INDIAN STATISTICAL INSTITUTE

Fourier Analysis : M. Math. 2nd year
Mid Semester Examination: 2017-18
September 04, 2017.

Maximum Marks 50 Maximum Time 2:30 hrs.

(1) (a) Does there exist any $g \in L^1(\mathbb{T})$ such that $f * g = f$ for all $f \in L^1(\mathbb{T})$? Justify your answer.

(b) Let $f \in L^1(\mathbb{T})$ be such that $\{\sigma_n(f)\}$ converges in $L^q(\mathbb{T})$ for some $q \in (1, \infty)$ (here $\sigma_n(f) = f * F_n$). Does this imply that $f \in L^q(\mathbb{T})$? Justify your answer.

(c) Does there exist any $q \in (1, \infty)$ such that for all $N \in \mathbb{N}$,

$$\|S_N\|_{q-q} = \|D_N\|_q?$$

justify your answer.

(d) If $a_1, a_2, \ldots, a_N$ are complex numbers of unit modulus and $P(e^{i\theta}) = \sum_{n=1}^{N} a_n e^{in\theta}$, $\theta \in [-\pi, \pi]$, then prove that $\|P\|_\infty \geq \sqrt{N}$. [4 × 3 = 12]

(2) If $f \in C^1(\mathbb{T})$ with $\hat{f}(0) = 0$ then prove that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \frac{\pi}{\sqrt{3}} \|f'\|_2$. [4]

(3) (a) If P and Q are trigonometric polynomials then prove that for sufficiently large $m \in \mathbb{N}$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) Q(e^{im\theta}) d\theta = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) d\theta \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} Q(e^{i\theta}) d\theta \right).$$

(b) If $g \in C(\mathbb{T})$ and P is a trigonometric polynomial then prove that

$$\lim_{m \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) g(e^{im\theta}) d\theta = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) d\theta \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta}) d\theta \right).$$

[2 × 4 = 8]

[P.T.O]

(4) Suppose $f \in C(\mathbb{T})$ is such that the sequence of functions $\{S_N(f)\}$ converges to zero uniformly on $[-\pi, \pi]$. Prove that $f(\theta) = 0$ for all $\theta \in [-\pi, \pi]$. (Hint: Apply Plancherel theorem on a suitable function.) [6]

(5) Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{T})$. Prove that $\text{span}\{\tau_\theta f \mid \theta \in [-\pi, \pi]\}$ is dense in $L^p(\mathbb{T})$ if and only if all Fourier coefficients of f are nonzero. Is the same true for $L^\infty(\mathbb{T})$? Justify your answer. [8]

(6) If D_N, $N \in \mathbb{N}$ denotes the Dirichlet kernel then define

$$F_N(\theta) = \frac{D_0(\theta) + D_1(\theta) + \ldots + D_{N-1}(\theta)}{N}, \quad \theta \in [-\pi, \pi], N \geq 1.$$

Prove that $\{F_N \mid N \in \mathbb{N}\}$ is an approximate identity. [8]

(7) If $f \in L^1(\mathbb{T})$ then prove that for every interval $(a, b) \subset [-\pi, \pi]$,

$$\lim_{N \to \infty} \int_a^b \sigma_N(f)(\theta) d\theta = \int_a^b f(\theta) d\theta.$$

(Hint: $\chi_{(a,b)} * F_N$, $N \geq 1$ are uniformly bounded.) [8]

NUMBER THEORY

MID-SEMESTER EXAM - 6 SEPT 2017
M.MATH - II YEAR
DURATION 3 HOURS, MAX MARKS 35

1) Prove that

$$\sum_{m|n} d(m)^3 = \left(\sum_{m|n} d(m)\right)^2$$

where $d(m)$ is the divisor function. (5 points)

2) Let $\varphi(n)$ be the Euler totient function. Prove that

$$\varphi(n) = \sum_{d|n} \mu(d)\frac{n}{d}.$$

Using this (or otherwise) and the fact that $\sum_{d=1}^{\infty} \mu(d)/d^2 = 6/\pi^2$ prove that

$$\sum_{n \leqslant x} \varphi(n) = \frac{3}{\pi^2}x^2 + O(x^{1+\varepsilon})$$

(5 + 8 points)

3) Let $\pi(x)$ be the number of primes smaller than x. Show that

$$\pi(x) \ll \frac{x}{\log x}.$$

(8 points)

4) Let χ be a primitive character modulo a prime M. Let

$$g(\chi) = \sum_{m=1}^{M} \chi(m)e^{2\pi i m/M}$$

be the Gauss sum associated with χ. Show that $|g(\chi)| \ll M^{1/2}$. (6 points)

5) Let $p > 3$ be a prime, and let $(\frac{\cdot}{p})$ be the quadratic residue symbol modulo p. Show that the number of $(x, y) \in \{1, 2, \dots, p-1\}$ satisfying the equation

$$y^2 = x(3x+2)$$

is $p - 2 - (\frac{3}{p})$. (8 points)

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2016-17
M. Math.-II Year
Differential Topology

Date : 07. 09. 2017 | Maximum Score : 50 | Time : 3:00 Hours

Any result that you use should be stated clearly.

(1) **(a):** Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$. Describe local parametrizations covering S^2 to prove that $S^2 \subset \mathbb{R}^3$ is a 2-dimensional manifold.
(b): Prove that if X and Y are manifolds of dimensions k and l, respectively, then $X \times Y$ is a manifold of dimension $k+l$.
(c): Let $f : X \to Y$ be a smooth map of manifolds. Let

$$graph(f) = \{(x, f(x)) : x \in X\}.$$

Prove that $graph(f)$ is a manifold of dimension equal to the dimension of X.

[5+ 5 + 5 = 15]

(2) **(a):** Let $f : X \to Y$ be a smooth map of manifolds and Z a submanifold of Y. Explain what do you mean by f to be
(i) an immersion; (ii) a submersion; (iii) a local diffeomorphism; (iv) an embedding; (v) transversal to Z. Give an example in each case.
(b): Let $f : M \to \mathbb{R}^n$ be an immersion. Suppose that $\cup_{x\in M} df_x(T_xM)$ is a proper subset of $\mathbb{R}^n$. Prove that there exists an immersion $g : M \to \mathbb{R}^{n-1}$.

[15 + 10 = 25]

(3) **(a):** What is a regular value of a smooth map $f : X \longrightarrow Y$ of manifolds?
(b): State pre-image theorem and use it to prove that $O(n)$ is manifold. Determine the dimension of $O(n)$.

[2 + 2 + 6 = 10]

(4) **(a):** Show that a non-degenerate critical point of a function $f : \mathbb{R}^n \to \mathbb{R}$ is an isolated critical point.
(b): Let H denote the hyperboloid defined by the equation

$$x^2 + y^2 - z^2 = 1.$$

Consider the function $f : H \to \mathcal{R}$ given by $f(x,y,z) = x$. Prove that $(0,1,0)$ is a non-degenerate critical point of f.

[10 + 10 = 20]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2017-2018, First Semester

M-Math II

Basic Probability Theory

Date: 08·09·17 **Max. Marks 30** **Duration:** $2\frac{1}{2}$ **Hours**

Note: Total marks 35, maximum you can score is 30.

1. Two dice are thrown twice independently. Assuming that faces of each die have equal probability to appear uppermost, find the probability that two throws show the same configuration if (a) the dice are distinguishable; (b) the dice are indistinguishable.

[5+5]

OR

Suppose there are n boxes and r distinguishable balls. Show that the probability that exactly m boxes contain exactly k balls (each) is

$$p_{[m]} = \frac{(-1)^m n! r!}{m! n^r} \sum_j (-1)^j \frac{(n-j)^{r-jk}}{(j-m)!(n-j)!(r-jk)!(k!)^j}$$

where the sum is over $j \geq m$ such that $j \leq n$ and $jk \leq r$.

[10]

2. Suppose there are $m+1$ boxes numbered $0, 1, \ldots, m$. For each $k = 0, 1, \ldots, m$, the box numbered k contains k red and $m-k$ black balls. A box is chosen at random, and n balls are drawn from it successively and randomly, the ball being returned to the box after each draw. If each of these n draws yield red ball, find the conditional probability that the $(n+1)$st draw from the same box also yields red ball.

[5]

3. Two players A and B are playing a tournament consisting of a series of games which is assumed to be a sequence of Bernoulli trials with probability of success p which is the probability that A wins the game, and failure $q = 1-p$ which is the probability that B wins. The winning player scores one point at each game. Find the distribution of the number of games needed to end the tournament—find its probability mass function, expectation and variance.

[10]

4. Suppose the random time gaps between any two consecutive repairs of an electronic machine, as well as the waiting time of its first repair, are independent and identically distributed random variables, each having exponential distribution with parameter α. If X is the random variable denoting the number of repairs done in the time interval $[0, t]$, then find the distribution of X. Also, find its moment generating function.

[10]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2017-2018, First Semester
M-Math II
Basic Probability Theory

Date: 08.09.17 **Max. Marks 30** **Duration:** $2\frac{1}{2}$ **Hours**

Note: Total marks 35, maximum you can score is 30.

