INDIAN STATISTICAL INSTITUTE

End Semester Examination: 2019

Subject Name : **Basic Probability Theory** Course Name : M.Math II yr.
Maximum Score: 100 Date 29th Nov 2019 Duration: 180 minutes

Note: Attempt all questions. Marks are given in brackets. Total marks is 110, but you can score maximum 100. State the results clearly you use. Use separate page for each question.

Problem 1 (6 + 6 = 12). If $P(X = x, Y = y, Z = z) = f(x, z)g(y, z)$ then show that X and Y are conditionally independent given Z. State and prove the converse.

Problem 2 (12). Let X and Y be independent random variables, each having the geometric distribution with parameter p. Find $E(Y|X + Y = n)$, $n \geq 2$.

Problem 3 (10). Let X and Y be two random variables such that $X \geq Y$ with probability one and $E(X) = E(Y)$. Prove that $X = Y$ with probability one.

Problem 4 (10). Let X, Y and Z be three random variables such that $P(X > Y) = P(Y > Z) = 2/3$. Prove that $P(X \geq Z) \geq 1/3$.

Problem 5 (12). Let n be an odd integer and $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{ber}(p)$, $0 < p < 1/2$. Let $Y = X_1 + \cdots + X_n \mod 2$. Find out the distribution of Y.

Problem 6 (6+ 10 = 16). Define the total variation $\Delta(X, Y)$ between X and Y. Suppose X_1 is independent with X_2 and Y_1 is independent with Y_2. Show that

$$\Delta((X_1, X_2), (Y_1, Y_2)) \leq \Delta(X_1, Y_1) + \Delta(X_2, Y_2).$$

Problem 7 (10). Let $X \sim \text{Bin}(n, 1/2)$. Prove that $P(X \geq 3n/4) \leq 4/n$.

Problem 8 (16). Suppose m balls are thrown randomly to n boxes. For $m = n \log n$, show that with probability $1 - o(1)$ every bin contains $O(\log n)$ balls.

Problem 9 (12). A novel is viewed as a finite sequence of English letters $a, b, ..., z$ (ignoring all other symbols or replacing capital letters by corresponding smaller letters). Suppose a monkey is typing the letters at random indefinitely. What is the probability that the monkey at some point of time types (continuously) the novel Hamlet?

INDIAN STATISTICAL INSTITUTE

Fourier Analysis : M. Math. 2nd year
End Semester Examination: 2019-20
November 25, 2019.

Maximum Marks: 60 Maximum Time: 3 hrs.

(1) Given any finite measure μ on $\mathbb{R}^n$ define its Fourier transform by

$$\hat{\mu}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot y} d\mu(y).$$

Prove that $\hat{\mu} \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Justify whether $\hat{\mu} \in C_0(\mathbb{R}^n)$ for all finite measures μ. [6]

(2) Give an example of a function $f \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$. [6]

(3) If $f \in L^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ then prove that

$$\int_{\mathbb{R}^n} \hat{f}(y)\hat{g}(y)e^{2\pi i x\cdot y}dy = (f * g)(x).$$

for almost every $x \in \mathbb{R}^n$. [6]

(4) Consider the distribution T given by

$$T(\phi) = \phi'(0), \quad \phi \in C_c^\infty(\mathbb{R}).$$

Find a $\psi \in L^1_{\text{loc}}(\mathbb{R})$ such that $T_\psi'' = T$. [6]

(5) a) If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is such that for all $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\phi(x)dx = 0,$$

then prove that $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

b) Using $a)$ or otherwise prove that if $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$ and $\hat{f} \in L^1(\mathbb{R}^n)$ then

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y)e^{2\pi i x\cdot y}dy,$$

for almost every $x \in \mathbb{R}^n$. [4+4=8]

(6) If a translation invariant linear operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ is continuous with $1 < q < p < \infty$ then prove that $T \equiv 0$. [8]

(7) Suppose $f : \mathbb{R} \to \mathbb{C}$ is a nonzero, measurable function such that

$$\int_{\mathbb{R}} |f(x)| e^{|x|} dx < \infty.$$

Prove that $\text{span}\{\tau_y f \mid y \in \mathbb{R}\}$ is dense in $L^2(\mathbb{R})$. [8]

(8) For complex valued measurable functions f on $\mathbb{R}^2$ consider the strong maximal operator

$$M_s f(x,y) = \sup_{(x,y)\in\mathbb{R}} \frac{1}{|R|} \int_R f(u,v) du dv,$$

where R varies over rectangles with sides parallel to axes and $|R|$ is the area of R. Assuming measurability of M_s, prove that it is not weak type $(1,1)$ [8]

(9) Fix $R > 0$, $n \in \mathbb{N}$, $\alpha \in (0,n)$ and consider the function $g(x) = \|x\|^{-n+\alpha} \chi_{B(0,R)}(x)$, for $x \neq 0$. Prove that there does not exist any positive real number C such that

$$\|f * g\|_{\frac{n}{n-\alpha}} \leq C \|f\|_1,$$

for all $f \in L^1(\mathbb{R}^n)$. [8]

INDIAN STATISTICAL INSTITUTE

Fourier Analysis : M. Math, 2nd year
Back Paper Examination: 2019-20

, 2019.

