INDIAN STATISTICAL INSTITUTE
B.STAT.(HONS.) I YEAR:1999-2000
SEMESTRAL-I EXAMINATION
CALCULUS-I

te:29.11.99 Maximum Marks: 60 Time: 3 Hours

Note: Answer six questions.

(a) For a sequence $\{a_n\}$ of positive terms, show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n}$$

(b) Give an example of $\{a_n\}$ where each $a_n > 0$,

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 0, \text{ and}$$

$$\liminf_{n\to\infty} \sqrt[n]{a_n} = 1$$

[6+4=10]

(a) Let f be a continuous function on the closed and bounded interval $[a,b]$. Show that f is uniformly continuous on $[a,b]$.

(b) Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. [6+4=10]

Let f be a continuous function on $\mathbb{R}$ such that at all $x \in \mathbb{R}$,

$$\lim_{h\to 0+} \frac{f(x+h)+f(x-h)-2f(x)}{h} = g(x)$$

exists. (a) Show that

(i) If $f'(x)$ exists, then $g(x)=0$.

(ii) If x is a point of local maximum for f, then $g(x) \leq 0$.

(b) Give an example of a function f such that at some x, $g(x)$ exists and is nonzero. [3+3+4=10]

(a) Let f be a continuous function and n a positive integer. Show that if $0 \leq f(x) \leq 1$ for all $x \in [0,1]$, then there exists at least one point $c \in [0,1]$ for which $f(c)=c^n$.

(b) Give an example of a function f on an interval I such that f is not continuous on I but f has the intermediate value property: given $x, y \in I$, $x<y$ and any real no. α lying between $f(x)$ and $f(y)$, there exists ξ, $x<\xi<y$ such that $f(\xi)=\alpha$. [6+4=10]

5.(a) Show that for the series $\Sigma a_n x^n$, there exists a positive r[e]
number R such that if $|x|<R$, $\Sigma a_n x^n$ converges absolutely a[nd if]
$|x|>R$, $\Sigma a_n x^n$ does not converge.

(b) Give an example of an infinitely differentiable function g on $\mathbb{R}$ such that

$$\begin{aligned} g(x) &= 0 \quad \text{if} \quad x \leq 0 \\ &= 1 \quad \text{if} \quad x = \tfrac{1}{2} \\ &= ? \quad \text{if} \quad x \geq 1 \end{aligned}$$

[5+5=1

6.(a) Prove that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ ....

for $-1<x<0$.

(b) If Σa_n is an absolutely convergent series and, for all x, $a_n \neq -1$, then show that

$$\Sigma \frac{a_n}{1+a_n} \text{ is convergent.}$$

(c) Decide if the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ is convergent. [6+2+2

7.(a) Find

$$\lim_{n\to 0} \frac{\log(1+x)}{x}$$

$$\lim_{n\to\infty} \frac{n}{\log n} \left[n^{1/n}-1\right]$$

(b) Show that $\log(1+x) \leq x$ for all $x > 0$

(c) Show that $(1-x)^{-1} \leq e^{2x}$ for $0 \leq x \leq \frac{1}{2}$. [4+3+3

INDIAN STATISTICAL INSTITUTE

Semestral Examination : Semester I (1999-2000)

B. Stat.(Hons) I Year

Vectors and Matrices - I

2 December 1999 Maximum Marks: 100 Duration: $3\frac{1}{2}$ hrs.

Note: Answer as much as you can. The whole paper carries 114 marks but the maximum you can score is 100. Marks allotted to each question are indicated near the right margin. While solving problems, you may use any theorem proved in the class after stating it clearly. $\rho(\mathbf{A})$ denotes the rank of the matrix $\mathbf{A}$.