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[10]

TOPICS IN NUMBER THEORY

MID-SEMESTER EXAM - 11 SEPT 2017
M.MATH - II YEAR
DURATION 3 HOURS, FULL MARKS 40

1) Let Δ_j be the j-th forward differencing operator. Show that

$$\Delta_j(x^k; b_1, \dots, b_j) = b_1 b_2 \dots b_j p_j(x)$$

where p_j is a polynomial of degree $k-j$ with leading coefficient $k!/(k-j)!$. (7 pts)

2) Let $\{c_n\}$ be any sequence of complex numbers, and let $F \in C_1[0, X]$. Show that

$$\sum_{1 \leqslant m \leqslant X} c_m F(m) = F(X) \sum_{m \leqslant X} c_m - \int_0^X F'(y) \sum_{m \leqslant y} c_m \, dy.$$

(5 pts)

3) Let

$$S(q,a) = \sum_{m=1}^{q} e(am^k/q), \quad \text{and} \quad S(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (q^{-1}S(q,a))^s e(-an/q)$$

for some integer $s > 0$. Show that $S(q)$ is multiplicative. (8 pts)

4) Let p be an odd prime. Show that the number of k-th power residues (i.e. residues of the form x^k with $p \nmid x$) modulo p^t is $\varphi(p^t)/(k, \varphi(p^t))$. (5 pts)

5) Let

$$S(q,a,b) = \sum_{m=1}^{q} e((am^k + bm)/q)$$

with $k \geqslant 2$. Suppose $q = p^{2j}$ with p odd prime, j positive integer and $p \nmid a$. Show that $S(p^{2j}, a, b) \ll p^j (p^{2j}, b)$. (8 pts)

6) Let

$$v(\beta) = \sum_{m=1}^{n} e(\beta m), \quad \text{and set} \quad J(n) = \int_{-1/2}^{1/2} v(\beta)^3 e(-\beta n) d\beta.$$

Show that $J(n) = (n-1)(n-2)/2$. (7 pts)

INDIAN STATISTICAL INSTITUTE
Semestral Examination : 2017-18
M. Math.-II Year
Differential Topology

Date : 20. 11. 2017 | Maximum Score : 50 | Time : 3:00 Hours

Any result that you use should be stated clearly.

(1) **(a):** Define a manifold with boundary.
(b): Prove that if X is a manifold with boundary of dimension k, then the boundary ∂X is a manifold without boundary of dimension $k-1$.
[4 + 10 = 14]

(2) **(a):** Explain what you mean by an oriented vector space. Using this notion define orientation on a manifold.
(b): What is an orientable atlas on a manifold?
(c): Prove that if a manifold X is orientable, then it admits an orientable atlas.
(d): Is the manifold $S^2 \times T^2$ orientable? Justify your answer.
[6 + 3 + 5 + 6 = 20]

(3) **(a):** What is an alternating p-tensor on a vector space? Give an example of an alternating n-tensor on $\mathbb{R}^n$.
(b): Define the notion of a smooth p-form on a manifold.
(c): Let 0 be a regular value of a smooth map $f : \mathbb{R}^3 \to \mathbb{R}$. Let $M = f^{-1}(0)$ and $f_x,\ f_y,\ f_z$ denote the partial derivatives of f with respect to $x,\ y,\ z$, respectively. Prove that the equalities

$$\frac{dx \wedge dy}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$$

hold on M whenever they make sense, and therefore the three 2-forms piece together to give a global smooth 2-form on M.
(d): Consider $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$ as a submanifold in $\mathbb{R}^3$. Compute $\int_S \omega$, where ω is a 2-form on S defined by

$$\omega = \begin{cases} \frac{dy\wedge dz}{x} & \text{for } x \neq 0 \\ \frac{dz\wedge dx}{y} & \text{for } y \neq 0 \\ \frac{dx\wedge dy}{z} & \text{for } z \neq 0. \end{cases}$$

[4 + 4 + 4 + 6 = 18]

(4) Calculate the exterior derivatives of the following forms in $\mathbb{R}^3$:
(a): $z^2 dx \wedge dy + (z^2 + 2y) dx \wedge dz$.
(b): $f dg$, where f and g are smooth real valued functions on $\mathbb{R}^3$.
[4 + 4 = 8]

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2017-18

M. MATH. II YEAR
Commutative Algebra I

Date : 22.11.17 Maximum Marks : 70 Duration : 3 Hours

Answer any 5 questions from Groups A and any 2 from Group B.
Clearly state the results that you use.

Group A
ATTEMPT ANY FIVE QUESTIONS.
R denotes a commutative ring with unity.

1. Let $I, P_1, P_2, \cdots, P_n$ be ideals of R such that P_i is prime $\forall\ i \geq 3$. Prove that if I is not contained in any of the P_i, then there exists $x \in I$ that is not contained in any P_i. [10]

2. Let I be an ideal of an integral domain R. Prove that if both $I + (x)$ and $(I : x)$ are principal ideals, then I is a principal ideal. [10]

3. Let I be a finitely generated ideal of R satisfying $I^2 = I$. Prove that there exists $f \in I$ for which $f^2 = f$ and $I = (f)$. [10]

4. Let M and N be finitely generated R-modules such that $M \otimes_R N = 0$. Prove that $Ann_R M + Ann_R N = R$. [10]

5. Let M be an R-module. Prove that if P is an ideal of R that is maximal among all annihilators of non-zero elements of M then P is a prime ideal of R. [10]

6. Let $B = \mathbb{R}[X, Y]/(X^2+Y^2-1)$ and M the ideal $(x, y-1)B$ where x and y denote respectively the images of X and Y in B. Prove that M is a projective B-module. [10]

7. Prove that the ring $R = \mathbb{C}[X, Y]/(Y^2 - X^2 - X^3)$ is not normal. [10]

8. Suppose that V is an affine algebraic set in $\mathbb{C}^n$ such that $(\mathbb{C}[V])^\star = \mathbb{C}^\star$. Show that any non-constant polynomial function $f : V \to \mathbb{C}$ must be surjective. [10]

GROUP B
ATTEMPT ANY TWO QUESTIONS.
k denotes a field.

1. Prove that $\mathbb{C}[X, Y, Z, W]/(X^2 + Y^2 + Z^2 + W^2 - 1)$ is a UFD. [12]

2. Prove that any k-subalgebra of the polynomial ring $k[X]$ is a finitely generated k-algebra. [12]

3. (i) Let A be an integral domain which is finitely generated as a k-algebra and L the field of fractions of A. Show that if $A \neq L$, then $A[1/f] \neq L$ for any $f \in A$.

   (ii) Give an example of a proper subring A of $\mathbb{Q}$ and an element $f \in A$ such that $A[1/f] = \mathbb{Q}$.

   [8+4=12]

INDIAN STATISTICAL INSTITUTE
First Semester Examination : 2017-18
M-MATH, Year II (non-B-Stat, non-B-Math)
BASIC PROBABILITY THEORY

Date: 24.10.'17 **Max. Marks 50** **Duration: 3 Hours**

Note: Answer all questions. Total marks: 54.
Maximum you can score: 50

1. 3 **indistinguishable** dice are rolled twice independently. If each face of each die has equal probability to appear uppermost, what is the probability that both the throws show the same configuration? [6]

   (b) The probability of a family chosen at random having exactly $k (k \geq 0)$ children is αp^k where $0 < p < 1$ and $\alpha = 1 - p$. Suppose that the probability that any child has blue eyes is b, $0 < b < 1$ independently of others. What is the probability that a family chosen at random has exactly $r (r \geq 0)$ children with blue eyes? [4]

2. **(a)** Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} (2x)/\pi^2 & \text{if } 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

   Let $Y = \sin X$. Find the p.d.f. of Y. [6]

   (b) Let X and Y be two discrete random variables with respective probability distributions $P(X = x_1) = p_1$, $P(X = x_2) = 1 - p_1$; and $P(Y = y_1) = p_2$, $P(Y = y_2) = 1 - p_2$. Show that if they are uncorrelated then they are independent. [6]

3. **(a)** Let X and Y be independent random variables with common probability mass function $P(X = k) = p(1-p)^k$, $k = 0, 1, \dots,$ $0 < p < 1$. Let $M := \max\{X, Y\}$. Find the joint distribution of M and X, the marginal distribution of M, and the conditional distribution of X given M. [10]

   (b) Let X and Y be iid random variables with common p.d.f. f:

$$f(x) = \begin{cases} (x\sqrt{2\pi})^{-1} e^{-(1/2)(\log x)^2} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the p.d.f. of $Z = XY$. [10]

4. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables such that $P(X_n = \pm n2^n) = 1/4$, and $P(X_n = 0) = 1/2$, $\forall n = 1, 2, \ldots$.

(i) Show that if $X_n > 0$ then $S_n := \sum_{j=1}^{n} X_j \geq 2^n$.

(ii) Using (i) show that $(S_n/2^n) \not\to 0$ as $n \to \infty$ with probability 1.

(iii) Find a sequence of positive numbers $\{B_n\}$ such that $S_n/B_n \to 0$ *a.s.* as $n \to \infty$ [4+4+4]

INDIAN STATISTICAL INSTITUTE

Fourier Analysis : M. Math. 2nd year
End Semester Examination: 2017-18
November 27, 2017.

Maximum Marks: 60 Maximum Time: 3 hrs.