Maximum Marks: 100 | Maximum Time: 3 hrs.

(1) Prove that $f(x) = \frac{\sin x}{x}\chi_{[-1,1]}(x)$, $x \in \mathbb{R}$, is the Fourier transform of an L^2 function but not of an L^1 function. [6]

(2) Can the Hardy-Littlewood maximal operator M satisfy an inequality of the form $\|Mf\|_q \leq C\|f\|_p$, for all $f \in S(\mathbb{R}^n)$ and some $p \neq q$? Justify your answer. [6]

(3) If $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ then prove that $f * g \in C_0(\mathbb{R})$. [8]

(4) If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then using the Fourier inversion prove that

$$\int_{\mathbb{R}^n} f(x)^2 dx = \int_{\mathbb{R}^n} \hat{f}(y)^2 dy.$$

[8]

(5) If μ is a finite Borel measure on $\mathbb{R}^n$ then prove that the operator T_μ given by

$$T_\mu f(x) = \int_{\mathbb{R}^n} f(x-y)d\mu(y)$$

is a continuous, translation invariant, linear operator from $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Can this operator be extended as a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$? Justify your answer. [8]

(6) Given any $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, prove that the map $y \mapsto \tau_y f$ is continuous from $\mathbb{R}^n$ to $L^p(\mathbb{R}^n)$. Justify whether the same is true for $p = \infty$. [8]

(7) For a nonnegative $\phi \in L^1(\mathbb{R}^n)$ with $\|\phi\|_1 = 1$ and ϵ positive define $\phi_\epsilon(x) = \epsilon^{-n}\phi\left(\frac{x}{\epsilon}\right)$. Prove that $f * \phi_\epsilon \to f$ in $L^p(\mathbb{R}^n)$, as $\epsilon \to 0$, for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Discuss the case $p = \infty$. [8]

[P.T.O.]

(8) Suppose M is a closed translation invariant, linear subspace of $L^2(\mathbb{R})$ and

$$\hat{M} = \{\hat{\phi} \mid \phi \in M\}.$$

If $P : L^2(\mathbb{R}) \to \hat{M}$ is the orthogonal projection then prove that for all $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, $f(x)Pg(x) = g(x)Pf(x)$ for almost every $x \in \mathbb{R}$. [8]

(9) Suppose $f : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ is a measurable function such that $|f(x)| \leq e^{-x^2}$ for all $x \in \mathbb{R}$. If $g \in L^1(\mathbb{R})$ is such that $f * g \equiv 0$ then prove that $g(x) = 0$ for almost every $x \in \mathbb{R}$. [8]

(10) (a) If $\phi \in C_c^\infty(\mathbb{R})$ is supported in $[-r, r]$ and if

(0. 1) $$f(z) = \int_{\mathbb{R}} \phi(t) e^{-itz} dt,$$

then prove that f is an entire function and for every $N \in \mathbb{N}$ there exists a positive real number C_N such that for all $z \in \mathbb{C}$

(0. 2) $$|f(z)| \leq C_N (1 + |z|)^{-N} e^{r|\mathrm{Im} z|},$$

where $\mathrm{Im} z$ denotes the imaginary part of z. [8]

(b) Given any entire function f satisfying (0. 2) prove that there exists a unique $\phi \in C_c^\infty(\mathbb{R})$ supported in $[-r, r]$ such that (0. 1) holds. [8]

(11) (a) If $f \in L^1(\mathbb{R})$ then prove that the series

$$F(x) = \sum_{n \in \mathbb{Z}} f(x + 2n\pi), x \in \mathbb{R}$$

defines an integrable function on $\mathbb{T}$.

(b) Express the Fourier coefficients of F in terms of $\hat{f}$.

(c) If for all large $|x|$, $|\hat{f}(x)| \leq (1 + x^2)$, then prove that

$$\sum_{n \in \mathbb{Z}} f(2n\pi) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

[6+4+6]

INDIAN STATISTICAL INSTITUTE

Semestral Examination 2019–2020

M.Math (Second year, First Semester)

Differential Topology

Maximum Marks: 60 Date: 18 November, 2019

Duration: 3 hours

Answer all questions.

State clearly any result that you use in your answer.

(1) Show that if $f : \mathbb{C} \to \mathbb{C}$ is an analytic function then f (viewed as a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$) is orientation preserving at all $z \in \mathbb{C}$ where $f'(z) \neq 0$. 5

(2) (a) Let M be a smooth manifold and Z a compact oriented n-dimensional submanifold of M without boundary. Define a map $\phi_Z : H^n_{deR}(M) \to \mathbb{R}$ by $\phi_Z([\omega]) = \int_Z \omega$, where $[\omega]$ denotes the de-Rham cohomology class of an n-form ω on M. Prove that ϕ_Z is well-defined and is a linear map. 5

(b) Suppose that Z_1 and Z_2 are two n-dimensional submanifolds of M which are cobordant. Then show that $\phi_{Z_1} = \phi_{Z_2}$. 5

(3) (a) Consider the 1-manifold C defined by the equation $2x^2 + 3y^2 = 1$ (oriented as the boundary of the enclosed region). Determine the value of the integral $\int_C \omega$, where

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \quad \text{on } \mathbb{R}^2 \setminus (0,0).$$

Prove that ω is closed but not exact. 5+5

(b) Let C_1 denote the circle $(x-2)^2 + y^2 = 1$ oriented as the boundary of the enclosed disc. What is the value of the integral $\int_{C_1} \omega$? Justify your answer. 5

(4) Let M be the manifold of all $n \times n$ real orthogonal matrices.

