1992-93 [224]

INDIAN STATISTICAL INSTITUTE

B.Stat. (Hons.) II Year : 1992-93

Probability Theory its Applications III

Semestral-I Examination

Date : 16.11.1992 Maximum Marks : 100 Time : 3 Hours

Answer as many questions as you can. The whole question paper carries 110 marks. The maximum you can score is 100.

1. Let X_1, X_2, X_3, X_4 be independent random variables having densities $f_1(x), f_2(x), f_3(x), f_4(x)$ respectively where

$$f_i(x) = \frac{\alpha^{r_i}}{\Gamma(r_i)} e^{-\alpha x} x^{r_i - 1} \quad \text{for } x > 0$$
$$= 0 \quad \text{o.w.}$$

$i = 1,2,3,4.$

Let $Y_i = \frac{X_i}{X_1+X_2+X_3+X_4}$ $\quad i = 1,2,3.$

Show that (Y_1, Y_2, Y_3) has the Dirichlet density $D(r_1, r_2, r_3; r_4)$. Find the conditional density of $Y_3 | Y_1 = a, Y_2 = b$.

[7+8]=[15]

2. Let $X_0, X_1, X_2, \ldots$ be iid random variables with $X_i \sim \mathcal{E}(\alpha)$, $\alpha > \infty$. Let N be an integer valued random variable defined by :

$N = n$ if and only if $X_1 \le X_0, X_2 \le X_0, \ldots X_{n-1} \le X_0$ and $X_n > X_0$ $\quad n = 1,2,\ldots$

(a) Show that $P[N=n] = \frac{1}{n(n+1)}$ $\quad n = 1,2,\ldots$

(b) Show that

$$P[N=n, X_N \le x] = \frac{1}{n(n+1)} (1-e^{-\alpha x})^{n+1} \quad n = 1,2,\ldots \quad x > 0$$

Hence find the probability distribution of the random variable X_N.

[5+10]=[15]

3.(a) State and prove the 1st and the 2nd Borel-Cantelli Lemmas.

p.t.o.

3.(b) Let X_n, $n \geq 1$ be iid random variables with $E(|X_1|) < \infty$.
Let for $n \geq 1$

$$Y_n = X_n \quad \text{if } |X_n| \leq n$$
$$\quad = 0 \quad \text{if } |X_n| > n$$

Show that $\frac{1}{n} \sum_{1}^{n} X_i \to 0$ a.s. if and only if

$\frac{1}{n} \sum_{1}^{n} Y_i \to 0$ a.s. [10 +10]=[20]

4. Define convergence in distribution. Show that if $X_n \xrightarrow{P} X$ then $X_n \xrightarrow{\mathcal{L}} X$. [5+ 10]=[15]

5.(a) Find the probability distribution whose characteristic function (c.f) is $\phi(t) = \cos^2 t$.

(b) Show that the c.f $\phi(t)$ of a random variable is real if and only if X is symmetrically distributed around zero.

(c) Show that for any random variable X with c.f. $\phi(t)$ and $E(|X|) < \infty$

$$|\phi(t) - 1 - it\,E(X)| \leq E\left[\,\text{Min}\left(\frac{|t|^2 X^2}{2},\ 2|tX|\right)\right].$$

(d) Let X_n, $n \geq 1$ be iid random variables with $E(|X_1|) < \infty$, $E(X_1) = 0$. Let the c.f. of $\frac{X_1 + \ldots + X_n}{n}$ be denoted by $\psi_n(t)$. Using (c) above, show that $\psi_n(t) \to 1$ as $n \to \infty$ for every t. Indicate how this gives an alternative proof of Weak law of Large Numbers. [5+5+10+15]=[35]

6. Let X_n, $n \geq 1$ be iid random variables with mean α and variance σ^2. Let f be a real-valued function defined on $\mathbb{R}$ such that f has a non-zero derivative at $x = \alpha$. Use central limit theorem to show that

$$\frac{\sqrt{n}\,(f(\bar{X}_n) - f(\alpha))}{\sigma f'(\alpha)} \xrightarrow{\mathcal{L}} N(0,1)$$

where $\bar{X}_n = \frac{1}{n} \sum_{1}^{n} X_i$. [10]

INDIAN STATISTICAL INSTITUTE

B. Stat. (Hons.) II Year : 1992-93

Economics I

Semestral-I Examination

Date : 20.11.1992 Maximum Marks : 100 Time : 3 Hours

This paper carries 120 marks. You may answer as many questions as you can. The maximum you can score is 100.

1. Prove that a competitive equilibrium is Pareto Optimal. Is this result, in your opinion, sufficient to justify a system of competitive capitalism ?

[20]

2. Consider a Cournot duopoly model where the two firms have constant unit costs c and c^* respectively and where the market demand function is given by $P = a-bq$. Let $c<c^*$.
   (a) What is the condition which guarantees that both firms produce positive quantities ?
   (b) Compute the equilibrium prices, quantities and profits when both firms are in the market.

[5+15=20]

3. Show, in terms of an appropriate model, that a durable good monopolist would prefer leasing to outright sales. Under what conditions may leasing be not preferable ?

[20]

4. Establish the relationship between returns to scale and the long run average and marginal costs using the 'function coefficient'.

[20]

5.(a) A consumer behaves perfectly competitively in both labour and product markets. Eight hours of each day constitute the time for sleeping, eating etc. The remaining 16 hours in each day he is free to work for wages or enjoy leisure. His utility function for those 16 hours is

$$U = \sqrt{XY}$$

where X : quantity of goods consumed, Y : no.of leisure hours.

(i) Derive the consumer's labour supply as a function of real wage rate.

(ii) Derive his demand for X.

5.(b) Mr. X, with an income of Rs.5000/- per year, spends 10% of his income on savings for which the income elasticity is 2. If his income rises to Rs.6000/- per year he will spend 12% of his income on savings' - prove or disprove.

