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UNIQUENESS OF UNBOUNDED OCCUPIED AND VACANT COMPONENTS IN BOOLEAN MODELS

BY RONALD MEESTER AND RAHUL ROY

University of Utrecht and Indian Statistical Institute

We consider Boolean models in d-dimensional Euclidean space. Each point of a stationary, ergodic point process is the center of a ball with random radius. In this way, the space is partitioned into an occupied and a vacant region. We are interested in the number of unbounded occupied or vacant components that can coexist. We show that under very general conditions on the distribution of the radius random variable, there can be at most one unbounded component of each type. In case the point process is Poisson, we obtain uniqueness of the unbounded components without imposing any condition at all. Although we do not prove the necessity of the conditions to prove uniqueness, we obtain examples of stationary, ergodic point processes where the unbounded components are not unique when the conditions are violated. Finally, we discuss more general random shapes than just balls which are centered at the points of the point process.

1. Introduction. The geometric properties of Boolean models have been studied quite extensively by probabilists. The main reference here is the excellent book of Hall (1988) on Boolean models and their geometric properties. The primary focus of the book—and in fact of most of the relevant literture—is on Boolean models driven by a Poisson point process. Percolation properties of Boolean models driven by a Poisson point process were first studied by Gilbert (1961) to model the transmission of radio waves. This *continuum percolation model*, in which each point of the point process is the center of a ball with random radius, was later studied by Hall (1985), Menshikov (1986), Roy (1990) and others.

A question of both geometric and percolation theoretic interest in Boolean models is the number of unbounded components admitted by the process. In Boolean models, unbounded components can be either the unbounded connected components in the region formed by the random balls, or the unbounded connected components in the complement of this region. The former are called the *occupied* components, and the latter are the *vacant* components. The number of unbounded vacant components is also asked by Sznitman (1991a, b). He studies the path of a Brownian motion in a space with obstacles, where the obstacles arise as balls centered at points of a Poisson point process.

In discrete percolation models, the unbounded occupied cluster is unique under very general conditions. Harris (1960) gave the first uniqueness result

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(in two dimensions), and later uniqueness was established in higher dimensions and under weaker conditions in Aizenmann, Newman and Kesten (1987), Gandolfi, Keane and Russo (1988) and Burton and Keane (1989). In a variant of the continuum percolation model introduced by Penrose (1991), Burton and Meester (1993) obtained uniqueness under very general conditions. In this model, any two points of the point process are connected with each other with a probability which only depends on their distance, independently of all other pairs. The uniqueness result of Burton and Meester only requires the point process to be stationary, and the connection rule should be long-range, that is, the probability for any pair of points to be connected to each other should be positive. Vacant components do not have any meaning in their model.

In this paper, we consider the uniqueness problem in Boolean models driven by stationary, ergodic point processes and where each ball has a random radius, independent of all other balls and the driving point process. We study uniqueness for unbounded occupied as well as vacant components. Since our percolation process depends on two independent random processes, namely, the point process and the process governing the radii, we have two different variables governing the unbounded components. We obtain sufficient conditions for uniqueness of the occupied and vacant components. In addition, we construct examples of Boolean models which admit more than one unbounded occupied or vacant component when the sufficient conditions are not satisfied.

Many of our results go through if the shapes are not necessarily balls, but arbitrary shapes subject to certain conditions. For the sake of clarity, however, throughout Sections 2–7 we consider models with balls only. We indicate, in brief, how the results can be generalized to more general shapes. The proofs of these generalizations will be clear from what follows in the subsequent sections.

In the general setup, we take a "basic" shape containing the origin, and we center at each point of the point process such a basic shape multiplied by a certain random factor. We require the basic shape to contain some ball in its interior. The statement of Theorem 2.1 remains true if we require the shapes to be arbitrarily large (i.e., to contain arbitrary large balls) with positive probability. Although convexity is used in the proof of Lemma 4.2, it is easy to see we can do without it here. For Theorem 2.2, we require that the shapes be arbitrarily small (i.e., contained in arbitrary small balls) with positive probability. We also need the boundary of the shape to have finite length here. Note that condition (2.2) might not be true for "strange" shapes. For Theorem 2.3, no further conditions on the shapes are required.

In the next section we introduce the model and state our main results. Sections 3–6 are devoted to the proofs of these results. The last section is devoted to examples.

2. The model and main results. Let $\mathbb{R}^d$ be d-dimensional Euclidean space, let $\mathscr{B}^d$ be the collection of Borel sets in $\mathbb{R}^d$ and let μ_d be the Lebesgue measure in $\mathbb{R}^d$. Let N be the set of all Radon counting measures on $(\mathbb{R}^d, \mathscr{B}^d)$.

Then N can be identified with the set of all finite and infinite configurations of points in $\mathbb{R}^d$ without limit points. Let $\mathscr{N}$ be the σ-algebra on N generated by sets of the form $\{\nu \in N \mid \nu(A) = k\}$, for all integers k and bounded Borel sets A. A *point process* is a measurable mapping X from a probability space $(\Omega, \mathscr{F}_1, P_1)$ into $(N, \mathscr{N})$.

By considering rectangles with rational coordinates only, we may assume that $\mathscr{F}_1$ is countably generated. For $A \in \mathscr{B}^d$, we denote by $X(A)$ the (random) number of points in A. For any $t \in \mathbb{R}^d$, let T_t: $\mathbb{R}^d \to \mathbb{R}^d$ be defined by $T_t(x) = x + t$, for all $x \in \mathbb{R}^d$. We only consider processes which are stationary, that is, the finite-dimensional distributions are invariant under all translations T_t, $t \in \mathbb{R}^d$. Furthermore, we restrict our attention to processes which are ergodic under the whole group $\{T_t, t \in \mathbb{R}^d\}$. This amounts to saying that any event which is invariant under the whole group has measure 0 or 1. By ergodic decomposition, this is not a real loss of generality. According to Pugh and Shub (1971), if $\mathbb{R}^d$ acts ergodically on a countably generated probability space, then all elements of $\mathbb{R}^d$, off a countable family of hyperplanes, are ergodic. (Here, a hyperplane is not assumed to contain the origin.) This implies that we may choose an orthonormal base $\{t_1, \dots, t_d\}$ of $\mathbb{R}^d$ such that, for all but countably many values of λ, $T_{\lambda t_i}$ acts ergodically for all i, and we denote by I the set of values for λ for which this is the case. We may assume that $1 \in I$. For all $\lambda \in I$, any event which is invariant under one of the translations $T_{\lambda t_1}, \dots, T_{\lambda t_d}$, has P_1-measure 0 or 1; from now on, all coordinates of elements in $\mathbb{R}^d$ are with respect to this base. We will need this in many places.