1. (a) State the modular law. [4]

   (b) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis of a vector space V and let S be the linear span of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ where $k < n$.

      i. If $k < n/2$, show that S cannot have two complements which are virtually disjoint. [4]

      ii. If $k \geq n/2$, show that $T := \mathrm{Sp}(\{\mathbf{x}_{k+1}, \mathbf{x}_{k+2}, \ldots, \mathbf{x}_n\})$ and $W := \mathrm{Sp}(\{\mathbf{x}_{k+1} + \mathbf{x}_1, \mathbf{x}_{k+2} + \mathbf{x}_2, \ldots, \mathbf{x}_n + \mathbf{x}_{n-k}\})$ are complements of S and that $T \cap W = \{\mathbf{0}\}$. [12]

2. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   and S the set of matrices which can be expressed as polynomials in $\mathbf{A}$. Find which matrices belong to S and find the dimension of S as a subspace of $F^{n \times n}$. [12]

3. (a) Give an infinite set of 2×2 singular matrices over $\mathbb{R}$ which forms a group under matrix multiplication and prove that it forms a group. [7]

   (b) If a set $\mathcal{G}$ of $n \times n$ matrices over a field F forms a group under multiplication, prove that any two matrices belonging to $\mathcal{G}$ have the same rank. [8]

4. (a) Prove or disprove: the rank of a non-null square matrix $\mathbf{A}$ is the maximum k for which $\mathbf{A}$ has a $k \times k$ non-singular principal submatrix. [4]

   (b) Prove or disprove: the rank of a non-null symmetric matrix $\mathbf{A}$ is the maximum k for which $\mathbf{A}$ has a $k \times k$ non-singular principal submatrix. [7]

5. Let $\mathbf{I} = \mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_k$ where each $\mathbf{A}_i$ is a matrix of order n. Show that the following statements are equivalent: [18]

   (a) Each $\mathbf{A}_i$ is idempotent.

   (b) $\mathbf{A}_i\mathbf{A}_j = \mathbf{0}$ whenever $i \neq j$.

   (c) $\rho(\mathbf{A}_1) + \rho(\mathbf{A}_2) + \cdots + \rho(\mathbf{A}_k) = n$.

6. (a) Let

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

   i. Suppose $\mathbf{A}$ is non-singular. Then show that $\mathbf{M}$ is non-singular iff $\mathbf{F} := \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is non-singular. Also then write down an expression for $\mathbf{M}^{-1}$ using $\mathbf{A}^{-1}$ and $\mathbf{F}^{-1}$. [9 + 4 = 13]

   ii. State the corresponding results when $\mathbf{D}$ is non-singular instead of $\mathbf{A}$. [4]

   (b) Let $\mathbf{A}$ be a non-singular matrix. Then show that $\mathbf{A} + \mathbf{u}\mathbf{v}^T$ is non-singular iff $\alpha := 1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u} \neq 0$. Also then show that

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{A}^{-1} - \frac{1}{\alpha}\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} \qquad [11]$$

   (c) Let $\mathbf{A}_{\alpha,\beta}$ be the $n \times n$ matrix with diagonal entries α and off-diagonal entries β. Using (b), find when $\mathbf{A}_{\alpha,\beta}$ is non-singular. Also then show that $\mathbf{A}^{-1}_{\alpha,\beta}$ is $\mathbf{A}_{\gamma,\delta}$ for some γ and δ and find γ and δ. [10]

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INDIAN STATISTICAL INSTITUTE
First Semestral Examination : 1999-2000
B.Stat.(Hons) - Ist Year
Probability Theory and its Applications I
Maximum Score : 120 pts

Date : 6.12.99 Time : 4 Hours

Note : This paper carries questions worth a **total of 135 points**. Answer as much as you can. The **maximum you can score is 120 points**.

1. A deck of bridge cards is randomly dealt among four players.

   (a) What is the probability that each player gets an ace?

   (b) Given that North has got four spades, what is the probability that he and his partner together have ten spades between themselves? <10+10> = [20]

2. In the context of (symmetric) random walk, prove that for any $r > 0$ and $k \leq r$, $P\left(S_n = k \text{ and } \max_{i \leq n} S_i = r\right) = P(S_n = 2r - k) - P(S_n = 2r - k + 2)$. Hence find the probability $P\left(\max_{i \leq 10} S_i = 5\right)$. <10+10> = [20]

3. I send a query to five different persons through e-mail. Two of the recipients have the probability 0.4 each to respond to my query, while the other three have the probability 0.5 each to respond. Assume that the five recipients act independently of one another. Let X denote the number of responses I receive.

   (a) Find the distribution of X.

