INDIAN STATISTICAL INSTITUTE
B. STAT. (HONS.) II YEAR: 1994-95
CALCULUS III
SEMESTRAL-I BACK PAPER EXAMINATION

Date: 30.12.94 | Maximum Marks: 100 | Time: 3 Hours

Note: Answer all questions.

1. Suppose $0<\delta<\pi$, $f(x)=1$ if $|x|\le\delta$, $f(x)=0$ if $\delta<|x|\le\pi$, and $f(x+2\pi)=f(x)$ for all x.

(a) Compute the Fourier Coefficients of f.

(b) Conclude that

$$\sum_{n=1}^{\infty}\frac{\sin(n\delta)}{n}=\frac{\pi-\delta}{2},\quad (0<\delta<\pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta}=\frac{\pi-\delta}{2}$$

(d) Let $\delta\rightarrow 0$ and prove that

$$\int_0^{\infty}\left(\frac{\sin x}{x}\right)^2 dx=\pi/2.$$ [25]

2. (a) State and prove Taylor's theorem for a function of n variables. [10]

(b) If $f(x,y)=\sqrt{|x|\,|y|}$, show that

$$f_x(x,y)=\frac{1}{2}\frac{\sqrt{|y|}}{\sqrt{|x|}}\quad \text{if } x>0$$

$$=-\frac{1}{2}\frac{\sqrt{|y|}}{\sqrt{|x|}}\quad \text{if } x<0$$

$$f_y(x,y)=\frac{1}{2}\frac{\sqrt{|x|}}{\sqrt{|y|}}\quad \text{if } y>0$$

$$=-\frac{1}{2}\frac{\sqrt{|x|}}{\sqrt{|y|}}\quad \text{if } y<0.$$

Prove that Taylor's expansion of $f(x,y)$ is not valid about the point (0,0). [15]

P.T.O.

3. (a) Evaluate

$$\iiint (a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2)\,dx\,dy\,dz$$

taken throughout the region

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1. \qquad [15]$$

(b) If $|a|<1$, show that

$$\int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\,dx = \pi \sin^{-1} a. \qquad [10]$$

4. (a) By the use of Stokes' theorem, compute the surface integral $\iint_S \text{curl } F.n\, dS$ where $F(x,y,z)=(xz,-y,x^2y)$ and S consists of three faces not in the xz-plane of the tetrahedron bounded by the three coordinate planes and the plane $3x+y+3z=6$. The normal n is the unit normal pointing out of the tetrahedron.

(b) Let $V(t)$ be a solid sphere of radins $t>0$ with centre at a point $\underline{a}$ in $\mathbb{R}^3$ and let $S(t)$ denote the boundary of $V(t)$. Let F be a vector field that is continuously differentiable on $V(t)$. If $|V(t)|$ derotes the volume of $V(t)$ and if n denot the outer normal to $S(t)$, show that

$$\text{div } F(\underline{a}) = \lim_{t\to 0} \frac{1}{|V(t)|} \iint_{S(t)} F.n\,dS \qquad [13]$$

INDIAN STATISTICAL INSTITUTE
B.Stat. (Hons.) II Year : 1994 - 95
SEMESTRAL - I BACKPAPER EXAMINATION
Economics I

Date: 29.12.1994 Maximum Marks: 65 Time: 2 hrs. 15 mints.

Note: The paper carries 70 marks. The maximum you can score is 65.

1. Show that the sum of all price and income elasticities for a single good is zero. (You may assume, for simplicity, that there are only two goods). [9]

2. A firm has the following long-run production function $x = a K^{.5} L^{.25} P^{.25}$, Where x = output, a is a constant and K, L, P are inputs. The prices of K, L, P are 1, 9 and 8 respectively (i) Derive the firm's long-run marginal cost function, (ii) In the short-run when P is fixed, derive an equation in the form P = f(x) showing the optimum quantity of the fixed factor P for the firm to acquire, as a function of the intended output x.

(11+10) = [21]

3. Prove that a competitive equilibrium is Pareto optimal. Is this result sufficient to defend a system of competitive capitalism?

(15+5) = [20]

4. Explain why in a Bertrand equilibrium firms earn competitive profits. How can tacit collusion explain the existence of positive profits even under Bertrand competition ?