[14+6=20]

6.(a) Prove that under perfect competition in the short run a tax on profits is borne entirely by the owners of firms, but in the long run the tax is shifted wholly to consumers in the form of price increases. Explain your reasoning.

(b) Let $f(x)$ be the production function for a firm with a constant-returns-to-scale technology. Suppose each factor x_i is paid its value marginal product $w_i = p.\partial f(x)/\partial x_i$. Show that profits must be zero.

[15+5=20]

INDIAN STATISTICAL INSTITUTE
B.Stat.(Hons.) II Year : 1992-93
Calculus III
Semestral-I Examination

Date : 23.11.1992 Maximum Marks : 100 Time : $3\frac{1}{2}$ Hours

The paper carries 110 marks. Answer all questions. Maximum you can score is 100. Justify your steps precisely.

1. Let a curve C in $\mathbb{R}^3$ be parametrized by

$$\alpha(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, 1\right), \ t \in \mathbb{R}$$

Prove that the angle between $\alpha(t)$ and $\alpha'(t)$ is a constant.

Can you identify the curve as a familiar one ?

[6+4= 10]

2.(a) Let $\underline{r}(x,y,z) = (x,y,z)$ and $r = \|\underline{r}\|$. Let $g : \mathbb{R}^+ \to \mathbb{R}$ be differentiable. Let $S = \mathbb{R}^3 \setminus \{0\}$. Define a scalar field ϕ on S by $\phi(\underline{r}) = g(r)$. Show that

$$\nabla \phi(\underline{r}) = \frac{g'(r)}{r}\underline{r}.$$

(b) Let $f(\underline{r}) = r^p \underline{r}$ on S, where $p \in \mathbb{R}$. Show that f is a gradient on S and find a corresponding potential.

(c) Show that $\phi(\underline{r}) = \frac{1}{r}$ is harmonic on S, i.e. $\nabla^2 \phi = 0$.

[5+4+6 = 15]

3. Find the shortest distance from the point (0,b) on the y-axis to the parabola $x^2 = 4y$. [10]

4. Let R be the region in $\mathbb{R}^2$ enclosed by the curve whose polar equation is given by

$$r = 1 - \cos\theta, \qquad \theta \in [0, 2\pi].$$

Sketch the region in xy- plane and find the integral of the function $f(x,y) = x^2$ over R.

[5+10 = 15]

p.t.o.

5. Let D denote the open unit disc in $\mathbb{R}^2$. Let f be harmonic on D. Assuming continuity of the mixed partial derivatives show that for any $r \in (0,1)$,

$$f(0,0) = \frac{1}{2\pi} \int_0^{2\pi} f(r \cos \theta, r \sin \theta)\, d\theta$$

[Hint : Define the RHS as a function of r, find its derivative by differentiating under the integral sign (justify this) and evaluating it using the Green's Theorem on the disc of radius r. [20]

6. On $\mathbb{R}^2$ or $\mathbb{R}^3$, let r denote the distance from the origin.

Let $n \geq 0$ be an integer. Let $I_n(a,b)$ denote the integral of $f(\underline{r}) = \frac{1}{r^n}$ on S, where

(a) S is the region in $\mathbb{R}^2$ between two concentric circles of radius a and b respectively, $0 < a < b$.

(b) S is the similar volume in $\mathbb{R}^3$ between two concentric spheres.

In both the cases, find values of n for which $I_n(a,b)$ tends to a limit as $a \to 0$.

[7+7+3+3 = 20]

7. Let C be a closed curve which is the boundary of a surface S. For scalar fields f and g on S, prove that

(a) $\int_C (f \nabla g) \cdot d\alpha = \iint_S [(\nabla f) \times (\nabla g)] \cdot n\, dS$

(b) $\int_C (f \nabla g + g \nabla f) \cdot d\alpha = 0.$

[12+8=20]

INDIAN STATISTICAL INSTITUTE
B.Stat.(Hons.) II Year : 1992-93
Statistical Methods III
Semestral-I Examination

Date : 25.11.1992 Maximum Marks : 60 Time : 3 Hours

(a) This paper carries sixty(60) marks of which two(2) marks have been reserved for neatness and clarity of presentation. Another ten(10) marks are reserved for Practical Note Book.

(b) Attempt altogether six(6) questions, taking at least two(2) from each GROUP.

GROUP A

1.(a) Based on a sample $(X_1,\ldots,X_n)$ of size n from $N(\mu,\sigma^2)$ population and considering all homogeneous linear unbiased estimators of the mean μ, show that the sample mean $\bar{X}$ satisfies :

(i) $E(\bar{X}-\mu)^2$ is the least;

(ii) $E|\bar{X}-\mu|$ is the least;

(iii) $P_r[|\bar{X}-\mu| < \varepsilon]$ is the maximum, uniformly in $\varepsilon > 0$.

(b) In the same set-up as in (a), let $h(X_1,\ldots,X_n)$ be any function of X_i's for which $E(h(.)) \equiv 0$. Show that

$$E(h(X_1,\ldots,X_n)(\bar{X}-\mu)) \equiv 0.$$

Hence, establish that

$$E(T(X_1,\ldots,X_n)-\mu)^2 \geq E(\bar{X}-\mu)^2 \;\forall\; \mu, \sigma^2$$

where T(.) is any other unbiased estimator of μ.

[(1+1½+1½)+4=8]

2.(a) Let $f(x|\alpha) \propto \alpha^x$, $x=0,1,2,3$.
Let $\Theta = \{\alpha : 0 < \alpha \leq 1\}$. Find mle of α when $x = 1$.

(b) A coin is tossed once. If it shows H(T), it is tossed further till it produces a T (H). Let N be the total number of tosses in this experiment.

(i) Find the distribution of N.

(ii) Based on the observed value of N, derive the mle of p = Probability of showing H in a single toss. Comment on your result.

p.t.o.