(1) For any given $p \in [1, \infty)$ construct a function $f \in L^p(\mathbb{R})$ such that $f \notin L^q(\mathbb{R})$ for any $q \neq p$, $q \in [1, \infty]$. [6]

(2) Suppose $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ is such that for all intervals I of length less than 1/2

$$\int_I f(x)dx = 0.$$

Does this imply that $f(x) = 0$ for almost every $x \in \mathbb{R}$. Justify your answer. [6]

(3) If for an $f \in L^1(\mathbb{R}^n)$ one has $Mf \in L^1(\mathbb{R}^n)$ then prove that $f(x) = 0$ for almost every $x \in \mathbb{R}^n$. Here Mf is the Hardy-Littlewood maximal function of f. [6]

(4) Suppose $f : \mathbb{R} \to \mathbb{C}$ is a measurable function such that for some positive real number y_0, $f/\widehat{P_{y_0}} \in L^1(\mathbb{R})$ where P_y denotes the Poisson kernel of $\mathbb{R}^2_+$. If $\hat{f}(1/n) = 0$ for all $n \in \mathbb{N}$ then prove that $f(x) = 0$ for almost every $x \in \mathbb{R}$. [6]

(5) Let a, b are positive real numbers and $f = \chi_{[a,b]}$ the indicator function of $[a, b]$. Wrie down the Hilbert transform Hf of f at all $x > b$ explicitly and using this prove that $Hf \notin L^1(\mathbb{R})$. [8]

(6) (a) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ then prove that

$$T_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx$$

defines a tempered distribution. [4]

(b) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then prove that $\widehat{(T_f)} = T_{\hat{f}}$. [4]

[P.T.O]

(7) Let M be a closed linear subspace of $L^2(\mathbb{R})$ such that if $f \in M$ then $\tau_x f \in M$ for all $x \in \mathbb{R}$. Let $\widehat{M} = \{\hat{f} \mid f \in M\}$.

(a) Prove that $\widehat{M}$ is a closed linear subspace of M with the property that if $f \in \widehat{M}$ then $e_y f \in \widehat{M}$ for all $y \in \mathbb{R}$ where $e_y(x) = e^{-2\pi i x y}$. [4]

(b) If $P : L^2(\mathbb{R}) \to \widehat{M}$ is the orthogonal projection then prove that for all f and g in $L^2(\mathbb{R})$,

$$f(x)Pg(x) = g(x)Pf(x),$$

for almost every $x \in \mathbb{R}$. [6]

(c) Hence or otherwise prove that for all $f \in L^2(\mathbb{R})$, $Pf = \phi.f$ such that $\phi(x) = 0$ or 1 for almost every $x \in \mathbb{R}$. [4]

(8) Consider the Poisson kernel P_y, $y > 0$, defined by

$$P_y(x) = c_n \frac{y}{(y^2 + \|x\|^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n,$$

where c_n is a positive real number such that $\|P_y\|_1 = 1$. Prove that there exists a positive number α such that for all nonnegative $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$

$$\sup_{y>0} f * P_y(x) \le \alpha M f(x), \quad \text{for almost every } x \in \mathbb{R}^n,$$

were Mf denotes the Hardy Littlewood maximal function. [12]

TOPICS IN NUMBER THEORY

END-SEMESTER EXAM - 29 NOVEMBER 2017
M.MATH - II YEAR
DURATION 3 HOURS, FULL MARKS 60

This is an open notes exam

1) (a) Give an example of a sequence $A = \{a_n\}$ with $a_n \in \mathbb{Z}$, which has lower asymptotic density 0 and upper asymptotic density 1.

(b) State Roth's theorem and Furstenberg's theorem.

(c) Let

$$A = \{m_1^3 + m_2^3 + m_3^3 + m_4^3 : m_i \in \mathbb{Z}, m_i > 0\}$$

be the set of positive integers expressible as sum of four cubes. It is known that

$$A(n) = \#\{a \in A : a \leqslant n\} = n + O(n^{9/10}).$$

Assuming this fact show that the equation

$$m_1^3 + m_2^3 + m_3^3 + m_4^3 + m_5^3 + m_6^3 + m_7^3 + m_8^3 + m_9^2 = 0$$

has infinitely many integral solutions with m_9 not a cube of an integer. (5 +4+ 11=20 pts)

2) Using the delta method give a complete proof of the fact that every sufficiently large positive integer is a sum of 10 integer squares, i.e., there exists $X > 0$ such for any $n \in [X, \infty) \cap \mathbb{Z}$, we have integers $m_i \in \mathbb{Z}$ $i = 1, 2, \ldots, 10$, such that $m_1^2 + \cdots + m_{10}^2 = n$. (25 pts)

3) (a) For f a compactly supported function on $\mathbb{R}$, prove the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n).$$

(b) Let w be smooth bump function i.e. Supp$(w) = [1, 2]$ and $w^{(j)} \ll_j 1$. Let χ be a primitive Dirichlet character modulo M. Using Poisson summation formula or otherwise show that

$$\sum_{n \in \mathbb{Z}} \chi(n) w\left(\frac{n}{N}\right) \ll M^{1/2+\varepsilon}.$$

(10+10 points)

Indian Statistical Institute

First Semestral Examination: 2017-2018

Programme: Master of Mathematics
Subject: Number Theory

Date: December 1, 2017 Duration: Three Hours Maximum Marks: 100

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1. Assuming that $L(1,\chi) \neq 0$ for any nonprincipal character $\chi \pmod q$, prove Dirichlet's Theorem on primes in arithmetic progressions. (17 marks)

2. Prove that $L(1,\chi) \neq 0$ for any nonprincipal character $\chi \pmod q$ taking non-real values. (17 marks)

3. Suppose $\{a_n\}_{n\geq 1}$ is a sequence of complex numbers. Show that the Dirichlet series $D(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ is absolutely convergent for $\mathrm{Re}(s) > 1$ iff

$$\sum_{n\leq X} |a_n| = O(X^{1+\varepsilon}) \text{ for any } \varepsilon > 0. \qquad \text{(17 marks)}$$

4. Prove that for any complex number s, the integral

$$f(s) := \int_1^\infty x^s (\sum_{n=1}^{\infty} e^{-\pi n^2 x}) \mathrm{d}x$$

converges and f so defined is an entire function in the variable s. (17 marks)

5. Starting from the definition, prove that $\zeta(s)$ admits meromorphic continuation to the half-plane $\mathrm{Re}(s) > 0$, determine its poles in this plane and compute the residues. (12 marks)

6. Prove that the Dirichlet series

$$\sum_{n\geq 2} \frac{\Lambda(n)}{n^s} \quad (\Lambda = \text{von Mangoldt function})$$

admits meromorphic continuation to the whole complex plane. Determine its poles along with residues the half-plane $\mathrm{Re}(s) \geq 1$. You may assume properties of $\zeta(s)$. (12 marks)

7. Suppose $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ are two characters and $(q_1, q_2) = 1$. Suppose further that $\chi = \chi_1\chi_2$ is a real character (i.e., takes real values only). Show that χ_1 and χ_2 both must be real characters. (17 marks)

Indian Statistical Institute

First Semestral Back-Paper Examination: 2017-2018

Programme: Master of Mathematics
Subject: Number Theory

Date: 27/12/2017 Duration: Three Hours Maximum Marks: 100

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1. Suppose G is a finite abelian group and $g \in G$ is some element other than the identity element of G. Show that there is a character χ of G such that

$$\chi(g) \neq 1.$$

(15 marks)

2. Define 'primitive character'. Prove that any Dirichlet character χ is induced from a primitive character that is uniquely determined by χ. How many primitive characters are there modulo 72? (4+6+5=15 marks)

3. Suppose $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ are two primitive characters and that $(q_1, q_2) = 1$. Show that $\chi = \chi_1\chi_2$ is a primitive character modulo q_1q_2. (25 marks)

4. Prove that $\zeta(1 + it) \neq 0$ for any real t. (20 marks)

5. Prove that the Dirichlet series

$$\sum_{n\geq 2} \frac{\Lambda(n)}{n^s} \quad (\Lambda = \text{von Mangoldt function})$$

admits meromorphic continuation to the whole complex plane. Find the poles and compute the residues in the half-plane $\mathrm{Re}(s) \geq 1$. You may assume facts about $\zeta(s)$. (12 marks)

6. Suppose $p = 2^{2^n} + 1$ is a prime, where $n > 1$. Show that 3 is a primitive root for p. (20 marks)

INDIAN STATISTICAL INSTITUTE

Fourier Analysis : M. Math, 2nd year
Back Paper Examination: 2017-18

DATE : 28/12/2017 , 2017.

Maximum Marks: 100 Maximum Time: 3 hrs.