(10+10) = [20]

INDIAN STATISTICAL INSTITUTE
B.STAT.(HONS.) II YEAR: 1994-95
PROBABILITY THEORY AND ITS APPLICATIONS III
SEMESTRAL-I BACK PAPER EXAMINATION

Date: 26.12.94 Maximum Marks:100 Time: 3 Hours

Note: 1. Total Marks=100, 2. Answer all questions.

1. Let $Y_1, \ldots, Y_{k+1}$ be independent standard normal variables.

   For $1 \le i \le k$, put $X_i = Y_i^2 / \sum_1^{k+1} Y_i^2$

   Show that $(X_1, \ldots, X_k)$ is Dirichlet. Calculate the parameters of the Dirichlet distribution. [10]

2. (a) Calculate the characterstic functions of the standard Cauchy random Variable.

   (b) Show that the average of n independent standard Cauchy Variables is again a Cauchy Variable. (10+5)=[15]

3. (a) For any characterstic function (t) show that

   $$1-\text{Re}\ (2t) \le 4[1-\text{Re}\ (t)]$$

   (b) Let , $_1$, $_2$,.... be characterstic functions and

   $$\ _k^{k}(t) \rightarrow\ (t) \text{ for all } t. \text{ Show that for all } t,$$

   $$|\ _k(t)| \longrightarrow 1$$

   (c) Show that an infinitely dirisible characteristic function never vanishes [characteristic function of an infinitely divisible random variable is called an infinitely divisible characteristic function]. (6+6+3)=[15]

4. If $\sum_{n=1}^{\infty} E|X_n - X|^2 < \infty$, show that $X_n \xrightarrow{a.e} X$. [10]

5. Let $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y$ and be a realvalued continuous function on R^2. Stating precisely the theorems you use, show that

   $(X_n, Y_n) \longrightarrow\ (X,Y)$. [10]

6. Let X, Y be independent uniform (0,1) variables. Calculate the conditional distribution of X given the maximum of X and Y. [15]

p.t.o.

7. Suppox $F_n \Longrightarrow F$ and for each n, u_n is a median of F_n. Let u be a limit point of $\{u_n, n \geq 1\}$. Show that u is a median of F.

8. Make the following statement précise and then sketch a proof: In a sequence of independent tosses of a coin with chance of heads $p (0 < p < 1)$ in each toss, the pattern "HTTHHT" occurs infinitely many times.

INDIAN STATISTICAL INSTITUTE

B. Stat.(Hons) II Year: 1994-95

STATISTICAL METHODS III

BACK PAPER EXAMINATION

Date: 26.12.1994] Maximum Marks: 100 [Time: 3 hours.

- *The figures in brackets [ ] indicate full marks.*
- *The paper is of 110 marks. The maximum marks you can score is 100.*
- *Question 6 (class presentation) is compulsory.*

1. Table I gives the frequencies of the eggs laid by gallflies on flower heads. The count of the flower heads with no eggs was not recorded.

Table I: Frequency Distribution of the Eggs Laid

No of Eggs	1	2	3	4	5	6	7	8	9
No of Flower heads	22	18	18	11	9	6	3	0	1

   (a) Assuming an appropriate model, estimate the average number of eggs laid on a flower head by the method of maximum likelihood. Calculate the standard error of your estimate. [20]

   (b) Test whether the model you have assumed in part a) for the data in Table I is justified at 0.05 level of significance. [10]

2. (a) A random sample of size n was drawn from an exponential population with mean θ. The detailed data were lost accidentally, but the statistician could remember that Y of the n observations were greater than 10. On the basis of Y, find the MLE of θ. Find an expression for the sample variance of the MLE you have computed and compare it with the variance of the MLE of θ one would have obtained had the complete data been available. Comment on your findings. [10]

   (b) Suppose $X_1, \cdots, X_n$ are i.i.d. Bin$(1, \theta)$. Obtain MLE of θ^2 and show that it is biased. Revise your estimator in order to reduce the bias. What is the bias of the new estimator you have constructed? [15]

3. $X_1, \cdots, X_n$ are random variables with p.d.f. $f(x;\theta) = exp(-(x-\theta)), x > \theta > 0$.

   (a) Derive the likelihood ratio test of size α to test the hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta > \theta_0$.

   (b) Construct a confidence interval of level (1-α) for θ. [12]

4. $X_1, \cdots, X_n$ is a random sample from a population with distribution given by $N(\theta, \sigma^2)$, σ^2 known. Find UMVUE of θ^2. Check if the variance of the UMVUE attains the Cramer-Rao Lower Bound. [15]

Anthropoidea are called ? ________ and ____________.