(a) What is the tangent space of M at the identity matrix I as a subset of the space of all real $n \times n$ matrices? 2

(b) Let $X \in T_I M$. Define a vector field $\tilde{X} : M \to TM$ by

$$\tilde{X}(g) = d\ell_g(X) \quad \text{for all } g \in M,$$

where $\ell_g : M \to M$ is the left multiplication by g on M.

Find the integral curve γ of $\tilde{X}$ satisfying $\gamma(0) = I$. Hence obtain the flow of $\tilde{X}$. 4+4

P.T.O

(5) Suppose that M is an orientable manifold. Prove that the product orientation on $M \times M$ is independent of the choice of the orientation on M. 5

(6) (a) Suppose we orient the 2-sphere S^2 as the boundary of the unit ball D^3 in $\mathbb{R}^3$, where D^3 has the standard orientation. Define a smooth non-vanishing 2-form Ω on S^2 such that $\Omega_x(v, v') > 0$ for every positively oriented basis v, v' of $T_x S^2$. 5

(b) Consider the antipodal map $a : S^2 \to S^2$ which takes a point x to $-x$. Prove that a is orientation reversing. 5

(c) Can $a : S^2 \to S^2$ be homotopic to the identity map? justify your answer. 5

(7) Determine the de Rham cohomology groups of an annulus. 10

Indian Statistical Institute

Semestral Examination : (2019–20)

M. Math II year, first semester

Differential Geometry

Date : 29.11.19 Maximum marks : 50 Duration : 3 hours.

Answer ALL questions. The marks indicated in brackets add up to 55. You score will be taken as the minimum of the marks obtained and 50.

Note the following :

(i) All the manifolds considered here are smooth (C^∞) manifolds and vector fields are smooth vector fields.

(ii) $\chi(M)$ denotes the set of smooth vector fields on M.

(iii) For $p \geq 1$, $\Omega^p(M)$ denotes the space of smooth p-forms on M. $\Omega(M) = \oplus_{p \geq 0} \Omega^p(M)$ (with $\Omega^0(M) \equiv C^\infty(M)$) denotes the algebra of smooth forms on M.

(1) Let M be the upper half-plane in R^2, i.e. $\{(x,y) \ : \ y > 0\}$, equipped with the Riemannian metric $< .,. >$ defined by:

$< \frac{\partial}{\partial x}, \frac{\partial}{\partial x} >=< \frac{\partial}{\partial y}, \frac{\partial}{\partial y} >= \frac{1}{y^2}, < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >= 0.$

We can consider M as a subset of the complex plane by identifying $(x,y) \in M$ with $z = x + iy$.

Let G denote the Lie group $SL(2,R)$. Define a left action L_g of G on M by setting

$$L_g(z) = \frac{az+b}{cz+d},$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

$a, b, c, d \in R$ with $ad - bc = 1$.

Verify the following:

(a) $Im(L_g(z)) > 0$ whenever $Im(z) > 0$; thus L_g indeed defines a map from M to M.

(b) L_g defines a smooth left action of the Lie group G on the manifold M.

(c) For every $p \in M$, $dL_g|_p$ is an isometry with respect to the Riemannian structure of M.

[3+3+6=12]

(2) Let M be an n dimensional manifold and $\mathcal{F}$ be a k dimensional $(k < n)$ smooth distribution. For any $p \geq 1$, define the following subspace $\mathcal{I}^p$ of the algebra $\Omega(M)$:

$$\mathcal{I}^p = \{\omega \in \Omega^p(M) \ : \ \omega(X_1, ..., X_p) = 0 \text{ for all smooth vector fields } X_1, ..., X_p \in \mathcal{F}\}.$$

For $p = 0$, set $\mathcal{I}^0 = \{f \in \Omega^0(M) \ : \ Xf = 0 \ \forall \text{ smooth vector field } X \in \mathcal{F}\}$;

$\mathcal{I} = \oplus_{p \geq 0} \mathcal{I}^p \subseteq \Omega(M)$.

Prove the following:

(a) $\mathcal{I}$ is an ideal of $\Omega(M)$, that is, $\omega \wedge \eta \in \mathcal{I}_{p+q}$ whenever $\omega \in \mathcal{I}^p$, $\eta \in \Omega^q(M)$.

(b) $\mathcal{F}$ is integrable if and only if $d(\mathcal{I}^p) \subseteq \mathcal{I}^{p+1} \ \forall \ p$.

Hint : use the intrinsic definition of the exterior derivative d.