(1) Give an example of an integrable function f on $\mathbb{R}$ such that $\hat{f} \notin L^1(\mathbb{R})$. [6]

(2) Prove that $L^2(\mathbb{R})$ is strictly contained in $L^{2,\infty}(\mathbb{R})$. [6]

(3) Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$. Prove that the function $F : \mathbb{R}^n \to L^p(\mathbb{R}^n)$ given by $F(y) = \tau_y f$ is uniformly continuous. [8]

(4) For a suiable function f on $\mathbb{R}^n$ let Mf denote the Hardy-Littlewood maximal function. Suppose that

$$\|Mf\|_q \leq C\|f\|_p,$$

for all $f \in L^p(\mathbb{R}^n)$ where C is independent of f. Prove that $q = p$. [6]

(5) Suppose f is a Lebesgue integrable function on $\mathbb{R}$.
 (a) Prove that $\hat{f}$ is uniformly continuous on $\mathbb{R}$. [4]
 (b) Prove that $\lim_{|x|\to\infty} |\hat{f}(x)| = 0$. [6]

(6) (a) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then prove that $f * g \in C_0(\mathbb{R}^n)$. Does the same conclusion hold if $p = 1$? Justify your answer. [6+2=8]

 (b) If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then prove that the function

$$g(x) = \int_{[x,x+1]} f(t)dt, \quad x \in \mathbb{R}$$

is in $C_0(\mathbb{R})$. Is this true if $p = \infty$? Justify your answer. [4]

[P.T.O]

(7) Suppose $f \in L^1(\mathbb{R})$ and $\phi \in C_c^\infty(\mathbb{R})$. If $f * \phi(x) = 0$ for almost every $x \in \mathbb{R}$ then prove that $f(x) = 0$ for almost every $x \in \mathbb{R}$. [8].

(8) Let ϕ be a non negative integrable function on $\mathbb{R}^n$ with $\|\phi\|_1 = 1$ and $g : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a bounded measurable function. For $\epsilon > 0$ and $x \in \mathbb{R}^n$, define,

$$\phi_\epsilon(x) = \epsilon^{-n}\phi\left(\frac{x}{\epsilon}\right), \quad F_\epsilon(x) = \phi_\epsilon(x)g(x,\epsilon), \quad c_\epsilon = \|F_\epsilon\|_1.$$

If $\lim_{\epsilon \to 0} c_\epsilon = 1$ then prove that for all $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$, the functions $f * F_\epsilon$ converges to f in $L^p(\mathbb{R}^n)$ as ϵ goes to zero. [10].

(9) If $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ then prove that $\hat{f}(x)\hat{g}(x) = \widehat{(f * g)}(x)$ for almost every $x \in \mathbb{R}^n$. Prove that the same conclusion remains valid if $g \in L^p(\mathbb{R}^n)$, $1 < p < 2$. [6+6=12]

(10) For a given positive number λ consider the set

$$A_\lambda = \{f \in L^1(\mathbb{R}) \mid \|f\|_\infty \le \lambda\}.$$

Prove that there exists a $C > 0$ such that for all $f \in A_\lambda$

$$m(\{x \in \mathbb{R} \mid |Hf(x)| > \lambda\}) \le \frac{C}{\lambda}\|f\|_1,$$

where m is the Lebesgue measure and H denotes the Hilbert transform. [8]

(11) Define $h(x) = e^{-\pi\|x\|^2}$ and $h_t(x) = t^{-\frac{n}{2}}h\left(\frac{x}{\sqrt{t}}\right)$, $x \in \mathbb{R}^n$. Consider the sublinear operator defined by

$$Tf(x) = \sup_{t>0}|f * h_t(x)|, \quad x \in \mathbb{R}^n.$$

Prove that there exists $C > 0$ such that for all $f \in L^1(\mathbb{R}^n)$,

$$\|Tf\|_{1,\infty} \le C\|f\|_1.$$

(Hint: Show that $|Tf(x)| \le A\ Mf(x)$, for all $x \in \mathbb{R}^n$, where M is the Hardy Littlewood maximal operator.) [14]

INDIAN STATISTICAL INSTITUTE
First Semester Backpaper Examination: 2017-18

M. MATH. II YEAR
Commutative Algebra I

Date 29.12.17 Maximum Marks : 100 Duration : 3 Hours

R denotes a commutative ring with 1.

1. Let R be an integral domain with field of fractions K. Prove that R is the intersection of the local rings $R_m(\subset K)$, as m varies over the set of maximal ideals of R. [14]

2. Prove that R is a reduced ring if and only if R_P is a reduced ring for every prime ideal P of R. (A ring is called reduced if 0 is the only nilpotent element of the ring.) [14]

3. Let $f = a_0 + a_1X + \cdots a_nX^n$ be an element of $R[X]$ ($a_i \in R \ \forall\ i$). Prove that f is a unit in $R[X]$ if and only if a_0 is a unit and a_i is nilpotent for each $i \geq 1$. [12]

4. (i) Let I be an ideal and c an element of R. Show that if $I + cR$ and $I : cR$ are both finitely generated ideals of R then I is a finitely generated ideal of R.

   (ii) Deduce that if every prime ideal of R is finitely generated then R is Noetherian. [7+7=14]

5. Let $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$ be a graded ring. Show that if R is Noetherian, then R is a finitely generated algebra over R_0. [14]

6. Let $R \subset A$ be commutative rings with A integral over R. Let P be a prime ideal in A and $Q = P \cap R$. Show that P is a maximal ideal of A if and only if Q is a maximal ideal of R. [10]

7. Let G be a finite subgroup of the group of ring automorphisms of a ring D and let $A = \{x \in D \mid \sigma(x) = x \ \forall\ \sigma \in G\}$. Show that D is integral over A. [10]

8. (i) Show that the maximal ideals of the ring $\mathbb{C}[X,Y]/(X^3 - Y^2)$ are in one-one correspondence with the set $\{(a,b) \in \mathbb{C}^2 \mid a^3 = b^2\}$. Clearly state all the results that you use.

   (ii) Let $\{f_\alpha \mid \alpha \in \Delta\}$ be a collection of polynomials in $\mathbb{Q}[X_1, \cdots, X_n]$. Show that if there exist complex numbers $z_1, \cdots, z_n$ such that $f_\alpha(z_1, \cdots, z_n) = 0 \ \forall\ \alpha \in \Delta$, then there exist algebraic numbers $a_1, \cdots, a_n$ such that $f_\alpha(a_1, \cdots, a_n) = 0 \ \forall\ \alpha \in \Delta$. [6+6=12]

Indian Statistical Institute

Mid-Semester Examination: 2017-18

Course: M.Math II year | Subject : Combinatorics and Graph Theory

Date: 19.02.2018 | Maximum Marks: 50 | Duration: 2hrs 30mins

Section - A

1. Prove that for every prime p, positive integer n, there exists a finite field with p^n elements. [10]

2. Construct a 3-error correcting BCH code with $n = 15$. What is the information rate and how would you encode information using this method. [10]

Section - B

3. (a) Let $d_1, d_2, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, d_2, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$. [5]

(b) Prove that if $E(G) > \frac{1}{4}(1 + \sqrt{4n-3})n$, then G contains a 4-cycle. [5]

4. (a) Draw the tree whose Prüfer code is (1,1,1,1,6,5). Briefly explain your steps. [4]

(b) The k-dimensional hypercube Q_k is a simple graph whose vertices are the k-tuples with entries in $\{0,1\}$ and whose edges are the pairs of k-tuples that differ in exactly one position. A j-dimensional subcube of Q_k is a subgraph of Q_k isomorphic to Q_j.

(1) Find the number of edges of Q_k.

(2) Find the number of j-dimensional subcubes of Q_k.

(3) Between any two vertices of Q_k which do not agree in any co-ordinate, construct k internally disjoint paths. [2+2+2 = 6]

5. (a) Let $z_1, \ldots, z_n$ be points in the plane, such that every three of them can be covered by a circle of radius 1. Then prove that all the points can be covered by a circle of radius 1. [3]

(b) Define a Horton set and prove that for every $n \geq 1$ an n-point Horton set exists. [3]

(c) Prove that no Horton set contains a 7-hole. [4]

INDIAN STATISTICAL INSTITUTE

Midsemester Examination

M.Math 2016-18

2nd year, 2nd semester

Project on Knot Theory

Date: 19 February, 2018 Maximum Marks: 20

Time: 2:30 pm Duration 1 hour

Write complete answers to all questions.

Explain clearly the notations and definitions that you use.

(1) (a) Describe (p, q) torus knots, where p, q are integers. Why are they called torus knots?

(b) Show that $(n, 1)$ torus knots are trivial for every integer n. 5+5

(2) (a) Draw a Trefoil knot and a figure eight knot.

(b) How do you show the difference between the above two knots using (at least two) knot invariants? 6+4

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2017-18 (Second Semester)

M. MATH. II YEAR
Commutative Algebra II

Date : 19.02.18 | Maximum Marks : 40 | Duration : 2 Hours

GROUP A
Attempt ANY TWO
Each question carries 15 marks.

1. (i) Let $R = \mathbb{C}[X,Y]/(X^2 - Y^3)$ and K denote the field of fractions of R. Show that there exist exactly one valuation ring of K containing $\mathbb{C}$ but not containing R.
(ii) Let L be a finite extension of $\mathbb{C}(X)$ and let S be the set of all discrete valuations v of L containing $\mathbb{C}$. Prove that for any $f \in L$, $\sum_{v \in S} v(f) = 0$.

2. Prove that the ring $A = \mathbb{R}[X,Y]/(X^2 + Y^2 - 1)$ is a Dedekind domain.
Let $R = \mathbb{C}[X,Y]/(XY)$ and $x = \overline{X}$. Compute $Ass_R\,(R/xR)$ and $Ass_R\,(R/x^2R)$.