We define the *density* $\lambda(X)$ of X as follows: Let $B_n \subset \mathbb{R}^d$ be the d-dimensional cube centered at the origin and with side length $2n$. By the ergodic theorem, the limit

$$\lim_{n \to \infty} \frac{X(B_n)}{(2n)^d}$$

exists and is constant almost surely. The density $\lambda(X)$ is defined to be equal to this almost sure limit.

Next, we consider the random balls associated with the model. Consider a second probability space $\Sigma = (\mathbb{R}^+)^{\mathbb{R}^d}$ equipped with a product measure (to be denoted by P_2) and on which we define nonnegative i.i.d. random variables $\rho_x(\sigma) := \sigma_x$, for all $x \in \mathbb{R}^d$ and $\sigma \in \Sigma$. We denote by ρ a random variable with the same distribution as the ρ_x's. If the occurrences of X are $(x_1, x_2, \dots)$, we write $\rho_i := \rho_{x_i}$, for all i. Let, for $x \in \mathbb{R}^d$ and $a \geq 0$, $S(x, a)$ be the closed ball with radius a centered at x, that is, $S(x, a) = \{y \in \mathbb{R}^d \mid |y - x| \leq a\}$.

We make the following definitions:

1. The *occupied region* C is defined as $C := \bigcup_{i=1}^{\infty} S(x_i, \rho_i)$.
2. The *vacant region* V is defined as $V := \mathbb{R}^d \setminus C$.

The Boolean model just described will be denoted by (X, ρ). The model is thus defined on the probability space $(\Omega \times \Sigma)$ with product measure $P = P_1 \times P_2$. Thus, the radii are independent of the point process X.

In the case where X is a Poisson process with finite density and ρ is a bounded random variable, Menshikov (1986) has shown that there exists a critical density $0 < \lambda_c < \infty$ such that, for $\lambda > \lambda_c$, C contains an unbounded component a.s. Furthermore, in two dimensions and the same Poisson setup, Roy (1990) has shown that there exists a $0 < \lambda_c < \infty$ such that, for $\lambda > \lambda_c$, C contains an unbounded component but V does not and, for $\lambda < \lambda_c$, V contains an unbounded component, but C does not. In three or more dimensions, in analogy with discrete percolation, one expects to have two critical intensities λ_c and λ_c^* such that $0 < \lambda_c < \lambda_c^* < \infty$, and the following hold:

1. for $\lambda < \lambda_c^*$, V contains an unbounded component, while C does not;
2. for $\lambda > \lambda_c$, C contains an unbounded component, while V does not;
3. for $\lambda \in (\lambda_c, \lambda_c^*)$, both C and V contain unbounded components.

We will be concerned with the *number* of unbounded components in C or V. To this end, we pick a vector t such that T_t acts ergodically, and we observe that the map T_t induces a transformation from Ω into itself (to be denoted by $T_{t,1}$) defined by the requirement that the occurrences of $X(T_{t,1}(\omega))$ are $T_t(x_1), T_t(x_2), \dots$. Of course T_t also induces a transformation from Σ into itself (to be denoted by $T_{t,2}$) defined by $(T_{t,2}\,\sigma)_x = \sigma_{T_t^{-1}x}$, for all $x \in \mathbb{R}^d$. Hence, the definition of T_t can be extended to $\Omega \times \Sigma$ by $T_t(\omega, \sigma) = (T_{t,1}\omega, T_{t,2}\sigma)$ and this transformation corresponds to shifting a whole configuration of points and associated balls in space over the vector t. Now note that $T_{t,1}$ acts ergodically on Ω by assumption, and $T_{t,2}$ is a mixing transformation on Σ. It is a standard result in ergodic theory [see, e.g., Petersen (1983), Theorem 6.1] that this implies that T_t acts ergodically on $\Omega \times \Sigma$ and any event which is invariant under all translations T_t has probability 0 or 1. In particular, the number of unbounded components is invariant, and it is thus a consequence of ergodicity that this number is equal to a constant a.s. Our main results are the following.

THEOREM 2.1. *Consider the Boolean model (X, ρ) with bounded radii in $\mathbb{R}^d$, and suppose that*

$$P(\rho > R) > 0 \quad \text{for all } R > 0. \tag{2.1}$$

Then C contains at most one unbounded component a.s.

THEOREM 2.2. *Consider a Boolean model (X, ρ) in $\mathbb{R}^d$ with $\lambda(X) < \infty$. Suppose that, for some $a > 0$,*

$$E\big(\text{number of components of } V \cap [0, a]^d\big) < \infty \tag{2.2}$$

and furthermore that

$$P(\rho < \varepsilon) > 0 \quad \text{for all } \varepsilon > 0. \tag{2.3}$$

Then V contains at most one unbounded component. In the case $d = 2$, condition (2.2) is not needed to obtain uniqueness.

Condition (2.2) seems to be restrictive and we do not know how to remove it. However, all "nice" processes like the Poisson point process satisfy the condition, as we show in the following proposition.

PROPOSITION 2.1. *Consider a Boolean model* (X, ρ) *in* $\mathbb{R}^d$. *If*

$$E\Big(X\big([0, a]^d\big)^d\Big) < \infty, \tag{2.4}$$

then (2.2) *holds.*

Another way of removing condition (2.2) is to impose some restrictions on the radius random variable.

PROPOSITION 2.2. *In a Boolean model* (X, ρ) *with* X *an arbitrary stationary point process, if* ρ *is such that, for some* $0 < \alpha < 1$ *and* $b > 0$, *we have*

$$\inf_{z>0} P(\rho > z + b \mid \rho > z) > \alpha > 0, \tag{2.5}$$

then (2.2) *holds for suitable* a.

In the case where the Boolean model is driven by a stationary Poisson process, we do not need the assumption (2.1), (2.2) or (2.3) to obtain uniqueness of either the occupied or the vacant component. The independence property of the Poisson point process is sufficient to replace these conditions.