(c) For $f(x|\theta) = 1, \theta - \frac{1}{2} \leq x \leq \theta + \frac{1}{2}$
$= 0$ otherwise

find an mle of θ and also an estimator based on the method of moments, from a sample of size n.

(d) For $f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}$, $-\infty<x<\infty$, $-\infty<\theta<\infty$,
find the mle of θ based on a sample of size n.

[2+(1+$\frac{1}{2}$ + $\frac{1}{2}$)+ 2 + 2 = 8]

3.(a) Using brief but clear arguments, write down the distribution of

$$U = \frac{\bar{X}_n - X_{n+1}}{s_n \sqrt{1+\frac{1}{n}}}.$$

where $X_1,\ldots,X_{n+1}$ i.i.d $N(\mu,\sigma^2)$,

$\bar{X}_n = (X_1+\ldots+X_n)/n$,

and $s_n^2 = (\Sigma X_i^2 - n\bar{X}_n^2)/(n-1)$, $n\geq 2$.

Hence, deduce limits a(.), b(.) such that

$P_r[a(X_1,\ldots,X_n)\leq X_{n+1}\leq b(X_1,\ldots,X_n)] = 1 - \alpha$.

Interpret your result.

(b) In normal samples with non-zero mean, deduce the sampling distribution of $T = \sum_1^n X_i^2$. (You may assume the corresponding result for the zero mean case).

Hence show that $\bar{F}(t)$ in the non-zero case exceeds $\bar{F}(t)$ in the zero-mean case for all $t > 0$.

[(2+1$\frac{1}{2}$ + $\frac{1}{2}$)+(2+2) = 8]

Contd.....

4.(a) Develop an exact level-α test for the hypothesis

$$H_0 : \lambda_1 = \lambda_2 \text{ Vs. } H_1 : \lambda_1 < \lambda_2$$

where $X_i \sim$ Poisson (λ_i), $i = 1,2$; X_1, X_2 are independent. Carry out the test when $x_1 = 4$ and $x_2 = 6$ are the observations on X_1 and X_2 respectively. You may take $\alpha = 0.05$.

(b) Ascertain the overall conclusion based on the following independent p-values: 0.23, 0.08, 0.01, 0.03.

(c) Check the following table :

Item	Price/Unit (in Rs.)	Utility/Unit (in same unit)	Availability (number of units)
A	0.30	0.20	20
B	1.20	0.60	10
C	1.50	0.80	5
D	4.50	1.20	1
E	0.75	0.50	2

You have Rs.9/- to purchase items from the above list. Develop a strategy to maximize the total utility of the items purchased. Derive the solution explicitly.

[(2+1)+(2)+(2+1)=8]

GROUP B

5. Twelve hogs were fed on diet A and fifteen on diet B with the following results for the gain in weight (over the same period) in lbs.

A -> 2.5 3.0 2.8 3.2 3.5 2.2 2.8 3.3 2.6 3.0 2.4 2.7

B -> 4.4 3.7 2.7 3.0 2.6 2.9 3.2 3.4 2.8 3.0 2.6 2.8 3.1 2.9 3.2

(a) Test if the assumption of common variance is tenable or not.

(b) Assuming common variance but different means, find 90% confidence limits for the average increase in gain of B over A. [4+4=8]

Contd.....

6. Consider a sample of 10 observations from

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x>0,\ \theta>0.$$

sample : 10.2 11.5 12.8 10.7 11.2 12.4 11.7 10.9 10.4 11.3
Based on the mle, test $H_0 : \theta = 10$ Vs. $H_1 : \theta > 10$.
Draw the power curve of the test. Is it UMP test ?

[3+4+1= 8]

7.(a) Determine the sample size (n) necessary to estimate the true mean with an error not exceeding 1% of it and with confidence at least 99% when the underlying population is normal with C.V. = 1%.

(b) A chemist repeats protein analysis 20 items with the following results about protein content (x) :

$$\Sigma x_i = 196.40, \quad \Sigma x_i^2 = 1928.65$$

Another chemist does the same analysis on the same sampl with the following results about protein content (denoted by y) :

$$\Sigma y_i = 205.16, \quad \Sigma y_i^2 = 4036.71$$

Examine who is more precise.

[4+4=8]

8. It is desired to test
H_0 : Coefficient of Variation (C.V.) of height distribution Males and Females are equal.

Towards this, data on heights of 10 males (x) and 15 females (y) were collected :

Males $\Sigma x_i = 703.4$ inch $\Sigma x_i^2 = 50,030$ sq.inch

Females $\Sigma y_i = 995.6$ inch $\Sigma y_i^2 = 103820$ sq.inch

It is known that the mean height of males is 68.64 inch. while that of females is 63.87 inch. Test the above hypothesis and also provide 95% confidence limits for the ratio of the C.V.'s.

[4+4=8]

INDIAN STATISTICAL INSTITUTE
B.Stat.(Hons.) II III Year: 1992-93
sociology
Semestral-I Examination

Date : 27.11.1992 Maximum Marks : 100 Time : 3 Hours

Attempt any FIVE questions - at least two from each Group . Questions carry equal marks.

GROUP A

1. Durkhein, Max Weber and Karl Marx are considered as three founders of sociological schools of thought. Choose any two of them and compare their central ideas about society.

2. What is social change ? How does it differ from social mobility ? Describe an approach for studying social change in India.

3. Define Polygyny and Polyandry. How do you account for these practices ? Mention the ways of acquiring a mate in Indian tribal society.

4. Critically review the approach of any sociologist who has studied Indian society.

GROUP B

1. Compare and contrast the importance of qualitative and quantitative methods in sociology with suitable examples.

2. Consider some villages without any all-weather road connecting them with an urban market town. Suppose, the road is made all-weather and buses, trucks, etc. begin to move frequently. An administration is interested to find out the impact of improved road and transport connection on community life.

   (a) How could you define "impact" ?
   (b) How would you like to assess this impact ?