   (b) Find the expected value and variance of X. <10+10> = [20]

4. Each time I send an e-mail, it has a chance of 0.02 of getting lost, independently of the others. If I send 500 e-mails over a certain period of time, what roughly are the chances that at most 2 of my e-mails will get lost? You are required to use an appropriate Poisson approximation to get a simple answer. [10]

5. Six boys and four girls appear in a math test and after that they are ranked from 1 to 10 according to their performance in the test. Assume that there are no ties and that all possible rankings are equally likely.

   (a) Let X denote the highest rank secured by a boy and Y the highest rank secured by a girl. (Rank 1 is considered higher than rank 2, etc.) Find the joint distribution of (X, Y).

   (b) With the same notation as in part (a), find the probabilities $P(X = 3)$ and $P(X - Y \leq 2)$.

   (c) Find the distribution of the total number of girls who secure ranks higher than *all* the boys. <10+10+10> = [30]

6. An urn has 20 black balls and 10 red balls. I roll a die, note the point on the face that shows up and then pick up that many balls at random from the urn without replacement. Let X denote the number of black balls in my sample and Y the number of red balls. Find the conditional distribution of Y given $X = 3$. [10]

7. Whenever I need a volunteer to invigilate an examination on my behalf, I request one of six available persons at random (and, fortunately, my requests have not been turned down so far!). After eight such occassions, I found out that only four of the six persons have been called so far at different times. What is the probability that one of then has been called four times? [10]

8. A coin, when flipped, lands 'head' up with probability p and 'tail' up with probability $q = 1 - p$. Assume that $0 < p < 1$ and let T_r denote the number of times the coin has to be flipped in order to get r heads in a row.

   (a) Find $E(T_r \mid T_{r-1} = k)$.

   (b) Find a relationship between $E(T_r)$ and $E(T_{r-1})$.

   (c) Using the answer to part (b) or otherwise, find the expected number of flips required to get three heads in a row. <5+5+5> = [15]

INDIAN STATISTICAL INSTITUTE

B-Stat (Hons.)– 1st Year, Semester I

Statistical Methods I

Semestral Examination (December 8, 1999)

TOTAL MARKS : 100 TIME ALLOWED : 3 hours

Answer All Questions.

(1). Consider the following data set on the heights (in feet) and weights (in kilograms) of 8 people :

$$(4.2, 48.5), (3.9, 40.2), (5.1, 51.6), (5.2, 52.4),$$

$$(4.8, 47.6), (4.9, 49.0), (5.0, 50.5), (4.4, 48.6)$$

(a). Compute two appropriate measures of the centre and one appropriate measure of the spread for this bivariate data.

[5 + 5 = 10]

(b). Also, fit an equation of the form *weight* $= \beta \times$ *height* to this data by the method of least squares and by the method of least absolute deviations. (You need to obtain explicit numerical values for β for the two methods).

[7 + 8 = 15]

(2). State and prove Chebyshev's inequality.

[10]

(3). Let M, μ and σ denote the median, the mean and the standard deviation for a univariate data set. Show that

$$\mu - \sigma \leq M \leq \mu + \sigma .$$

[10]

(4). Suppose that we have 4 independent measurements Y_1, Y_2, Y_3 and Y_4 such that

$$E(Y_1) = E(Y_2) = E(Y_3) = E(Y_4) = \mu$$

and

$$Var(Y_1) = \sigma^2\ ,\ Var(Y_2) = 3\sigma^2\ ,\ Var(Y_3) = 5\sigma^2\ ,\ Var(Y_4) = 2\sigma^2\ .$$

Here both μ and σ are unknown. Derive (with full justification) the best linear unbiased estimate for μ based on these 4 measurements.

[15]

(5). A box contains 10 marbles and their colours are unknown to you. One person tells you that there are 4 red and 6 white marbles. Another person tells you that there are 2 red and 8 white marbles. Assume that one of the two persons is telling the truth. 3 marbles will be drawn from the box at random without replacement, and you will be allowed to observe their colours. Use your knowledge of statistical hypotheses and tests to suggest an appropriate procedure to decide who is *most likely* telling the truth in this case. (You need to give proper formulation of the hypotheses and a clear description of your procedure). Also obtain the probabilities for making wrong decisions in such a situation.

[25]

(6). Assignments.

[15]

INDIAN STATISTICAL INSTITUTE
B.STAT.-I YEAR, 1999-2000
SEMESTRAL EXAMINATION - I
SUB : COMP. TECH. & PROGG. I

Date : 10.12.99 Full Marks : 100 Duration : 4 Hours

Note : This paper carries 114 marks. Answer as musch as you can. The maximum marks that you can get is 100.

