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OPTIMAL ASYMPTOTIC TESTS OF COMPOSITE HYPOTHESES FOR CONTINUOUS TIME STOCHASTIC PROCESSES

By B. L. S. PRAKASA RAO
Indian Statistical Institute

SUMMARY. Consider a stochastic process $\{X_t, t \geq 0\}$ whose distributions depend on an unknown parameter $(\gamma, \boldsymbol{\theta})$. A locally asymptotically most powerful test, for testing the composite hypothesis $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$ in the presence of a nuisance parameter $\boldsymbol{\theta}$ is developed following the concept of $C(\alpha)$–tests introduced by Neyman. Results are illustrated by means of example of process $\{X(t), t \geq 0\}$ satisfying the linear stochastic differential equation $dX(t) = (\gamma X(t) + \theta)dt + dW(t), t \geq 0$.

1. Introduction

Neyman (1959) developed the notion of $C(\alpha)$–tests for testing composite statistical hypotheses. He suggested a method by which a locally asymptotically most powerful test can be constructed for testing a composite hypothesis $H_0 : \gamma = \gamma_0$ against the alternative $H_1 : \gamma \neq \gamma_0$ when the observations $\{X_k, 1 \leq k \leq n\}$ are independent and identically distributed whose distributions $F(x; \gamma, \boldsymbol{\theta})$ depending on an unknown scalar parameter γ and an unknown nuisance parameter vector $\boldsymbol{\theta}$. Neyman (1979) gave an extensive review of $C(\alpha)$–tests and their use. These results were extended to other types of probability structures such as when the observations are independent but not identically distributed in Bartoo and Puri (1967). Sarma (1968) studied the case when the observations are made on a stationary Markov process. Bhat and Kulkarni (1972) generalized the results to discrete time stochastic processes. For an exposition of some of these results, see Basawa and Prakasa Rao (1980).

Our aim in this paper is to develop optimal asymptotic tests of composite hypotheses for continuous time stochastic processes. The problem is formulated in Section 2. Martingale test statistics useful in constructing asymptotic tests of hypotheses are discussed in Section 3. The asymptotic power of such tests is

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investigated in Section 4. Optimality of these tests in a special case is studied in Section 5. Results are illustrated by deriving an optimal asymptotic test for testing the hypotheses $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$ in a linear stochastic differential equation

$$dX(t) = (\gamma X(t) + \theta)dt + dW(t), t \geq 0$$

where $\{W(t), t \geq 0)\}$ is the standard Wiener processes, $X(0) = 0, \gamma$ and $\theta \in R$. Here θ is the nuisance parameter.

2. Formulation of the problem

Let Γ be an open interval containing the origin and Θ be an open set contained in $\mathcal{R}^k$ and let $\zeta = \Gamma \times \Theta$. For every $(\gamma, \boldsymbol{\theta}) \in \zeta$, let $\{X_t(\gamma, \boldsymbol{\theta}), t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $P^{(T)}_{\gamma,\theta}$ be the probability measure induced by the process $\boldsymbol{X}_T(\gamma, \boldsymbol{\theta}) = \{X_t(\gamma, \boldsymbol{\theta}), 0 \leq t \leq T\}$ on a suitable function space $\mathcal{X}_T$ with an associated σ-algebra $\mathcal{B}_T$. Suppose $P^{(T)}_{\gamma,\theta} \ll \mu^T$ where μ^T is a probability measure on $(\boldsymbol{X}_T, \mathcal{B}_T)$. We assume that $\mathcal{X}_T$ is independent of $(\gamma, \boldsymbol{\theta}) \in \zeta$ and that the σ-algebras $\mathcal{B}_T$ are nested.

The problem of interest is to construct an optimal asymptotic test of the hypothesis $H_0 : \gamma = \gamma_0 \in \Gamma$ against the alternative $H_1 : \gamma \neq \gamma_0$. Here optimality is in a suitable sense to be defined later. We now define what we mean by an asymptotic test of the hypothesis H_0 against H_1.

Definition 2.1. Let $0 < \alpha < 1$. Let $\{R_T\}$ be a family of measurable subsets of $\{\mathcal{X}_T\}$ for $T > 0$. The family is said to define *an asymptotic test* of level α of the hypothesis $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$ if

$$\lim_{T\to\infty} P^{(T)}_{\gamma_0,\theta}[\boldsymbol{X}_t(\gamma_0, \boldsymbol{\theta}) \in R_T] = \alpha$$

for all $\boldsymbol{\theta} \in \Theta$.

Let $K(\alpha)$ be a class of asymptotic level α tests of the hypothesis $H_0 : \gamma = \gamma_0$. Let $\gamma^* = \{\gamma_T\}$ be a collection in Γ converging to γ_0 as $T \to \infty$. Let $\mathcal{D}$ denote a family of such collections γ^* and $\{R^0_T\} \in K(\alpha)$.

Definition 2.2. An asymptotic level α test $\{R^0_T\} \in K(\alpha)$ *is optimal within the class* $K(\alpha)$ if for any collection $\{\gamma_T\} \in \mathcal{D}$ and for any $\boldsymbol{\theta} \in \Theta$ ·

$$\lim_{T\to\infty} P^{(T)}_{\gamma_0,\theta}[\boldsymbol{X}_t(\gamma_0, \boldsymbol{\theta}) \in R_T] - P^{(T)}_{\gamma_T,\theta}[\boldsymbol{X}_T(\gamma_T, \boldsymbol{\theta}) \in R_T]\} \geq 0$$

for all $\{R_T\} \in K(\alpha)$.

Suppose the process $\{X_t(\gamma, \boldsymbol{\theta}), t \geq 0\}$ is $\mathcal{F}_t$-adapted for every $t \geq 0$ and for every $(\gamma, \boldsymbol{\theta}) \in \zeta$. Let $\{\hat{\boldsymbol{\theta}}_t\}$ be an $\mathcal{F}_t$-adapted process. $\hat{\boldsymbol{\theta}}_t$ may or may not be a function of $\{X_s, (\gamma, \boldsymbol{\theta}), 0 \leq s \leq t\}$.