3. What is Panel Study ? How does it differ from a Case Study ? What are the merits and limitations of Panel Study ?

4. Define Content Analysis. What kind of data is usually used for it ? Describe with illustration the various steps that are involved in content analysis.

INDIAN STATISTICAL INSTITUTE
B. Stat. (Hons.) II Year : 1992-93
Economics I
Semestral-I Examination (Backpaper)

Date : 9.1.1993 Maximum Marks : 100 Time : 3 Hours

This paper carries 120 marks. You may answer as many questions as you can. The maximum you can score is 100.

1. Prove that a competitive equilibrium is Pareto Optimal. Is this result, in your opinion, sufficient to justify a system of competitive capitalism ? [20]

2. Consider a Cournot duopoly model where the two firms have constant unit costs c and c^* respectively and where the market demand function is given by $P = a-bq$. Let $c<c^*$.
   (a) What is the condition which guarantees that both firms produce positive quantities ?
   (b) Compute the equilibrium prices, quantities and profits when both firms are in the market.
   [5+15=20]

3. Show, in terms of an appropriate model, that a durable good monopolist would prefer leasing to outright sales. Under what conditions may leasing be not preferable ? [20]

4. Establish the relationship between returns to scale and the long run average and marginal costs using the 'function coefficient'. [20]

5.(a) A consumer behaves perfectly competitively in both labour and product markets. Eight hours of each day constitute the time for sleeping, eating etc. The remaining 16 hours in each day he is free to work for wages or enjoy leisure. His utility function for those 16 hours is

$$U = \sqrt{XY}$$

where X : quantity of goods consumed, Y : no. of leisure hours.
   (i) Derive the consumer's labour supply as a function of <u>real</u> wage rate.
   (ii) Derive his demand for X.

p.t.o.

5.(b) Mr. X, with an income of Rs.5000/- per year, spends 10% of his income on savings for which the income elasticity is 2. If his income rises to Rs.6000/- per year he will spend 12% of his income on savings' – prove or disprove.

[14+6=20]

6.(a) Prove that under perfect competition in the short run a tax on profits is borne entirely by the owners of firms, but in the long run the tax is shifted wholly to consumers in the form of price increases. Explain your reasoning.

(b) Let $f(x)$ be the production function for a firm with a constant-returns-to-scale technology. Suppose each factor x_i is paid its value marginal product $w_i = p.\partial f(x)/\partial x_i$. Show that profits must be zero.

[15+5=20]

INDIAN STATISTICAL INSTITUTE
B.Stat.(Hons.) II Year : 1992-93
Statistical Methods III
Semestral-I Backpaper Examination

Date : 11.1.1993 Maximum Marks : 100 Time : 3 Hours

All questions carry equal marks. Attempt all questions.

1. Explain the method of maximum likelihood for estimation of a real-valued parameter. Suppose X $\sim$ Truncated Binomial with parameter p excluding x = 0. Indicate the nature of the maximum likelihood equation, and a way to solve it.

   Illustrate a situation where the above distribution is feasible.

2. Define the One-parameter exponential family of distributions. Show that for such a distribution, the mle and the estimate based on the method of moments coincide. Is such an estimate unbiased ? Explain with illustrations.

3. If $x_1, \ldots, x_n$ constitute a random sample from $N(\theta_1+\theta_2, \sigma^2)$ population and if $y_1, y_2, \ldots, y_m$ constitute another random sample from $N(\theta_1-\theta_2, \sigma^2)$ population, find the mle of θ_1, θ_2 and σ^2 and examine their unbiasedness. You may assume the populations to be independent.

4. If $X \sim B(n,\pi)$ and $Y \sim B(m,\delta)$ and if X and Y are independent, develop a test for $H_0: \pi = \delta$ vs. $H_1: \pi \neq \delta$. Carry out the test when n = 10, x = 3, m = 8, y = 2 at level $\alpha = 0.05$.

5. What is meant by power of a statistical test of a hypothesis ? Draw the power curve of the two-sided equal-tail test of a normal mean with s.d. 2 and sample size n = 25 when the test is based on the sample mean.

6. There are two independent normal populations with unknown parameters. The following results are available based on randomly drawn observations from the two populations :

Popl^n	sample size	Σx_i	Σx_i^2
1	$n_1 = 20$	150	1200
2	$n_2 = 25$	190	1550

Construct 95% confidence interval for the ratio of the variances of the two populations. Hence test the hypothesis of equality of the variances.

p.t.o.

7. With reference to the data in Problem 6, suppose it is desired to test the hypothesis H_0: mean of Popl^n I is 1.2 times mean of Popl^n II. Under suitable assumptions (to be stated by you), test the above hypothesis at 5% level of significance.

8. Determine the sample size necessary to retain the first kind of error at α and the second kind of error at β while testing for the mean of a normal population against both-sided alternatives, assuming the population s.d. to be 2, in the following cases :

$\alpha = 0.10$, $\beta = 0.10$

$\alpha = 0.05$, $\beta = 0.05$

$\alpha = 0.01$, $\beta = 0.05$.

9. There are k independent normal populations with the same unknown mean m and with unknown and unequal variances $\sigma_1^2, \ldots, \sigma_k^2$ respectively. Given n_i sample observations from the ith population, $1 \leq i \leq k$, derive the mles of m and $\sigma_1^2, \ldots,$ Is the mle of m unbiased ?

10. Derive non-central t-distribution. Illustrate an application of the above in problems of statistical inference.

1992-93 224(b)

INDIAN STATISTICAL INSTITUTE
B.Stat.(Hons.) II Year : 1992-93
Probability Theory and its Applications
Semestral-I Backpaper Examination

Date : 16.1.1993   Maximum Marks : 100   Time : 3 Hours

Answer as many questions as you can. The whole question paper carries 110 marks. The maximum you can score is 100.