THEOREM 2.3. *For a Boolean model* (X, ρ) *driven by a stationary Poisson point process, we have the following*:

(i) C *contains at most one unbounded component a.s.*
(ii) V *contains at most one unbounded component a.s.*

The original proof of Burton and Keane (1989) for uniqueness in discrete percolation was based on the fact that it is impossible to accommodate a tree-like structure in the integer lattice in a stationary way; there are just not enough points. The whole idea works there because different clusters contain different vertices (of course) and the density of the vertices in space is finite. This idea will be used here as well, but as we are working in $\mathbb{R}^d$ rather than in $\mathbb{Z}^d$, we have to find other objects which have the property that their density is finite and which cannot belong to two different components. In the case of occupied components, these objects will be connected components of a small part of occupied regions; in the case of vacant components, they will be parts of the boundaries of the balls or local vacant components.

As will be clear from the proofs of the theorems, the unique unbounded occupied component, if it exists, when condition (2.1) is satisfied, will have an infinite volume. However, the same cannot be said about the unique unbounded vacant component, if it exists, when conditions (2.2) and (2.3) are satisfied. It could be that the vacant component is a long thin region in $\mathbb{R}^d$

whose volume is finite, even though it is unbounded. In the case when the driving point process is Poisson, it is relatively easy to see, using the independence property of the Poisson point process, that the vacant unbounded component will necessarily have infinite volume. It is precisely for this reason that we need the extra condition (2.2) in Theorem 2.2.

We will see in the last section that (2.1) and (2.3) cannot be removed from the statements of the theorems. We believe that (2.2) is not necessary for Theorem 2.2; however, we do not have any rigorous argument for this. Note that in Theorem 2.1 there is no restriction on X at all, but in Theorem 2.2 we need a moment condition on the point process. As a rule, it turns out that the vacancy is a little more delicate than occupancy, and we do not know how to get rid of the moment condition in Theorem 2.2.

3. Preliminary results. In this section, we prove two results which turn out to be very useful, but which are also interesting in their own right. The first result tells us that it is not a loss of generality to assume that the expected Lebesgue measure of the random balls is finite, as the whole space is covered a.s. if this expectation is infinite. This extends a result in Hall (1988) for Poisson point processes. The second result shows that if balls can be arbitrarily small, then the existence of vacancy is equivalent to the fact that any bounded subset of $\mathbb{R}^d$ has nonempty intersections with only finitely many balls a.s.

PROPOSITION 3.1. *Consider the model* (X, ρ) *in* $\mathbb{R}^d$. *If* $E\rho^d = \infty$, *then* $P(C = \mathbb{R}^d) = 1$.

PROPOSITION 3.2. *Consider the model* (X, ρ) *in* $\mathbb{R}^d$, *and suppose that* (2.3) *holds. Then the following two statements* (i) *and* (ii) *are equivalent*:

(i) $P(C = \mathbb{R}^d) < 1$.
(ii) *Any bounded region is intersected by only finitely many balls a.s.*

PROOF OF PROPOSITION 3.1. If X has infinite density, we can "thin" the process in some stationary way to obtain a finite-density process. If we prove the proposition for this process, then it is certainly true for the original infinite-density process. Therefore, we may assume that the density $\lambda(X)$ of X is finite and is equal to 1. Let C_n be the ball centered at the origin and with radius $2^{n/d}$, $n \in \mathbb{N}$. (C_n is nonrandom.) Note that $\mu_d(C_{n+1}) = c_d 2^{n+1} = 2\mu_d(C_n)$, where c_d is a constant depending only on the dimension. From the ergodic theorem, we have that, for n large enough (depending on the realization, $\frac{3}{4}V_n \le X(C_n) \le \frac{5}{4}V_n$, where $V_n = \mu_d(C_n)$. Now we write, for large enough n, $X(C_{n+1} \setminus C_n) = X(C_{n+1}) - X(C_n) \ge \frac{3}{4}V_{n+1} - \frac{5}{4}V_n = \frac{6}{4}V_n - \frac{5}{4}V_n = \frac{1}{4}V_n$. Hence, for n large enough, the "annulus" $C_n \setminus C_{n-1}$ contains at least $b_d 2^{n+1}$ points of the point process, where b_d is another constant depending only on the dimension.

Now let E_n be the event that C_0 is *not* completely covered by a ball which is centered in $C_n \setminus C_{n-1}$. Furthermore, let A_m be the event that m is the first index such that $X(C_n \setminus C_{n-1}) \geq b_d 2^{n+1}$, for all $n \geq m$. It follows from the above that the A_m's form a partition of the probability space. Now write

$$
\begin{aligned}
P\left(\bigcap_{k=m}^{\infty} E_k \middle| A_m \right)
&\leq P\Bigg(\bigcap_{k=m}^{\infty} (\text{all balls centered in } C_k \setminus C_{k-1} \\
&\qquad\qquad \text{have radius at most } 2^{k/d} + 1) \Big| A_m \Bigg) \\
&\leq \prod_{k=m}^{\infty} P(\rho \leq 2^{k/d} + 1)^{b_d 2^{k+1}},
\end{aligned}
$$

where the last inequality follows from the independence of the radii and the point process. It suffices to show that this expression equals zero. For k large enough, $2^{k/d} + 1 \leq 2^{(k+1)/d}$, so if we replace $k+1$ by k, for k large, the terms in the product are at most $P(\rho \leq 2^{k/d})^{b_d 2^k}$. Now, for m large enough, we find

$$
\begin{aligned}
&\prod_{k=m}^{\infty} P(\rho \leq 2^{k/d})^{b_d 2^k} \\
&\quad = \prod_{k=m}^{\infty} P(\rho^d \leq 2^k)^{b_d 2^k} \\
&\quad \leq \prod_{k=m}^{\infty} \left\{ P(\rho^d \leq 2^k) \cdot P(\rho^d \leq 2^k + 1) \cdots P(\rho^d \leq 2^{k+1} - 1) \right\}^{b_d} \\
&\quad = \left\{ \prod_{k=2^m}^{\infty} P(\rho^d \leq k) \right\}^{b_d} \\
&\quad = \left\{ \prod_{k=2^m}^{\infty} (1 - P(\rho^d > k)) \right\}^{b_d}.
\end{aligned}
$$

This expression is zero if and only if $\sum_{k=2^m}^{\infty} P(\rho^d > k) = \infty$, which is equivalent to $E\rho^d = \infty$, proving the proposition. □

PROOF OF PROPOSITION 3.2. We first show that (i) follows from (ii). Statement (ii) implies that there exists a $k \in \mathbb{N}$ such that the unit cube is intersected by exactly k balls with positive probability. Hence there is a set of indices $\{i_1, \dots, i_k\}$ such that, with positive probability, the only balls intersecting the unit cube are $S(x_{i_1}, \rho_{i_1}), \dots, S(x_{i_k}, \rho_{i_k})$. Finally, there exists a number $\delta > 0$ such that there is a positive probability that, in addition, $|x_{i_j} - x_{i_l}| > \delta$, for all $j \neq l$. Leaving the rest of the realization as it is, we may now reduce the radii of the designated balls to a number smaller than $\frac{1}{3}\delta$.