Definition 2.3. Consider an $\mathcal{F}_t$-adapted proccess $\{\mathbf{v}^{(t)}, t \geq 0\}$ possibly dependent on $(\gamma, \boldsymbol{\theta})$ such that $\mathbf{v}^{(t)} \xrightarrow{p} \infty$ as $t - \infty$ under $P^{(t)}_{\gamma,\theta}$. $\hat{\boldsymbol{\theta}}_t$ is said to be *locally* $\mathbf{v}^{(t)}$*-consistent estimator* of $\boldsymbol{\theta}$ if there exists $A_j \neq 0$ such that

$$v_j^{(t)} \mid \hat{\theta}_{jt} - \theta_j - A_j(\gamma - \gamma_0) \mid = O_p(1), 1 \leq j \leq k$$

under $\{P^{(t)}_{\gamma,\theta}\}$ for all γ and $\boldsymbol{\theta}$. If $A_j = 0$ for $1 \leq j \leq k$, then $\hat{\boldsymbol{\theta}}_t$ is called a $\mathbf{v}^{(t)}$*-consistent estimator of* $\boldsymbol{\theta}$.

In the following, we shall denote the probability measure induced by the process $\{X_t(\gamma, \boldsymbol{\theta}), t \geq 0\}$ by $P_{\gamma,\theta}$.

3. Martingale test statistics

Suppose $\{f_t(\boldsymbol{X}_t(\gamma, \boldsymbol{\theta}); \boldsymbol{\theta}), t \geq 0\}$ is an $\mathcal{F}_t$-adapted stochastic process such that

$$\{f_t(\boldsymbol{X}_t(\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta}), t \geq 0\}$$

is zero mean square integrable martingale under $P_{\gamma_0,\theta}$. From the definition of a square integrable martingale, it follows that

$$\sup_{0 \leq t < \infty} E_{\gamma_0,\theta}[f_t(\boldsymbol{X}_t(\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta})]^2 < \infty.$$

Suppose the function $f_t(x; \boldsymbol{\theta})$ is differentiable thrice with respect to $\theta_j, 1 \leq j \leq k$. Let f_{jt} denote the first partial derivative of f_t with respect to θ_j and f_{jlt} denote the second partial derivative with respect to θ_j and θ_l respectively. We assume that $f_{jlt} = f_{ljt}$. Let

$$\sigma_f^2(\boldsymbol{\theta}; T) = < f_T(\boldsymbol{X}_t(\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta}) >_T$$

where $< Y >_T$ denotes the quadratic variation of a square integrable martingale $\{Y_T\}$.

Suppose that

$$Z_T((\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta}) \equiv \frac{f_T(\boldsymbol{X}_T(\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta})}{\sigma_f(\theta; T)} \xrightarrow{L} N(0, 1) \text{ as } T \to \infty$$

where $(\gamma_0, \boldsymbol{\theta})$ is the true parameter. Sufficient conditions ensuring this asymptotic behaviour are given in Helland (1982). If $\boldsymbol{\theta}$ is known, then $Z_T((\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta})$ can be used as a test statistic for testing $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$. Since Z_T depends on the unknown parameter $\boldsymbol{\theta}, Z_T((\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta})$ is not computable.We now study sufficient conditions on f under which the asymptotic behaviour of $Z_T((\gamma_0, \boldsymbol{\theta}); (\hat{\boldsymbol{\theta}}_T)$ is the same as that of $Z_T((\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta})$ where $\hat{\boldsymbol{\theta}}_T$ is a suitable

$\mathcal{F}_T$ -adapted estimator of $\boldsymbol{\theta}$. It is clear that $\{Z_T((\gamma_0, \boldsymbol{\theta}); \hat{\boldsymbol{\theta}}_T)\}$ is a well defined $\mathcal{F}_T$-adapted process.

For notational convenience, we write $f_T^{(\gamma_0)}$ for $f_T(\boldsymbol{X}_T(\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta})$ and σ_{fT} for $\sigma_f(\boldsymbol{\theta}; T)$ in the following discussion. Assume that the following conditions hold:

(A0) $\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}$ is twice continuously differentiable in $\boldsymbol{\theta}$;

(A1) $\boldsymbol{\theta}_T$ is $\mathbf{v}^{(T)}$-consistent for $\boldsymbol{\theta}$ as $T \to \infty$ under $(\gamma_0, \boldsymbol{\theta})$ in the sense of Definition 2.3;

(A2) (i) $E_{\gamma_0,\theta}[\frac{\partial}{\partial\theta_j}(\frac{f_T^{(\gamma_0)}}{\sigma_{fT}})] < \infty, 1 \le j \le k$;

(ii) $\frac{1}{v_j^{(T)}}\{\frac{\partial}{\partial\theta_j}[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}] - E_{\gamma_0,\theta}[\frac{\partial}{\partial\theta_j}(\frac{f_T^{(\gamma_0)}}{\sigma_{fT}})]\} \xrightarrow{p} 0$ as $T \to \infty$

under $P_{\gamma_0,\theta}^{(T)}$ for $1 \le j \le k$;

(iii) for every $\boldsymbol{\theta} \in \Theta$, there exists a neighbourhood $U_\theta(\gamma_0)$ of $\boldsymbol{\theta}$ such that

$$\sup_{\theta \in U_\theta(\gamma_0)} | \frac{\partial^2}{\partial\theta_j \partial\theta_l}[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}] | \le h_{jl}^{(T)}$$

where

$$E_{\gamma_0,\theta}[h_{jl}^{(T)}(\boldsymbol{X}_T(\gamma_0, \boldsymbol{\theta}))/\min(v_j^{(T)}, v_l^{(T)})] = O(1), 1 \le j, l \le k;$$

and

(iv) $\{\min(v_l^{(T)}, v_j^{(T)})\}^{-1}\{h_{jl}^{(T)} - E_{\gamma_0,\theta}[h_{jl}^{(T)}]\} \xrightarrow{p} 0$ as $T \to \infty$. under $P_{\gamma_0,\theta}^{(T)}$.

Here $h_{jl}^{(T)}$ stands for $h_{jl}^{(T)}(\boldsymbol{X}_T(\gamma_0, \boldsymbol{\theta}))$.