1. Let $(X_1, X_2, \ldots X_n)$ be a n-variate symmetric Normal random vector with $E(X_i) = \mu$, $i = 1,2,\ldots n$ and variance-covariance matrix

$$\Sigma = \sigma^2 \begin{pmatrix} 1, & \rho, & \ldots & \rho \\ \rho, & 1, & \ldots & \rho \\ \vdots & & & \\ \rho, & \rho, & \ldots & \rho, 1 \end{pmatrix}.$$

Let $\bar{X} = \frac{1}{n}\sum_1^n X_i$, $S^2 = \sum_1^n (X_i - \bar{X})^2$

By using a suitable orthogonal transformation show that $\bar{X}$ and S^2 are independently distributed. Show that $\frac{S^2}{(1-\rho)\sigma^2}$ follows χ^2_{n-1}. [15]

2. Let X,Y be independent U(0,1) random variables. Let U = Max (X,Y). Find the conditional distribution of $U|X=a$. Find $E(U|X=a)$ for $0<a<1$. [15]

3. (a) Show that $X_n \to X$ almost surely if and only if for for every $\epsilon > 0$

$$P(\limsup_n A_n(\epsilon)) = 0$$

where $A_n(\epsilon) = \{w : |X_n(w)| > \epsilon\}$

Hence conclude that $X_n \to X$ a.s. implies that $X_n \xrightarrow{P} X$.

(b) State and prove the 1st Borel-Cantelli Lemma.

(c) Let X_n, $n \geq 1$ be 0-1 valued random variable defined on the same sample space with

$$P[X_n=1]=p_n=1-P(X_i = 0), \; n = 1,2,\ldots$$

Show that $X_n \to 0$ a.s. if and only if

$$\sum_1^\infty p_n < \infty.$$

[10+8+7]=[25]

p.t.o.

4. Define convergence in distribution. Show that if $X_n \xrightarrow{P} 0$ and Y_n converges in distribution to X (where X_n, Y_n are defined or the same sample space) then $X_n + Y_n \xrightarrow{\mathcal{L}} X$. [5+10] =[15]

5.(a) Show that the characteristic function $\phi(t)$ of any random variable is continuous. Show that the c.f. $\phi(t)$ of any random variable satisfies the following :
for any integer $n \geq 1$, real numbers $t_1, t_2, \ldots t_n$ and complex numbers $z_1, z_2, \ldots, z_n$

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \phi(t_j - t_k) . z_j \bar{z}_k \geq 0.$$

(b) If the c.f. $\phi(t)$ satisfies

$$\int_{-\infty}^{\infty} |\phi(t)| dt < \infty$$

then we have a formula for the density function $f(x)$ corresponding to this $\phi(t)$. Write down this formula. Use this to compute the c.f. for the density

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}.$$

[10+5]=[25]

6. State and prove the central limit theorem for a sequence of iid random variables with finite variance.

[15]

INDIAN STATISTICAL INSTITUTE
B.Stat. (Hons.) II Year : 1992-93
SEMESTRAL II EXAMINATION

Statistical Methods IV

Date: 26.4.1993 Maximum Marks: 100 Time: 3 hours

Note: Attempt ALL questions.

1. A random sample of size n is drawn from a two-way cross-classified (infinite) population with binary response for each of the two attributes A and B.

In usual notations, let

$$\theta = \pi(AB)\,\pi(\bar{A}\bar{B})/\pi(A\bar{B})\,\pi(\bar{A}B).$$

(i) Show that $p_\theta(x|s,t)$ defined by

$$p_\theta(x|s,t) = \text{Prob.}(n(AB) = x|n(A) = s, n(B) = t)$$

has the representation

$$p_\theta(x|s,t) = \binom{s}{x}\binom{n-s}{t-x}\theta^x / \varphi_t(\theta)$$

where

$$\varphi_t(\theta) = \sum_x \binom{s}{x}\binom{n-s}{t-x}\theta^x.$$

(ii) Show that

$$\sum_{x \geq a} p_\theta(x|s,t) \text{ is increasing in } \theta$$

for all $s, t \geq 1$, $a \geq 1$.

(iii) Hence suggest an unbiased test of independence against positive dependence of the attributes A and B.

(iv) For $n = 20$, $n(AB) = 13$, $n(A) = 16$, $n(B) = 16$, carry out the test of independence at 5% level of significance.

(5+5+5+5) = [20]

2. Assume that X and Y have a bivariate normal distribution with

$$\sigma_X = 3, \quad \sigma_Y = 4, \quad \rho = -0.35$$

while the means m_X and m_Y are unknown. It is required to obtain a confidence region for the means m_X and m_Y based on a random sample of size n.

contd..... 2/-

(i) Suggest a 95% confidence region for (m_X, m_Y) based on their mle's.

(ii) What is the area of this region ?

(iii) How is the area affected by the sign of ρ ?

(iv) What happens to the area if $\rho = 0$?

(10+5+5+5) = [25]

3.(a) In a random sample of size n from a bivariate normal population with <u>all</u> parameters unknown, show that the distribution of r, the sample correlation coefficient, depends only on ρ, the population correlation coefficient.

(b) What is z-transformation of r ? Illustrate its usefulness in statistical inference.

(c) Deduce the sampling distribution of r when $\rho = 0$. Hence show that a suitable function of r follows (central) t-distribution when $\rho = 0$. Find the degrees of freedom of the t-statistic. Show how this is used to test H_0: $\rho = 0$.

(5+5+15) = [25]

4. In a random sample of size n = 15 from the U(0,θ) - population (i.e., uniform over (0,θ)), it is found that the largest observation is 3.25. Stating the main results you use (no derivation required):

(i) find an unbiased estimate of θ;

(ii) examine if H_0: θ = 4 is tenable or not at 5% level;

(iii) deduce a 95% confidence interval for θ.

(5+5+5) = [15]

5. In a random sample of size 30 from a Cauchy population with median θ, it is found that the sample median = 25. Work out 95% confidence limits to θ, based on the sample median. You are required to state (without proof) the main results you use.