[5+10=15]

(3) Let M be a smooth manifold with a Riemannian structure $< \cdot, \cdot >$ and f be a smooth real-valued function such that the vecor field grad(f) satisfies $< \mathrm{grad(f)}, \mathrm{grad(f)} >_{\mathrm{p}} = 1$ for all $p \in M$, where grad(f) is defined by

$$< \mathrm{grad}(f), X >_p = X(p)(f),$$

for any smooth (possibly locally defined) vector field X on M. Prove that the integral curves of grad(f) are geodesics with respect to the given Riemannian structure. [15]

(4) Let M be a manifold and ∇, ∇' be two affine connections on M. Define a map $A : \chi(M) \times \chi(M) \to \chi(M)$ (where $\chi(M)$ denotes the vector space of all smooth vector fields on M) by

$$A(X, Y) = \nabla_X(Y) - \nabla'_X(Y).$$

Prove that:

(i) $A(\cdot, \cdot)$ is $C^\infty(M)$-linear in both the arguments.

(ii) The following are equivalent:

(a) $A(X, Y) = -A(Y, X)$ for all $X, Y \in \chi(M)$;

(b) ∇ and ∇' have the the same set of geodesics, that is, a smooth curve $\gamma(t)(a < t < b)$ is a geodesic for ∇ if and only if it is a geodesic for ∇'.

[4+9=13]

INDIAN STATISTICAL INSTITUTE
End-Semestral Examination: 2019-20 (First Semester)

M. MATH. II YEAR
Commutative Algebra I

Date: 27/11/2019 Maximum Marks: 70 Duration: 3 Hours

Throughout the paper, k will denote a field and R a commutative ring with unity.

GROUP A
Answer ANY FOUR

1. Let A be a finitely generated k-algebra. Let G be a finite group of automorphisms of A and $R = A^G = \{x \in A \mid \sigma(x) = x \ \forall \sigma \in G\}$, the subring of invariants. Prove that

   (i) A is integral over R.

   (ii) R is a finitely generated k-algebra. [5+9=14]

2. (i) Let M be an R-module and $f : M \to M$ an R-linear map. Show that if M is Artinian and f is injective, then f is an isomorphism.

   (ii) Let $A = \mathbb{Q}[X_1, X_2, \cdots, X_n]$ and $g, f_1, f_2, \cdots, f_n \in A$. Suppose that $g(p) = 0$ whenever $f_1(p) = f_2(p) = \cdots = f_n(p) = 0$ for any point $p \in \mathbb{C}^n$. Prove that there exist $g_1, \ldots, g_n \in A$ such that $g^m = f_1g_1 + f_2g_2 + \cdots + f_ng_n$ for some integer $m \geq 0$.

   [6+8=14]

3. (i) Let R be a subring of an integral domain B such that B is a finitely generated R-algebra. Prove that there exist $s(\neq 0) \in R$ and $y_1, \ldots, y_n \in B$, which are algebraically independent over R such that B_s is integral over $R_s[y_1, \ldots, y_n]$.

   (ii) Show that the ring $R = \mathbb{C}[X, Y]/(Y^2 - X^2 - X^3)$ is a Noetherian domain whose normalisation is of the form $\mathbb{C}[t]$, where $t \notin R$ and display an explicit monic polynomial $f(X) \in R[X]$ of which t is a root. [7+7=14]

4. Let $R = k[X, Y, Z]/(XY - Z^2)$. Let x, y, z denote the images in R of X, Y, Z respectively. Let $P = (x, z)R$ and $m = (x, y, z)$.

   (i) Show that P is a prime ideal of R.

   (ii) Prove that $P^2 = (x) \cap m^2$ is an irredundant primary decomposition of P^2.

   [4+10=14]

5. Let R be a ring. Suppose that R_P is an integral domain for every prime ideal P of R. Prove that R is isomorphic to a finite product of integral domains. [14]

6. Give an example of the following with brief justification.

   (i) A finitely generated ideal J of a non-Noetherian ring R such that R/J is a Noetherian ring.

   (ii) An R-module M such that $\text{Ass}_R(M)$ is empty. [7+7=14]

GROUP B
Answer ANY FIVE

State whether the following statements are TRUE or FALSE with brief justification.

(i) Any finitely generated flat module over a Noetherian ring R is a projective module.

(ii) If $R[X]$ is a normal ring for an indeterminate X over R, then R is a normal ring.

(iii) For any primary ideal Q of a ring A and a ring homomorphism $\pi : R \to A$, $\pi^{-1}(Q)$ is a primary ideal of R.

(iv) For an R module M and a multiplicatively closed subset S of R, $S^{-1}\mathrm{Ann}_R(M) = \mathrm{Ann}_{S^{-1}R}S^{-1}M$.

(v) Any infinite integral domain with finitely many units has infinitely many maximal ideals.

(vi) If L_1 and L_2 are algebraic extensions of a field k then $L_1 \otimes_k L_2$ is an integral domain.

(vii) The ring $A = \{\frac{f(X,Y)}{g(X,Y)} | f(X,Y), g(X,Y) \in \mathbb{R}[X,Y]; g(0,0) \neq 0; g(1,1) \neq 0\}$ is Noetherian.

(viii) If A is a finite integral extension of an integral domain R, then A is faithfully flat over R. [$5 \times 4 = 20$]

INDIAN STATISTICAL INSTITUTE
Back Paper Semestral Examination: 2019-20 (First Semester)

M. MATH. II YEAR
Commutative Algebra I

Date: 10/01/2020 Maximum Marks: 100 Duration: 3 Hours

Throughout the paper, k will denote a field and R a commutative ring with unity.