Let us expand $Z_T((\gamma_0, \boldsymbol{\theta}); \hat{\boldsymbol{\theta}}_T)$ around the point γ_0 under the conditions (A0) to (A2). Then

$$\begin{aligned}
&Z_T((\gamma_0, \boldsymbol{\theta}); \hat{\boldsymbol{\theta}}_T) - Z_T((\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta}) \\
&= \sum_{j=1}^{k} (\hat{\theta}_{jT} - \theta_j) v_j^T \frac{1}{v_j^{(T)}} \left\{ \frac{\partial}{\partial\theta_j} \left[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}\right] - E_{\gamma_0,\theta}\left(\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}\right]\right)\right\} \\
&+ \Sigma_{j=1}^{k} (\widehat{\theta}_{jT} - \theta_j) v_j^T \frac{1}{v_j^{(T)}} \left\{ E_{\gamma_0,\theta}\left(\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}\right]\right)\right\} \\
&+ \frac{1}{2} \sum_{j=1}^{k} \sum_{\ell=1}^{k} (\hat{\theta}_{jT} - \theta_j)(\hat{\theta}_{lT} - \theta_l) v_l^{(T)} v_j^{(T)} \frac{1}{v_j^{(T)} v_l^{(T)}} \left\{ \frac{\partial^2}{\partial\theta_j \partial\theta_l} \left[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}\right] \mid \boldsymbol{\theta} = \boldsymbol{\theta}^* \right\}
\end{aligned}$$

where $\| \boldsymbol{\theta} - \boldsymbol{\theta}^* \| \le \| \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_T \|$ and hence

$$\begin{aligned}
&Z_T((\gamma_0, \boldsymbol{\theta}); \hat{\boldsymbol{\theta}}_T) - Z_T((\gamma_0, \boldsymbol{\theta}); \boldsymbol{\theta}) \\
&\qquad = \sum_{j=1}^{k} O_p(1) o_p(1) + \sum_{j=1}^{k} O_p(1) E_{\gamma_0,\theta}\left(\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}\right]\right) \\
&\qquad\quad + \frac{1}{2} \sum_{j=1}^{k} \sum_{l=1}^{k} O_p(1) O_p(1) \frac{1}{[\min(v_j^{(T)}, v_l^{(T)})]^2} h_{jl}^{(T)}
\end{aligned}$$

under the assumptions (A0) to (A2). It can be checked that the above expression is $o_p(1)$ as $T \to \infty$ provided

$$E_{\gamma_0,\theta}(\frac{\partial}{\partial\theta_j}[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}})] = 0, 1 \leq j \leq k,$$

since $\min(v_j^{(T)}, v_t^{(T)}) \xrightarrow{P} \infty$ under $P_{\gamma_0,\theta}^{(T)}$-measure. Hence we have the following result.

Theorem 3.1. *Suppose the conditions* (A0) to (A2) *hold for a zero mean square integrable martingale* $\{f_t(\boldsymbol{X}_t(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta}), t \geq 0\}$. *Define*

$$Z_T((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta}) \equiv \frac{f_T(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}{\sigma_f(\boldsymbol{\theta};T)} \quad \ldots (3.1)$$

where

$$\sigma_f^2(\boldsymbol{\theta},T) = < f_T(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta}) >_T .$$

Then

$$Z_T((\gamma_0,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) - Z_T((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta}) \xrightarrow{P} 0 \text{ as } T \to \infty$$

under $\{P_{\gamma_0,\theta}^{(T)}\}$ *provided*

$$E_{\gamma_0,\theta}\left[\frac{\partial}{\partial\theta_j}\left(\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}\right)\right] = 0, 1 \leq j \leq k. \quad \ldots (3.2)$$

In particular, if

$$Z_T((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta}) \xrightarrow{\mathcal{L}} N(0,1) \text{ as } T \to \infty \text{ under } \{P_{\gamma_0,\theta}^{(T)}\}$$

then

$$Z_T((\gamma_0,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) \xrightarrow{\mathcal{L}} N(0,1) \text{ as } T \to \infty \text{ under } \{P_{\gamma_0,\theta}^{(T)}\}$$

under the conditions (A0) to (A2) provided (3.2) holds.

Example 3.1. Consider the stochastic process $\{X_t\}$ defined by the stochastic differential equation

$$dX_t = b(X_t;\gamma,\theta)dt + \sigma(X_t)dW_t, t \geq 0, X_0 = 0$$

where θ is the nuisance parameter and we would like to test the hypothesis $H_0 : \gamma = \gamma_0$. It is known that, under some smoothness conditions, the loglikelihood function given the observation $\{X_s, 0 \leq s \leq t\}$ is given by

$$\ell_t(\gamma,\theta) = \int_0^t \frac{b(X_s;\gamma,\theta)}{\sigma^2(X_s)} dX_s - \frac{1}{2}\int_0^t \frac{b^2(X_s;\gamma,\theta)}{\sigma^2(X_s)} ds. \quad \ldots (3.3)$$

Using the Ito's formula, it can be seen that (cf. Lanska (1979)),

$$\ell_t(\gamma,\theta) = H(X_t;\gamma,\theta) - H(X_0;\gamma,\theta) + \int_0^t h(X_s;\gamma,\theta)ds \qquad \ldots(3.4)$$

where

$$H(x;\gamma,\theta) = \int_0^x \frac{b(y;\gamma,\theta)}{\sigma^2(y)}dy$$

and

$$h(x;\gamma,\theta) = -\frac{1}{2}\Big(\frac{b^2(x;\gamma,\theta)}{\sigma^2(x)} + \frac{\partial b(x;\gamma,\theta)}{\partial x}\Big) + \frac{b(x;\gamma,\theta)\frac{d\sigma(x)}{dx}}{\sigma(x)}.$$

Let $\ell_t'(\gamma,\theta)$ denote the derivative of $\ell_t(\gamma,\theta)$ with respect to θ. Then

$$\ell_t'(\gamma,\theta) = H'(X_t;\gamma,\theta) - H'(X_0;\gamma,\theta) + \int_0^t h'(X_s;\gamma,\theta)ds \qquad \ldots(3.5)$$

$$= \int_0^t \frac{b'(X_s;\gamma,\theta)}{\sigma^2(X_s)}dX_s - \int_0^t \frac{b'(X_s;\gamma,\theta)b(X_s;\gamma,\theta)}{\sigma^2(X_s)}ds. \qquad \ldots(3.6)$$

It is important to note that the expression (3.5) does not involve any stochastic integral and $\{\ell_t'(\gamma,\theta); t \geq 0\}$ forms a martingale with respect to the natural family of σ-algebras $\{\mathcal{F}_t\}$ from (3.6). Hence, if $\{\hat{\theta}_t, t \geq 0\}$ is $\mathcal{F}_t$–adapted, then the process $\{\ell_t(\gamma,\hat{\theta}_t); t \geq 0\}$ process is well defined and one can expand it via Taylor's expansion using (3.4) as was done in Lanska (1979). This gives an example of a martingale test statistic obtained from the martingale $\{\ell_t'(\gamma,\theta); t \geq 0\}$. One can construct a similar class of martingale statistics by choosing suitable $\mathcal{F}_t$–adapted processes G and g so that

$$G'(X_t;\gamma,\theta) - G'(X_0;\gamma,\theta) + \int_0^t g'(X_s;\gamma,\theta)ds, s \geq 0$$

forms a $\mathcal{F}_t$–adapted martingale from which martingale test statistics can be formed.