[15]

:bcc:

INDIAN STATISTICAL INSTITUTE
B.Stat. (Hons.) II Year : 1992-93
SEMESTRAL II EXAMINATION

Elements of Algebraic Structures

Date: 30.4.1993 Maximum Marks: 100 Time: 3 hours

Note: You may answer ALL questions.

1.(a) If a group of order 28 has a normal subgroup of order 4, show that the group is Abelian. [10]

(b) In a group G of order 385 show that the 7-sylow subgroup is normal and lies in the centre of G. [15]

2.(a) Show that a Euclidean domain is a principal ideal domain. [10]

(b) R is a ring such that its only right ideals are (0) and R. Show that either every non-zero element of R is invertible or R has a prime number of elements, product of any two of which is zero. [15]

3.(a) F, K are fields, $F \subseteq K$; a,b are elements of K. I is the set of all polynomials in F[x] having a and b as zeros. Show that I is an ideal in F(x), consisting of multiples of a fixed polynomial in I. [10]

(b) V is a finite dimensional vector space over a field F, and W is a subspace of V. Show that W is finite dimensional and that $\dim_F V/W = \dim_F V - \dim_F W$. ($\dim_F V$ denotes dimension of V over F). [10]

4. Prove or disprove:

(a) R is a ring with multiplicative identity and without zero divisors and $f(x) \in R[x]$. If f(x) has degree n, then the number of elements a in R such that f(a) = 0, is atmost n. [10]

(b) F is a field and $f(x) \in F[x]$. If I is the ideal generated by f(x) in F[x], then F[x]/I is a finite dimensional vector space over F. [10]

(c) The multiplicative group of a field is cyclic. [10]

p.t.o.

5. Given that $x^4 + x + 1$ is an irreducible polynomial over GF(2), construct a field F of 16 elements. Find a primitive element α in F and express all powers of α as vectors over GF(2).

Determine the minimum polynomial of α^5 over GF(2).

[20]

:bcc:

INDIAN STATISTICAL INSTITUTE
B.Stat. (Hons.) I Year : 1992-93
SEMESTRAL II EXAMINATION

Vectors and Matrices II

Date: 30.4.1993 Maximum Marks: 100 Time: $3\frac{1}{2}$ hours

Note: Answer as much as you can. The whole paper carries 116 marks but the maximum you can score is 100. Marks allotted to each question are shown in square brackets near the right margin.

1.(a) Show that the following statements about an $n \times n$ matrix A are equivalent:

(i) $A^2 = A$

(ii) $\mathcal{N}(A) = \mathcal{C}(I - A)$

(iii) $\rho(A) + \rho(I - A) = n$. [11]

Here $\mathcal{N}$, $\mathcal{C}$ and ρ denote null space, column space and rank.

(b) Show that if $A^2 = A$ then A is the projector into some subspace S along some complement T of S. [6]

2.(a) Give with proof a formula for the general solution of a consistent system $Ax = b$ when a matrix G such that $AGA = A$ is known. (Prove the properties of G you need. You may use the result of question 1.(a)) [12]

(b) If $[H : B : d]$ is obtained from $[A : I : b]$ by elementary row operations, where H is in Hermite canonical form, how will you find a g-inverse of A and a general solution of $Ax = b$ (assuming it is consistent) ? Give proof assuming that $H^2 = H$. [7]

3.(a) Assuming the elementary properties of determinant including those on expansion by a row and the determinant of a product, prove that the square matrix A has an inverse if and only if $|A| \neq 0$. [7]

(b) Show that the following statements about an $n \times n$ matrix A over the domain $\mathbb{Z}$ of integers are equivalent:

(i) for each $b \in \mathbb{Z}^n$, the system $Ax = b$ has a unique solution over $\mathbb{R}$ and the solution belongs to $\mathbb{Z}^n$,

contd..... 2/-

(ii) A has an inverse over $\mathbb{Z}$,

(iii) $|A| = 1$ or -1. [9]

4.(a) Prove that every orthonormal set of vectors is linearly independent. [4]

(b) Prove that the following statements about an orthonormal set $B = \{x_1, \ldots, x_k\}$ in a finite-dimensional inner product space V are equivalent:

(i) B is a maximal orthonormal set

(ii) B is a basis of V

(iii) $x = \sum_{i=1}^{k} \langle x, x_i \rangle x_i$ for all $x \in V$

(iv) $\langle x, y \rangle = \sum_{i=1}^{k} \langle x, x_i \rangle \langle x_i, y \rangle$ for all $x, y \in V$

(v) $\|x\|^2 = \sum_{i=1}^{k} |\langle x, x_i \rangle|^2$ for all $x \in V$.

(Hint: prove cyclic implication) [12]

5.(a) Show that the eigenvalues of a real symmetric matrix are all real. [5]

(b) Prove that a real symmetric matrix is orthogonally similar to a diagonal matrix (you have to prove any result of a similar nature which you want to use). [9]

(c) State (without proof) the spectral form of a real symmetric matrix. Give the properties and significance of the scalars and matrices involved. [5]

6. Reduce the following quadratic form to a diagonal form and find its rank and signature. What is its definiteness category ?

$$5x_1^2 + 35x_2^2 + 14x_3^2 - 10x_1x_2 - 8x_1x_3 + 44x_2x_3.$$ [11]

7.(a) State the determinantal criteria for a real symmetric matrix to be (i) p.d., (ii) n.n.d. and (iii) indefinite. [6]

(b) Find the determinant of the real $n \times n$ matrix A with each diagonal entry 1 and each off-diagonal entry α. Show that A is p.d. iff $-\frac{1}{n-1} < \alpha < 1$. [12]

:bcc:

1992-93 | 228

INDIAN STATISTICAL INSTITUTE
B.Stat. (Hons.) II Year : 1992-93
SEMESTRAL II EXAMINATION

Economics and Official Statistics

Date: 3.5.1993 Maximum Marks: 100 Time: $3\frac{1}{2}$ hours

Note: Answer any TWO questions from Questions No.1 to 3 and ALL the rest.