[1] p.t.o.

5. (a) On the basis of n random observations from a normal distribution with mean θ and variance θ^2, derive the most powerful test of size $\alpha = 0.05$ to test the null hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$. What can you say about the performance of this test if the alternative is $H_1 : \theta > 1$? [10]

(b) The number of defective articles manufactured using two different processes is given in Table II. Do the data confirm the superiority of process 2 to process 1? Clearly state the assumptions you have made. [8]

TABLE II: Number of Defective Articles

Process	Defective	Non-Defective	Total
1	7	3	10
2	5	5	10
Total	12	8	20

6. Class presentation. [10]

INDIAN STATISTICAL INSTITUTE

B. Stat.(Hons) II Year: 1994-95

STATISTICAL METHODS III

SEMESTRAL EXAMINATION

Date: November 25, 1994] Maximum Marks: 100 [Time: $3\frac{1}{2}$ hours.

- *Figures in brackets [ ] indicate full marks.*
- *The paper is of 120 marks. The maximum marks you can score is 100.*
- *Question 7 (class presentation) is compulsory.*

1. Suppose $Y_1, \cdots, Y_n$ are n observations such that $Y_i = \beta X_i + \sigma X_i^{\alpha} \epsilon_i, i = 1, \cdots, n$, where $\alpha, \sigma > 0$ and $X_1, \cdots, X_n$ are fixed positive numbers. Assume that ϵ_i are i.i.d. N(0,1). Discuss how you would compute the MLEs of α, β and σ from a given data.
(Hint: It will be an iterative scheme of estimation. Appropriate use of the residuals in the iterative scheme is needed.) [20]

2. In a factory, the items are produced in lots of 6 items each. Due to some machine defect, defective items were being produced with proportion θ. To estimate the value of θ, the following inspection scheme was implemented:

   From each lot, select an item at random and check if it is defective. If the item is not defective, pass the lot without further inspection. If the item is defective, inspect the whole lot and record the number of defective items present in that lot.

   Suppose Y is the number of defective items in a lot and the value of Y observed for four lots was 3, 1, 2, 1. Formulate an appropriate model for Y and compute UMVUE of θ. [20]

3. (For the questions asked below, detailed proofs need not be given. Short answers can be given if you approach in the right direction)

   (a) $X_1, \cdots, X_n$ are i.i.d. U$(0,\theta)$. Show that $W = \sum a_i X_i / \sum b_i X_i$ and $X_{(n)}$ are independent, where $a_i, b_i > 0, i = 1, \cdots, n$.

   (b) X is distributed either as U(0,1) or U$(\frac{1}{2}, \frac{3}{2})$. Find a sufficient statistic (other than X) for this two member probability family, based on one observation on X.

   (c) Suppose $X_1, \cdots, X_n$ are i.i.d. Poisson(λ) and λ follows a distribution belonging to the conjugate family of Poisson (which is Gamma). Is $\bar{X}$ a Bayes estimator? [15]

4. A manufacturer cuts metal sheets of length a and width b to get the sheets of area A, known. It is found that due to error variation, the exact length and width are not always achieved and that length of the sheets is $a(1 + \epsilon_a)$ and width is $b(1 + \epsilon_b)$, where the random errors (ϵ_a, ϵ_b) are jointly Normal$(0,0,\sigma_a^2,\sigma_b^2,0)$,

Anthropoidea are called ________ and ____________.

[1] p.t.o.

σ_a^2, σ_b^2 unknown. Further, error variation depends on a and b and it is assumed that $\sigma_h^2 = h^\lambda, \lambda > 0, h = a, b$. Show that whatever be λ, cutting square sheets is the optimal method of cutting sheets of area A in the sense of minimizing relative MSE for the area A.

(Relative MSE of T for θ= MSE(T, θ)/θ^2). [15]

5. (a) On the basis of n random observations from the Pareto distribution with p.d.f. $f(x;\theta,\beta) = \frac{\beta\theta^\beta}{x^{\beta+1}}, 0 < \theta \leq x < \infty$, find the most powerful test of size $\alpha, 0 < \alpha < 1$ to test $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$, where θ_0, θ_1 are both known, $\theta_1 \neq \theta_0$. [15]

   (b) The data below are the times between successive failures of air conditioning equipment in a Boeing 727 airplane:

   74, 57, 48, 29, 50, 12, 70, 21, 29, 38, 59, 27, 153, 27, 86, 26.