The assumptions imply that this can be done with positive probability, and after the reduction there is a set with positive Lebesgue measure vacant in the unit cube. Hence there is a positive probability that vacancy exists in the unit cube and the ergodic theorem implies that vacancy exists a.s.

To show that (ii) follows from (i), we assume that there are infinitely many balls intersecting the unit circle C_0 with positive probability. Now a contradiction arises as follows. If infinitely many balls intersect C_0, there must exist a half line ℓ through the origin such that there is a subsequence $i_1, i_2, \dots$ such that $S(x_{i_k}, \rho_{i_k}) \cap C_0 \neq \emptyset$, for all k, and such that the angle between ℓ and the line through the origin and x_{i_k} tends to zero as k tends to infinity. However, it is easy to see that this implies that there must be a half-space which is completely covered by balls. Now consider the random variables Y_n and Z_n, $n \in \mathbb{Z}$, defined as follows: $Y_n = 1$ if $[0,1]^{d-1} \times [n, n+1]$ is completely covered, otherwise $Y_n = 0$. Also, $Z_n = 1$ if $[n, n+1] \times [0,1]^{d-1}$ is completely covered and zero otherwise. Note that $\{Y_n\}$ and $\{Z_n\}$ are stationary and ergodic stochastic processes (here we need the fact that each of the T_{t_i}'s from Section 2 act ergodically), but if a half-space is completely covered, at least one of the processes ($\{Y_n\}$, say) satisfies $Y_n = 1$, for all n large enough, or $Y_n = 1$, for all $-n$ large enough. However, ergodicity then implies that $P(Y_n = 1) = 1$, which contradicts our assumption (i). □

4. Uniqueness for occupancy: Proof of Theorem 2.1. It is quite common to prove uniqueness by first showing that the number of unbounded components can be only 0, 1 or ∞ and then ruling out the possibility of having infinitely many unbounded components. We will follow this route, too, and start with the following lemma, the idea of which goes back to Newman and Schulman (1981).

LEMMA 4.1. *Under the conditions of Theorem* 2.1, *the occupied region* C *contains either* 0, 1 *or infinitely many unbounded components a.s.*

PROOF. The following reasoning is more or less standard [Newman and Schulman (1981)]. Suppose that the number of unbounded components in C is $K \geq 2$ a.s. Then, for n large enough, there is a positive probability that B_n intersects them all. Also, there is an index m such that the event $E := \{$all unbounded components intersect B_n, $x_m \in B_n\}$ has positive probability. (Note that neither n nor m is random.) Hence it follows from the assumptions that also the event $E^* = E \cap \{\rho_m > 2n\sqrt{d}\}$ has positive probability. On E^*, however, the number of unbounded components is just one, because the ball $S(x_m, \rho_m)$ connects all K former components. This is the desired contradiction. □

LEMMA 4.2. *Suppose condition* (2.1) *holds. Let* $A \subset \mathbb{R}^d$ *be a convex set with finite diameter, that is,* $\sup_{x, y \in A} |x - y| < K$ *for some* $K < \infty$. *Let* $C(A)$

denote the (random) region

$$C(A) := \bigcup_{x_i \in A} S(x_i, \rho_i).$$

Thus, $C(A)$ is the occupied region formed by points in A. Then the number of connected components in $A \cap C(A)$, to be dentoed by Y_A, has finite expectation.

PROOF. The set A is convex, and hence its intersection with any ball, if not empty, consists of one component exactly. Hence Y_A is at most k if $X(A) = k$. Furthermore, Y_A can only then be larger than 1 if all balls centered in A have radius at most K. Now condition on the number of occurrences in A to obtain

$$\begin{aligned} E\{Y_A|X(A) = k\} &= \sum_{n=1}^{\infty} nP(Y_A = n|X(A) = k) \\ &\le P(Y_A = 1|X(A) = k) + \sum_{n=2}^{k} n\{P(\rho \le K)\}^k \\ &\le 1 + \tfrac{1}{2}k(k+1)\{P(\rho \le K)\}^k. \end{aligned}$$

Hence

$$EY_A \le \sum_{k=0}^{\infty} \left(1 + \tfrac{1}{2}k(k+1)\{P(\rho \le K)\}^k\right)P(X(A) = k), \tag{4.1}$$

and this sum is finite because $P(\rho \le K) < 1$ by condition (2.1). □

We need one more lemma, the following combinatorial result of Gandolfi, Keane and Newman (1989).

LEMMA 4.3 [Gandolfi, Keane and Newman (1992)]. *Let S be a set and let R be a finite subset of S. Suppose that the following hold:*

(a) *For all $r \in R$, we have a family $(C_r^{(1)}, C_r^{(2)}, C_r^{(3)})$ of disjoint nonempty subsets of S, not containing r, and* $\mathrm{card}(C_r^{(i)}) \ge K$ *for all i and r.*

(b) *For all $r, r' \in R$, one of the following events occurs, writing C_r for $\bigcup_{i=1}^{3} C_r^{(i)}$.*

(i) $(\{r\} \cup C_r) \cap (\{r'\} \cup C_{r'}) = \emptyset$.

(ii) $\exists$ *i, j such that $C_r^{(i)} \supset \{r'\} \cup C_{r'} \setminus C_{r'}^{(j)}$ and $C_{r'}^{(j)} \supset \{r\} \cup C_r \setminus C_r^{(i)}$.*

Then $\mathrm{card}(S) \ge K(\mathrm{card}(R) + 2)$.