4. Asymptotic power

Define $\{f_t(\boldsymbol{X}_t(\gamma,\boldsymbol{\theta});\boldsymbol{\theta})\}$ as in Section 3 and suppose the Conditions (A0) to (A2) hold. Further assume that the equation (3.2) holds. We would like to study the asymptotic behaviour of $Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)$ as $T \to \infty$ where $\{\gamma_T\}$ is an arbitrary collection in Γ converging to γ_0 and $\hat{\boldsymbol{\theta}}_T$ is a $\mathbf{v}^{(T)}$–consistent estimator. Here

$$Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) = \frac{f_T(\boldsymbol{X}_T(\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)}{\sigma_f(\hat{\boldsymbol{\theta}}_T;T)}.$$

and

$$Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) \equiv \frac{f_T(\boldsymbol{X}_T(\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta})}{\sigma_f(\boldsymbol{\theta};T)}.$$

Let us write $f_T^{(\gamma)}$ for $f_T(\boldsymbol{X}_T(\gamma,\boldsymbol{\theta});\boldsymbol{\theta})$ and σ_{fT} for $\sigma_f(\boldsymbol{\theta};T)$.

In addition to the conditions (A0) to (A2) and the validity of the equation (3.2) for all $(\gamma,\boldsymbol{\theta})$, assume that the following conditions hold for $\{\gamma_T\}$:

(A3) (i) $\frac{1}{v_j^{(T)}}\{\frac{\partial}{\partial\theta_j}[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}]-E_{\gamma_T,\theta}[\frac{\partial}{\partial\theta_j}(\frac{f_T^{(\gamma_T)}}{\sigma_{fT}})]\}\xrightarrow{P}0$ in $P_{\gamma_T,\theta}^{(T)}$-measure as $T\to\infty$ for $1\le j\le k$;

(ii) there exists a neighbourhood Γ_0 of γ_0 and a neighbourhood $U_\theta(\gamma_0)$ of $\boldsymbol{\theta}$ such that

$$\sup_{\gamma\in\Gamma_0}\ \sup_{\boldsymbol{\theta}\in U_\theta(\gamma_0)} \left| \frac{\partial^2}{\partial\theta_j\partial\theta_l}\left[\frac{f_T^{(\gamma)}}{\sigma_{fT}}\right]\right| \le h_{jl}^*(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta}))$$

where

$$E_{\gamma_T,\theta}\left[h_{jl}^*(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta}))/(\min(v_l^{(T)},v_j^{(T)}))\right] = O(1)$$

for $1\le j,l\le k$;

(A4) $$\frac{1}{v_j^{(T)}}\left\{E_{\gamma_T,\theta}\left[\frac{\partial}{\partial\theta_j}\left(\frac{f_T^{(\gamma_T)}}{\sigma_{jT}}\right)\right] - E_{\gamma_0,\theta}\left[\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_0)}}{\sigma_{jT}}\right)\right]\right\}\xrightarrow{P}0$$

as $T\to\infty$ for $1\le j\le k$; and

(A5) $\max(v_j^{(T)},1\le j\le k)(\gamma_T-\gamma_0) = O_P(1)$.

Note that

$$\begin{aligned} &Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) - Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) \\ =\ & \sum_{j=1}^k(\hat{\theta}_{jT}-\theta_j)v_j^T\frac{1}{v_j^{(T)}}\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}\right] \\ &+\frac{1}{2}\Sigma_{j=1}^k\Sigma_{l=1}^k(\hat{\theta}_{jT}-\theta_j)(\hat{\theta}_{lT}-\theta_l)v_l^{(T)}v_j^{(T)}\frac{1}{v_j^{(T)}v_l^{(T)}}\frac{\partial^2}{\partial\theta_j\partial\theta_l}\left[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}\right]_{\boldsymbol{\theta}=\boldsymbol{\theta}_T^*} \\ =\ & \sum_{j=1}^k(\hat{\theta}_{jT}-\theta_j)v_j^T\frac{1}{v_j^{(T)}}\left(\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}\right] - E_{\gamma_T,\theta}\left\{\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}\right]\right\}\right)\end{aligned}$$

$$+\Sigma_{j=1}^{k}(\hat{\theta}_{jT}-\theta_j)E_{\gamma_T,\theta}\left\{\frac{\partial}{\partial\theta_j}\left[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}\right]\right\}$$

$$+\frac{1}{2}\Sigma_{j=1}^{k}\Sigma_{l=1}^{k}(\hat{\theta}_{jT}-\theta_j)(\hat{\theta}_{lT}-\theta_l)v_l^{(T)}v_j^{(T)}\frac{1}{v_j^{(T)}v_l^{(T)}}\frac{\partial^2}{\partial\theta_j\partial\theta_l}\left[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}\right]_{\theta=\theta_T^*}$$

From the $\mathbf{v}^{(T)}$–consistency of the estimator $\hat{\boldsymbol{\theta}}_T$ and the conditions (A1), (A3) to (A5), it follows that

$$Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)-Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta})$$

$$=\sum_{j=1}^{k}(\hat{\theta}_{jT}-\theta_j)v_j^{(T)}\frac{1}{v_j^{(T)}}E_{\gamma_T,\theta}\{\frac{\partial}{\partial\theta_j}[\frac{f_T^{(\gamma_T)}}{\sigma_{fT}}]\}+o_P(1)$$

$$=\sum_{j=1}^{k}(\hat{\theta}_{jT}-\theta_j)v_j^{(T)}\frac{1}{v_j^{(T)}}E_{\gamma_0,\theta}\{\frac{\partial}{\partial\theta_j}[\frac{f_T^{(\gamma_0)}}{\sigma_{fT}}]\}+o_P(1).$$

Equation (3.2) implies that

$$Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)-Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta})=o_P(1)$$

in $P_{\gamma T,\theta}^{(T)}$–probability as $T\to\infty$. Hence we have the following theorem.

Theorem 4.1. *Suppose the conditions (A0), (A1) and (A3) to (A5) hold in addition to (3.2). Then*

$$Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}_T)-Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta})\xrightarrow{P}0 \text{ in } P_{\gamma T,\theta}^{(T)} \textit{ probability as } T\to\infty.$$

In addition to (A0) to (A5), suppose the following conditions hold.