(1) Write a short note on various errors that may occur while using machines for calculations. [10]

(2) Suppose we want to find an $x>0$ such that $x = 1 + \frac{1}{x} + \frac{1}{x^2}$. Suppose we wish to apply the method of iteration starting with any point in the interval [1,10]. Do you think we will be approaching the solution always ? Justify your answer.

[10]

(3) We want to write a Computer program for finding $a^{1/10}$; $a>1$, using Newton Raphson method. The value for 'a' is read during the execution of the program.

(i) How does one formulate the above problem with the help of Newton Raphson method ?

(ii) The starting point of the process in the Newton Raphson method is taken to be 2 for every $a>1$. Is it necessarily true that the sequence, obtained by using Newton Raphson method, approaches $a^{1/10}$ for every $a>1$? Justify your answer.

[1 + 10]

(4) (a) Define the divided difference of ordern for a function f at $y_0, y_1, \ldots y_n$.

(b) Derive the Newton's formula for interpolation with the help of divided differences.

[3 + 10]

(5) Let f: $R \rightarrow R$ be such that f is known at x_0, x_1, x_2 and x_3 Let $x_i - x_{i-1} = h \ \forall_i = 1,2,3$ when $h>0$. Derive the Newton cotes formula of degree 3 for numerical integration of f in the interval $[x_0, x_3]$.

[15]

(6) Write down the algorithm of Binary search technique. Compute the complexity of the technique.

[8 + 2]

(7) Write down the algorithm of Insertion sort technique. Compute the computational complexity for the worst case as well as average case for this technique.

[6 + 2 + 2]

(8) We are given two one - dimensional arrays A and B, which are sorted in ascending order. Write a Fortran program to merge them into a single sorted array C that contains every item from A and B in ascending order. (you are not allowed to use any sorting algorithm).

(9) Write a Fortran program to approximate the value at a given point in using Lagrange Interpolation formula, based on the given (n+1) points, (x_k, y_k), k=0,1,...n.

[8]

(10) What is a Flow Chart ? Draw a Flow Chart to find out the least number among the four numbers a,b,c and d.

[2 + 6]

(11) (a) Give the exact output of the followings as will be given by a computer. Show clearly the blanks, and mark the column numbers in the output.

(i)

```
        A = -123.45

        L = 123

        C = -0.06

    Write (*,10) A,L,C

    10 Format (2X, F6.1, 2X, I7, T8, F6.3)
```

(ii)

```
    Character Str1 * 3, Str2 * 2

        Str1 = 'THE'

        Str2 = 'TO'

        Write (*,10) Str1, Str2, Str1, Str2

    10 Format (1X, A5, 3X, 'A3', 1X, A1, 'A1', A3, A)
```

[3 X 2]

(b) Show the storage configuration of the variable in COMMON statements of the following.

In main program

```
        Real A(4,3), B, C(15), D

        COMMON A,B,C,D
```

In subprogram Real I(8,2), X,Y,Z

```
        COMMON I, X,Y,Z
```

B-Stat (Hons.)
First Year, Second Semester
1999–2000
Statistical Methods II
Semestral II Examination (April 20, 2000)

Total Marks : 100 *Time Allowed : 3 Hours*

Answer All Questions

(1). Consider the following data on Probability, Calculus and Statistics Scores for 10 students in an undergraduate class :

Roll Number :	01	02	03	04	05	06	07	08	09	10
Probability :	84	93	58	69	98	88	66	73	53	79
Calculus :	87	99	75	81	91	93	78	74	67	54
Statistics :	75	97	63	57	89	95	82	68	71	83

(a). Compute the multiple correlation coefficient of Statistics Score with Probability and Calculus Scores.
(b). Compute the partial correlation coefficient between Statistics and Calculus Scores fixing the Probability Score.

[10 + 10 = 20]

(2). Suppose that the random variable X has p.d.f. $f(x) = cx^{3/2}(1-x)^{5/2}$, where $0 \leq x \leq 1$ and c is a normalizing constant such that $\int_0^1 f(x)dx = 1$. If you have access to a uniform random number generator (on the interval [0, 1]), describe how you would generate the values of the random variable X. Your generation procedure should be as efficient as possible and you must justify it.