1. Let $B = \mathbb{C}[X, Y, Z]/(XY - Z^2)$ and x the image of X in B.
   (i) Examine whether x is prime in B.
   (ii) Show that $B \cong \mathbb{C}[U^2, UV, V^2]$.
   (iii) Examine whether B is a UFD. [4+9+4]

2. Show that a unique factorisation domain is a normal ring. [10]

3. Let R and A be integral domains such that $R \subseteq A$ and A is integral over R.
   (i) Prove that an element $a \in R$ is a unit in R if and only if it is a unit in A.
   (ii) Deduce that R is a field if and only if A is a field. [10+5]

4. Let V be an affine algebraic set in $\mathbb{C}^n$ such that $\mathbb{C}[V]^* = \mathbb{C}^*$. Prove that any $f \in \mathbb{C}[V] \setminus \mathbb{C}$ induces a surjective polynomial function on V. Give an example of an affine algebraic set V in $\mathbb{C}^2$ and a non-constant polynomial function $f : V \to \mathbb{C}$ which is not surjective. [9+3]

5. Prove that if a finitely generated k-algebra A is a field then A is algebraic over k. [14]

6. Let $R = k[X, Y]/(X^2, XY)$ and x, y denote the image of the X, Y in R. Show that y^n is (x, y) primary in R for every $n \geq 1$. [10]

7. Prove that every ideal of a Noetherian ring R contains a product of prime ideals. [12]

8. Let R be a ring, M a module, and I an ideal. Suppose I is maximal in the set of annihilators of nonzero elements x of M. Prove that $I \in \mathrm{Ass}_R(M)$. [10]

Indian Statistical Institute
First Semester Examination: 2019-20

Course Name: M. Math, 2^{nd} year
Subject Name : Topology III
Maximum Marks: 50, Duration: 3 hours
Date: 20.11.2019

- Answer as many questions as you can.
- Maximum marks is 50.
- You may use any results proved in class. Any other results (including those in homework problem sets) require proof.

1. Prove that $H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \cong \mathbb{Z}_{2k}[\alpha, \beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$ for $|\alpha| = 1$, $|\beta| = 2$. 10
2. Prove that the homotopy groups of $\mathbb{C}P^n \times S^{2m+1}$ and $\mathbb{C}P^m \times S^{2n+1}$ are isomorphic but the spaces are not homotopy equivalent. 10
3. Let M, N be closed, oriented manifolds of dimension n. Suppose $f : M \to N$ such that $f_*([M]) \neq 0$. Prove that $f^* : H^k(N; \mathbb{Q}) \to H^k(M; \mathbb{Q})$ is injective for all k. 10
4. Let $P_{n,k}$ (for $n > k$) be the space $\mathbb{R}P^{n-1}/\mathbb{R}P^{k-1}$.
   a) Prove that $P_{n,k}$ is never weakly equivalent to $P_{n-2,k} \vee S^{n-2} \vee S^{n-1}$ for $k > 2$ and $n > k + 2$. 5
   b) Prove that $P_{7,4} \simeq S^4 \vee \Sigma^4\mathbb{R}P^2$. 5
5. Let M be a closed n-manifold with non-trivial torsion in $H_{n-2}(M)$. Prove that $\pi_1(M)$ is non-trivial. 10
6. Let M be a closed n-manifold.
   a) Prove that for every $i \leq n$, there is a class $v_i \in H^i(M; \mathbb{Z}/2)$ such that $Sq^i(\alpha) = v_i \cup \alpha$ for all $\alpha \in H^{n-i}(M; \mathbb{Z}/2)$. 4
   b) Prove that $v_i = 0$ if $i > \frac{n}{2}$. 2
   c) Compute all the v_i for $M = \mathbb{R}P^6$. 4

Some useful formulas

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{n+1}),\ |x| = 1,\ H^*(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}[y]/(2y, y^{n+1}),\ |y| = 2, & \text{if n is even} \\ \mathbb{Z}[y,z]/(2y, y^{n+1}, zy, z^2)\ |z| = n, |y| = 2 & \text{if n is odd} \end{cases}$$

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x],\ |x| = 1,\ H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[y]/(2y),\ |y| = 2$$

$$H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}),\ |x| = 2,\ H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x],\ |x| = 2.$$

$$H^*(L_{2n+1}(k); \mathbb{Z}) = \mathbb{Z}[y,z]/(ky, y^{n+1}, zy, z^2)\ |z| = 2n+1, |y| = 2,$$

$$H^*(L_{2n+1}(k); \mathbb{Z}/k) = \begin{cases} \mathbb{Z}/k[x,y]/(y^{n+1}, x^2)\ |x| = 1, |y| = 2, \text{if k is odd} \\ \mathbb{Z}/k[x,y]/(y^{n+1}, x^2 - \frac{k}{2}y)\ |x| = 1, |y| = 2, \quad \text{if k is even} \end{cases}$$

$$H^*(L_\infty(k);\mathbb{Z}) = \mathbb{Z}[y]/(ky)\ |y| = 2,\ H^*(L_\infty(k);\mathbb{Z}/k) = \begin{cases} \mathbb{Z}/k[x,y]/(x^2)\ |x| = 1, |y| = 2, \text{if k is odd} \\ \mathbb{Z}/k[x,y]/(x^2 - \frac{k}{2}y)\ |x| = 1, |y| = 2, \text{ if k is even} \end{cases}$$