(A6) Let $m_T(\gamma,\boldsymbol{\theta})=E_{\gamma,\theta}[f_T(\boldsymbol{X}_T(\gamma,\boldsymbol{\theta});\boldsymbol{\theta})]$. Assume that $m_T(\gamma,\boldsymbol{\theta})$ is twice continuously differentiable with respect to γ with uniformly bounded second derivatives.

Expanding around γ_0, we have

$$m_T(\gamma_T,\boldsymbol{\theta})=m_T(\gamma_0,\boldsymbol{\theta})+(\gamma_T-\gamma_0)\frac{\partial m_T(\gamma,\boldsymbol{\theta})}{\partial\gamma}\mid\gamma=\gamma_0$$

$$+\frac{(\gamma_T-\gamma_0)^2}{2}\frac{\partial^2 m_T(\gamma,\boldsymbol{\theta})}{\partial\gamma^2}\mid\gamma=\gamma_T^*$$

where $\mid\gamma_T^*-\gamma_0\mid\leq\mid\gamma_T-\gamma_0\mid$.

(A7) Suppose that

$$\frac{f_T(\boldsymbol{X}_T(\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) - m_T(\gamma_T,\boldsymbol{\theta})}{\sigma_{fT}((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta})} \xrightarrow{\mathcal{L}} N(0,1)$$

in $P^{(T)}_{\gamma_T,\theta}$-probability as $T \to \infty$ where

$$\sigma^2_{fT}((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) = < f_T(\boldsymbol{X}_T(\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) - m_T(\gamma_T,\boldsymbol{\theta}) >_T$$

and

(A8) $$\frac{\sigma_{fT}((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta})}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})} \xrightarrow{p} 1 \text{ as } T \to \infty \text{ in } P^{(T)}_{\gamma_T,\theta} - \text{probability.}$$

Then

$$\{f_T(\boldsymbol{X}_T(\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) - m_T(\gamma_T,\boldsymbol{\theta})\}/\sigma_{fT}((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) \xrightarrow{\mathcal{L}} N(0,1)$$

in $P^{(T)}_{\gamma_T,\theta}$-probability as $T \to \infty$. Define

$$Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) \equiv \frac{f_T(\boldsymbol{X}_T(\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta})}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}.$$

Hence

$$Z_T((\gamma_T,\boldsymbol{\theta});\boldsymbol{\theta}) - \frac{m_T(\gamma_T,\boldsymbol{\theta})}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})} \xrightarrow{\mathcal{L}} N(0,1)$$

in $P^{(T)}_{\gamma_T,\theta}$-probability as $T \to \infty$. Therefore, by Theorem 4.1,

$$Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) - \frac{m_T(\gamma_T,\boldsymbol{\theta})}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})} \xrightarrow{\mathcal{L}} N(0,1)$$

in $P^{(T)}_{\gamma_T,\theta}$-probability as $T \to \infty$ under the conditions (A0) to (A1) and (A3) to (A8) provided (3.2) holds and we have the following result.

Theorem 4.2. *Suppose the conditions (A0), (A1) and (A3) to (A8) hold. Then*

$$Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) - \frac{m_T(\gamma_T,\boldsymbol{\theta})}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})} \xrightarrow{\mathcal{L}} N(0,1)$$

as $T \to \infty$.

Special case. Let us consider the special case when $\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})$ is non-random and v_j^T is non-random for $1 \le j \le k$. Then

$$Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) \simeq N\left(\frac{m_T(\gamma_T,\boldsymbol{\theta})}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}, 1\right) \qquad \ldots(4.1)$$

as $T \to \infty$.

Furthermore

$$\frac{m_T(\gamma_T,\boldsymbol{\theta})}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})} = (\gamma_T-\gamma_0)\frac{1}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}\frac{\partial m_T(\gamma,\boldsymbol{\theta})}{\partial\gamma}|_{\gamma=\gamma_0}$$

$$+\frac{(\gamma_T-\gamma_0)^2}{2}\frac{1}{\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}\frac{\partial^2 m_T(\gamma,\boldsymbol{\theta})}{\partial\gamma^2}|_{\gamma=\gamma_T^*} \qquad \ldots(4.2)$$

where $|\gamma_T^* - \gamma_0| \leq |\gamma_T - \gamma_0|$. Let $\eta_T = \max(v_j^{(T)}, 1 \leq j \leq k)$. Suppose

$$\gamma_T = \gamma_0 + (\Delta\eta_T)^{-1} \text{ for fixed } \Delta > 0. \qquad \ldots(4.3)$$

Then

$$Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T) \simeq N\left(\frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}\frac{\partial m_T(\gamma,\boldsymbol{\theta})}{\partial\gamma}|_{\gamma=\gamma_0}, 1\right) \qquad \ldots(4.4)$$

from (4.1). Under the conditions on $m_T(\gamma,\boldsymbol{\theta})$ assumed above, it follows by arguments similar to those in Neyman (1959) that the asymptotic power of the test defined by the test statistic $Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)$ is obtained from the normal distribution with mean

$$\frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}\frac{\partial m_T(\gamma,\boldsymbol{\theta})}{\partial\gamma}|_{\gamma=\gamma_0} \qquad \ldots(4.5)$$

and variance unity.

Remarks. Assumptions (A0) to (A8) stated in this section are of the classical Cramer–Wald type. It may be explored whether they can be restated in terms of LAN or LAMN and L_2–differentiability conditions. We do not do this here.