1. Describe fully the Pearl-Reed method of fitting the logistic curve to a time series of equidistant observations on the population of a country. How are the initial estimates of the parameters obtained ? [20]

2. Explain any two of the following methods logical problems which arise in the estimation of demand functions from time series data: (a) multicollinearity, (b) aggregation and (c) identification.
Briefly indicate how the pooling of cross-section and time series data can help overcome these problems. [20]

3. Write short notes on any two of the following:
   (a) Comparative merits of fixed base and chain base systems in the construction of a series of price index numbers;
   (b) Lorenz curve of the lognormal distribution - its equation and properties;
   (c) Elasticity of substitution and the CES production function. [20]

4. The following shows the distribution of income units in Britain for 1953 over size classes of income. Fit a two-parameter lognormal distribution to the data. (You may estimate the parameters by any suitable method.)

contd..... 2/-

income class (£)	% of income units
0 - 99	5.3
100 - 199	15.3
200 - 299	12.6
300 - 399	16.3
400 - 499	17.2
500 - 599	12.3
600 - 699	11.3
700 - 799	2.6
800 - 999	3.7
1000 -1499	2.1
1500 -1999	0.5
2000 -	0.8

[20]

5. The following data are based on a sample survey in Sri Lanka. The data relate to the year 1965. Households were ranked in ascending order of per capita income and then grouped into five quintiles, each having 20% of the total population.

quintile group	average monthly per capita income (Rs.)	% of income spent on food
1	112.8	66
2	197.6	63
3	283.3	59
4	428.0	56
5	1303.2	40

Compute y = average monthly per capita expenditure on food for each group and regress it on x = average monthly per capita income for the same group. Assume that the Engel relation is of the constant elasticity form. Hence estimate the Engel elasticity for food, as a whole.

[25]

6. Practical Record. [15]

:bcc:

INDIAN STATISTICAL INSTITUTE
B.Stat. (Hons.) II Year : 1992 - 93
SEMESTRAL II EXAMINATION
Demography and SQCOR

Date: 5.5.1993 Maximum Marks: 100 Time: 3 hours

Note: Use separate answerscript for each group.
Answer ALL questions.

GROUP - A
(Demography).

Max. Marks: 50

1.(a) Describe briefly any one method for construction of abridged life tables.

(b) Derive the following relationship:

$$\mu_x = \frac{1}{2} [\text{Colog}_e\, p_x + \text{Colog}_e\, p_{x-1}],$$

symbols having their usual significance.

(c) Prove that in a stationary life table population the average age at death for those living at age x is an increasing function of x.

[5+4+5) = [14]

2.(a) Starting from suitable assumptions deduce the logistic law of population growth in the form

$$P_t = \frac{L}{1 + e^{(\beta-t)/\alpha}}.$$

(b) Explain the significance of the parameters L, α and β.

(6+4) = [10]

3.(a) Write notes on

(i) Crude birth rate
(ii) Net reproduction rate
(iii) Child woman ratio
(iv) Parity progression ratio.

(b) The percentage of ever-married women observed in successive five year age groups along with the average number of female children born (on completion of reproductive period) to ever married women in the same age groups are given below.

contd..... 2/-

Age group	Percentage of ever married women	Average no. of female children born (completed fertility) to ever married women
15 - 20	15	4.27
20 - 25	45	4.00
25 - 30	80	3.11
30 - 35	84	2.12
35 - 40	88	1.21
40 - 45	90	.40

Calculate the Gross Reproduction Rate.

(4x2+5) = [13]

4. Describe in detail the component method of population projection. Illustrate your answer with population projections for India 1991 - 2011.

OR

Explain the concept of stable population. Discuss the nature of the roots of the Lotka equation:

$$\int_0^\infty e^{-ra} p(a)\, m(a)\, da = 1.$$

Hence derive the form of the stable age distribution.

[13]

GROUP - B

Max. Marks: 50

(SQC and OR)

1.(a) Explain the meaning of 'Control'. When would you advocate the use of a variable Control Chart ?

(b) What is a single sampling acceptance plan ?

(c) Formulate the Dodge-Romig types of single sampling plans and indicate methods for determining their parameters.

[(4+4) + (4+8)] = [20]

2.(a) What are the different steps required in tackling an OR problem.

(b) Briefly describe the different types of OR models.

(c) Formulate a standard LP problem. What do you understand by: (i) basic feasible solution, (ii) optimal solution and (iii) sensitivity analysis.

(d) Discuss the different characteristics of customers in a queueing system.

[5+6 + (5+9) + 5] = [30]

:bcc:

INDIAN STATISTICAL INSTITUTE
B.Stat. (Hons.) II Year: 1992-93
SEMESTRAL II EXAMINATION
Economics II

Date: 7.5.93 Maximum Marks: 100 Time: 3 Hours

Note: Answer ANY FOUR questions. All questions carry equal marks. Marks allotted to the different parts of a question are shown in [ ].

1.(a) What do you mean by the following terms?
crowding out, accommodating monetary policy.

(b) In terms of a suitable model demonstrate the following alternative situations, in which 'crowding out' is (i) full, (ii) incomplete and (iii) absent. Add a few lines to describe the economic process involved in each case.

(c) Which one of the cases mentioned in (b) is the most likely alternative? Give reasons for your answer. [4+16+5]=[25]

2.(a) Show that in the context of a general macro economic system the classical and the Keynerian models may be viewed as providing alternative equations to close that system algebraically.

(b) Compare some of the basic results of the two models. (15+10)=[25]

3.(a) What is meant by the term 'marginal efficiency of capital (mec)'?

(b) Show why and how the mec will change when the level of investment changes.

(c) Explain the nature of the 'investment schedule', (i.e, the relation between the rate of interest and the level of investment).

