AN ESTIMATE OF INTER-GROUP VARIANCE IN ONE AND TWO-WAY DESIGNS

By K. R. SHAH

Indian Statistical Institute

SUMMARY. A procedure for estimation of inter-block variance has been suggested as an alternative to the usual procedure given by Yates and others. A computational procedure to obtain this estimate is also given. The variance of this estimate is compared with that of the usual estimate when ρ, the ratio of the inter-block variance to the intra-block variance is large, say $\rho > \rho_0$, the estimate considered here has a smaller sampling variance. A table showing these values of ρ_0 is given for all BIB designs listed by Fisher and Yates. Similar procedures for estimating inter-row and inter-column variances are given for designs where the experimental material is symbolically arranged in rows and columns to eliminate heterogeneity in two directions.

1. Introduction

Yates (1939, 1940) suggested the use of inter-block contrasts for estimating treatment effects. In order to combine these estimates with those obtained from intra-block contrasts, one needs knowledge or at least the estimates of variances of both inter and intra-block contrasts. The variance of a normalised intra-block contrast which we shall call intra-block variance is estimated by the error mean square in the ordinary analysis of variance. The variance of a normalised inter-block contrast which we shall call the inter-block variance was estimated by Yates (1939, 1940). Nair (1944) and Rao (1947, 1956) using the block sum of squares adjusted for treatments.

In the case of two-way designs where the experimental material is symbolically arranged in rows and columns, for the same purpose of utilising inter-row and inter-column contrasts for estimating treatment effects, Roy and Shah (1961) gave procedures for estimating inter-row and inter-column variances. The estimates considered there are analogous to the estimate of inter-block variance given by Yates.

In this paper an alternative estimation procedure is put forth for the estimation of inter-group variance, where the experimental material is grouped in blocks as in one-way designs or in rows and columns as in two-way designs. In the case of two-way designs this procedure turns out to be computationally simpler.

2. Preliminaries

Consider an experiment in which v treatments are tested on bk plots, arranged in b blocks of k plots each, such that each plot receives exactly one treatment, and each treatment is applied atmost once in a block and altogether in r blocks. Let y_{iu} denote the yield of the u-th plot in the i-th block. It is assumed that

$$y_{iu} = \mu+\beta_i+\sum_{j=1}^{v} \theta_j m_{jiu}+e_{iu} \qquad \dots \quad (2.1)$$

where μ is the general mean, β_i the effect of the i-th block, θ_j the effect of the j-th treatment, $m_{jiu} = 1$ or 0 according as the u-th plot in the i-th block does or does not receive the j-th treatment and ϵ_{iu} is the experimental error, $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, v$; $u = 1, 2, \ldots, k$. The general mean μ and the treatment effects θ_j are regarded as unknown constant parameters, subject to the restriction that $\Sigma\theta_j = 0$. The block effects β_i's and the experimental errors ϵ_{iu}'s are taken to be independent random variables, each with expectation zero and variances given by

$$V(\beta_i) = \sigma_\beta^2, \; V(\epsilon_{iu}) = \sigma_0^2 \quad \text{for all } (i, u), \qquad \ldots \quad (2.2)$$

we shall write

$$\sigma_1^2 = \sigma_0^2 + k\sigma_\beta^2, \; \rho = \sigma_1^2/\sigma_0^2. \qquad \ldots \quad (2.3)$$

Let $\sum_u m_{jiu} = n_{ji}$, the number of times the j-th treatment occurs on plots in the i-th block. Thus $n_{ji} = 1$ or 0 and $\sum_i n_{ji} = r, \sum_j n_{ji} = k$. The $v \times b$ matrix $N = ((n_{ji}))$ is called the incidence matrix of the design.

Denote by B_i, the total yield of the i-th block, by T_j, the total yield for the j-th treatment, and by G the total yield of all the plots. Thus

$$B_i = \sum_u y_{iu}, \; T_j = \sum_i \sum_u m_{jiu} y_{iu}, \quad G = \sum_i \sum_u y_{iu}; \qquad \ldots \quad (2.4)$$

we shall use the row vectors $B = (B_1, B_2, \ldots, B_b)$ and $T = (T_1, T_2, \ldots, T_v)$, we further define

$$Q = T - \frac{1}{k} BN', \; Q_1 = \frac{1}{k} BN' - \frac{rG}{bk} E_{1v}, \; P = B - \frac{1}{r} TN$$

$$C = rI - \frac{1}{k} NN', \; C_1 = \frac{1}{k} NN' - \frac{r^2}{bk} E_{vv}, \; D = kI - \frac{1}{r} N'N \qquad \ldots \quad (2.5)$$

where E_{mn} stands for a matrix with m rows and n columns with each element unity. It is easy to verify that

$$E(Q) = \theta C, \quad E(Q_1) = \theta C_1, \quad E(P) = 0,$$

$$V(Q) = C\sigma_0^2, \; V(Q_1) = C_1\sigma_1^2, \; V(P) = D\sigma_0^2 + D^2\sigma_\beta^2, \qquad \ldots \quad (2.6)$$

where $\theta = (\theta_1, \theta_2, \ldots, \theta_v)$ is the vector of treatment effects parameters.