5. Optimal tests in the non–random case

In the last section, we have derived a formula for computing the asymptotic power of the test statistic $Z_T((\gamma_T,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)$ where $\hat{\theta}_{jT}$ is a $v_j^{(T)}$–consistent estimator of $\theta_j, 1 \leq j \leq k$ and $\eta_{(T)}(\gamma_T-\gamma_0) = O(1)$ where $\eta_{(T)} = \max(v_j^{(T)}, 1 \leq j \leq k)$. Let

$$\phi^{(T)}(x;\boldsymbol{\theta}) = \log\frac{dP_{\gamma,\theta}^{(T)}}{d\mu^{(T)}}(x)|_{\gamma=\gamma_0}, x \in \mathcal{X}_T.$$

Assume that, for every $x \in \mathcal{X}_T$ and for every $(\gamma,\boldsymbol{\theta})$, the density $\frac{dP_{\gamma,\theta}^{(T)}}{d\mu^{(T)}}(x)$ is at least twice differentiable with respect to all the $(k+1)$ parameters. Let $\phi_\gamma^{(T)}(x;\boldsymbol{\theta})$

and $\phi_{\theta_j}^{(T)}(x;\boldsymbol{\theta})$ denote the derivatives of the log $\frac{dP_{\gamma,\theta}^{(T)}}{d\mu^{(T)}}(x)$ with respect to γ and θ_j respectively evaluated at $\gamma=\gamma_0$. Assume further that

$$\begin{aligned}\lambda_{\gamma_0\gamma_0}^{(T)} &\equiv E_{\gamma_0,\theta}[\phi_{\gamma_0}^{(T)}(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})]^2<\infty,\\ \lambda_{\gamma_0\theta_i}^{(T)} &\equiv E_{\gamma_0,\theta}[\phi_{\gamma_0}^{(T)}(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})\phi_{\theta_i}^{(T)}(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})]<\infty, 1\le i\le k\end{aligned}$$

and

$$\lambda_{\theta_i\theta_j}^{(T)}\equiv E_{\gamma_0,\theta}[\phi_{\theta_i}^{(T)}(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})\phi_{\theta_j}^{(T)}(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})]<\infty, 1\le i,j\le k.$$

Note that $\lambda_{\gamma_0\gamma_0}^{(T)}, \lambda_{\gamma_0\theta_i}^{(T)}$ and $\lambda_{\theta_i\theta_j}^{(T)}$ are all functions of $\boldsymbol{\theta}$ and γ_0. Let

$$\Lambda^{(T)}=\begin{bmatrix}\lambda_{\gamma_0\gamma_0}^{(T)} & \lambda_{\gamma_0\theta_1}^{(T)} & \cdots & \lambda_{\gamma_0\theta_k}^{(T)}\\ \lambda_{\gamma_0\theta_1}^{(T)} & \lambda_{\theta_1\theta_1}^{(T)} & \cdots & \lambda_{\theta_1\theta_k}^{(T)}\\ \vdots & \vdots & & \vdots\\ \lambda_{\gamma_0\theta_k}^{(T)} & \lambda_{\theta_1\theta_k}^{(T)} & \cdots & \lambda_{\theta_k\theta_k}^{(T)}\end{bmatrix}=\begin{bmatrix}\lambda_{\gamma_0\gamma_0}^{(T)} & \vdots & \Lambda_{12}^{(T)}\\ \cdots & & \cdots\\ \Lambda_{21}^{(T)} & \vdots & \Lambda_{22}^{(T)}\end{bmatrix}$$

and suppose that $\Lambda_{22}^{(T)}$ is invertible. Define

$$\Lambda_{12}^{(T)'}\Lambda_{22}^{(T)-1}=(a_1^{(T)}(\gamma_0,\boldsymbol{\theta}),\ldots,a_k^{(T)}(\gamma_0,\boldsymbol{\theta}))$$

where A' denotes transpose of a matrix A. Then the regression of $\phi_{\gamma_0}^{(T)}$ on $\phi_{\theta_i}^{(T)}, 1\le i\le k$ is

$$\sum_{i=1}^{k}a_i^{(T)}(\gamma_0,\boldsymbol{\theta})\phi_{\theta_i}^{(T)}.$$

Define

$$Y_T(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})=\phi_{\gamma_0}^{(T)}-\sum_{i=1}^{k}a_i^{(T)}(\gamma_0,\boldsymbol{\theta})\phi_{\theta_i}^{(T)} \qquad \ldots(5.1)$$

and

$$\tilde{\sigma}^2((\gamma_0,\boldsymbol{\theta});T)=E_{\gamma_0,\theta}[Y_T(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})]^2. \qquad \ldots(5.2)$$

Let

$$Z_T(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)=\frac{Y_T(\boldsymbol{X}_T(\gamma_0,\boldsymbol{\theta});\hat{\boldsymbol{\theta}}_T)}{\tilde{\sigma}((\gamma_0,\boldsymbol{\theta});T)}. \qquad \ldots(5.3)$$

(B1) Suppose that differentiability with respect to $(\gamma,\boldsymbol{\theta})$ under the integral sign in (5.4) is valid and that the support of $P_{\gamma,\theta}^{(T)}$ does not depend on $(\gamma,\boldsymbol{\theta})$. Note that

$$\int_{\chi^T} \frac{dP^T_{\gamma,\theta}}{d\mu^T} d\mu^T = 1, \qquad \ldots(5.4)$$

$$E_{\gamma_0,\theta}[\phi^{(T)}_{\gamma_0}] = 0 \text{ and } E_{\gamma_0,\theta}[\phi^{(T)}_{\theta_i}] = 0, \qquad \ldots(5.5)$$

and

$$E_{\gamma_0,\theta}[Y_T \phi^{(T)}_{\theta_i}] = 0 \qquad \ldots(5.6)$$

(B2) Suppose differentiation with respect to $\boldsymbol{\theta}$ under the integral sign is permissible in the equation (5.6).

It is easy to check that

$$E_{\gamma_0,\theta}[\frac{\partial Y_T}{\partial \theta_i} + Y_T \phi^{(T)}_{\theta_i}] = 0, 1 \leq i \leq k \qquad \ldots(5.7)$$

and hence

$$E_{\gamma_0,\theta}[\frac{\partial Y_T}{\partial \theta_i}] = 0 \qquad \ldots(5.8)$$

from (5.6) which implies that

$$E_{\gamma_0,\theta}\left\{\frac{\partial}{\partial \theta_i}\left[\frac{Y_T}{\tilde{\sigma}((\gamma_0,\boldsymbol{\theta});T)}\right]\right\} = 0 \qquad \ldots(5.9)$$

from (5.6) and (5.7). Let $m_T^{(f)}(\gamma,\boldsymbol{\theta})$ be the expectation of $f_T(\boldsymbol{X}_T(\gamma,\boldsymbol{\theta});\boldsymbol{\theta})$ under $P^{(T)}_{\gamma,\theta}$ as defined in Section 3 and $m_T(\gamma,\boldsymbol{\theta})$ be the corresponding expression for $Y_T(\boldsymbol{X}_T(\gamma,\boldsymbol{\theta});\boldsymbol{\theta})$ defined above. In order to compare the asymptotic powers of level α test statistics from these, let us compute from (4.5). Note that