3. (i) State Going Up and Going own Theorems for integral extensions.
(ii) Prove that if $f : R \to S$ is a ring homomorphism and Q is a primary ideal of S, then $f^{-1}(Q)$ is a primary ideal of R. Deduce that $Q = S^{-1}Q \cap R$ for any primary ideal Q of R and any multiplicatively closed subset S of R such that $S \cap Q = \phi$.

GROUP B
Attempt ANY TWO
Give examples of the following.
Each question carries 6 marks.

1. A non-Noetherian valuation ring

2. A prime ideal P of a Noetherian ring R such that P^2 is not a P-primary ideal.

3. An Artinian but not a Noetherian module over a Noetherian ring.

Indian Statistical Institute
Mid-Semestral Examination: 2017-2018
Programme: Master of Mathematics
Course: **Elliptic Curves**

Maximum Marks: 60
Duration: 2 Hours and 30 minutes Date: February 20, 2018

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1. Write the homogenous polynomial defining the projective line passing through the two points $(1 : 2 : 3)$ and $(2 : 3 : 4)$. (5 marks)

2. (a) Determine if the affine curve $C : x^2y + 4y - 8$ in $\mathbb{A}^2(\mathbb{R})$ is smooth. (b) Is the projectivization of this curve an elliptic curve over $\mathbb{R}$? Justify your answer. (4+8=12 marks)

3. Show that a smooth conic can not contain a flex. (12 marks)

4. (a) Show that an irreducible cubic in $\mathbb{P}^2(\mathbb{C})$ can have at most one singular point. (b) What can you say about the multiplicity of this singular point? Justify your answer. (6+6=12 marks)

5. Suppose C_1 and C_2 are two smooth conics in $\mathbb{P}^2(\mathbb{C})$; P_1, Q_1, R_1 are three distinct points on C_1 and P_2, Q_2, R_2 are three distinct points on C_2. Show that there is a projective transformation T that takes C_1 to C_2, P_1 to P_2, Q_1 to Q_2 and R_1 to R_2. (14 marks)

6. Show that the series

$$\sum_{(m,n)\neq(0,0)} \frac{1}{(z-m-in)^2} - \frac{1}{(m+in)^2}$$

is absolutely convergent for every $z \notin \mathbb{Z} + i\mathbb{Z}$. (10 marks)

INDIAN STATISTICAL INSTITUTE
Mid Semestral Examination 2017-18
M. Math II Year
Automata, Languages and Computations.

Date:21-02-18 Maximum Marks:80 Duration:2hours 30 mins.

Note: Answer as many as you can. Maximum score is 80
Unless otherwise stated, notation used is as defined in the class.

1. Define a non-deterministic finite automaton(NFA).
When is a string w said to be accepted by an NFA? Construct a finite automaton that accepts all string over $\{0,1\}$ containing an even number of occurrences of 01. Explain(without proof) your construction. [5+7]

2. Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be 3 DFAs accepting $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ respectively over the same alphabet. Construct a single DFA that accepts $(\mathcal{L}_1 - \mathcal{L}_2) \bigcap \mathcal{L}_3$. Explain, in brief, your construction [8]

3. State Kleene's Theorem.
Define regular expressions over an alphabet Σ.
Let $\mathbf{r}$ and $\mathbf{s}$ be two regular expressions. Prove the following identity

$$(\mathbf{r} + \mathbf{s})^* = (\mathbf{r}^*.\mathbf{s}^*)^*.$$

[2+3+7]

4. Write an algorithm to test, given a DFA $\mathcal{M}$, whether $\mathcal{L}(\mathcal{M})$is empty or not. Prove its correctness. [8]

5. Is the Language $\mathcal{L} = \{a^p : \text{ p is prime}\}$ regular? Justify. [7]

6. Given a language $\mathcal{L} \subseteq \Sigma$ define an equivalence relation on Σ^* as follows

$$x \equiv_{\mathcal{L}} y \text{ if } \forall w, xw \in \mathcal{L} \Longleftrightarrow yw \in \mathcal{L}.$$

Show that if $\mathcal{L}$ is regular, then $\equiv_{\mathcal{L}}$ has finite index i.e. has a finite number of equivalence classes. [10]

7. Find a context free grammar(CFG) that generates the following language

$$\mathcal{L} = \{a^n b^m c^k : n, m.k > 0\}$$

Find Chomsky Normal Form(CNF) grammar that generates the same language. [6+6]

8. Given a regular language $\mathcal{L}$ write down a regular grammar that generates $\mathcal{L}$ (No proof required) [6]

9. State Bar-Hillel's Pumping Lemma.
Use it to show that the language over $\{a, b, c\}$ with equal occurrences of a, b, c is not context-free. [3+7]

10. Write down an algorithm that will test a given a CFG G to determine whether $\mathcal{L}(G)$ is finite or infinite. Prove the correctness of the algorithm. [4+7]

INDIAN STATISTICAL INSTITUTE
Mid-Semesteral Examination : 2017-18
M. Math. - Second Year
Mathematical Logic

Assignment—1

You need not submit the solutions of Problems 3 to 6. But you should work out these because they will be used later in the class. Submit the solutions of the remaining questions by 5 March 2018.

(1) A subset X of an L-structure M is called a *generator* of M if M is the only substructure containing X. Let M, N be L-structures, X a generator M and $f, g : M \to N$ homomorphisms such that $f|X = g|X$. Show that $f = g$.

(2) (a) Let M and N be elementarily equivalent L-structure. For every positive integer n, show that $|M| = n \Rightarrow |N| = n$.
(b) Let T be a complete theory with an infinite model. Show that all models of T are infinite.

(3) Let L be a first order language. L-formulas $\varphi[\overline{x}]$, $\psi[\overline{x}]$ are called *tautologically equivalent*, written $\varphi \equiv \psi$, if forall L-structures M

$$M \models \forall\overline{x}(\varphi \leftrightarrow \psi).$$

Below φ, ψ, η denote L-formulas. Show the following:
(a) $\neg\neg\varphi \equiv \varphi$.
(b) $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$ and $\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$.
(c) $\varphi\wedge(\psi\vee\eta) \equiv (\varphi\wedge\psi)\vee(\varphi\wedge\eta)$ and $\varphi\vee(\psi\wedge\eta) \equiv (\varphi\vee\psi)\wedge(\varphi\vee\eta)$.

(4) An L-formula is called a *literal* if it is either an atomic formula or the negation of an atomic formula. An L-formula is said to be in *conjunctive normal form* (CNF in short) if it is in the form $\bigwedge_{i=0}^{k-1}\bigvee_{j=0}^{m_i-1}\varphi_{ij}$, where φ_{ij}s are literals. An L-formula is said to be in *disjunctive normal form* (DNF in short) if it is in the form $\bigvee_{i=0}^{k-1}\bigwedge_{j=0}^{m_i-1}\varphi_{ij}$, where φ_{ij}s are literals. Show that every open formula is tautologically equivalent to a formula in CNF as well as to a formula in DNF.

(5) Below φ and ψ denote L-formulas. Show the following.
(a) $\neg\exists x\varphi[x,\overline{x}] \equiv \forall x\neg\varphi[x,\overline{x}]$ and $\neg\forall x\varphi[x,\overline{x}] \equiv \exists x\neg\varphi[x,\overline{x}]$.
(b) $\exists x(\varphi[x,\overline{x}]\vee\psi[x,\overline{x}]) \equiv \exists x\varphi[x,\overline{x}]\vee\exists x\psi[x,\overline{x}]$ and $\forall x(\varphi[x,\overline{x}]\wedge\psi[x,\overline{x}]) \equiv \forall x\varphi[x,\overline{x}] \wedge \forall x\psi[x,\overline{x}]$.
(c) If a variable v does not occur in $\varphi[x,\overline{x}]$, then $\exists x\varphi \equiv \exists v\varphi_x[v]$ and $\forall x\varphi \equiv \forall v\varphi_x[v]$.

(d) If the variable x is not free in φ, then

$$\varphi \vee \exists x\psi[x, \overline{x}] \equiv \exists x(\varphi \vee \psi[x, \overline{x}]),$$
$$\varphi \vee \forall x\psi[x, \overline{x}] \equiv \forall x(\varphi \vee \psi[x, \overline{x}]),$$
$$\varphi \wedge \exists x\psi[x, \overline{x}] \equiv \exists x(\varphi \wedge \psi[x, \overline{x}])$$

and

$$\varphi \wedge \forall x\psi[x, \overline{x}] \equiv \forall x(\varphi \wedge \forall\psi[x, \overline{x}]).$$

(6) An L-formula is said to be in *prenex normal form* (PNF in short) if it is in the form $Q_0x_0 \cdots Q_{n-1}\varphi$, where $Q_i = \exists$ or $\forall$ and φ open. Show that every L-formula is tautologically equivalent to a formula in PNF.

(7) An *automorphism* of an L-structure M is an isomorphism of M onto itself. We use $Aut(M)$ to denote the set of all automorphisms of M. Satisfy yourself that $Aut(M)$ is a group with respect to the composition and that $Aut(M)$ acts canonically on M^n. Show that the number of orbits in $\mathbb{Q}^n$ under the action of $Aut(\mathbb{Q})$ is finite, where $\mathbb{Q}$ is regarded as a model of DLO.