(d) Suppose, some improvements in infrastructural facilities (like the construction of the Second Hooghly Bridge) help to reduce the marginal cost, of producing the investment good at all levels. Will the 'investment schedule alter?. If so, how? [4+12+4+5]=[25]

4.(a) Explain the following terms:
capital gain, speculative demand for money

(b) Using Tobin's portfolio balance approach show how the speculative demand for money of a particular type of investor will be inversely related to the rate of interest.

p.t.o.

4.(c) How will the demand function derived in (b) shift when the investor's estimate of risk associated with bond holding increases? [5+15+5] = [25]

5.(a) Develop a suitable model (stating the assumptions clearly) to show that if the government keeps its budget deficit at an unchanged level every period, money supply will keep on rising by a fixed amount every period, if there were no changes in other exogenous variables affecting money supply.

Also find the value of the money multiplier correspond[ing] to the given value of the budget deficit.

(b) Suppose in a period the government budget deficit is zero, but the economy succeeds in achieving a surplus on external trade account. Will it have any effect on the economy's money supply? If so, measure that effect and describe the process through which money supply will be affected. [16+9]

6.(a) What is referred to as the Phillips Curve in the literature? Show how the Phillips Curve may be derived from the relatio[ns] obtaining in the labour market. What are the policy implications of the Phillips Curve ?

(b) What is the criticism raised by Friedman against the original version of the Phillips Curve (i.e., the version mentioned in (a))? Develop a modified version of the Phillips Curve to take care of Friedman's criticism. Assess the policy implications mentioned in (a) in the context of the modified version. [12+13]=[25]

7.(a) Explain the following concepts in the context of an open economy:

(i) internal balance vs. external balance

(ii) fixed exchange rate system vs. flexible exchange rate system.

(b) What kind of difficulties is an economy likely to face if it adopts a fixed exchange rate system? What are the policies available to overcome these difficulties?

(c) Show how the equilibrium level of real income and external balance will be affected under each of the following cases:

(i) government expenditures increase;

(ii) exports increase;

(iii) country's propensity to import increases

(Assume, govt. expenditures, investment expenditures and exports are exogenously determined) [10+4+11] = [25]

1992-93 | 268/358

INDIAN STATISTICAL INSTITUTE
B.Stat.(Hons.) II and III Year : 1992-93
Semestral-II Examination
Differential Equation

Date : 7.5.1993 Maximum Marks : 100 Time : 3 Hours.

Answer any FIVE questions.

1. Solve the differential equations :

(i) $\frac{dy}{dx} - x^2y^2 + 4x^2 = 0$, (ii) $2(y^3+1)+3xy^2 \frac{dy}{dx} = 0$,

(iii) $x \cos x \frac{dy}{dx} + (x \sin x - \cos x) y = 0$,

(iv) $\frac{d^4y}{dx^4} - y = x \cos x$, (v) $\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 4y = e^{-2x}$.

2.(a) Solve the differential equations :

(i) $x^4 \frac{d^4y}{dx^4} + 5x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$

(ii) $(x-1) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0$ (You may use the result that $y=x$ is one solution).

(b) Given that $y = \sec x$ and $y = \tan x$ are solutions of the equation

$$L(y) \equiv \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x) y = 0,$$

find the solution of $L(y) = \sin x$ such that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$.
(You need not find the functions $p(x)$, $q(x)$).

3. (a) Find the solution of the ODE

$$\frac{d^2y}{dx^2} + (x-1)^2 \frac{dy}{dx} - (x-1)y = 0$$

in the form $y(x) = \sum_{n=0}^{\infty} c_n(x-1)^n$ which satisfies $y(0) = 0$, $y'(0) = 1$.

(b) Find two independent solutions of the ODE

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \nu(\nu+1) y = 0$$

in the region $|x|<1$. If ν is a non-negative integer n, show that there is a polynomial solution of degree n.

p.t.o.

4. Show that if the solution of the ODE

$$2x \frac{d^2y}{dx^2} + (3-2x) \frac{dy}{dx} + 2y = 0$$

is expressed in the form $y = x^{\rho} \sum_{n=0}^{\infty} a_n x^n$,

ρ can take two possible values. Find the relation between a_n and a_{n+1}, and show that one solution reduces to a polynomial. Find the first four terms of the other solution.

5.(a)(i) Show that the function $f(x,y) = y^{1/3}$ does not satisfy a Lipschitz condition on the set $S = \{(x,y) : |x| \le 1, 0 \le y$ [illegible] but it satisfies a Lipschitz condition on the set $R = \{(x,y) : |x| \le 1, \frac{1}{2} \le y \le \frac{1}{3}\}$.

(ii) Find two different solutions of the differential equation

$$\frac{dy}{dx} = y^{\alpha} \ (0<\alpha<1), \ y(1) = 0.$$

(b) For the ODE $\frac{dy}{dx} = 3y+1$ with $y(0) = 2$ compute Picard's first four iterates. Also compare these with the exact solution.

6.(a)(i) Find the form of the curve of shortest length joining two points on a plane.

(ii) Find the path of quickest descent of a particle moving under gravity between two points in the same vertical plane.

(b)(i) Find the solution of the PDE

$$(y-u) \frac{\partial u}{\partial x} + (u-x) \frac{\partial u}{\partial y} = x-y$$

with $u = 0$ when $y = 2x$.

(ii) Show that the PDE

$$u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 1$$

has no solution such that

$$u = \frac{1}{2}y \text{ when } x = \frac{1}{4}y^2.$$

7.(a) Transform the PDE

$$x^2 \frac{\partial^2 u}{\partial x^2} = y^2 \frac{\partial^2 u}{\partial y^2}$$

to canonical form. Hence show that its general solution is

$$u = f(xy) + xg\left(\frac{y}{x}\right)$$

where f and g are arbitrary functions.

contd.....

7.(b) Prove that the solution of the PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \text{ with } u(x,0) = f(x),\ \frac{\partial u}{\partial t}(x,0) =$$

g(x) (f, g are given functions) is

$$u = \frac{1}{2}\left[\, f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$