PROOF OF THEOREM 2.1. In order to reach a contradiction, we assume that there exist infinitely many unbounded occupied components a.s. Define, for each integer n and $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$,

$$B_n^z = B_n^0 + (z_1, \dots, z_d),$$

where of course

$$B_n^0 = \{x \in \mathbb{R}^d | -n \le x_i \le n, i = 1, \dots, d\}.$$

(Note that in this notation $B_n = B_n^0$.) As in Burton and Meester (1993), we call the connected components of $B_n^z \cap C(B_n^z)$ *local components of size n.*

For N large enough, it follows from the proof of Lemma 4.1 that the following event has positive probability, say, η:

$$\begin{aligned} E^0(N) = \{&B_N^0 \text{ is covered by a ball centered in } B_N^0 \text{, and from} \\ &\text{the boundary of this ball at least three disjoint} \qquad (4.2) \\ &\text{occupied components ("branches") radiate to infinity}\}. \end{aligned}$$

We choose N such that $2N \in I$ (see Section 2) in order to make sure that we can apply the ergodic theorem. Now we choose K very large; we shall see at the end of the proof how large exactly. We choose M so large that the following event has probability at least $\frac{1}{2}\eta$:

$$\begin{aligned} E^0(N, M) = E^0(N) \\ \cap \{&\textit{All three branches of } B_N^0 \textit{ contain balls centered} \\ &\textit{in at least } K \textit{ different boxes } B_N^{2Nz} \subset B_{MN}^0, z \ne (0, \dots, 0)\}. \end{aligned}$$

The events $E^z(N)$ and $E^z(N, M)$ are defined by translating this event over the vector z.

It follows from the ergodic theorem that, for L sufficiently large (depending again on the realization), the set

$$R = \{z | B_{MN}^{2Nz} \subset B_{LN}^0, E^{2Nz}(N, M) \text{ occurs}\}$$

has cardinality at least $\frac{1}{4}\eta L^d$. For $z \in R$, we have $Y_{B_N^{2Nz}} = 1$ by definition (remember the definition of Y_A in Lemma 4.2), and if we denote by $C_z^{(1)}$, $C_z^{(2)}$ and $C_z^{(3)}$ the set of all local components of size N in each of its three branches contained in B_{MN}^{2Nz}, then $C_z^{(i)} \cap C_z^{(j)} = \varnothing$, for $i \ne j$, and $\text{card}(C_z^{(i)}) \ge K$, for all i. Furthermore, for $z, z' \in R$, $z \ne z'$, if we identify z and z' with the only local component of B_N^{2Nz}, it is not difficult to check that (i) in Lemma 4.3 occurs if the points of X in B_N^{2Nz} are in a different component than the points of X in $B_N^{2Nz'}$, and that (ii) occurs otherwise. Hence, we conclude from Lemma 4.3 that

$$\sum_{\{z | B_N^{2Nz} \subset B_{LN}^0\}} Y_{B_N^{2Nz}} \ge K\left(\frac{\eta}{4} L^d + 2\right). \qquad (4.3)$$

To derive a contradiction, we remark that it follows from Lemma 4.2 that $E\{Y_{B_N^0}\} < \infty$ and from the ergodic theorem that, for L sufficiently large, we have

$$\sum_{\{z | B_N^{2Nz} \subset B_{LN}^0\}} Y_{B_N^{2Nz}} \le 2L^d E\{Y_{B_N^0}\}. \qquad (4.4)$$

Hence we find from (4.3), (4.4) and Lemma 4.2 that, for L large enough,

$$K\left(\frac{\eta}{4}L^d + 2\right) \leq 2L^d E\{Y_{B_N^0}\} < \infty,$$

which gives the desired contradiction if we choose K larger than $8\eta^{-1}E\{Y_{B_N^0}\}$. □

5. Uniqueness for vacancy: Proof of Theorem 2.2. As already indicated at the end of Section 2, the vacancy case is a little trickier than the occupancy case. We start with three lemmas which, although not in precisely the same form, are in spirit the analogues of Lemma 4.1, 4.2 and 4.3, respectively. We shall abuse notation a bit and write $\mu_{d-1}(\cdot)$ for the $(d-1)$-dimensional measure of unions of $(d-1)$-dimensional subsets of $\mathbb{R}^d$.

LEMMA 5.1. *Suppose condition* (2.3) *in Theorem* 2.2 *holds. Then the number of unbounded components in* V *equals* 0, 1 *or* ∞ *a.s.*

PROOF. The idea is the same as in Lemma 4.1, although a little bit more care is needed. Again, we suppose that there exist $K \geq 2$ unbounded vacant components a.s. and, again, n is chosen such that the box B_n has positive probability of intersecting them all. According to Proposition 3.2, there exists a finite set $A = \{a_1, \dots, a_m\}$ such that, in addition, with positive probability

$$A = \{i \in \mathbb{N} | S(x_i, \rho_i) \cap B_n \neq \varnothing\}.$$

Let F be the event

$$F = \{B_n \text{ intersects all unbounded vacant components}\}$$
$$\cap \{S(x_i, \rho_i) \cap B_n \neq \varnothing \text{ iff } i \in A\}.$$

Furthermore, there is a (nonrandom) number N such that

$$F_N := F \cap \{S(x_i, \rho_i) \subset B_N, i \in A\}$$

has positive probability, and, finally, there is a $\delta > 0$ such that

$$P(F_N, |x_i - x_j| > \delta, \text{for all } i \neq j \in A) > 0.$$

Now (2.2) allows us to reduce the radii of the balls $S(x_i, \rho_i)$, $i \in A$, to at most $\frac{1}{3}\delta$. The configuration outside B_N remains unchanged by this procedure, so no extra unbounded vacant components arise, but then the reduction of the radii creates a configuration with only one vacant component, a contradiction. □

LEMMA 5.2.

(i) *The expected number of balls intersecting the unit cube is finite.*
(ii) *It is the case that*

$$E\{\mu_{d-1}(\partial(C) \cap [0,1]^d)\} < \infty, \tag{5.1}$$

where $\partial(\cdot)$ *denotes the boundary of a set in* $\mathbb{R}^d$.

Note that the measure of the boundary causes no problems: Proposition 3.2 guarantees that in case of existence of vacancy, only finitely many balls intersect $[0, 1]^d$.