(8) For $A \subset M$, we set $Aut_A(M)$ the subgroup of $Aut(M)$ consisting of all automorphisms of M that fixes A pointwise. (If $\mathbb{K}$ is a subfield of a field $\mathbb{F}$, then $Aut_{\mathbb{K}}(\mathbb{F})$ is denoted by $G(\mathbb{F}, \mathbb{K})$ and is called the Galois group of $\mathbb{K}$). Note that $Aut(M) \subset M^M$. We equip M^M with the product of discrete topologies on M. For $\overline{a}, \overline{b} \in M^n$, set

$$\Sigma(\overline{a}, \overline{b}) = \{f \in Aut(M) : f(\overline{a}) = \overline{b}\}.$$

Sets of the form $\Sigma(\overline{a}, \overline{b})$, $\overline{a}, \overline{b} \in M^n$, $n = 0, 1, 2, \cdots$, form a base for the topology of $Aut(M)$. Show the following:

(a) $Aut(M)$ is a topological group.

(b) A subgroup G of $Aut(M)$ is open if and only if it contains $Aut_{\overline{a}}(M)$ for some $\overline{a} \in M^n$, $n = 0, 1, 2, \cdots$.

(c) A subgroup G of $Aut(M)$ is dense if and only if the orbit of any $\overline{a} \in M^n$, $n = 0, 1, 2, \cdots$, equals $\{g \cdot \overline{a} : g \in G\}$.

INDIAN STATISTICAL INSTITUTE

Mid-Semesteral Examination : 2017-18

M. Math. - Second Year

Mathematical Logic

Date : 23. 02. 2018 | Maximum Score : 100 | Time :3 Hours

The paper carries 115 marks. You are free to answer as many questions as you wish. Maximum score—100.

(1) Let X be an infinite set. Show the following.
  (a) For every positive integer n, $|X \times \{0, 1, \cdots n-1\}| = |X|$.
  (b) $|X \times X| = |X|$. [10 + 10]

(2) Assuming the continuum hypothesis show that there is a transfinite sequence $\{f_\alpha : \alpha < \aleph_1\}$ of distinct entire functions such that for every complex number z, the set $\{f_\alpha(z) : \alpha < \aleph_1\}$ is countable. [15]

(3) Let κ be an uncountable cardinal. Equip $\kappa \times \mathbb{Q}$ with lexicographic order. Also, take any $B \models DLO$ of cardinality κ and equip $\aleph_0 \times B$ with lexicographic order. Using these show that DLO is not κ-categorical (for any $\kappa > \aleph_0$). [10]

(4) Show that the theory of random graphs is $\aleph_0$-categorical. [20]

(5) Let L be a first order language and M a L-structure.
  (a) Let $N \sqsubset M$. Show that $N \prec M$ if and only if for every formula $\varphi[x, \overline{y}]$ and for every $\overline{a} \in N$,

$$M \models \exists x \varphi[x, \overline{a}] \Rightarrow N \models \exists x \varphi[x, \overline{a}].$$

  (b) For every $X \subset M$ show that there exists a $N \prec M$ such that $X \subset N$ and $|N| \leq \max\{|X|, |L|\}$. [15 + 15]

(6) A ordered field is a field $\mathbb{F}$ with a linear $<$ on $\mathbb{F}$ satisfying the following axioms:

$$\forall x \forall y \forall z (x < y \rightarrow x + z < y + z)$$

and

$$\forall x \forall y ((0 < x \wedge 0 < y) \rightarrow 0 < x \cdot y).$$

  (a) Show that an ordered field is not algebraically closed.
  (b) Show that the class of all ordered field satisfying the least upper bound axiom is not elementary. [10 + 10]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2017-18

M. Math. II Year

Representation Theory of Locally Compact Groups

Date: 24/02/2017 Maximum Marks: 40 Duration: 3 Hours

Note: **Give proper justification to all your answers.**

State clearly all the results you are using.

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(1) Show that a subgroup H of a topological group G is discreet if and only if it has an isolated point. **[6]**

(2) Let G be a locally compact group. Suppose G has a compact neighbourhood V of 1 such that $gVg^{-1} = V$ for every $g \in G$. Show that G is unimodular. **[6]**

[Hint. *Note that $\chi_V(gxg^{-1}) = \chi_V(x)$ for all $g, x \in G$, where χ_V is the characteristic function of G.*]

(3) Let μ be the Haar measure on the p-adic group $\mathbb{Q}_p$, $S = \{x \in \mathbb{Q}_p : |x|_p \geq 1\}$, and $\alpha \in \mathbb{R}$. Show that

$$\int_S |x|_p^\alpha \, d\mu(x)$$

exists if and only if $\alpha < -1$. **[6]**

(4) Let

$$G = \left\{ \begin{pmatrix} x_1 & 0 \\ x_2 & x_3 \end{pmatrix} : x_1 > 0, x_3 > 0, x_2 \in \mathbb{R} \right\}.$$

Show that G is a locally compact group under matrix multiplication and the subspace topology inherited from $\mathbb{R}^3$. Describe the left and right Haar integral on G. Also find the modular function of G. **[12]**

(5) Let G be a locally compact group with left Haar measure λ. Recall that the regular representations $\pi_L : G \to U(L^2(G, \lambda))$ and $\widetilde{\pi}_R : G \to U(L^2(G, \lambda))$ of G are defined by

$$[\pi_L(x)f](y) = f(x^{-1}y) \quad \text{and} \quad [\widetilde{\pi}_R(x)f](y) = \Delta(x)^{1/2} f(yx).$$

Suppose that G has more that 1 element. Show that π_L is reducible. **[12]**

[Hint. *Let $x \in G$ and $x \neq 1$. Show that $\widetilde{\pi}_R(x) \in C(\pi_L)$. To show $\widetilde{\pi}_R(x) \neq cI$, take a relatively compact neighbourhood V of 1 such that $V \cap Vx^{-1} = \emptyset$. Use the function χ_V.*]

Indian Statistical Institute

End-Semester Examination 2017-18

Course: M.Math II year | Subject: Combinatorics and Graph Theory
Date: 23/04/2018 | Maximum Marks: 100 | Duration: 3 hrs

Section: A

1.(a) Define 3-design and 2-design. [4]

.(b) Sow that a 3-design is necessarily a 2-design but not the vice-versa. [6]

.(c) Find any 2-design with 13 treatments. [10]

2.(a) Construct a double error correcting BCH code with $n=15$. [10]

.(b) Describe Berlekamp-Massey algorithm for decoding BCH codes. [10]

Section: B

3.(a) Find the spectrum of the Petersen graph and using this prove that the complete graph K_{10} cannot be decomposed into 3 Petersen graphs. [4+6 = 10]

.(b) Let f be a monotone decreasing Boolean function that is not identically 0. Associate with f the simplicial complex $K_f = \{supp(x): f(x) = 1\}$. Prove that if f is non-evasive, then K_f is contractible. Also show that if f is a non-constant monotone Boolean function with a transitive automorphism group Γ, then Γ acts on $G(K_f)$ and has no fixed point. [5+5 = 10]

4.(a) Use Euler's idea of looking at the zeros at suitable values to prove that $\zeta(4) = \frac{\pi^4}{90}$ [6]

.(b) Calculate the number of primitive Boolean strings of length n. [4]

.(c) A non-crossing partition of an ordered set S is a partition in which no two blocks "cross" each other, i.e., if a and b belong to one block and x and y to another, they are not arranged in the order $a\ x\ b\ y$. Find a recurrence relation to calculate the number of non-crossing partitions of an ordered set S of n elements. [10]

5.(a)Prove that connectedness is evasive. [6]

.(b) Consider a Sperner colouring of a triangle, i.e., consider a triangulation, the vertices of the external triangle are labelled 1,2,3, the internal vertices are labelled arbitrarily from 1,2 and 3 and the vertices on the edge 1-2 are labelled with either 1 or 2 and so on. Prove that there must be an internal triangle labelled 1-2-3. Also suppose that the external triangle in a Sperner labelling is labelled 1-2-3 in clockwise order. Let A be the number of internal 1-2-3 triangles oriented in the clockwise direction, and let B be the number of such triangles oriented in the counter clockwise direction. Then prove that A − B = 1. [3+5 = 8]

.(c) A lattice is a poset in which every pair of elements has both, a least upper bound and a greatest lower bound. Let L be a lattice and let $\{A, B, C\}$ be a partition of L such that (a) if $x \in A$ and $y \leq x$, then $y \in A$ and (b) if $x \in C$ and $x \leq y$, then $y \in C$. Prove that $1+\sum_{x\in A}\sum_{y\in C}\mu(x,y) = \sum_{x,y\in B}\mu(x,y)$, μ is the Mobius function. [6]

INDIAN STATISTICAL INSTITUTE
Second Semester Examination: 2017-18

M. MATH. II YEAR
Commutative Algebra II

Date: 23.4.2018 | Maximum Marks: 60 | Duration: 3 Hours

Attempt ANY FOUR questions.
Each question carries 16 marks.

1. (i) Let $A = \mathbb{C}[X, Y, Z, W]/(XY - ZW) = \mathbb{C}[x, y, z, w]$ (where x, y, z, w denote the images in A of X, Y, Z, W respectively). Find three elements a, b, c in A such that $B = \mathbb{C}[a, b, c]$ is isomorphic to the polynomial ring in three variables over $\mathbb{C}$ and A is integral over B. Write down explicit integral equations satisfied by x, y, z, w over B.

