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Higher-Dimensional Analogues of the Map Coloring Problem

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Abstract. After a brief discussion of the history of the problem, we propose a generalization of the map coloring problem to higher dimensions.

The map coloring problem was originally posed by Francis Guthrie in 1852. It asks for the minimum number of colors required to color all possible maps (real or imagined) if we wish to ensure that neighboring countries receive different colors. Clearly, we may imagine any number of countries sharing a common point in their boundary. So, in this problem, two countries are to be considered as neighbors only if their boundaries share an entire linear continuum. Four colors are required, since there are maps with four mutually adjacent countries (as in the usual planar drawing of a tetrahedron). Francis Guthrie guessed that four colors always suffice for his problem. This came to be known as the four color conjecture (see [**3, 16**]).

Beginning with Augustus de Morgan [**2, 6**] (to whom Francis' brother Frederick Guthrie first communicated the problem), innumerable mathematicians—both professionals and amateurs—tried their hands at the problem. Many possible approaches, reformulations, and partial results, as well as several wrong proofs resulted [**7**]. One preliminary observation that could be made was that six colors always suffice. To see this, proceed as follows. The map to be colored may well be taken as drawn on the surface of a ball (as in a globe) rather than on a sheet of paper. A little thought shows that, without loss of generality, the countries may be taken to be the faces of a convex polyhedron inscribed in the ball. We can then invoke Euler's formula $v - e + f = 2$, where v, e, f are the number of vertices, edges, and faces of the polyhedron. Using this formula, it is not hard to see that, in any map, there must be a country with five or fewer neighbors. Now, with six colors in hand, we may proceed to color all maps inductively, as follows. Assume that we already know how to color all maps with fewer countries using six colors. Pick out a country with five or fewer neighbors in the given map, and momentarily forget it. By our assumption, the rest of the map can be colored using the six colors. Since the forgotten country has at most five neighbors, at most five of the colors have been used to color these neighbors. So, there is a leftover color that can now be used to color the forgotten country, thus completing the six-coloring of the given map. For an alternative proof of this "six colors theorem", which avoids Euler's formula, read on.

The first significant progress on the four color conjecture was made by A. B. Kempe, when he published [**11**] an incorrect proof of the conjecture in 1879. But Kempe's work contained a very fruitful idea, which led to the eventual resolution of

the problem. In 1890, P. J. Heawood [9] pointed out the flaw in Kempe's argument, but he also showed that Kempe's logic may be modified to prove that five colors always suffice. The argument begins as in the proof of the six colors theorem sketched above. But, if the forgotten country has five neighbors who have received all five colors in the coloring of the rest of the map, then Kempe's argument gives a method to recolor them, so that one of the colors is freed for use on the forgotten country.

Finally, in 1976, K. Appel and W. Haken collaborated with a number of computers (taking more than a thousand hours of computer time) to prove the four color conjecture (see [3, 4, 5]). Their work made no use of the huge superstructure of theories created during the twentieth century. Instead, they went back to a vastly elaborated version (due to Heesch [10]) of Kempe's original idea. Perhaps there is a moral lurking here. Much has been made of the fact that Appel and Haken's proof is not a "human" one, and nobody can possibly verify it manually. But we think that the real problem lies elsewhere. If there was a prior theoretical proof that one only needed to check a million cases (say) to settle the questions one way or the other, then we think, most mathematicians would have happily left these million verifications to a machine. Unfortunately, the situation with the Appel–Haken proof is not like that. Until the computer stopped and came out with its verdict in favour of the four color conjecture, there was no guarantee that the program would ever halt. It was rather like the celebrated halting problem of Turing acted out in real life. The proof consists of finding a finite "unavoidable set of reducible configurations". A set of configurations is unavoidable if there is a proof showing that any hypothetical counterexample to the four color conjecture must contain one of them. A configuration is reducible if, whenever a counterexample contains it, there is a well-defined procedure to create a smaller counterexample by eliminating it. The program halted precisely because it found such a set. Of course, we may ignore the mechanical genesis of this list, and get its authenticity verified (necessarily by another machine, since the list is so large). Such a verification constitutes an airtight proof which ought to keep everybody happy. But the question remains: why does such a set exist? Is it mere happenstance, or is there a good theoretical reason for its existence? So, the search for a "human" proof is still active. (Compare the popular article [3] written by Appel and Haken; or else, for an abridged discussion of their proof, the reader may consult [16].)