Some useful results

Universal coefficient theorem for homology : For any abelian group A, we have the following short exact sequence which splits (but not naturally)

$$0 \to H_k(X) \otimes A \to H_k(X;A) \to Tor(H_{k-1}(X), A) \to 0$$

Universal coefficient theorem for cohomology : For any abelian group A, we have the following short exact sequence which splits (but not naturally)

$$0 \to Ext(H_{k-1}(X), A) \to H^k(X;A) \to Hom(H_k(X), A) \to 0$$

Long exact sequence for cohomology : For $Y \subset X$ an inclusion, there is a long exact sequence

$$\cdots H^n(X,Y) \to H^n(X) \to H^n(Y) \xrightarrow{\delta} H^{n+1}(X,Y) \to \cdots$$

Künneth-formula : Suppose $H^*(X)$ is torsion-free. Then for any Y, $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$ with the isomorphism given by external cup product.

Borsuk=Ulam theorem : There is no map $g : S^n \to S^{n-1}$ such that $g(-x) = -g(x)$.

Cellular approximation : A map between CW complexes is homotopic to a cellular map.

Long exact sequence for homotopy groups : For an inclusion $A \subset B \subset X$, there is a long exact sequence

$$\cdots \pi_n(B,A) \to \pi_n(X,A) \to \pi_n(X,B) \to \pi_{n-1}(B,A) \to \cdots$$

A fibre bundle is a Serre fibration.

Fibrations induce long exact sequences : If $F \to E \to X$ is a Serre fibration, one obtains a long exact sequence

$$\cdots \pi_k F \to \pi_k E \to \pi_k X \to \pi_{k-1} F \to \cdots$$

Blakers Massey Theorem : Suppose $X = A \cup B$ (of subcomplexes) with $C = A \cap B$. If (A,C) is m-connected and (B,C) is n-connected, then $(A,C) \to (X,B)$ is an $(m+n)$-equivalence.

Freudenthal suspension theorem : Suppose X is $(n-1)$-connected. Then, the map $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is isomorphism for $k \leq 2n-2$ and surjective for $k = 2n-1$.

Hurewicz Theorem : Suppose X is $(n-1)$-connected for $n \geq 2$. Then $\rho : \pi_n(X) \to H_n(x)$ is an isomorphism.

Cofibration sequence : $A \to X \to X \cup CA \to \Sigma A \to \Sigma X \to \cdots$.

Fibration sequence : $\cdots \Omega E \to \Omega X \to F \to E \to X$.

Steenrod squares: Sq^i are natural, additive operations $Sq^i : H^k(X : \mathbb{Z}/2) \to H^{k+i}(X;\mathbb{Z}/2)$. These satisfy

1) $Sq^0 = id$, $Sq^i(x) = 0$ if $i > |x|$, $Sq^{|x|}(x) = x^2$.
2) Sq^i commutes with suspension.
3) $Sq^k(xy) = \sum_{i+j=k} Sq^i(x) Sq^j(y)$.
4) Adem relations : If $a < 2b$, $Sq^a Sq^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$.

Thom isomorphism : If ξ is a n-dimensional oriented vector bundle over M, $\tilde{H}^{*+n}(Th(\xi)) \cong H^*(M)$. The isomorphism is given by cup product with the Thom class.

Poincaré duality : If M is a closed, oriented n-manifold, $H_k(M) \cong H^{n-k}(M)$ with the isomorphism given by cap product with the orientation class $[M]$.

Indian Statistical Institute
Back paper Examination: 2019-20

Course Name: M. Math, 2^{nd} year
Subject Name : Topology III
Total Marks: 100, Duration: 3 hours
Date: 16.01.2020

- Answer as many questions as you can.
- Maximum marks is 45.
- You may use any results proved in class. Any other results (including those in homework problem sets) require proof.

1. Let $i : S^6 \to \mathbb{C}P^8/\mathbb{C}P^2$ be the inclusion of the 6-skeleton. Prove that there is no retraction for i. 10

2. Let $\eta : S^3 \to S^2$ be the Hopf map, that is, the attaching map for $\mathbb{C}P^2$. Consider the map τ expressed as the composite
$$\tau : S^3 \to S^3 \vee S^3 \xrightarrow{\eta \vee \eta} S^2 \vee S^2,$$
where the map $S^3 \to S^3 \vee S^3$ is the map obtained by pinching the equatorial S^2 to a point. Let P be the mapping cone of τ.
a) Let $r_1 : S^2 \vee S^2 \to S^2$ be the map which quotients out the second factor. Prove that $r_1 \circ \tau$ is homotopic to η. Using this produce a map $P \to \mathbb{C}P^2$ which induces the identity map on H^4. 6
b) By constructing maps similar to the one in a), compute the ring structure on $H^*(P)$. 8
c) Is $P \simeq S^2 \times S^2$? 6