$$\begin{aligned}
\zeta_T^{(f)} &= \frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}\frac{\partial m_T^{(f)}(\gamma,\boldsymbol{\theta})}{\partial\gamma}|_{\gamma=\gamma_0} \\
&= \frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}E_{\gamma_0,\theta}\left[\frac{\partial f_T^{(\gamma)}}{\partial\gamma}|_{\gamma=\gamma_0}\right] \\
&= \frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}E_{\gamma_0,\theta}[f_T^{(\gamma_0)}\phi^{(T)}_{\gamma_0}] \\
&= \frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}E_{\gamma_0,\theta}[f_T^{(\gamma_0)}\{Y_T + \sum_{i=1}^k a_i^{(T)}\phi^{(T)}_{\theta_i}\}] \\
&= \frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}E_{\gamma_0,\theta}[f_T^{(\gamma_0)}Y_T]
\end{aligned}$$

since $E_{\gamma_0,\theta}[f_T^{(\gamma_0)}\phi^{(T)}_{\theta_i}] = 0$ by hypothesis which in turn follows from the fact

$$E_{\gamma_0,\theta}[f_T^{(\gamma_0)}\phi^{(T)}_{\theta_i}] = -E_{\gamma_0,\theta}\left[\frac{\partial f_T^{(\gamma)}}{\partial\theta_i}|_{\gamma=\gamma_0}\right] = 0$$

by (3.2) in the nonrandom case. All the above calculations can be justified under the additional assumption that (B3) the expression $E_{\gamma,\theta}[f_T(\boldsymbol{X}_T(\gamma,\boldsymbol{\theta});\boldsymbol{\theta})]$ can be differentiated under the integral sign with respect to γ and $\boldsymbol{\theta}$. Hence

$$
\begin{aligned}
\zeta_T^{(f)} &\le \frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}(E_{\gamma_0,\theta}[f_T^{(\gamma_0)}]^2 E_{\gamma_0,\theta}[Y_T]^2)^{1/2}\\
&= \frac{1}{\Delta\eta_T\sigma_{fT}((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})}[\sigma_{fT}^2((\gamma_0,\boldsymbol{\theta});\boldsymbol{\theta})\tilde{\sigma}^2((\gamma_0,\boldsymbol{\theta});T)]^{1/2}\\
&= \frac{1}{\Delta\eta_T}\tilde{\sigma}((\gamma_0,\boldsymbol{\theta});T)\\
&\doteq \frac{1}{\Delta\eta_T\tilde{\sigma}((\gamma_0,\boldsymbol{\theta});T)}E_{\gamma_0,\theta}[Y_T]^2\\
&= \frac{1}{\Delta\eta_T\tilde{\sigma}((\gamma_0,\boldsymbol{\theta});T)}E_{\gamma_0,\theta}[Y_T\phi_{\gamma_0}^{(T)}]\\
&\quad (\text{ since } E_{\gamma_0,\theta}[Y_T\phi_{\theta_i}^{(T)}]=0, 1\le i\le k)\\
&= \frac{1}{\Delta\eta_T\tilde{\sigma}((\gamma_0,\boldsymbol{\theta});T)}E_{\gamma_0,\theta}\left[\frac{\partial Y_T}{\partial\gamma}\Big|_{\gamma=\gamma_0}\right]\\
&= \frac{1}{\Delta\eta_T\tilde{\sigma}((\gamma_0,\boldsymbol{\theta});T)}\frac{\partial m_T(\gamma,\boldsymbol{\theta})}{\partial\gamma}\Big|_{\gamma=\gamma_0}.
\end{aligned}
$$

Now, following arguments similar to those given in Bhat and Kulkarni (1972) (cf. Basawa and Prakasa Rao (1980)), it can be shown that the $C(\alpha)$–test based on Y_T is optimal in the sub–class of critical regions symmetric about $\gamma=\gamma_0$ for testing $H_0:\gamma=\gamma_0$ against $H_1:\gamma\neq\gamma_0$.

6. Example

Consider the diffusion process defined by the stochastic differential equation

$$dX(t)=(\gamma X(t)+\theta)dt+dW(t), t\ge 0, X(0)=0 \qquad \ldots(6.1)$$

where $\{W(t),t\ge 0\}$ is the standard Wiener process.

The problem is to test the hypothesis

$$H_0:\gamma=\gamma_0 \text{ against } H_1:\gamma\neq\gamma_0$$

in the presence of the nuisance parameter θ. Let $P_{\gamma,\theta}^{(T)}$ be the probability measure generated by the process $\{X(t),0\le t\le T\}$ on $C[0,T]$ when (γ,θ) is the true parameter. Here $C[0,T]$ is the space of all real–valued continuous functions on $[0,T]$ endowed with the supremum norm. Let μ^T be the measure generated by the standard Wiener process on $C[0,T]$. Note that

$$P_{\gamma,\theta}^{(T)}\left(\int_0^T X^2(t)dt<\infty\right)=1 \text{ for all } T>0. \qquad \ldots(6.2)$$

Hence $P_{\gamma,\theta}^{(T)}$ is absolutely continuous with respect to μ^T and

$$\log \frac{dP^{(T)}_{\gamma,\theta}}{d\mu^T} = \int_0^T (\gamma X(t) + \theta) dX(t) - \frac{1}{2}\int_0^T (\gamma X(t) + \theta)^2 dt \qquad \ldots (6.3)$$

(cf. Basawa and Prakasa Rao (1980)). It is easy to check that

$$\phi^{(T)}_{\gamma_0} = \int_0^T X(t) dX(t) - \int_0^T (\gamma_0 X(t) + \theta) X(t) dt \qquad \ldots (6.4)$$

and

$$\phi^{(T)}_{\theta} = \int_0^T dX(t) - \int_0^T (\gamma_0 X(t) + \theta) X(t) dt \qquad \ldots (6.5)$$

when (γ_0, θ) is the true parameter. In view of (6.1), it can be seen that

$$\phi^{(T)}_{\gamma_0} = \int_0^T X(t) dW(t)$$

and

$$\phi^{(T)}_{\theta} = \int_0^T dW(t) = W(T).$$

In particular

$$\begin{aligned}
\lambda^{(T)}_{\gamma_0\gamma_0} &= E_{\gamma_0,\theta}[\textstyle\int_0^T X(t) dW(t)]^2 \\
&= \textstyle\int_0^T E_{\gamma_0,\theta}[X(t)]^2 dt, \\
\lambda^{(T)}_{\gamma_0\theta} &= E_{\gamma_0,\theta}[\textstyle\int_0^T X(t) dW(t) \int_0^T dW(t)] \\
&= \textstyle\int_0^T E_{\gamma_0,\theta}[X(t)] dt,
\end{aligned}$$