   (ii) Let R be a normal domain, K the field of fractions of R, L a finite Galois extension of K with Galois group G and A the integral closure of R in L. Let P and Q be prime ideals of A such that $P \cap R = Q \cap R$. Prove that there exists $\sigma \in G$ such that $\sigma(P) = Q$.

   [8 + 8]

2. (i) Let $P \subsetneq Q$ be prime ideals in a Noetherian ring R. Show that if there exists one prime ideal P_1 in R with $P \subsetneq P_1 \subsetneq Q$, then there exist infinitely many prime ideals P_i in R such that $P \subsetneq P_i \subsetneq Q$.

   (ii) Determine all maximal ideals m of the ring $R = \mathbb{C}[X, Y, Z]/(X^2Y + Z^2 + Z^3)$ for which R_m is not regular.

   [8 + 8]

3. Let R be a Noetherian domain.

   (i) Prove that if x is a nonzero non-unit in R, then the height of xR is one.

   (ii) Prove that R is a unique factorisation domain if every prime ideal minimal over a principal ideal is itself principal.

   [10 + 6]

4. Let R be a ring and I be an ideal of R.

   (i) Define the I-adic completion of R.

   (ii) Prove that if R is Noetherian then the I-adic completion of R is Noetherian.

   (iii) If M is a finitely generated R-module, $\hat{M}$ the I-adic completion of M and $\hat{R}$ the I-adic completion of R, then prove that $\hat{M} = M \otimes_R \hat{R}$.

   [3 + 7 + 6]

5. Answer ANY FOUR of the following questions.

(i) Let V be an affine algebraic set in C^n. Suppose that $(\mathbb{C}[V])^* = \mathbb{C}^*$. Show that any $f \in \mathbb{C}[V] \setminus \mathbb{C}$ induces a *surjective* polynomial function $f : V \to \mathbb{C}$.

(ii) Give an explicit example of a maximal ideal of height one in $\mathbb{C}[[X]][Y]$.

(iii) Give an example of a sequence of parameters in a regular local ring which is not a regular sequence of parameters.

(iv) Let R be a Noetherian ring, and $I \subset J$ ideals of R. Prove that if R is I-adically complete, it is also J-adically complete.

(v) Determine all prime numbers p such that the polynomial $X^2 + 1$ has a root in the ring $\mathbb{Z}_p[X]$, where $\mathbb{Z}_p$ is the ring of p-adic integers.

[$4 \times 4 = 16$]

6. State whether the following statements are TRUE or FALSE with brief justification. Answer ANY FOUR.

(i) All the maximal ideals of a finitely generated k-algebra have the same height.

(ii) If $A \subset B$ is an integral extension and I is an ideal of B, then ht$I \cap A >$ htI.

(iii) If A is the normalisation of a one-dimensional Noetherian domain and m a maximal ideal of A, then A_m is a discrete valuation ring.

(iv) A Noetherian domain R is separated in the I-adic topology with respect to any ideal I of R.

(v) If x_n is a sequence in a filtered module M which is Hausdorff and complete with respect to the topology defined by the filtration then the series $\sum_n x_n$ converges in M.

[$4 \times 4 = 16$]

INDIAN STATISTICAL INSTITUTE

Second Semester Examination : 2017-18

Course Name : M. Math II year

Subject Name : Special Topics : Riemann Surfaces

Date : 27.04.18 Maximum Marks : 60 Duration : 3 hours

Note, if any :

1. Let X be a Riemann surface such that the universal covering of X is biholomorphic to $\mathbb{C}$. Show that the fundamental group of X is abelian. (6 marks)

2. Let L be a lattice in $\mathbb{C}$, and let

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

denote the Weierstrass $\mathcal{P}$-function of the lattice L. The functions $\mathcal{P}, \mathcal{P}'$ are meromorphic L-periodic functions on $\mathbb{C}$ and may be considered as meromorphic functions on the Riemann surface $\mathbb{C}/L$. Show that any meromorphic function on $\mathbb{C}/L$ is a rational function of $\mathcal{P}$ and $\mathcal{P}'$, i.e. $\mathcal{M}(\mathbb{C}/L) = \mathbb{C}(\mathcal{P}, \mathcal{P}')$, where $\mathcal{M}(\mathbb{C}/L)$ denotes the field of meromorphic functions on $\mathbb{C}/L$. (10 marks)

3. Let $\lambda \in \mathbb{C} - \{0, 1\}$. Show that the projective algebraic curve $C = \{[X : Y : Z] \in \mathbb{CP}^2 | Y^2 Z = X(X - Z)(X - \lambda Z)\}$ is nonsingular. (6 marks)

4. Let X be a Riemann surface and let ω be a C^∞ 1-form on X. Show that ω is closed, i.e. $d\omega = 0$, if and only if $\int_X df \wedge \omega = 0$ for all $f \in C_c^\infty(X)$. (6 marks)

5. Let X be a Riemann surface and let $z = x + iy : U \subset X \to \mathbb{C}$ be a local coordinate. Let $f : U \to \mathbb{C}$ be a C^∞ function on U. Let $M_i : \mathbb{C} \to \mathbb{C}, w \mapsto iw$ denote the multiplication by i map, and for $p \in U$ define a linear map $J_p : T_pX \to T_pX$ by $J_p(v) = (dz_p)^{-1} M_i dz_p(v), v \in T_pX$.

(i) Show that f is holomorphic on U if and only if for all $p \in U$ and $v \in T_pX$, we have $df_p(J_pv) = idf_p(v)$.

(ii) Show that for all $p \in U$,

$$df_p = \frac{\partial f}{\partial z}(p)dz_p + \frac{\partial f}{\partial \bar{z}}(p)d\bar{z}_p$$

where the differential operators $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ are defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

(iii) Show that the 1-form $\omega = fdz$ on U is closed if and only if f is holomorphic.

(6+6+6 = 18 marks)

6. Let X be a Riemann surface, and let H denote the space of harmonic 1-forms on X, $\mathcal{H}$ the space of holomorphic 1-forms on X, and $\overline{\mathcal{H}}$ the space of anti-holomorphic 1-forms on X. Show that

$$H = \mathcal{H} \oplus \overline{\mathcal{H}}$$

(6 marks)

7. Let X be a compact Riemann surface of genus g, and let $a_1, \dots, a_g, b_1, \dots, b_g$ be simple closed curves on X forming a canonical homology basis of X. Let ω be a harmonic 1-form on X.

(i) Show that if U is an open subset of X and $f : U \to \mathbb{C}$ is a C^∞ function such that $\omega = df$ on U, then f is a harmonic function on U.

(ii) Show that if

$$\int_{a_j} \omega = \int_{b_j} \omega = 0$$

for all $j = 1, \dots, g$, then $\omega = 0$. (6+6 = 12 marks)

8. Let ω be a non-zero meromorphic 1-form on $\hat{\mathbb{C}}$, and for $p \in \hat{\mathbb{C}}$ let $m(\omega, p) \in \mathbb{Z}$ denote the order of ω at p. Show that

$$\sum_{p \in \hat{\mathbb{C}}} m(\omega, p) = -2$$

(6 marks)

INDIAN STATISTICAL INSTITUTE
Semesteral Examination : 2017-18
M. Math. - Second Year
Mathematical Logic

Date : 30. 04. 2018   Maximum Score : 100   Time :3 Hours

You are free to use any result from set theory and algebra. However, you must state any such result that you use at least once.

The paper carries 120 marks. You are free to answer as many questions as you wish. Maximum score—100.

(1) (a) State and prove the compactness theorem for first order theories.

(b) Let M_1 and M_2 be elementarily equivalent L-structures. Show that there is an elementary extension M of M_1 in which M_2 is elementarily embedded.

(c) Show that the class C of all fields of characteristic $\neq 0$ is not elementary.

[(3 + 12) + 15 + 15]

(2) (a) Let κ be an infinite cardinal. Show that every consistent, κ-categorical theory all whose models are infinite is complete.

(b) Show that $ACF(p)$, $p = 0$ or prime, is a complete theory.

(c) Let φ be a first order sentence in the language of rings. Show that $ACF(0) \models \varphi$ if and only if there is an infinite set P of prime numbers such that for every $p \in P$ there is a model $\mathbb{K} \models ACF(p)$ with $\mathbb{K} \models \varphi$.

[10 + 10 + 15]

(3) (a) Show that ACF and RCF have quantifier elimination.

(b) For a field $\mathbb{K}$, $X \subset \mathbb{K}^n$ and $J \subset \mathbb{K}[X_1, \cdots, X_n]$, set

$$\mathcal{V}(J) = \{\overline{a} \in \mathbb{K}^n : \forall f \in J(f(\overline{a}) = 0)\}$$

and

$$\mathcal{I}(X) = \{f \in \mathbb{K}[X_1, \cdots, X_n] : \forall \overline{a} \in X(f(\overline{a}) = 0)\}.$$

Using model theory show that if $\mathbb{K}$ is algebraically closed and I an ideal in $\mathbb{K}[X_1, \cdots, X_n]$, then for every $f \in \mathcal{I}(\mathcal{V}(I))$ there is a $n \geq 1$ such that $f^n \in I$.