As is well known (Rao, 1947) the intra-block equations for estimating treatment differences are

$$Q = \theta C \qquad \ldots \quad (2.7)$$

giving us $\hat{\theta} = QC^*$ as a solution, where we put A^* for the pseudo-inverse (Rao, 1955) of the matrix A. It is also shown by Rao (1947) that the combined inter and intra-block equations are

$$Q + \frac{1}{\rho} Q_1 = \theta \left(C + \frac{1}{\rho} C_1 \right). \qquad \ldots \quad (2.8)$$

Since ρ is usually unknown, to solve these equations one uses an estimate of ρ obtained by taking the ratio of the estimate of σ_1^2 to that of σ_0^2.[(Yates, 1939, 1940); (Rao, 1947)]. An estimate of σ_0^2 is provided by MS_e the error mean square in the ordinary analysis of variance. We now consider the problem of estimating σ_1^2.

3. Estimates of σ_1^2

Estimate of σ_1^2 using sum of squares due to blocks adjusted for treatments was obtained by Yates (1939, 1940) for special designs and was later adopted by Rao (1947) for general incomplete block designs. This can be put in the form

$$E_1 = \frac{kPD^*P' - (v-k)MS_e}{v(r-1)}. \quad \ldots \quad (3.1)$$

Another estimate is
$$E_2 = \frac{(B-\hat{\theta}N)(I-\frac{1}{b}E_{bb})(B-\hat{\theta}N)' - \alpha MS_e}{k(b-1)} \quad \ldots \quad (3.2)$$

where α is the trace of the matrix C^*NN', which can be shown to be equal to $k(v-1)$ $(1-E)/E$ where E is the efficiency factor of the design (Kempthorne, 1956; Roy, 1958).

Under the additional assumption that the random variables β_i's and ϵ_{iu}'s are jointly normally distributed we shall derive the variances of E_1 and E_2. Now, we state the following lemma used in deriving $V(E_1)$ and $V(E_2)$.

Lemma : *If $x = (x_1, x_2, \ldots, x_n)$ has multivariate normal distribution with mean 0 and dispersion matrix Λ and if Δ is a matrix such that there exists an orthogonal matrix P satisfying*

$$P\Lambda P' = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad P\,\Delta\,P' = \text{diag}\,(\delta_1, \delta_2, \ldots, \delta_n)$$

then
$$E(x\Delta x') = \sum_i \delta_i\lambda_i$$
$$V(x\Delta x') = 2\sum_i \delta_i^2\lambda_i. \quad \ldots \quad (3.3)$$

The proof is immediate if we transform from x to $Z = xP'$.

First, we observe that MS_e is distributed independently of P, B and $\hat{\theta}$. We also note that B and $\hat{\theta}$ are independently distributed. Hence $V(B-\hat{\theta}N) = kI\sigma_1^2 + N'C^*N\sigma_0^2$.

Using the Lemma we get

$$V(E_1) = \frac{2}{v^2(r-1)^2}\left[k^2\sum_{i=1}^{b-1}(x_i\sigma_\beta^2+\sigma_0^2)^2 + \frac{1}{e}(v-k)^2\sigma_0^4\right] \quad \ldots \quad (3.4)$$

and
$$V(E_2) = \frac{2}{k^2(b-1)^2}\left[k^4\sum_{i=1}^{b-1}(\sigma_\beta^2+\sigma_0^2/x_i)^2 + \frac{1}{e}(\alpha^2\sigma_0^4)\right]$$

where $x_1, x_2, \ldots, x_{b-1}$ are the non-zero latent roots of D (when all treatment contrasts are estimable, i.e., when the design is connected, the matrix D has exactly one latent root zero. Here, we shall consider only connected designs) and e stands for the number of error degrees of freedom in the intra-block analysis.

Since $\bar{x} = \Sigma x_i/(b-1) = v(r-1)/(b-1)$, it is easy to see that

$$V(E_1)-V(E_2) = \frac{2k^2}{(b-1)^2\bar{x}^2}\Big[\sum_{i=1}\Big\{(x_i\sigma_\beta^2+\sigma_0^2)^2-\bar{x}^2(\sigma_\beta^2+\sigma_0^2/x_i)^2\Big\}$$

$$+\Big\{\frac{(v-k)^2}{k^2}-\frac{\alpha^2\bar{x}^2}{k^4}\Big\}\frac{\sigma_0^4}{e}\Big]. \quad \ldots \quad (3.5)$$