[15]

(3). Assume that you have 3 independent observations from a bivariate normal distribution with zero correlation coefficient. Obtain the probability distributions of Spearman's and Kendall's rank correlation coefficients computed from these observations.

[15]

(4). Consider the k-sample problem, where you have independent observations X_{ij}'s with $1 \leq i \leq k$, $1 \leq j \leq n_i$ and X_{ij} having $N(\mu_i, \sigma^2)$ distribution. Show that the between sample sum of squares and the within sample sum of squares are independently distributed, and the within sample sum of squares has a χ^2 distribution with $\sum_1^k n_i - k$ degrees of freedom.

[15]

(5). For a bivariate data set involving two variables X and Y, it is known that the variances of X and Y are 7.28 and 3.49 respectively, and the correlation coefficient between X and Y is 0.83. Determine (with adequate justification) the values of a and b such that $a \geq 0$, $b \geq 0$ and $a + b = 1$ for which $aX + bY$ will have the highest and the lowest variances.

[15]

(6). Suppose that you have independent and identically distributed observations $X_1, X_2, \ldots, X_n$ having a uniform distribution on the interval $[\theta - 1, \theta + 1]$, where $-\infty < \theta < \infty$ is an unknown parameter. Obtain the maximum likelihood estimate of θ based on these observations. Is it unique ? Justify your answers.

[10]

(7). Assignments

[10]

INDIAN STATISTICAL INSTITUTE

SECOND SEMESTRAL EXAMINATION (1999-2000)
B.STAT I Year
CALCULUS - II

Answer five questions.
Maximum Marks : 100

Date 24 April 2000 Duration : 3 hrs.

1. a) Let f be Riemann integrable on [a, b]. Show that, given $\epsilon > 0$, there exists a step function s on [a, b] such that

$$\int_a^b |f(x) - s(x)|\,dx < \epsilon$$

b) Let f be a bounded integrable function, <u>not necessarily continuous,</u> on the interval [a,b] such that $f(t) = G'(t)$ for some differentiable function G for all $t \in [a,b]$.

Show that

$$\int_a^b f(t)\,dt = G(b) - G(a)$$

c) State Lebesgue's theorem on Riemann integrability of a function.

[7+10+3=20]

2. a) Show that.

$$\frac{1}{e^{x^2}}\int_0^x e^{t^2}\,dt \to 0 \quad as \quad x \to \infty.$$

b) Show that

$$\log n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

has a limit as $n \to \infty$.

c) Show that

$$h(u) = \frac{1}{u^2}(u - \log(1+u))$$

is a monotonically decreasing function from -1 to ∞.

[8+6+6=20]

3. a) Suppose $\{f_n\}$ $\{g_n\}$ are defined on an interval [a,b], and

(i) $\left|\sum f_n(x)\right| \leq M$, for some $M > 0$, for all k and all $x \in [a,b]$

(ii) $g_1(x) \geq g_2(x) \geq \ldots\ldots \geq g_n(x) \ldots\ldots$ for each $x \in [a,b]$ and g_n converges uniformly to the zero function on [a,b].

Show that $\Sigma f_n(x)\, g_n(x)$ converges uniformly on [a,b].

b) Show that, if $\{U\alpha\}$ is an open cover of the bounded closed set $E \subseteq \Re$, then there exist $\alpha_1, \ldots\ldots, \alpha_k$ such that

$$\bigcup_{i=1}^{k} U_{\alpha_i} \supseteq E.$$

[You may assume the result to be true when E is a closed and bounded interval]

[12+8=20]

4. a) Let $f_0 : [0, a] \to \Re$ be a continuous function. Show that the sequence of functions defined by

$$f_{n+1}(x) = \int_0^x f_n(t)\, dt, \qquad n = 0, 1, 2, \ldots$$

converges uniformly to the zero function on $[0,a]$

b) Show that $\Gamma(t)$, $t \in (0, \infty)$ is a continuous function of t where $\Gamma(t)$ is the Gamma function.

a) Show that if f is a continuous function on $[a,b]$ such that

$$\int_a^b f(t)\, t^n\, dt = 0, \quad n = 0, 1, 2, \cdots$$

Then $f(t) = 0$ for all $t \in [a,b]$

[7+8+5=20]

a) Show that an absolutely continuous function on a closed bounded interval is of bounded variation.

