INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination – Semester I : 2013-2014

M.Stat. I Year

Measure Theoretic Probability

Date : 26.08.13 | Maximum Score : 40 | Time : $2\frac{1}{2}$ Hours

Note : This paper carries questions worth a **total** of **50 MARKS**. Answer as much as you can. The **MAXIMUM** you can score is **40**.

1. (a) Define what is meant by a semifield on a non-empty set Ω.
(b) Show that if $\mathcal{S}$ is a semifield, then the class of sets obtained by taking of all possible finite disjoint unions of sets in $\mathcal{S}$ is closed under finite intersections and complementations.

(3+6)=[**9**]

2. Let $\mathcal{C}$ be a class of subsets of a non-empty set Ω. Let $\mathcal{A}$ be the collection of all those sets $A \subset \Omega$ that belong to the σ-field generated by *some* countable subclass of $\mathcal{C}$, that is, $\mathcal{A} = \bigcup\{\sigma(\mathcal{D}) : \mathcal{D} \text{ countable}, \mathcal{D} \subset \mathcal{C}\}$.
(a) Show that $\mathcal{A}$ is a σ-field.
(b) Deduce that $\mathcal{A} = \sigma(\mathcal{C})$.

(6+4)=[**10**]

3. (a) Let $\mathcal{A}$ be a σ-field on a non-empty set Ω. Suppose f is a real-valued function on Ω such that for every real number a, the set $\{\omega \in \Omega : f(\omega) < a\}$ belongs to the σ-field $\mathcal{A}$. Show then that, for every borel set $B \subset \mathbb{R}$, the set $\{w \in \Omega : f(\omega) \in B\}$ belongs to $\mathcal{A}$.
(b) Use (a) to deduce that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then, for every borel set $B \subset \mathbb{R}$, the set $\{w \in \Omega : f(\omega) \in B\}$ is a borel subset of $\mathbb{R}$.

(6+4)=[**10**]

4. (a) Define what is meant by a measure on a field.
(b) Let $\mathcal{F}$ be a field on a non-empty set Ω and μ a non-negative finitely additive set-function on $\mathcal{F}$. Suppose μ satisfies the property: $A_n \in \mathcal{F},\ n \geq 1,\ A_n \downarrow \emptyset \Longrightarrow \mu(A_n) \to 0$. Show that μ is a measure on $\mathcal{F}$.

(3+6)=[**9**]

5. (a) Let μ be a measure on the Borel σ-field $\mathcal{B}$ on $\mathbb{R}$ satisfying $\mu(x + B) = \mu(B)$ for every $B \in \mathcal{B}$ and $x \in \mathbb{R}$. Suppose $\mu((0,1]) = c < \infty$.
(a) Show that for any two integers $a < b$, $\mu((a,b]) = c \cdot (b-a)$.
(b) Show that for any integer $n \geq 1$, $\mu((0,\frac{1}{n}]) = c/n$ and hence conclude that for any two rationals $a < b$, $\mu((a,b]) = c \cdot (b-a)$.
(c) By using Caretheodory Extension Theorem suitably (or otherwise), conclude that $\mu(B) = c \cdot \lambda(B)$ for all $B \in \mathcal{B}$, where λ denotes the Lebesgue measure.

(4+5+3)=[**12**]

Indian Statistical Institute
Midterm Examination
Large Sample Statistical Methods
M.Stat. First Year
First Semester (2013-2014)

Date: 30.08.13 Maximum Marks: 40 Duration: 2 hours and 45 minutes

This paper contains questions worth a total of 44 marks. Answer as many questions as you can. The maximum you can score is 40.

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution F with density f, defined as $f(x) = \frac{dF(x)}{dx}$. Here both F and f are unknown. Fix a point $x_0 \in R$. Supopose you construct the naive estimator of $f(x_0)$ defined as $\hat{f}(x_0) = \frac{F_n(x_0+h_n)-F_n(x_0-h_n)}{2h_n}$, where $h_n > 0$. Here $F_n(.)$ is the empirical distribution function. Can you give a simple sufficient condition on h_n that ensures consistency of $\hat{f}(x_0)$ as an estimator of $f(x_0)$ as $n \to \infty$? Justify your asnswer. [5]

2. Let $X_1, \ldots, X_n$ be a random sample from $N(0,1)$. Derive the asymptotic distribution of the fourth central sample moment after appropriate centering and scaling. [6]