and

$$\lambda^{(T)}_{\theta\theta} = E_{\gamma_0,\theta}[W^2(T)] = T.$$

Hence

$$\Lambda^{(T)} = \begin{bmatrix} \lambda^{(T)}_{\gamma_0,\gamma_0} & \lambda^{(T)}_{\gamma_0\theta} \\ \lambda^{(T)}_{\gamma_0\theta} & \lambda^{(T)}_{\theta\theta} \end{bmatrix} \qquad \ldots (6.6)$$

$$= \begin{bmatrix} \int_0^T E_{\gamma_0\theta}[X^2(t)] dt & \int_0^T E_{\gamma_0\theta}[X(t)] dt \\ \int_0^T E_{\gamma_0\theta}[X(t)] dt & T \end{bmatrix}$$

and

$$\lambda_{\gamma_0\theta}^{(T)}\lambda_{\theta\theta}^{(T)-1} = \frac{1}{T}\int_0^T E_{\gamma_0,\theta}[X(t)]dt \equiv a(\gamma_0,\theta) \text{ (say)} .$$

Furthermore the regression of $\phi_{\gamma_0}^{(T)}$ on $\phi_\theta^{(T)}$ is given by

$$(\frac{1}{T}\int_0^T E_{\gamma_0,\theta}[X(t)]dt)\phi_\theta^{(T)}.$$

Define

$$\begin{aligned} Y_T \equiv Y_T(\gamma_0,\theta) &= \phi_{\gamma_0}^{(T)} - a(\gamma_0,\theta)\phi_\theta^{(T)} \\ &= \int_0^T X(t)dW(t) - a(\gamma_0,\theta)\int_0^T dW(t) \\ &= \int_0^T [X(t) - a(\gamma_0,\theta)]dW(t). \end{aligned} \qquad \ldots(6.7)$$

Then

$$\begin{aligned} \tilde{\sigma}^2((\gamma_0,\theta);T) &= E_{\gamma_0,\theta}[Y_T]^2 \\ &= \int_0^T E_{\gamma_0,\theta}[X(t) - a(\gamma_0,\theta)]^2 dt \\ &= \int_0^T E_{\gamma_0,\theta}[X^2(t)]dt - Ta^2(\gamma_0,\theta) \\ &= \int_0^T E_{\gamma_0,\theta}[X^2(t)]dt - \tfrac{1}{T}(\int_0^T E_{\gamma_0,\theta}[X(t)]dt)^2. \end{aligned} \qquad \ldots(6.8)$$

Suppose there exists $0 < \beta_T \to \infty$ independent of θ such that

$$\frac{1}{\beta_T}\int_0^T E_{\gamma_0,\theta}[X(t) - a(\gamma_0,\theta)]^2 dt \to b(\gamma_0,\theta) \text{ as } T \to \infty \qquad \ldots(6.9)$$

for some $0 < b(\gamma_0,\theta) < \infty$. By the central limit theorem for stochastic integrals (cf. Basawa and Prakasa Rao (1980) or Kutoyants (1984)), it follows that

$$\frac{Y_T(\gamma_0,\theta)}{\beta_T^{\frac{1}{2}} b^{1/2}(\gamma_0,\theta)} \xrightarrow{\mathcal{L}} N(0,1) \text{ as } T \to \infty. \qquad \ldots(6.10)$$

It is easy to see that

$$E_{\gamma_0,\theta}[Y_T\phi_\theta^{(T)}] = 0. \qquad \ldots(6.11)$$

In order to use

$$Z_T(\gamma_0,\theta) = \frac{Y_T(\gamma_0,\theta)}{\beta_T^{1/2} b^{1/2}(\gamma_0,\theta)} \qquad \ldots(6.12)$$

as a test statistic for testing $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma \neq \gamma_0$, we need a v_T- consistent estimator of θ for some $v_T \to \infty$ as $T \to \infty$. It is easy to check from (6.3) that

$$\frac{\partial}{\partial\gamma}\left[\log\frac{dP^T_{\gamma,\theta}}{d\mu^T}\right] = \int_0^T X(t)dX(t) - \int_0^T (\gamma X(t)+\theta)X(t)dt \qquad \ldots(6.13)$$

and

$$\frac{\partial}{\partial\theta}\left[\log\frac{dP^T_{\gamma,\theta}}{d\mu^T}\right] = \int_0^T dX(t) - \int_0^T (\gamma X(t)+\theta)dt. \qquad \ldots(6.14)$$

The likelihood equations are

$$\frac{\partial}{\partial\gamma}\left[\log\frac{dP^T_{\gamma,\theta}}{d\mu^T}\right] = 0 = \frac{\partial}{\partial\theta}\left[\log\frac{dP^T_{\gamma,\theta}}{d\mu^T}\right]. \qquad \ldots(6.15)$$

They lead to the estimators

$$\hat{\theta}_T = \frac{X(T) - \hat{\gamma}_T\int_0^T X(t)dt}{T} \qquad \ldots(6.16)$$

and

$$\hat{\gamma}_T = \frac{\int_0^T X(t)dX(t) - \hat{\theta}_T\int_0^T X(t)dt}{\int_0^T X^2(t)dt}. \qquad \ldots(6.17)$$

However, if (γ_0, θ) is the true parameter and γ_0 is known, then

$$X(T) = \gamma_0\int_0^T X(t)dt + \theta T + W(T) \qquad \ldots(6.18)$$

and the maximum likelihood estimator $\hat{\theta}_T$ of θ satisfies the relation

$$\hat{\theta}_T - \theta = \frac{W(T)}{T}. \qquad \ldots(6.19)$$

Note that $\sqrt{T}(\hat{\theta}_T - \theta)$ is normal with mean zero and variance one. Hence $\hat{\theta}_T$ is a v_T–consistent estimator of θ with $v_T = T^{1/2}$. The statistic

$$Z_T(\gamma_0, \theta_T) = \frac{Y_T(\gamma_0, \hat{\theta}_T)}{\beta_T^{1/2} b^{1/2}(\gamma_0, \hat{\theta}_T)}$$

can be used as a test statistic for testing $H_0 : \gamma = \gamma_0$ against the alternative $H_1 : \gamma \neq \gamma_0$ and it is an optimal asymptotic test.

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INDIAN STATISTICAL INSTITUTE
DELHI CENTRE
7. S.J.S. SANSANWAL MARG
NEW DELHI
INDIA