b) Let

$$F(x) = \int_0^x \log t\, dt, \quad 0 \le x \le 1$$

Decide if F is an absolutely continuous functin.

c) Let f be integrable on $[0,x]$ for every $x > 0$ and suppose that $\int_0^\infty |f(x)|\, dx$ exists. Show that

$$\int_0^\infty f(x) \sin x\, dx$$

exists

[6+8+6= 20]

a) Show that

$$\log\left|\sin \frac{x}{2}\right| = -\log 2 - \sum_{n=1}^{\infty} \frac{\cos nx}{n}, \quad x \neq 0$$

b) Find

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

) Let f be an integrable function on the interval $[a,b]$. Show that

$$\int_a^b f(x) \sin nx\, dx \to 0 \quad as \quad n \to \infty.$$

(You may use the Riemann-Lebesgue Lemma)

[10+7+3=20]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: (1999-2000)

B. Stat.(Hons) I Year

Vectors and Matrices-II

2 May 2000 Maximum Marks: 100 Duration: $3\frac{1}{2}$ hrs.

Note: Answer all questions. The whole paper carries 116 marks but the maximum you can score is 100. Marks allotted to each question are indicated near the right margin. While answering problems, state clearly the theorems you are using.

1. Let $\mathbf{A}$ be an $n \times n$ integral matrix (i.e., the entries are integers). Prove that the following three statements are equivalent: [15]

   (a) $\mathbf{A}^{-1}$ is integral,

   (b) $|\mathbf{A}| = 1$ or -1,

   (c) For all integral $n \times 1$ vectors $\mathbf{b}$, $\mathbf{Ax} = \mathbf{b}$ has an integral solution.

2. (a) Prove that if $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{B})$ and $\mathcal{C}(\mathbf{C}) \subseteq \mathcal{C}(\mathbf{B})$ then $\mathbf{AB}^-\mathbf{C}$ is invariant under different choices of $\mathbf{B}^-$. [5]

   (b) Prove that for any complex $m \times n$ matrix $\mathbf{A}$, $\mathbf{Q} = \mathbf{A}(\mathbf{A}^*\mathbf{A})^-\mathbf{A}^*$ is hermitian and idempotent and show that for all $\mathbf{x} \in \mathbb{C}^m$, $\mathbf{Qx} \in \mathcal{C}(\mathbf{A})$ and $\mathbf{x} - \mathbf{Qx} \perp \mathcal{C}(\mathbf{A})$. [16]

3. (a) Show that the minimal polynomial of a complex square matrix $\mathbf{A}$ divides every annihilating polynomial of $\mathbf{A}$. [5]

   (b) Show that a complex number α is a root of the minimal polynomial of $\mathbf{A}$ iff it is a characteristic root of $\mathbf{A}$. [9]

   (c) Given that the characteristic polynomial of a 3×3 matrix $\mathbf{A}$ is $\lambda^3 - 2\lambda^2$, show that the rank of $\mathbf{A}$ is either 1 or 2. Give an example to show that each of these values is attained. [10]

4. (a) Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the distinct eigenvalues of a real symmetric matrix $\mathbf{A}$ and $\mathbf{P}$ an orthogonal matrix such that $\mathbf{P}^T\mathbf{AP} = \text{diag}(\alpha_1\mathbf{I}_{n_1}, \alpha_2\mathbf{I}_{n_2}, \ldots, \alpha_k\mathbf{I}_{n_k})$. Identify the eigenspaces corresponding to $\alpha_1, \alpha_2, \ldots, \alpha_k$ in terms of $\mathbf{P}$ and deduce that the eigenspaces corresponding to α_i and α_j are orthogonal if $i \neq j$. [8]

   (b) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^T\mathbf{AP} = \mathbf{D}$ where

$$\mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$ [15]

P.T.O

5. (a) Prove that if A is p.d., then $|A| \leq a_{11}a_{22}\cdots a_{nn}$ and that equality holds iff A is diagonal. [9]

(b) Deduce from (a) that if A is a real $n \times n$ matrix such that $|a_{ij}| \leq 1$ for all i and j then $|\det A| \leq n^{n/2}$. Show also that if equality holds then each entry in A is 1 or -1. [8]