PROOF OF LEMMA 5.2. According to Proposition 3.1, we can assume that $E\rho^d < \infty$. First we show that the expected number of balls which intersect the unit cube is finite. Writing I^d for the unit cube, $x + A$ for the set $\{x + y | y \in A\}$ and $\mathbf{O}$ for the origin in $\mathbb{R}^d$,

$$
\begin{aligned}
E\left\{\sum_{i=1}^{\infty} 1_{\{S(x_i, \rho_i) \cap I^d \neq \varnothing\}}\right\} &= E\left\{\sum_{i=1}^{\infty} 1_{\{x_i \in I^d + S(\mathbf{O}, \rho_i)\}}\right\} \\
&= E\left\{\sum_{i=1}^{\infty} 1_{\{x_i \in I^d + S(\mathbf{O}, \rho)\}}\right\} \\
&= E\left\{X\left(I^d + S(\mathbf{O}, \rho)\right)\right\} \\
&= E\left\{E\left\{X\left(I^d + S(\mathbf{O}, \rho)\right) | \rho\right\}\right\} \\
&= E\left\{\lambda(X)\mu_d\left(I^d + S(\mathbf{O}, \rho)\right)\right\} \\
&\leq \lambda(X) E\left\{(2\rho + 1)^d\right\} < \infty.
\end{aligned}
$$

Since, for every x and $a > 0$, $I^d \cap \partial S(x, a)$ encloses a convex region in I^d, we have that $\mu_{d-1}(I^d \cap \partial S(x, a)) \leq \mu_{d-1}(I^d) = 2d$, uniformly in x and a. Hence the left-hand side of (5.1) is bounded from above by $\lambda(X)E\{(2\rho + 1)^d\}2d < \infty$ as required. □

The final ingredient is another version of the combinatorial result in Lemma 4.3.

LEMMA 5.3. *Let R be a finite set, and let μ be any measure in $\mathbb{R}^d$. With each $r \in R$, we associate a family $(C_r^{(1)}, C_r^{(2)}, C_r^{(3)})$ of disjoint (up to μ-measure zero) nonempty subsets of $\mathbb{R}^d$ such that $\mu(C_r^{(i)}) \geq K$ for all i and r. Suppose that, for all $r, r' \in R$, one of the following events occurs, writing C_r for $\bigcup_{i=1}^{3} C_r^{(i)}$:*

(i) $C_r \cap C_{r'} = \varnothing$.
(ii) $\exists\ i, j$ *such that* $C_r^{(i)} \supset C_{r'} \setminus C_{r'}^{(j)}$ *and* $C_{r'}^{(j)} \supset C_r \setminus C_r^{(i)}$.

Then $\mu(\bigcup_{r \in R} C_r) \geq K(\mathrm{card}(R) + 2)$.

PROOF. The proof is just a copy of the proof of the lemma in Burton and Keane (1989). □

PROOF OF THEOREM 2.2. Now we have gathered all the machinery we need to prove Theorem 2.2. In fact, the proof proceeds more or less like the proof of Theorem 2.1. We assume again that the number of unbounded vacant components is infinite a.s. For N large enough, it follows from the proof of Lemma 5.1 that the following event has positive probability, say; η:

$$D^0(N) = \{B_N^0 \text{ intersects an unbounded vacant component } U \text{ such that } U \setminus B_N^0 \text{ consists of at least three unbounded components ("branches")}\}. \tag{5.2}$$

Again, $D^z(N)$ is this event translated over the vector a. At this point, we distinguish between $d = 2$ and $d \geq 3$. We start with $d = 2$. Define $D^z(N, M)$ similar to the corresponding event $E^z(N, M)$ in Section 4, but here the requirement is that the boundary of each of the branches has one-dimensional Lebesgue measure at least K within B_{MN}^z. (In two dimensions, any unbounded branch must have an infinite boundary.) We now define R as

$$R = \{z | B_{MN}^{2Nz} \subset B_{LN}^0,\ D^{2Nz}(N, M) \text{ occurs}\}.$$

Note that there might be more than one unbounded vacant component for which the event $D^0(N)$ occurs. In such a case, we just choose one of them.

Using Lemma 5.3 and the fact that any two branches do not share any boundary part of positive one-dimensional measure, we see that the arguments which led to (4.3) here lead to

$$\sum_{\{z | B_N^{2Nz} \subset B_{LN}^0\}} \mu_1(\partial(C) \cap B_N^{2Nz}) \geq K\left(\frac{\eta}{4}L^2 + 2\right). \tag{5.3}$$

In this situation, (4.4) is replaced by

$$\sum_{\{z | B_N^{2Nz} \subset B_{LN}^0\}} \mu_1(\partial(C) \cap B_N^{2Nz}) \leq 2L^2 E\{\mu_1(\partial(C) \cap B_N^0)\} < \infty, \tag{5.4}$$

where the last inequality is just Lemma 5.2(ii). From (5.3) and (5.4) we now find that, for large L, we have

$$K\left(\frac{\eta}{4}L^2 + 2\right) \leq 2L^2 E\{\mu_1(\partial(C) \cap B_N^0)\} < \infty,$$

and the contradiction again arises by taking K large enough. Note that condition (2.2) has not been used in this part of the proof!

For $d \geq 3$, we cannot assume that an unbounded component will have a boundary of infinite $(d-1)$-dimensional Lebesgue measure. Here we use condition (2.2) to obtain a local vacant component argument, which is analogous to the local occupied component argument in the proof of Theorem 2.1. As such we omit the details of the proof. □

Next, we indicate how to prove Propositions 2.1 and 2.2. For both these propositions we need the following geometric result.

PROPOSITION 5.4. *If k d-dimensional balls intersect $[0, 1]^d$, then the vacant region inside $[0, 1]^d$ has at most $C_d k^d$ components, where C_d is a constant depending only on the dimension.*

PROOF. For case of exposition, we first give the proof for the case $d = 3$. Without loss of generality we can assume that no ball is contained in the union of the others since adding this particular ball would not affect the number of vacant components. So consider three balls, none of which is contained in the union of the other two. We claim that the intersection of the boundaries of these balls consists of at most two points. To see this, note that if the intersection of the boundaries of the first two balls is not empty, it is the boundary γ of some circle. Denote the plane containing this circle by H. The intersection of all three boundaries can only then be larger than two points if the intersection of the third ball with H is γ. It is easy to check that in this case there is a ball which is contained in the union of the other two, a contradiction. We call a point in the intersection of three boundaries a triple point. Now each vacant component in $[0,1]^d$ which does not intersect the boundary of the cube contains at least one triple point on its boundary and a triple point can belong to only one vacant component. Hence there are at most $2\binom{k}{3}$ vacant components which do not intersect the boundary of the unit cube. Next, we look at the intersection of vacant components with the faces of the cube. The intersection of a three-dimensional ball with a face is a two-dimensional ball. Any vacant component which intersects a face of the unit cube but no edge must contain at least one point of intersection of two such two-dimensional balls. For each face therefore there can be at most $2\binom{k}{2}$ such vacant components, giving a total number of at most $12\binom{k}{2}$. Finally, an analogous argument shows that the number of vacant components which intersect an edge is at most $12(k+1)$. Hence, the total number of vacant components is bounded by $C_3 k^3$ for a suitable constant C_3. The argument in higher dimensions is essentially the same and is omitted. □

Now Proposition 2.1 is immediate from Proposition 5.4, and Proposition 2.2 follows from Proposition 5.4 after simple calculations similar to the calculations in the proof of Lemma 4.2.