   Assuming that the data have arisen from an exponential distribution with mean θ, construct an exact $100(1-\alpha)\%$ level confidence interval of equal probability tails for θ. Examine whether the interval you have constructed is the shortest length confidence interval of its level. [10]

6. A chemical compound containing 12.5% of iron was given to two technicians A and B for chemical analysis. A made 15 determinations and B made 10 determinations of the percentage of iron. Their results are given in Table II. Compare the accuracy of the chemical assays made by A and B. Clearly state the assumptions you have made. [15]

Percentage of Iron in a Compound

Determinations by A			Determinations by B	
12.46	12.43	12.77	12.05	12.33
11.89	12.12	12.33	12.22	12.45
12.76	11.85	12.56	12.45	12.39
11.95	12.24	12.65	11.97	12.37
12.77	12.28	12.12	12.21	12.65

7. Class Presentation. [10]

INDIAN STATISTICAL INSTITUTE
B.Stat.(Hons.)II & III Yr:1994-95
Anthropology
Semestral-I Examination

Date : 23.11.1994 Maximum Marks : 100 Time : 3 Hours.

GROUP A

1. State Lamarckism. Describe Palaeontological evidence of organic evolution. Compare biological evolution of man with that of the cultural with respect to six important attributes. [2+4+6]

2. Compare an Asiatic ape with an African on six important biological features. State two important changes that have taken place in pelvis, in skull and in foot of man for the attainment of erect posture. [6+6]

3. Write short notes on any two of the following :
   (i) Marriage and its types
   (ii) Kinship and its terminology
   (iii) Natural selection
   (iv) Neolithic culture. [8+8]

4. Fill up the following gaps :
   (a) Two of the three species of chimpanzee are named as ______ and ______. [1]
   (b) The earliest stone age culture is known as ______ which is characterized by the use of ______ [1]
   (c) Palaeolithic period dates back to ______ years B.P. and characterized by the tools called ______ [1]
   (d) Second and fourth glacial periods are called ______ and ______ respectively. [1]
   (e) Use of bone tools known as ______, characterize ______ people of upper Palaeolithic period. [1]
   (f) Besides Hominoidea, other two superfamilies of Anthropoidea are called ______ and ______. [1]

p.t.o.

4.(g) Theory of mutation was discovered by ____________ where as the theory of genetic drift has been developed by ________. [1]

(h) First use of sledge characterizes the ________ people when the climate was __________ [1]

(i) When the value of morphological facial index of male individual varies from 84.0 - 87.9 the individual is classified as ____________. [1]

(j) Mesorrhine people are with nasal index which varies from ______ to ________ and height of pygmy statured man varies from ______ to ______. [1]

GROUP B

Answer any five questions.

1. Describe Mendel's laws of inheritance with suitable examples. [10]

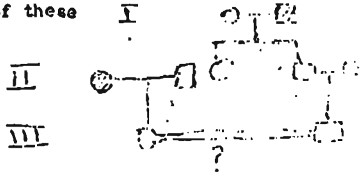
2. What do you mean by ABO blood groups in Man. Describe the inheritance pattern of the same. [10]

3. State with suitable examples the criteria of inheritance of traits due to autosomal dominant and recessive gene. [10]

4.(a) Define Hardy-Weinberg Principle. [4]

(b) Assume a population in which the blood group genes O, A and B are in the proportions 0.5, 0.2 and 0.3. If marriages occur at random, what will be the frequencies of persons with the four blood groups. [6]

5. The pedigree shown below is for the autosomal dominant trait achondroplasia, a rare form of dwarfism. Assuming complete penetrance, the probability that mating between III-1 and III-2 will have an affected child is (i) zero (ii) 1/4, (iii) 1/2 (iv) 1/6 (v) 1/8 (vi) 1/32 and (vii) none of these



[10]

6. Write short notes on any two of the followings:

(a) Chromosome
(b) Mongolism
(c) Random Genetic Drift
(d) Klinefelter's Syndrome.

[10]

INDIAN STATISTICAL INSTITUTE
B.STAT.(HONS.) II YEAR: 1994-95
ECONOMICS-I
SEMESTRAL-I EXAMINATION

Date: 21.11.94 Maximum Marks: 65 Time: 2-15 minutes

Note: This is an open book examination. The paper carries 75 marks. The maximum you can score is 65.