Early in the game, mathematicians realized that the infinite variety of shapes and sizes of countries is not relevant to the problem. What matters is the knowledge of which pairs of countries are neighbors. In order to forget the inessential, they used the notion of a graph. In simple terms, a graph is a picture consisting of finitely many dots (called vertices) in which some pairs of dots are joined by (possibly curved) line segments, called edges. Obviously, an edge joining the vertices x and y intersects an edge joining the vertices x and z, at least in the common vertex x. A graph drawn in the plane is called planar if the edges have no further intersections. Each country in a map may be represented by a dot in its interior (perhaps its capital city). Clearly, whenever two countries are neighbors, the corresponding dots may be joined by a line segment (perhaps curved), which meets their common boundary at a point not lying on any other country, otherwise staying in the interior of these two countries. A planar graph results. If we can color the vertices of this graph in four colors so that neighboring vertices (i.e., those joined by an edge) receive different colors, then we may transfer the colors from the vertices to the corresponding countries, resulting in a proper four-coloring of the map. Conversely, given any planar graph, we may suitably blow up the vertices into countries, to create a corresponding map. Thus, we have the mathematician's favourite version of the map coloring problem: Show that the vertices of any planar graph can be properly colored using four colors.

So what should be a 3-dimensional version of this problem? We might try to define a "spatial graph" as a graph that may be drawn in three-dimensional space without undue intersections. Unfortunately, a moment's thought shows that all graphs are "spatial". Indeed, we may take any (finite) number of points in space "in general position", i.e., such that no four of them are on a common plane. (For instance, we may choose these points arbitrarily on the "moment curve" $t \mapsto (t, t^2, t^3)$.) Then we may join each pair of these points by a straight line segment. No two of these segments will then meet, except at a common vertex. The same observation goes for all higher dimensions, since the 3-dimensional space embeds in all Euclidean spaces of higher dimensions.

Is that the end of the road for our search for a higher-dimensional analogue of the map coloring problem? Not quite! After all, the attempt to abstract away the irrelevant geometric details of shape and size by going to planar graphs is not a very successful one. Indeed, any (sufficiently complicated) planar graph is visually indistinguishable from a map. Consider the following problem instead. Take a finite set of nonoverlapping discs in the plane. That is, any two of them are either disjoint or (externally) tangential. In other words, no two of them have any common interior point. We pose the following question: What is the minimum number of colors needed if we are to color these discs in such a way that touching discs are given different colors? Does this not look like a much simpler problem which should have a fairly straightforward answer? Notice that, given such a set of discs, we can again form a planar graph (apparently of a very special kind) by joining the centers of each pair of touching discs by straight line segments. So, if Appel and Haken are to be believed, then the answer to this problem should be four. (We can easily have four mutually touching discs in the plane, say by taking three equal mutually touching discs and then placing a fourth small disc in the niché created by them.) An amazing theorem due to Paul Koebe, E. M. Andreev, and William P. Thurston (see [**1, 12, 15**]) says that every planar graph can be redrawn as the graph of a set of nonoverlapping discs. Thus, our innocent-looking problem of coloring discs is equivalent to the map coloring problem! Now, it is obvious how we may reprove the "six colors theorem" (albeit using the powerful K-A-T theorem). Given any finite set of nonoverlapping discs in the plane, choose the smallest one. Say its radius is of unit length. Then, the discs touching it are equal or larger, so that it is obvious that at most six discs touch the smallest one. Moreover, if six discs do touch the smallest disc, then these neighboring discs must be unit discs as well. We may assume that our set of discs is "connected" (in the sense that a point particle may go from one disc to any other, all the time staying within the discs in the set). Now, is it possible that all the discs have six or more discs touching them? If so, then the above argument shows that all the discs must be of the same size, and each must have exactly six neighbors touching it. But this is impossible since it is intuitively clear that, in this case, a disc in the periphery (technically, a disc touching the boundary of the convex hull of all the discs) would have at most four neighbors! Thus, we have shown that, given any finite configuration of nonoverlapping discs in the plane, there is at least one disc that touches at most five others. Now we may complete the proof of the "six colors theorem" as before. As promised, we have avoided any use of Euler's formula.

