqslant 3$. Since $F \in Q$, it follows that $v(F) > 0$ and IF^{-1} is a P-matrix of exact order 2 of the first category. For $\bar{q} \in R^k_+$, LCP$(\bar{q}, F^{-1})$ has a unique solution from the proof of Theorem 4.5, which is given by $(\bar{w} = \bar{q}, \bar{z} = 0)$.

Now, define a vector $q \in R^n$ as

$$q_i = \begin{cases} (-F\bar{q})_i & \text{if } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

LCP(q, A^{-1}) has a solution, which is as follows:

$$w_i = 0; \qquad z_i = \bar{q}_i \quad \text{for } i \in J,$$

$$w_i = \sum_j a^{ij} z_j; \qquad z_i = 0, \quad \text{for } i \notin J.$$

As $a^{ij} > 0$, for $i \notin J$, $j \in J$, (w, z) is a solution, to LCP(q, A^{-1}). We claim that LCP(q, A^{-1}) has no other solution. Suppose LCP(q, A^{-1}) has a solution (w^1, z^1) which can be written as

$$\begin{bmatrix} -A^{LL} & 0 \\ -A^{\bar{L}L} & I^{\bar{L}} \end{bmatrix} \begin{bmatrix} z'_L \\ w'_{\bar{L}} \end{bmatrix} = \begin{bmatrix} q_L \\ q_{\bar{L}} \end{bmatrix}$$

where $\phi \neq L \subseteq \{1, \ldots, n\}$, with $L \cap J \neq \phi$, and $L \neq J$.

But q_L has a zero entry, corresponding to which A^{LL} has a row of positive entries; thus there does not exist a vector

$$z_L^1 \geqslant 0 \quad \text{such that } -A_{LL} z_L^1 = q_L.$$

Hence, LCP(q, A^{-1}) has a unique solution, which is nondegenerate. This implies that deg $P_{A^{-1}}$ and in turn, deg P_A is ± 1.

But as in the proof of Theorem 4.4, by considering the principal subdeterminants of the solutions of LCP(q, A) for a $q > 0$, q nondegenerate with respect to A, we see that

$$\deg P_A = \begin{cases} -r+1 & \text{if } A \text{ is a } P\text{-matrix of the exact order 2,} \\ r-1 & \text{if } A \text{ is an } N\text{-matrix of the exact order 2,} \end{cases}$$

where $|L| = r$.

As $r \geqslant 1$, deg P_A is ± 1 only when $r = 2$, that is $|L| = 2$. This completes the proof of the theorem. □

Now, a complete characterization of the matrices of exact order 2 regarding their Q-nature can be stated as follows:

THEOREM 4.9. *Let $A \in R^{n\times n}$, $n \geqslant 5$ be a matrix of exact order 2 with $v(A) > 0$. Then $A \in Q$ if and only if either $L = \phi$, or the cardinality of the set L is 2, where*

$$L = \{i\colon B_i \nless 0,\ B_i \text{ is of the second category},\ 1 \leqslant i \leqslant n\}.$$

Finally we present two examples to illustrate some of the results proved in this section.

EXAMPLE 4.3. Let

$$A = \begin{bmatrix} -5.3846 & 1.5385 & 1.5385 & 1.5385 & -20 \\ 1.5385 & -0.1538 & -0.6538 & -0.6538 & .1 \\ 1.5385 & -0.6538 & -0.1538 & -0.6538 & 1 \\ 1.5385 & -0.6538 & -0.6538 & -0.1538 & .7 \\ -30 & 2 & 2 & 4 & -1 \end{bmatrix}.$$

In this example of an N-matrix of exact order 2, B_1 is of the first category, while B_i, for $2 \leqslant i \leqslant 5$, is of the second category. We find that

$$A^{-1} = \begin{bmatrix} 0.0025 & -0.1439 & -0.1439 & 0.0418 & -0.0464 \\ -0.5732 & 0.3918 & -1.6032 & -1.4688 & -0.0348 \\ -0.5732 & -1.6082 & 0.3918 & -1.4688 & -0.0348 \\ 0.5685 & -0.4941 & -0.4941 & 1.7564 & -0.0626 \\ -0.0951 & -0.0928 & -0.0928 & -0.1021 & 0.0023 \end{bmatrix}$$

Note that $v(A) < 0$; Now since $B_i \nless 0$ are of the second category, for $2 \leqslant i \leqslant 5$, we find that $a^{ij} < 0$, for $i \neq j$, $2 \leqslant i$, $j \leqslant 5$, as anticipated by Lemma 4.5. Also, as asserted by Lemma 4.7, the first two and first column of A^{-1}, each contains a positive entry.

Now take $D \in R^{4\times 4}$ to be the principal submatrix of A^{-1}, leaving the first row and first column. Clearly, D is a Z-matrix, and it can be verified that D^{-1} is a P-matrix of exact order 2 of the second category, with $v(D) < 0$, as stated in Theorem 4.5.

EXAMPLE 4.4. Consider the matrix

$$A^{-1} = \begin{bmatrix} 1+\epsilon & 2 & 3 & 4 & 5 \\ 6 & 7+\epsilon & 8 & 9.1 & 10 \\ 110 & 120 & 130+\epsilon & 140 & 150 \\ 16 & 17.1 & 18 & 19+\epsilon & 20 \\ 21 & 22 & 23 & 24 & 25+\epsilon \end{bmatrix}$$

where ϵ is taken to be 0.0795766.