(c) Let $f \in \mathbb{R}(X_1, \cdots, X_n)$ be not a sum of squares in $\mathbb{R}(X_1, \cdots, X_n)$. Using model theory how that there is a $\overline{a} \in \mathbb{R}^n$ such that $f(\overline{a}) < 0$.

[(10 + 10) + 10 + 10]

Indian Statistical Institute

Semestral Examination: 2017-2018

Programme: Master of Mathematics

Course: **Elliptic Curves**

Maximum Marks: 100

Duration: 3 Hours *Date*: May 2, *2018*

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1. Suppose $P = (x, y)$ is a point on an elliptic curve E with integer coefficients such that both P and $2P$ have integer coordinates. Show that either $y = 0$ or $y | D_E$, where D_E is the discriminant of the elliptic curve E.
You may assume general facts about the discriminant of a polynomial.

(12 marks)

2. Find all rational points of order 4 on the elliptic curve

$$y^2 = x^3 + 4x.$$

(13 marks)

3. (a) If $E : y^2 = x^3 + ax^2 + bx + c$, with $a, b, c \in \mathbb{Z}$, is an elliptic curve and if $P = (x, y)$ is a rational point on E then show that P must be of the form $(m/r^2, n/r^3)$, where m, n, r are integers, $r \geq 1$, $(m, r) = (n, r) = 1$.
(b) Using the above, prove that the logarithmic height h on E has the property that given any point $P_0 \in E(\mathbb{Q})$, there is a constant κ_0 that depends only on P_0 and the curve, so that

$$h(P + P_0) \leq 2h(P) + \kappa_0,$$

for any point $P \in E(\mathbb{Q})$. (10+20=35 marks)

4. Determine if the point $(-2, 3)$ is a torsion point on the elliptic curve $y^2 = x^3 + 17$. (15 marks)

5. Suppose $\Lambda \subset \mathbb{C}$ is a lattice of rank 2 and let f be an elliptic function relative to Λ.
(a) Show that the sum of the orders of vanishing of f at points inside a fundamental parallelogram for the lattice Λ is zero.
(b) Define the Weierstrass $\wp$-function. Determine all the poles with corresponding residues of $\wp(z)$.
(c) Show that $\wp(z)$ is an even function.
(d) Show that every elliptic function is a rational function of $\wp(z)$ and $\wp'(z)$.

(3+7+5+15=30 marks)

INDIAN STATISTICAL INSTITUTE

Second Semester Examination: 2017-18

M. MATH. II YEAR

Representation Theory of Locally Compact Groups

Date: 05/05/2018   Maximum Marks: 60   Duration: $2\frac{1}{2}$ Hours

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Give proper justification to all your answers. State clearly all the results you are using.

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(1) Is every cyclic representation of a locally compact group irreducible? [7]

(2) For a Hilbert space H, let H^* denote its dual, that is, the space of all bounded linear functionals on H. For an operator $A : H \to H$, define $A^t : H^* \to H^*$ by

$$(A^t f)(u) = f(Au), \quad f \in H^*, u \in H.$$

Let π be a unitary representation of a locally compact group G on a Hilbert space H and define $\pi^c(x) = \pi(x^{-1})^t$, $x \in G$. Show that π^c is a unitary representation of G on H^*. [8]

(3) Let G be a compact group. For any unitary representation π of G on a Hilbert space H_π, let $\varphi_{u,v}(x) = \langle \pi(x)u, v\rangle$ and $\mathcal{E}_\pi = \text{span}\{\varphi_{u,v} : u, v \in H_\pi\}$. Define $\mathcal{E} =$ linear span of $\bigcup_{[\pi]\in\hat{G}} \mathcal{E}_\pi$.
Show that if $f, g \in \mathcal{E}$, then $fg \in \mathcal{E}$. [15]

(4) For a finite-dimensional representation π of a compact group G, let χ_π denote the character of π, that is, $\chi_\pi(x) = \text{tr } \pi(x), x \in G$.

Let $n \geq 0$ be an integer and π_n be the unitary representation of $G = SU(2)$ on the space H_n of homogeneous polynomials of degree n in two complex variables defined by

$$[\pi_n(g)f](z) = f(zg), \quad g \in SU(2), z \in \mathbb{C}^2, f \in H_n.$$

(a) Compute the $L^2(G)$-norm of χ_{π_n}.

(b) Let $m \geq n \geq 0$. Show that $\chi_{\pi_m \otimes \pi_n} = \sum_{k=0}^{n} \chi_{\pi_{m+n-2k}}$. [5 + 10]

(5) Let $T = \left\{t(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z = x + iy, (x, y) \in \mathbb{R}^2\right\}$ be the translation subgroup of the Euclidean motion group $M(2)$. For $a \in \mathbb{R}^2$, consider the unitary irreducible representation σ^a of T on $\mathbb{C}$ defined by $\sigma^a(t(z)) = e^{i\langle z,a\rangle}$.

Let $\tau^a = \text{ind}_T^{M(2)}\sigma^a$, that is, τ^a is the representation of $M(2)$ induced from the representation σ^a of T. Show that τ^a is unitarily equivalent to the principal series representation π^a of $M(2)$.

Recall that π^a is a unitary representation on the Hilbert space $L^2(K)$, where $K = \left\{r(\alpha) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} : \alpha \in \mathbb{R}\right\}$, and π^a is defined by

$$[\pi^a(g)F](s) = e^{i\langle z,sa\rangle}F\left(r(\alpha)^{-1}s\right),$$

where $g = h(z, \alpha) = t(z)r(\alpha)$, $F \in L^2(K)$ and $s \in K$. [15]

Examination: Semester II(2017-18)
M. Math II Year
Automata, Languages and Computations.

Date: 07.05.18 Maximum Marks:100 Duration:3 hours

Note: Answer as many as you can. Maximum score is 100
Unless otherwise stated, notation used is as defined in the class.

1. A Context Free Grammar(CFG) is said to be ambiguous if there is a string w with more than one leftmost(rightmost) derivations.
Consider the following grammar **G**:

$$S \to aB/bA$$

$$A \to a/aS/bAA$$

$$B \to b/bS/aBB$$

Is the grammar ambiguous? Justify. [6]

2. When is a PDA said to be deterministic? Show that if $\mathcal{L}$ is a language accepted by a deterministic PDA by empty stack, then there exists a deterministic PDA $\mathcal{M}'$ that accepts $\mathcal{L}$ by final states and $\mathcal{L}$ has the prefix property, *i.e.*, no proper prefix of a string in $\mathcal{L}$ is in $\mathcal{L}$. [8]

3. Show that if $\mathcal{L}$ is a context-free language and $\mathcal{R}$ is regular, then $\mathcal{L} \bigcap \mathcal{R}$ is context-free by constructing a suitable PDA.(Show only the construction.) [5]

4. (a) Design (i)single-tape (ii) multitape Turing machines(TMs) that accept the set of all strings over $\{a, b\}$ containing an equal numbers of a's and b's. Compare the number of moves made by the respective TMs on accepting a string of length n.
   (b) Also, construct a TM that uses only $\lceil \log n \rceil$ cells (not counting the cells on the input tape) that accepts all binary strings in which the number of 0's is greater than the number of 1's. [(8+7)+8]

5. (a) When is a function $f : I\!N^k \to I\!N$ said to be recursive? What is its relation with Turing-computable functions?
   Show that the following functions and predicates are recursive.

i. $f(n,m) = [n/m] = \begin{cases} 0 & \text{if } m = 0 \\ \text{the largest integer} \leq n/m & \text{if } m > 0 \end{cases}$

ii. $Prime(n) \longleftrightarrow n$ is prime.

iii. $Divide(n,m) \longleftrightarrow m$ divides n.

iv. $p(n) =$ the nth prime number.

v. $lcm(n,m) = \begin{cases} 0 & \text{if } n = 0 \text{ or m} = 0 \\ \text{the least common multiple of } n, m & \text{otherwise} \end{cases}$

vi. $gcd(n,m) = \begin{cases} 0 & \text{if } n = 0 \text{ or m} = 0 \\ \text{the greatest common divisor of } n, m & \text{otherwise} \end{cases}$

vii. $f(n) = \lceil \log n \rceil$

(b) Consider the following function.

$$B(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ } or \text{ } y = 0 \\ x \dot{-} (\mu z \leq x(\exists w \leq x(w.(x \dot{-} z) = x) & \\ \quad and \ \exists u \leq y(u.(x \dot{-} z) = y))) & \text{otherwise} \end{cases}$$

Here "'$\mu z \leq xP(x)$"' means "'the smallest $z \leq x$ such that $P(x)$ holds"'. Describe in simple English the function B. [(5+28)+4]

6. Describe an encoding of a Turing machine $\mathcal{M}$ whose tape alphabet is $\{0, 1, B\}$ and hence, give a recursive enumeration of all such TMs. Hence, or otherwise, give an example of a recursively enumerable language that is not recursive. Prove that it is not recursive. [7+10]

7. (a) Define the classes $\mathcal{P}$ and $\mathcal{NP}$.

   (b) When is a language said to be $\mathcal{NP}$-complete? Show that if an $\mathcal{NP}$-complete language is in $\mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.

   (c) Describe the CHROMATIC NUMBER problem. Assuming that 3-SAT is $\mathcal{NP}$- complete, prove that the CHROMATIC NUMBER problem is also $\mathcal{NP}$-complete. [6+8+10]