On simplification this gives

$$V(E_1)-V(E_2) = \frac{2k^2}{(b-1)^2\bar{x}^2}\Big[\sigma_\beta^4\Big\{\sum_i x_i^2-(b-1)\bar{x}^2\Big\}+2\sigma_\beta^2\sigma_0^2\Big\{\Sigma x_i-\bar{x}^2\Sigma\frac{1}{x}\Big\}$$

$$+\sigma_0^4\Big\{b-1-\bar{x}^2\Sigma\frac{1}{x_i^2}+\frac{(v-k)^2}{k^2e}-\frac{\alpha^2\bar{x}^2}{k^4e}\Big\}\Big]. \quad \ldots \quad (3.6)$$

In this expression σ_β^4 has positive coefficient while σ_0^4 and $\sigma_0^2\sigma_\beta^2$ have negative coefficients. Hence $V(E_1)-V(E_2)$ is positive for somewhat large values of $\sigma_\beta^2/\sigma_0^2$ or equivalently for somewhat large values of $\rho = \sigma_1^2/\sigma_0^2$. By ρ_0, we denote the value of ρ such that for $\rho > \rho_0$, $V(E_1)-V(E_2)$ is positive i.e., the usual estimate has larger variance than the new estimate. The values of ρ_0 are given below in the table for all BIB designs listed by Fisher and Yates (1957). For each of these designs ρ_0 happens to lie between 4 and 5.

VALUES OF ρ_0 FOR ALL BIBD BY FISHER AND YATES
(Other than Symmetrical Designs)

k	r	b	v	ρ_0	k	r	b	v	ρ_0
3	6	10	5	4.3429	5	10	18	9	4.1795
3	5	10	6	4.3828	5	9	18	10	4.1875
3	4	12	9	4.4179	5	7	21	15	4.2225
3	6	26	13	4.6270	5	6	30	25	4.1900
					5	10	82	41	4.3435
3	9	30	10	4.6801					
					6	8	12	9	4.0822
3	7	35	15	4.7080	6	9	15	10	4.1062
3	9	57	19	4.8091	6	9	24	16	4.1534
3	10	70	21	4.8463	6	8	28	21	4.1560
					6	9	69	46	4.2142
4	7	14	8	4.2520	6	10	85	51	4.2400
4	10	15	6	4.2044	7	10	30	21	4.1304
4	6	15	10	4.2640	7	9	36	28	4.1290
4	8	18	9	4.3016	7	8	56	49	4.1157
4	5	20	16	4.2584	8	10	45	36	4.1040
4	8	50	25	4.4464	8	9	72	64	4.0970
4	9	63	28	4.4832	9	10	90	81	4.0672

It may be noted that the two estimates are identical in the case of $x_1 = x_2 = \cdots = x_{b-1}$ i.e. when the design is a dual of a BIB design. Hence in particular for symmetrical BIB designs the two estimates are identical.

Other properties of E_2 will be discussed in a subsequent communication. We derive here a simple computational procedure to obtain E_2 after the intra-block analysis is performed. It is easy to see that

$$(B-\hat{\theta}N)\left(I-\frac{1}{b}E_{bb}\right)(B-\hat{\theta}N)' = kSS_B^*-BN'\hat{\theta}'+\hat{\theta}NN'\hat{\theta}' \qquad \ldots \quad (3.7)$$

where SS_B^* denotes unadjusted block sum of squares. On simplification this reduces to $k[SS_B^*-(T-r\hat{\theta})\hat{\theta}']$. Once, the intra-block estimates $\hat{\theta}$ are obtained, this is easily computed.

4. Designs with two-way elimination of heterogeneity

A similar problem for designs with two-way elimination of heterogeneity would necessitate estimates of three variances. Here, if the experimental material is arranged into a two-way array of rows and columns one has to estimate inter-row variance, inter-column variance in addition to the usual error variance which one may call interaction variance. As usual, the estimate of interaction variance is provided by MS, the error mean square in the ordinary analysis of variance. A method of obtaining estimates of inter-row and inter-column variances was given by Roy and Shah (1961). The estimates given there are analogous to the estimate of inter-block variance given by E_1 and to compute them one has to carry out two additional one-way analysis, one with rows and treatments ignoring columns and the other with columns and treatments ignoring rows. In this case, it would be simpler to compute the estimates of inter-row and inter-column variances corresponding to E_2 in the one-way case.

If σ_R^2 denotes the variance of a normalised inter-row contrast an estimate of σ_R^2 is given by

$$\hat{\sigma}_R^2 = \frac{(R-tM)\left(I-\frac{1}{m}E_{mm}\right)(R-tM)'-tr(K^*MM')MS_e}{n(m-1)} \qquad \ldots \quad (4.1)$$

where all the quantities defined in the R.H.S. are as defined by Roy and Shah (1961). Using the same notations, the estimate of σ_c^2, the variance of a normalised inter-column contrast is given by

$$\hat{\sigma}_C^2 = \frac{(C-tN)\left(I-\frac{1}{n}E_{nn}\right)(C-tN)'-tr(K^*NN')MS_e}{m(n-1)}. \qquad \ldots \quad (4.2)$$

After the interaction analysis is performed $\hat{\sigma}_R^2$ and $\hat{\sigma}_c^2$ can be easily calculated.

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