6. The Poisson case: Proof of Theorem 2.3. We begin with the occupancy case. Observe that in the proof of Theorem 2.1, we needed condition (2.1) in three different places, namely, (i) to prove Lemma 4.1, (ii) to show that $EY_A < \infty$ in (4.1) and (iii) to show that the event in (4.2) has positive probability. We shall use the independence property of the Poisson process to show all the above, without taking recourse to condition (2.1). This, coupled with the other arguments of Section 4, will prove (i) of Theorem 2.3.

For (ii), indeed, if X is a Poisson point process of intensity λ, from (4.1) we have

$$\begin{aligned} EY_A &\le \sum_{k=0}^{\infty} \left(1 + \tfrac{1}{2}k(k+1)\right)\left(P(\rho \le K)\right)^k P(X(A) = k) \\ &= \sum_{k=0}^{\infty} \left(1 + \tfrac{1}{2}k(k+1)\right)\left(P(\rho \le K)\right)^k (k!)^{-1} \exp(-\lambda\mu_d(A))(\lambda\mu_d(A))^k \\ &< \infty. \end{aligned}$$

For (i), we need to show the following.

LEMMA 6.1. *For a Boolean model (X, ρ) driven by a Poisson point process of intensity λ, C contains either* 0, 1 *or infinitely many unbounded components a.s.*

PROOF. In view of Lemma 4.1, we only need to consider the case when the random variable ρ is bounded. Thus let $R > 0$ be such that

$$P(\rho > R) = 0,$$
$$P(R - \eta < \rho \leq R) > 0 \quad \text{for any } \eta > 0.$$

Now suppose (X, ρ) admits $K \geq 2$ unbounded occupied components a.s. If we remove all the balls centered inside a box B, then the resulting picture should contain at least K unbounded occupied components a.s. In other words, using the notation from Section 4, $C(B^c)$ admits at least K unbounded occupied components a.s. (but, of course, not infinitely many).

Given a box B and $\varepsilon > 0$, consider the event

$$A(B, \varepsilon) := \{d(U, B) \leq R - \varepsilon \text{ for any unbounded occupied component } U \text{ in } C(B^c)\},$$

where $d(\cdot)$ denotes Euclidean distance in $\mathbb{R}^d$. Now partition the box into cubic cells with edge length $a > 0$, and let $\mathscr{C}_a = \{W_1, \dots, W_N\}$ denote the collection of all the cells which are adjacent to the boundary of B. Clearly, for a box B and an $\varepsilon > 0$ for which $A(B, \varepsilon)$ occurs, we can find $a = a(B, \varepsilon) > 0$ and $\eta = \eta(a) > 0$ such that, for any point $x \notin B$ with $d(x, B) \leq R - \varepsilon/2$, there exists a cell $W = W(x) \in \mathscr{C}_a$ for which we have $\sup_{y \in W} d(x, y) \leq R - 2\eta$. This means that if we center in each cell of $\mathscr{C}_a$ a ball with radius between $R - \eta$ and R, then the region $\{x \notin B: d(x, B) \leq R - \varepsilon/2\}$ will be completely covered by these balls.

Let $E = E(a, \eta)$ be the event that each cell in $\mathscr{C}_a$ contains at least one Poisson point with an associated ball of radius between $R - \eta$ and R. Since E depends on the configuration inside the box B, $A(B, \varepsilon)$ depends on the configuration outside the box B and the radii are independent of the Poisson process, we have

$$P(A(B, \varepsilon) \cap E) = P(A(B, \varepsilon))P(E).$$

Now if both $A(B, \varepsilon)$ and E occur, then there is only one unbounded occupied component. Now $P(E) > 0$, so in order to arrive at a contradiction, we need to show that there exist a box B and an $\varepsilon > 0$ such that $P(A(B, \varepsilon)) > 0$.

Since (X, ρ) admits $K \geq 2$ unbounded occupied components, we can find a box B so large that, with positive probability, $d(U, B) < R$ for every unbounded component U of C. Also, the radius of any ball is at most R, so, with positive probability, $d(U, B) < R$ for every unbounded occupied component U in $C(B^c)$. Thus for this B we can find $\varepsilon > 0$ such that $A(B, \varepsilon)$ occurs with positive probability. □

Finally, for (iii) we use arguments as in the last lemma to show that the following event has positive probability and replaces $E^0(N)$ in (4.2):

> $\{B_N^0$ is covered by balls centered in B_N^0 and, from the boundary of the union of these balls, at least three disjoint occupied "branches" radiate to infinity$\}$.

This is enough to prove Theorem 2.3(i).

For the proof of (ii) of Theorem 2.3, we first observe that (2.4) holds for any bounded set B, and thus (2.2) holds. Moreover, to prove Theorem 2.2 we needed condition (2.3) in Proposition 3.2, in Lemma 5.1 and to show that $D^0(N)$ in (5.2) has positive probability. However, in the proof of Proposition 3.2 we note that we do not need condition (2.3) to show that whenever $V \neq \emptyset$ a.s. there can be at most finitely many balls having nonempty intersection with a fixed bounded region. We shall use only this part of Proposition 3.2 and as such condition (2.3) is irrelevant here. Lemma 5.1, however, needs to be reproved because it crucially uses (2.3). Although we use the independence property in more or less the same way as in Lemma 6.1—removing all balls from a box B instead of adding as in Lemma 6.1—we have to be more careful because balls centered outside the box could have nonempty intersection inside the box and disallow us from connecting the vacant components in the box. It will be clear from the proof of the following lemma that $P(D^0(N)) > 0$ remains true in this Poisson setup. Thus, to complete the proof of Theorem 2.3, we need the following lemma.

LEMMA 6.2. *For a Boolean model* (X, ρ) *driven by a stationary Poisson point process, the vacant region* V *contains either* 0, 1 *or infinitely many unbounded components a.s.*

PROOF. Suppose there are $K \geq 2$ unbounded components of V. For $A \subset \mathbb{R}^d$, let $V(A)$ denote the vacant region in $\mathbb{R}^d$ which we obtain after removing all points (and associated balls) in A^c. Let $B_n = [-n, n]^d$, for some integer $n > 0$, be a box big enough such that, for some $\eta > 0$,

$$P\,(\text{all unbounded vacant components in } V(B_n^c) \text{ have nonempty intersection with } B_n) > \eta.$$

There exist integers $N > 0$ and $R > 0$ such that

$$P\,(\text{all unbounded vacant components in } V(B_n^c) \text{ have nonempty intersection with } B_n \text{ and there are at most } N \text{ balls each with a radius less than or equal to } R \text{ which have nonempty intersection with } B_n) > \frac{\eta}{2}.$$

Now consider the annulus $A = [-n - R, n + R]^d \setminus B_n$ and partition this by the integer lattice. Let $\mathscr{C}$ be the finite collection of all cells of this lattice in A.