3. Suppose $X_1, \ldots, X_n$ are iid from a distribution F given by $F(x) = xI_{(0 \leq x < \frac{1}{2})} + (2x - \frac{1}{2})I_{(\frac{1}{2} \leq x \leq \frac{3}{4})}$. Let $\hat{\eta}_{\frac{1}{2},n}$ denote the smallest sample median based on $X_1, \ldots, X_n$. For $n = 10000$, how will you approximate the probability that the random variable $\hat{\eta}_{\frac{1}{2},n}$ is larger than 0.51 ? Justify your answer. [5]

4. Show that a sequence of random variables $\{X_n\}$ is stochastically bounded if and only if for every real sequence $\{k_n\}$ tending to ∞ as $n \to \infty$, $P(|X_n| > k_n) \to 0$ as $n \to \infty$. [5]

5. Suppose $X_n \sim N(0,1)$ and $Y_n = X_n I(|X_n| \leq n)$ for all $n \geq 1$. Does $\text{Sin}(X_n) - \text{Sin}(Y_n)$ converge in probability ? Prove your answer. [4]

6. Suppose $F_n(a_n x + b_n) \to G(x)\ \forall x \in C_G$, where $\{F_n\}$ and G are distribution functions on R and $\{a_n\}$ and $\{b_n\}$ are real sequences. Show that $a_n > 0$ for all sufficiently large n. [4]

7. Let P and Q be two probability measures on $(\Omega, \mathcal{A})$. Then show that statement (a) below implies statement (b).

   (a) For any $A \in \mathcal{A}$ with $P(A) = 0$, one has $Q(A) = 0$.

   (b) For each $\epsilon > 0$, there exists $h_\epsilon > 0$ such that for any $A \in \mathcal{A}$ with $P(A) < h_\epsilon$, one has $Q(A) < \epsilon$. [5]

8. Suppose $\{X_n\}$ is a sequence of random variables on a common probability space. Show that one can find a sequence of *strictly positive* real constants $\{b_n\}$ such that $\{b_n X_n\}$ converges to zero almost surely. [5]

9. Suppose $X_n \sim \text{Beta}(\frac{1}{n}, \frac{1}{n})$, for $n \geq 1$. Does $\{X_n\}$ converge in distribution ? Prove your answer. [5]

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination: (2013–2014)

M. Stat First Year

Statistical Inference I

Date: 02/09/13 Marks: ..40.. Duration: 2 hours.

Attempt all questions

1. Suppose that for every $m = 1, 2, \ldots$,

$$f_{X_1,\ldots,X_m}(x_1, \ldots, x_m) = \frac{1}{10^m} \sum_{i=0}^{10} a_i i^x (10 - i)^{m-x}$$

if all $x_i \in \{0, 1\}$, and 0 otherwise, where $x = \sum_{j=1}^m x_i$. The numbers a_i are non-negative and add to 1. Let $\Theta = \lim_{n\to\infty} \sum_{i=1}^n X_i/n$. Obtain the prior distribution of Θ. **Marks**: 10

2. Suppose that the posterior distribution of Θ is denoted as $F_{\Theta|X}(\theta)$. Define $L(\theta, \delta(x))$ for a nonrandomized rule as

$$L(\theta, \delta(x)) = a(\theta - \delta)I\{\theta \geq \delta\} + b(\delta - \theta)I\{\theta < \delta\},$$

where $a > 0$, $b > 0$ are known constants and $I(\cdot)$ is the indicator function.

1. Find δ that minimizes the posterior risk $r(\delta|x)$.
2. "If $a = b$ then δ is the posterior mean"-true or false? Justify.

Marks: 6+4=10

3. Let the parameter space and the action space be $\Omega = (0, \infty)$ and $\aleph = [0, \infty)$, respectively. Let the loss be $L(\theta, a) = (\theta - a)^2$. Let $X \sim U(0, \theta)$, and let λ be the $U(0, c)$ distribution for $c > 0$. Then

(i) Obtain the formal Bayes rules for this problem.

(ii) Are all the Bayes rules admissible? Justify.