3. Let n be odd. Prove that for any map $\mathbb{R}P^{n+1} \to \mathbb{R}P^n$ the composite $\mathbb{R}P^n \to \mathbb{R}P^{n+1} \to \mathbb{R}P^n$ is null-homotopic. 10

4. a) Prove that $H^2(K(\mathbb{Z},2);\mathbb{Z}/2) \cong \mathbb{Z}/2$. Use the generator to obtain a map $K(\mathbb{Z},2) \to K(\mathbb{Z}/2,2)$. Prove that the homotopy fibre of the map is homotopy equivalent to $K(\mathbb{Z},2)$. 4
b) In the above fibration $(K(\mathbb{Z},2) \xrightarrow{f} K(\mathbb{Z},2) \to K(\mathbb{Z}/2,2))$, prove that we get an induced map $q : Cone(f) \to K(\mathbb{Z}/2,2)$. Prove that q_* is surjective on homotopy groups. 4
c) Prove that q is a 4-equivalence. Use this to deduce that $H^4(K(\mathbb{Z}/2,2) \cong \mathbb{Z}/2$. 5
d) Compute $H^{n+2}(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ for all n. 5
e) Compute the maps
$$Sq^1 : H^{n+1}(K(\mathbb{Z}/2,n)) \to H^{n+2}(K(\mathbb{Z}/2,n))$$
and
$$Sq^2 : H^n(K(\mathbb{Z}/2,n)) \to H^{n+2}(K(\mathbb{Z}/2,n))$$
for all n. 5
f) Deduce the Adem relation $Sq^1 Sq^1 = 0$ from the computations in (b). 2

5. Let M be an oriented manifold of dimension n with fundamental class $[M]$. Let $f : S^n \to M$ be a continuous map such that $f_*(\iota_n) = q[M]$ where ι_n is a fundamental class of S^n.
a) Show that $f_* : H_k(S^n;\mathbb{Z}/p) \to H_k(M;\mathbb{Z}/p)$ is an isomorphism if p does not divide q. 8
b) Show that multiplication by q is 0 in $H_i(M;\mathbb{Z})$ for $0 < i < n$. 7

6. Let M be a simply connected 7-manifold. Prove that the torsion subgroup of $H_2(M)$ is isomorphic to the torsion subgroup of $H_4(M)$. 10

7. Let M be a n-manifold with non-trivial torsion in $H_{n-1}(M)$. Prove that $\pi_1(M)$ is non-trivial. 10

Some useful formulas

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{n+1}),\ |x| = 1,\ H^*(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}[y]/(2y, y^{n+1}),\ |y| = 2,\ \text{if n is even} \\ \mathbb{Z}[y,z]/(2y, y^{n+1}, zy, z^2)\ |z| = n, |y| = 2 \text{ if n is odd} \end{cases}$$

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[x],\ |x| = 1,\ H^*(\mathbb{R}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[y]/(2y),\ |y| = 2$$

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}),\ |x| = 2,\ H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x],\ |x| = 2.$$

$$H^*(L_{2n+1}(k);\mathbb{Z}) = \mathbb{Z}[y,z]/(ky, y^{n+1}, zy, z^2)\ |z| = 2n+1, |y| = 2,$$

$$H^*(L_{2n+1}(k);\mathbb{Z}/k) = \begin{cases} \mathbb{Z}/k[x,y]/(y^{n+1}, x^2)\ |x| = 1, |y| = 2, \text{if k is odd} \\ \mathbb{Z}/k[x,y]/(y^{n+1}, x^2 - \frac{k}{2}y)\ |x| = 1, |y| = 2, \text{ if k is even} \end{cases}$$

$$H^*(L_\infty(k);\mathbb{Z}) = \mathbb{Z}[y]/(ky)\ |y| = 2,\ H^*(L_\infty(k);\mathbb{Z}/k) = \begin{cases} \mathbb{Z}/k[x,y]/(x^2)\ |x| = 1, |y| = 2, \text{if k is odd} \\ \mathbb{Z}/k[x,y]/(x^2 - \frac{k}{2}y)\ |x| = 1, |y| = 2, \text{ if k is even} \end{cases}$$

Some useful results

Universal coefficient theorem for homology : For any abelian group A, we have the following short exact sequence which splits (but not naturally)

$$0 \to H_k(X) \otimes A \to H_k(X;A) \to Tor(H_{k-1}(X), A) \to 0$$

Universal coefficient theorem for cohomology : For any abelian group A, we have the following short exact sequence which splits (but not naturally)

$$0 \to Ext(H_{k-1}(X), A) \to H^k(X;A) \to Hom(H_k(X), A) \to 0$$

Long exact sequence for cohomology : For $Y \subset X$ an inclusion, there is a long exact sequence

$$\cdots H^n(X,Y) \to H^n(X) \to H^n(Y) \xrightarrow{\delta} H^{n+1}(X,Y) \to \cdots$$

Künneth-formula : Suppose $H^*(X)$ is torsion-free. Then for any Y, $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$ with the isomorphism given by external cup product.

Borsuk=Ulam theorem : There is no map $g : S^n \to S^{n-1}$ such that $g(-x) = -g(x)$.

Cellular approximation : A map between CW complexes is homotopic to a cellular map.