Since $\mathscr{C}$ is finite, we can find (nonrandom) cells $W_1, \dots, W_N$ in $\mathscr{C}$ such that, for $W := W_1 \cup \cdots \cup W_N$,

$$P\Big(\text{all unbounded vacant components in } V\big((B_n \cup W)^c\big) \text{ have nonempty intersection with } B_n \text{ and } B_n \text{ has empty intersection with any ball not centered in } B_n \cup W\Big) > 0.$$

Denote the event in the last expression by E. Then E depends only on the points outside $B_n \cup W$ and the associated radii. Hence it follows from the independence property of the Poisson point process that

$$P(E \cap \{X(B_n \cup W) = 0\}) = P(E)P(X(B_n \cup W) = 0) > 0.$$

However, the event in the previous expression implies that all the K vacant components are connected by the vacant region B_n, that is, there exists only one vacant component. This contradiction establishes Lemma 6.2. □

7. Examples. We shall indicate how one can construct some examples to show that (2.1) and (2.3) cannot be removed from the statements of our results. Our first example will be the situation in which the radii are bounded from below and above. We will show that, for any integer K, we can construct a point process which admits K unbounded occupied and $K + 1$ unbounded vacant components. In the second example, we only require the radii to be bounded from above. There, we shall construct a point process which admits the coexistence of exactly two unbounded occupied components.

EXAMPLE 1 (The case $\delta < \rho < R$ a.s. for some $\delta,\ R > 0$). Without loss of generality, let us assume that $R < 1$. We borrow a bit from Burton and Keane (1991) here. They considered discrete percolation in $\mathbb{Z}^2$, and they showed that there exists for any K a stationary and ergodic (under the actions of $\mathbb{Z}^2$) measure μ_K on $\{0,1\}^{\mathbb{Z}^2}$ such that the following event has μ_K-measure 1: "There exist exactly K occupied components $C_1, C_2, \dots, C_k$, which are all infinite and such that $\min_{z \in C_i,\ z' \in C_j}\{|z - z'|\} \geq 2$ for all $i \neq j$, and such that

$$C_i = \{\dots, z_{-1}^i, z_0^i, z_1^i, \dots\},$$

where $|z_j^i - z_{j+1}^i| = 1$, for all j, and $|z_j^i - z_k^i| \geq 2$, for all j, k with $|j - k| > 1$." In words, the occupied components are all infinite and just look like doubly infinite strips of width 1.

We construct a stationary point process as follows. Consider any straight-line segment of length 1 between two occupied neighbors in $\mathbb{Z}^2$. We put $\lceil 3\delta \rceil$ points evenly spaced on these line segments. Finally, we shift the whole configuration over a vector (u, v), where u and v are chosen randomly according to Lebesgue measure on the unit interval. This procedure results in a stationary and ergodic point process, and it is easy to check that under the assumptions on ρ, each of the clusters C_i give rise to one unbounded occupied component, while the complement of all these components form $K + 1$ vacant unbounded components.

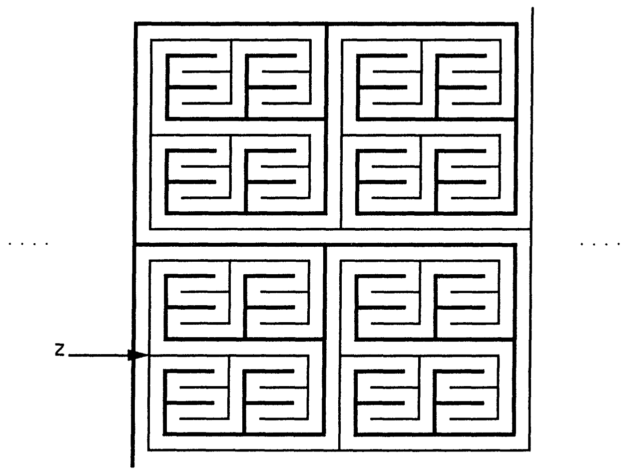

FIG. 1. *The two tree-like structures in the plane. The picture is to be continued in the obvious self-similar way.*

EXAMPLE 2 (The case $\rho < R$ a.s.). We are not going to be very precise here, as constructions of the type we will consider already appear in different places in the literature [e.g., Rudolph (1979), Burton, Choi and Meester (1992) or Burton and Meester (1993)]. The idea is to imbed in the plane in a stationary (and ergodic) way, two tree-like structures as in Figure 1. We make sure that the two trees never get closer than $3R$ to each other. The trees consist of straight-line segments of various lengths, and we order these lengths in the natural increasing order, say, $l_1, l_2, \dots$. Depending on the precise distribution of ρ, we define a_n as the smallest positive integer with the following property: If we put a_n points evenly spaced on a line segment of length l_n, the probability that the whole line segment is contained in the union of all balls centered at these a_n points is at least $1 - 2^{-n}$. Having defined a_n for all n, the point process is obtained by putting a_n points, evenly spaced, on each line segment of length l_n. Now consider the point z in Figure 1. There is a unique sequence of line segments along which we can radiate to infinity starting from z, and we denote this sequence by $k_1(z), k_2(z), \dots$. From the Borel–Cantelli lemma, it follows that with probability 1, for all n large enough, the line segment $k_n(z)$ is completely covered by balls centered at $k_n(z)$. Thus this gives rise to at least one unbounded occupied component. Obviously, the two trees give rise to different unbounded occupied components because the radii are never larger than R. However, because of the fact that, for z and z' of the same tree, $k_n(z) = k_n(z')$ for all n large enough (depending on z and z' of course), each tree gives rise to exactly one unbounded occupied component. We conclude that there are exactly two unbounded occupied components a.s.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTRECHT
P.O. BOX 80.010
3508 TA UTRECHT
THE NETHERLANDS

INDIAN STATISTICAL INSTITUTE
7 S.J.S. SANSANWAL MARG
NEW DELHI–110016
INDIA