6. (a) Show that if A and B are $n \times n$ real symmetric matrices and if A is p.d., then there exists a nonsingular matrix P such that both P^TAP and P^TBP are diagonal. [8]

(b) Show that if A and B are $n \times n$ p.d. matrices and $A - B$ is n.n.d., then $B^{-1} - A^{-1}$ is ~~p.d.~~ nnd [8]

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Indian Statistical Institute

Semester-2 1999-2000

B.Stat.(H) - First Year

Semestral Examination (3rd May, 2000)

Subject: Computational Techniques and Programming II

Maximum Marks: 100 Answer as much as you can. Duration 3.5 hrs.

1. Derive a formula for finding the first derivative of a function in terms of the operator Δ. Use the formula to find an approximate value of the first derivative of the function f at 0, where $f(0) = 15, f(1) = 0, f(2) = -1, f(3) = 0$ and $f(4) = 15$. [5 + 7]

2. Let
$$A = \begin{pmatrix} 2 & a & 0 \\ 1 & 3 & b \\ 0 & 1 & 2 \end{pmatrix}$$
be a non-singular matrix. Let $\vec{c} \in \Re^3$. Show that Gauss-Seidel's procedure for finding solutions to the system of equations $A\vec{x} = \vec{c}$ converges for any $\vec{c}$ if and only if $|a + b| < 6$. [15]

3. State the Newton-Raphson procedure for finding the inverse of a matrix. State the conditions under which the procedure converges and provide the proof of convergence. [2 + 2 + 6]

4. State Jacobi's method for finding the eigenvalues of a real symmetric matrix. Show that the method indeed provides the eigen values. [5 + 10]

5. Let
$$A = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
Let $\vec{x}_0, \vec{y}_0 \in \Re^3$.
Let $\vec{x}_i = A\vec{x}_{i-1} \quad i = 1, 2, 3, \ldots$
Let $\vec{y}_i = A\vec{y}_{i-1} \quad i = 1, 2, 3, \ldots$

   (a) Show that the sequence $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ converges for every $\vec{x}_0 \in \Re^3$. Show also that
   $$\lim_{n\to\infty} \vec{x}_n = \lim_{n\to\infty} \vec{y}_n$$

   (b) Find $\lim_{n\to\infty} \vec{x}_n$. [(7 + 3) + 5]

P.T.O.

6. (a) What is the difference between *defining* and *declaring* a variable?

   (b) Give an example to show how the `typedef` operator can be used in a C program.

   (c) Give an example to illustrate the use of the `break` statement in a C program.

   (d) Let `s`, `n` be two variables containing respectively the name of a student, and his %-age score in a course. Write a printf statement that would print a message like the following:

   `Score of Subal Basu is 83%.`

   (e) What is the final value of `x` below?

   `int x = 3, y = 4; x *= y / 3 + 1;`

   (f) Given the following: `int a[10], *ip, i; ip = a;`

   What is the difference between `*(ip + i)` and `*ip + i`?

   (g) Given the following:

   ```
   #define ADD1(x) x + 1
   int x = 4, y = 8;
   ```

   What is the value of the expression `x * ADD1(y)`? [2 + 1 + 2 + 2 + 2 + 2.5 + 2.5]

7. Explain what the following function does (`s` and `t` are strings). [6]

```
void fn(char *s, char *t)
{ while (*s++ = *t++); }
```

8. Write a function that takes an integer as input parameter and displays its binary representation on the screen. [6]

9. Write a function to compute the correlation coefficient between two sets of numbers. Your function should take three input parameters: `X`, `Y` and `n`, where `X` and `Y` are two arrays of floating point numbers, and `n` is the number of elements in each array. The return value should be the correlation coefficient between `X[0],...,X[n-1]` and `Y[0],...,Y[n-1]`. [9]

10. (a) We want to represent a point in the X-Y plane using a C structure. Define a C structure `point` that stores the x and y coordinates for a given point.

    (b) Now, define a C structure to represent a triangle in this plane (a triangle can be represented by its 3 vertices).