1. A consumer spends his entire income on the consumption of two goods x_1 and x_2. The utility function satisfies a separability condition in the following sense: the marginal utility of $x_1(x_2)$ is independent of $x_2(x_1)$. Marginal utility of x_1 is constant but marginal utility of x_2 falls as x_2 rises. What can you infer about the following:
(a) The slope of the indifference curve;
(b) The curvature of the indifference curve;
(c) The price elasticity of demand for x_1. (5x3=15)

2. A firm producing x from factors K and L can use either of two production processes: Process I : $x = a K^{.25} L^{.75}$, Process II: $x = b K^{.75} L^{.25}$. The price of K is Re.1.00. Calculate the price of L as a function of a and b at which the firm will be indifferent between the two production processes. (10)

3. The expenditure elasticity with respect to price of a good is defined as the proportionate change in total expenditure on the good in response to 1 per cent change in price. Establise a relationship between this elasticity and the own price elasticity of demand of the good. (5)

4. Consider a two-factor production function $Q=f(K,L)$ that exhibits constant returns to scale. Show that if $MP_L > AP_L$, then $MP_K < 0$. (7)

5. Suppose sugar is supplied by a competitive industry. Analyze the effects of the following alternative policies of the government to cope with increased scarcity and higher price of sugar (in terms of supply-demand response in the short and long run) on consumers' and producers' surplus and the black market for sugar:
(a) Price freeze and rationing by queue.
(b) Price freeze and rationing by coupon.
(c) A tax on the consumption of sugar.
(d) A tax on the export of sugar. (4x4=16)

p.t.o.

6. Show that a non-linear tariff yields higher profits than a two part tariff for a monopolist who faces two types of consumers. (10)

7. Two firms with zero fixed costs and marginal costs C_1 and C_2 are located at the two extreme points of a linear city with unit length. Consumers are uniformly distributed along the line. Transportation costs are given by tx^2 where t is a constant and x is the distance travelled. Each firm chooses its price to maximize profits.

   (i) Find out the range of values of C_1, C_2 such that both firms sell positive amounts.

   (ii) If firm 1 is capacity constrained, i.e. cannot sell more than y units, determine the equilibrium prices and quantities. (6x2=12

INDIAN STATISTICAL INSTITUTE

Stat. (Hons.) II Year : 1994-95

SEMESTRAL - I EXAMINATION

Probability Theory and its Applns. III

Date: 17.11.1994 Maximum Marks: 60 Time: 4 hours

Note: Answer ALL questions.

1. Let $Y_1, \ldots, Y_{k+1}$ be independent random variables with $Y_i \sim$ Gamma (α_i). For $1 \leq i \leq k$, define

$$X_i = \frac{Y_i}{\sum_1^{k+1} Y_i}.$$

Show that $(X_1, \ldots, X_k)$ is Dirichlet. Calculate the parameters of the Dirichlet distribution.

[8]

2. Let X_1, X_2, X_3, X_4 be independent standard normal variables. Show that $X_1 X_2 - X_3 X_4$ is a double exponential variable.

[10]

3. Let X be a r.v. with characterstic function $\phi(t)$. For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, suppose $\phi(\lambda) = 1$. Show that

$$P[X = 0, \pm\frac{2\pi}{\lambda}, \pm\frac{4\pi}{\lambda}, \ldots] = 1.$$ [8]

4. Suppose $\{X_n\}$ is a sequence of random variables such that $X_n \xrightarrow{d} m$. Show that $X_n \xrightarrow{P} m$. [8]

5. Suppose f is a real continuous function on [0, 1]. Show that

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1\, dx_2 \ldots dx_n = f\left(\frac{1}{2}\right).$$

8]

6. Let (X, Y) be uniformly distributed in the interior of the triangle with vertices (0, 0), (2, 0) and (1, 1). Calculate the conditional distribution of X given Y.

[8]

p.t.o.

7. Let $X_n \sim B(n, p)$ and Y_n be the integer part of X_n.

Stating precisely the theorems you are using, show that

$$\frac{Y_n}{\sqrt{npq}} \xrightarrow{d} N(0, 1)$$

[10]

(Here it is assumed that $0 < p < 1$)

INDIAN STATISTICAL INSTITUTE

B.Stat.(Hons.) II Year : 1994-95

Calculus III

Periodical Examination

Date : 14.11.1994 Maximum Marks : 100 Time : 3 Hours.