Now $A = (A^{-1})^{-1}$, can be verified to be an N-matrix of exact order 2. In A, B_1 and B_5 are of the second category, while B_2, B_3, B_4 are of the first category. Note that $v(A) > 0$, as $A^{-1} > 0$.

This example shows that the converse of Lemma 4.6 is not true. We notice that though $A^{-1} > 0$ and $A^{1'} > 0$, B_1 is of the second category.

5. Algorithms and concluding remarks. In this section, we consider the question of finding an algorithm to compute a solution to LCP(q, A) where A is a matrix of exact order $0, 1, \ldots, 2$. Algorithms for some of the subclasses are already known. In this section we sum up the known results and present some results new to the literature.

It is known that Lemke's algorithm (Lemke 1974) will find a solution to the LCP(q, A) for any $q \in R^n$ when A is a P-matrix. See Murty (1972). It is also known that a solution to LCP(q, A) can be obtained from a solution to LCP$(-A^{-1}q, A^{-1})$ computed by Lemke's algorithm when A is an N-matrix of the first category. We refer to Saigal (1972). This result also takes care of the case where A is a P-matrix of exact order 1 of the first category since such a matrix is just the inverse of an N-matrix of the first category.

For N-matrices exact order 1 of the first category we have the following result.

THEOREM 5.1. *Let A be an N-matrix of exact order 1 of the first category. Let the rth coordinate of $A_{\cdot 1}$ be positive and let $d = \lambda_1(-A_{\cdot 1}) + \Sigma_{i \neq s} I_{\cdot j} + \mu I)_{\cdot s}$ where $1 \leqslant s \leqslant n$, $s \neq r$, λ_1 is a fixed number such that $\lambda_1(-a_{r1}) + 1 < 0$ and μ is sufficiently large. Then Lemke's algorithm initiated with the vector d in the complementary cone pos(B) where $B = ((-A_{\cdot 1}, I_{\cdot s}, 2 \leqslant s \leqslant n))$ computes a solution to LCP(q, A) for any $q \in R^n$.*

PROOF. Notice that the rth coordinate of d, d_r, is negative for any $\mu > 0$. Thus $d \notin \text{pos}(I)$, for any $\mu > 0$; Also, since $I_{\cdot 1} \notin \text{pos}(-A)$, it follows that there is a $\mu_0 > 0$ such that for $\mu > \mu_0$, $d \notin \text{pos}(-A)$: Thus for μ sufficiently large by Theorem 3.3, LCP(d, A) has a unique solution and the theorem follows. □

REMARK 5.1. Using standard methodology and the above theorem, we can develop a computational scheme for computing a solution to LCP(q, A) whenever A is an N-matrix of exact order 1 of the first category.

The following result can be easily seen for P-matrices of exact order 1 of the first category.

THEOREM 5.2. *Let A be a P-matrix of exact order 1 of the first category. Then for any $q \in R^n$, solution to LCP(q, A) can be computed by using Lemke's algorithm initiating it with any positive vector d.*

PROOF. This follows from Theorem 2.5, as the inverse of an N-matrix is a P-matrix of exact order 1. □

We now consider matrices of exact order 2.

THEOREM 5.3. *Let A be a P-matrix of exact order 2 of the first category. Then a solution to LCP(q, A) can be computed by applying Lemke's algorithm, initiating it with any positive vector d.*

PROOF. This follows from the proof of Theorem 4.5. □

THEOREM 5.4. *Suppose A is a matrix of exact order 2 of the second category with $B_i \nless 0$, for $1 \leqslant i \leqslant n$. Then a solution to LCP$(q, A)$ if one exists, can be computed by obtaining a solution to LCP$(-A^{-1}q, A^{-1})$, in at most n steps.*

PROOF. As $v(A) < 0$ from Theorem 4.6, it follows that $A^{-1} \in Z$. Now there are a number of methods to solve $(-A^{-1}q, A^{-1})$ which will produce a solution if it exists, or show that there is no solution in at most n steps. See Chandrasekaran (1970), Saigal (1971), Mohan (1976) and Ramamurthy (1986).

When A is of exact order 2 of the second category, we proved in Theorem 4.7 that $-A$ is a Q-matrix. For this class of matrices we can notice that for each $q \in R^n$, q nondegenerate with respect to A, the LCP$(q, -A)$ has more than one solution; this asserts the fact that Saigal's result and Todd's condition for proving Q-nature of a matrix are improvements over the earlier known results.

As we go up the hierarchy in the classes of exact order matrices, the results we derived here require more calculations. In a similar manner as done in §4, one can classify the exact order k matrices into three different categories, based on the exact order 1 principal submatrices present in them. Then it is easily seen that exact order k matrices of the first category are Q-matrices. As the size of the matrices under consideration needs to be $\geqslant (k + 3)$, the task of studying the classes of exact order k matrices becomes difficult. In fact, the problems we looked at in this paper, remain open for the general exact order k matrices.

Acknowledgements. The inspiration and motivation for this work came from the papers of Professor K. G. Murty and Romesh Saigal, written on the linear complementarity problem. We are thankful to Professor C. Olech for his suggestions. The presentation of this paper has been greatly improved by the suggestions and comments of the unknown referees. In particular, one of the referees has suggested the use of degree theory in the proof of Theorem 4.7. We are also grateful to Professor M. S. Gowda for sending us some details and references on the use of degree theory and to Professor R. B. Bapat for suggesting Example 2.1.

The third author was supported by Dr. K. S. Krishnan (DAE) Fellowship, BARC, Bombay.

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