Long exact sequence for homotopy groups : For an inclusion $A \subset B \subset X$, there is a long exact sequence

$$\cdots \pi_n(B,A) \to \pi_n(X,A) \to \pi_n(X,B) \to \pi_{n-1}(B,A) \to \cdots$$

A fibre bundle is a Serre fibration.

Fibrations induce long exact sequences : If $F \to E \to X$ is a Serre fibration, one obtains a long exact sequence

$$\cdots \pi_k F \to \pi_k E \to \pi_k X \to \pi_{k-1} F \to \cdots$$

Blakers Massey Theorem : Suppose $X = A \cup B$ (of subcomplexes) with $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected, then $(A, C) \to (X, B)$ is an $(m+n)$-equivalence.

Freudenthal suspension theorem : Suppose X is $(n-1)$-connected. Then, the map $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is isomorphism for $k \leq 2n-2$ and surjective for $k = 2n-1$.

Hurewicz Theorem : Suppose X is $(n-1)$-connected for $n \geq 2$. Then $\rho : \pi_n(X) \to H_n(x)$ is an isomorphism.

Cofibration sequence : $A \to X \to X \cup CA \to \Sigma A \to \Sigma X \to \cdots$.

Fibration sequence : $\cdots \Omega E \to \Omega X \to F \to E \to X$.

Steenrod squares: Sq^i are natural, additive operations $Sq^i : H^k(X : \mathbb{Z}/2) \to H^{k+i}(X;\mathbb{Z}/2)$. These satisfy

1) $Sq^0 = id$, $Sq^i(x) = 0$ if $i > |x|$, $Sq^{|x|}(x) = x^2$.
2) Sq^i commutes with suspension.
3) $Sq^k(xy) = \sum_{i+j=k} Sq^i(x)Sq^j(y)$.
4) Adem relations : If $a < 2b$, $Sq^a Sq^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$.

Thom isomorphism : If ξ is a n-dimensional oriented vector bundle over M, $\tilde{H}^{*+n}(Th(\xi)) \cong H^*(M)$. The isomorphism is given by cup product with the Thom class.

Poincaré duality : If M is a closed, oriented n-manifold, $H_k(M) \cong H^{n-k}(M)$ with the isomorphism given by cap product with the orientation class $[M]$.

NUMBER THEORY
FINAL EXAMINATION
M.MATH II YR
DURATION 3 HOURS

This is a closed-book and closed-note exam. Maximum marks: 60

1. (a) Define primitive root.
(b) Prove that 3 is a primitive root modulo p if p is a prime of the form $2^n + 1$ for some $n > 1$.
(c) Prove that the sum of primitive roots modulo a prime p is congruent to $\mu(p-1)$ modulo p. **(2+6+5 marks)**

2. (a) State Liouville's theorem about approximation of algebraic numbers.
(b) Let $\xi \in \mathbb{R}$. Assume there exists a sequence $(x_n, y_n) \in \mathbb{Z} \times \mathbb{N}$, $n = 1, 2, 3, \ldots$ such that $x_n/y_n \neq \xi$ for infinitely many n, and

of distinct pairs - ok.

$$|x_n - \xi y_n| \to 0 \quad \text{as } n \to \infty.$$

Prove that $\xi \in \mathbb{R} - \mathbb{Q}$.

(c) Let

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$

Using (b) show that e is irrational. **(3+4+6 marks)**

3. A positive integer is called a triangular number if it is of the form $n(n+1)/2$ for some $n \in \mathbb{N}$. A positive integer is called a square number if it is of the form m^2 for some $m \in \mathbb{N}$.
(a) Show that

$$\sqrt{2} = [1; 2, 2, 2, \ldots].$$

(b) Suppose N is both triangular and a square, i.e.

$$N = \frac{n(n+1)}{2} = m^2$$

for some $n, m \in \mathbb{N}$. Show that $(2n+1, 2m)$ is a solution of the equation

$$x^2 - 2y^2 = 1.$$

(c) Use continued fraction to find all such N.
(4+2+7 marks)

4. Let χ be a quadratic character modulo $q > 1$. Let

$$S = \sum_{n=1}^{q} n\chi(n).$$

(a) If $(a, q) = 1$, prove that

$$a\chi(a)S \equiv S \bmod q.$$

(b) Write $q = 2^j r$ with r odd. Show that there exists $a \in \mathbb{Z}$ with $(a, q) = 1$ such that $a \equiv 3 \bmod 2^j$ and $a \equiv 2 \bmod r$.

(c) Using (a) and (b) show that

$$12S \equiv 0 \bmod q.$$

(4+3+6 marks)

5. Let $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ be two sequences of positive real numbers.

(a) Suppose $\mathbf{x}$ is uniformly distributed modulo 1. Suppose α is a positive real number. Is $\alpha\mathbf{x} = (\alpha x_n)$ necessarily uniformly distributed? - modulo 1

(b) Suppose $\mathbf{x}$ is uniformly distributed modulo 1 and the sequence $(\{y_n\})$ is convergent. Show that $\mathbf{x} + \mathbf{y}$ is uniformly distributed modulo 1. fractional part?

(c) Suppose $(\{x_n\})$ converges as $n \to \infty$. Show that $\mathbf{x}$ is not uniformly distributed modulo 1. **(3+6+5 marks)**