Answer as many questions as you can. The maximum score you can obtain is 100 marks.

1.(a) Show that when $-\pi \le x \le \pi$,

$$\cos kx = \frac{\sin k\pi}{\pi}\left(\frac{1}{k} - \frac{2k\cos x}{k^2-1^2} + \frac{2k\cos 2x}{k^2-2^2} - \ldots\ldots\right),$$

k being non-integral. Deduce that

$$\pi \cot k\pi = \frac{1}{k} + 2k\sum_{n=1}^{\infty}\frac{1}{k^2-n^2}$$

and $$\frac{\pi}{\sin k\pi} = \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n+k} + \frac{1}{n+1-k}\right).$$

[8+3+3]

(b) Show that the following expansion is valid :

$$\log\left|\tan\frac{x}{2}\right| = -2\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{2n-1}, \quad (x \ne k\pi, k$$

an integer). [6]

2.(a) Let $F(y) = \int_0^{\infty} e^{-x^2}\cos 2xy\,dx$, $y \in \mathbb{R}$. Show that F satisfies the differential equation $F'(y) + 2y\,F(y) = 0$ and deduce that $F(y) = \frac{1}{2}\sqrt{\pi}\,e^{-y^2}$.

[10]

(b) Prove that the gamma function $\Gamma(x)$ satisfies

$$\Gamma(2x) = \frac{1}{\sqrt{\pi}}\,2^{2x-1}\,\Gamma(x)\,\Gamma(x+\tfrac{1}{2}), \quad x>0.$$

[10]

3.(a) Let $a_{2n-1} = -\frac{1}{\sqrt{n}}$, $a_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n}$ for $n = 1,2,\ldots$.

Show that $\prod_{n\ge 2}(1+a_n)$ converges but that $\sum_{n=1}^{\infty} a_n$ diverges.

[4+4]

(b) Let p_n denote the n^{th} prime integer. If $s>1$, prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{k=1}^{\infty}\frac{1}{1-p_k^{-s}}.$$

[12]

p.t.o.

4.(a) Evaluate

$$\iiint (1-x-y-z)^{\ell-1}\, x^{m-1}\, y^{n-1}\, z^{p-1}\, dx\,dy\,dz,$$

$(\ell, m, n, p > 0)$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$, $x+y+z=1$. The answer should be expressed in terms of the gamma function.
(Hint : employ the transformation $x+y+z=u$, $x+y=uv$, $x=uvw$)

[15]

(b) Prove that if f is continuous and such that $\int_0^1 f(x)\,dx = 0$, then

$$\int_0^1 f(x)\,dx \int_0^x dy \int_0^y f(z)\,dz < 0.$$

unless $f \equiv 0$.

[10]

(a) Let $R = \{(t_1, t_2) : t_1^2 + t_2^2 \le r\}$ where $r > 0$.
Define $x(t) = t_1 e_1 + t_2 e_2 + \sqrt{r^2 - t_1^2 - t_2^2}\; e_3$

$$y(t) = t_1 e_1 - t_2 e_2 - \sqrt{r^2 - t_1^2 - t_2^2}\; e_3$$

if $(t_1, t_2) \in R$ where e_1, e_2, e_3 are standard unit vectors along the coordinate axes. Let n_1, n_2 be the unit normals to the two surfaces thus defined. Let $f(\underline{X}) = \underline{X}/|\underline{X}|^3$ if $\underline{X} \neq 0$, $f(0) = 0$ where $\underline{X} = (x_1, x_2, x_3) \in \mathbb{R}^3$. If $S_1 = x(R)$ and $S_2 = y(R)$, show that

$$\iint_{S_1} f \cdot n_1 \, dS = \iint_{S_2} f \cdot n_2 \, dS = 2\pi$$

where dS is the element of surface area.

[10]

(b) Let $F = \left(\frac{x}{\rho^2}, \frac{y}{\rho^2}, \frac{z-c}{\rho^2}\right)$, $\rho^2 = x^2 + y^2 + (z-c)^2$, where $0 < c < 1$ and let B be the unit ball

$$\{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$$

Show that

$$\iiint_B \operatorname{div} F \, dV = 2\pi\left[1 + \frac{1}{2c}(1-c^2)\log\frac{1+c}{1-c}\right].$$

[15]