Now it should be clear how we can generalize the map coloring problem to arbitrary dimensions. For $d \geq 1$, let $\chi(d)$ be the smallest number such that any (finite) set of d-dimensional nonoverlapping closed balls (not necessarily of the same size) in d-dimensional Euclidean space may be colored in $\chi(d)$ colors, so that any two touching balls receive different colors. Thus, we have $\chi(1) = 2$ (trivial) and $\chi(2) = 4$ (Appel and Haken). What is the value of $\chi(3)$?

There is a small but important question that needs to be answered. How do we know that $\chi(d)$ is finite? In other words, is it possible that in some dimension there are arbi-

trarily complicated sets of nonoverlapping balls requiring unboundedly large number of colors? No! To see this, recall that the kissing number $\tau(d)$ of d-dimensional Euclidean space is defined as the maximum number of nonoverlapping equal balls that may touch a given ball of the same size. It is intuitively clear that $\tau(d)$ is finite for each d. Indeed, the $\tau(d) + 1$ unit balls in such a kissing configuration are contained in a ball of radius 3 (concentric with the central ball). Comparing (d-dimensional) volumes, we then see that $\tau(d) + 1 \leq 3^d$. Thus, $\tau(d) \leq 3^d - 1$ for all d. This is a very crude bound. For improved bounds, see Conway and Sloane [8]. The exact value of $\tau(d)$ is known only for $d = 1, 2, 3, 4, 8$, and 24. Indeed, we have $\tau(1) = 2$, $\tau(2) = 6$, $\tau(3) = 12$, $\tau(4) = 24$, $\tau(8) = 240$, and $\tau(24) = 196560$ (see [8, 13, 14]). Isn't that a surprise?

Now, given any finite set of nonoverlapping balls in d-dimensional space, the neighbors of the smallest ball (say, a unit ball) may be shrunk into unit balls, still remaining nonoverlapping and still touching the smallest ball. This shows that there is at least one ball (namely the smallest) with at most $\tau(d)$ neighbors. Therefore, as before, we may prove by induction on the number of balls that any finite set of nonoverlapping balls in d-space may be properly colored using at most $\tau(d) + 1$ colors. Thus, $\chi(d) \leq \tau(d) + 1$. Also, in space of dimension $d \geq 2$, we can construct a set of $d + 2$ mutually touching balls (take $d + 1$ equal balls with centers at the vertices of a regular simplex, having the side length of the simplex as their diameters, and then place a small ball in the middle touching all of them). Thus, $\chi(d) \geq d + 2$ for all $d \geq 2$. So, we have

$$d + 2 \leq \chi(d) \leq \tau(d) + 1 \leq 3^d, \quad \text{for all } d \geq 2.$$

In the magic dimensions $d = 1, 2, 8$, and 24, it is known that, up to congruence, there are unique configurations attaining the kissing numbers. In these dimensions we may argue (exactly as we have done in the case $d = 2$) that $\chi(d) \leq \tau(d)$, a slight improvement.

In our familiar three-dimensional space, we have $5 \leq \chi(3) \leq 13$. What is the true value? We won't even hazard a guess. Happy hunting!

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Stirling's Formula and Its Extension for the Gamma Function

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Abstract. We present new short proofs for both Stirling's formula and Stirling's formula for the Gamma function. Our approach in the first case relies upon analysis of Wallis' formula, while the second result follows from the log-convexity property of the Gamma function.

The well-known formula of Stirling asserts that

$$n! \approx \sqrt{2\pi}\, n^{n+1/2} e^{-n} \quad \text{as } n \to \infty, \tag{1}$$

in the sense that the ratio of the two sides tends to 1. This provides an efficient estimation to the factorial, used widely in probability theory and in statistical physics.

Articles treating Stirling's formula account for hundreds of items in JSTOR. A few of the most relevant references may be found in [**2**], [**3**], [**5**], and [**7**].

As was noticed by Stirling himself, the presence of π in the formula (1) is motivated by the Wallis formula,

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2n-1) \cdot (2n-1) \cdot (2n+1)},$$

that is,

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n+1}} \cdot \frac{(2n)!!}{(2n-1)!!} = \sqrt{\frac{\pi}{2}},$$

where $n!! = n \cdot (n-2) \cdots 4 \cdot 2$ if n is even, and $n \cdot (n-2) \cdots 3 \cdot 1$ if n is odd.