    (c) Write a function `area` that takes a triangle as input parameter and returns its area. (Hint: area of a triangle = $\sqrt{s(s-a)(s-b)(s-c)}$ where a, b, c are the sides of the triangle, and s is its semi-perimeter.) [1 + 1 + 6]

Indian Statistical Institute

Semester-2 1999-2000

B.Stat.(H) - First Year

Semestral Examination (2000)

Subject: Computational Techniques and Programming II

Maximum Marks: 100 Answer as much as you can. Duration ~~4~~ $3\frac{1}{2}$ hrs.

Date: 3.5.00

1. Derive a formula for finding the first derivative of a function in terms of the operator Δ. Use the formula to find an approximate value of the first derivative of the function f at 0, where $f(0) = 15, f(1) = 0, f(2) = -1, f(3) = 0$ and $f(4) = 15$. [5 + 7]

2. Let

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be a non-singular matrix. Let $\vec{c} \in \Re^3$. Show that Gauss-Seidel's procedure for finding solutions to the system of equations $A\vec{x} = \vec{c}$ converges for any $\vec{c}$ if and only if $|a + b| < 6$.

[15]

3. State the Newton-Raphson ~~type~~ procedure for finding the inverse of a matrix. State the conditions under which the procedure converges and provide the proof of convergence.

[2 + 2 + 6]

4. State Jacobi's method for finding the eigenvalues of a real symmetric matrix. Show that the method indeed provides the eigen value~~s~~. [5 + 10]

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$$\lim_{n\to\infty} \vec{x}_n = \lim_{n\to\infty} \vec{y}_n$$

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P.T.O.

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   ```
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INDIAN STATISTICAL INSTITUTE
Second Semestral Examination : 1999-2000
B.Stat.(Hons) - Ist Year
Probability Theory and its Applications II
Maximum Score : 120 pts

Date : 5.5.2000 Time : 4 Hours

Note : This paper carries questions worth a **total** of **140 points**. Answer as much as you can. The **maximum** you can score is **120 points**.

1. Let X be an exponentially distributed random variable with parameter λ. Find the probability density functions of the following random variables:

   (a) $Y = \left(X - \frac{1}{\lambda}\right)^2$, and,

   (b) $Z = \sqrt{X - [X]}$, where $[\cdot]$ denotes 'integer part'.

(10+10)=[20]

2. Let X be a random variable with probability density function given by

$$f(x) = 2|x|e^{-2|x|} \text{ for } -\infty < x < \infty.$$

   (a) Find the moment generating function of X, clearly stating where it exists.

   (b) Use the m.g.f obtained above to find all the moments of X that are finite.

(10+5)=[15]

3. Let (X, Y) be a pair of random variables with joint density given by

$$f(x,y) = \begin{cases} C(y^2 - x^2)e^{-y} & \text{for } -y < x < y,\ 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the constant C.

   (b) Find the probability $P[\,2X \geq Y\,]$.

   (c) Find the covariance between X and Y.

   (d) Find the conditional probabilities $P[\,-2 < X < 2 \mid Y = y\,]$ for $y = 1$ and $y = 3$.

(7+10+10+8)=[35]

P.T.O.

4. Let Y be a strictly positive random variable and let F be *any* probability distribution function on $\mathbb{R}$. Prove then that the function G defined as

$$G(x) = E[F(x\sqrt{Y})] \quad \text{for } x \in \mathbb{R},$$

is a probability distribution function and that, if F is continuous everywhere, then so also is G.

[15]

5. Let X and Y be independent random variables with X distributed uniformly on the interval $(0, 1)$ and Y distributed uniformly on the interval $(0, 2)$. Find the joint distribution of the pair $(X + Y,\ X/Y)$.

[10]

6. Let X and Y be independent and identically distributed exponential random variables with common parameter $\lambda = 1$. Find the conditional distribution of $X \vee Y$ given X and hence find $E[X \vee Y \mid X]$.

[15]

7. X is a random variable with $P[X = n] = \dfrac{2^{n-1}}{3^n}$, $n = 1, 2, \ldots$ and Y is a random variable whose conditional distribution, given $X = n$, is uniform on the interval $(n - 1, n + 1)$.

   (a) Find the distribution function of Y and hence find its probability density function.

   (b) Find $E\left[\dfrac{3Y^2 - 1}{X}\right]$. [Hint: Try conditioning on X.]

   (c) Find the conditional distribution of X given $Y = y$.

(10+10+10)=[30]