Marks: 6+4=10

4. Suppose that $X \sim N(\mu, \sigma^2)$. Let $\theta = (\mu, \sigma)$ be the quantity of interest. Let the loss be $L(\theta, a) = (\mu - a)^2$. Prove that $\delta(x) = x$ is admissible. (*If you are using any known results, state them clearly; but proofs of the known results are not required*). **Marks**: 10

INDIAN STATISTICAL INSTITUTE

M. Stat (1st Year) B – Stream,2013-14

Midsemestral Examination

Subject: Applied Stochastic Process

Date: 04 / 09 / 2013 Full Marks: 40

Duration: 1 hr 15 mins

<u>The paper contains questions of 45 marks. Attempt all questions.</u>
<u>Maximum marks you can score is 40.</u>

1. a) Discuss the genotypes and their phenotypes for the colourblindness problem.

   b) Let a colourblind man marries a non – colourblind woman whose father was colourblind. Find the probability of their son to be colourblind and the probability of their daughter to be colourblind.

   [3 + 5]

2. Let there be two types of particles A and B. At time 0, let there be only one particle and that is of type A. At the end of each minute each particle A is replaced by 2 A particles or 1 A particle and 1 B particle or 2 B particles with probabilities 1/4, 2/3 and 1/12 respectively. Each particle B lives for 1 minute and then vanishes

P. T. O.

without reproducing. Assuming independence, find the expected number of B particles those have appeared or died by the end of 10 minutes. Find the probability of eventual extinction of the process.

[6 + 6]

3. Let A, a be two alleles for some X – linked gene. At each generation, 100μ % A mutate to a and 100v % a mutate to A. $0 < \mu, v < 1$. At generation 0, proportions in females AA: Aa: aa = r_0: $2s_0$: t_0 and proportions in males A : a = p_0 : q_0. Find the limiting proportions of the genotypes in males and in females through generations.

[10]

4. Let $\{X_n\}_{n\geq 0}$ be a branching process with $X_0 = 1$ and at each generation n, the offspring distribution depends on n. Let it be Poisson (λ_n).

   a) If $\lambda_n = 1 - 1/(n+1)$, find the probability of extinction of the process.

   b) If $\lambda_n = (1 + 1/(n+1))^2$, is the probability of extinction 1 or less than 1? Justify your answer.

[5 + 10]

INDIAN STATISTICAL INSTITUTE
MID-SEMESTRAL EXAMINATION 2013-2014

M.STAT 1st year. Design of Experiments

Date 4.9.13 Total marks 20 Duration: One hour

Answer all questions.

Keep your answers brief and to the point.
Marks will be deducted for rambling answers.

1. Let d be a connected block design. When is d said to be (variance) balanced? Prove that if d is balanced then all the non-zero eigen values of the C matrix of d are equal. [1+4=4]

2. Consider a BIB design with parameters v, b, r, k, λ. Write down the conditions that must be satisfied by the parameters. (No proof needed). [3]

3. Does a BIB design with parameters $v = b = 22, r = k = 7, \lambda = 2$ exist? Justify your answer. [3]

4. Consider a BIB (v, b, r, k) design. Suppose each block of this design is augmented by all the v symbols applied once each, so that the resulting design still has b blocks, but now each is of size $k + v$. (a) Will this design be balanced? (b) Will the design in (b) above be orthogonal? Justify. [3]

5. Write down the initial blocks for constructing a BIB design with $v = 9, b = 12, r = 4, k = 3$. Without constructing the entire design show that if these initial blocks are developed, then the resulting design will be the required BIBD. [3]

6. Write down the steps to indicate how you can construct a Hadamard matrix of order 12. Actual construction not needed. (Given that 2 is a primitive element of GF(11)). [4]

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination : 2013-14

Course Name : M.Stat. 1st Year

Subject Name : Sample Survey and Design of Experiments Date : 06/09/2013

Note: Use separate answer sheets for two groups.

Group – Sample Survey. (Total Marks = 20) Duration : 1 *hrs.*

Answer any two.

1. (a) Prove that for any given sampling design, there exists a sampling scheme that realizes this design.

   (b) State and prove Godambe and Joshi (1965)'s theorem on the existence/non-existence of the uniformly minimum variance estimator for Y within the class of all unbiased estimators.

   $(6 + 4 = 10)$

2. (a) Given any design p, prove that

   (i) $\sum_{i=1}^{N} \pi_i = E(\nu(s))$, and

   (ii) $\sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \pi_{ij} = Var(\nu(s)) + E(\nu(s))(E(\nu(s)) - 1)$,

   where π_i and π_{ij}s are the first and second order inclusion probabilities and $\nu(s)$ is the effective sample size of a sample s.

   (b) For simple random sampling with replacement of n draws from a population of size N, find the expected number distinct units apprearing in a sample.

   $(6 + 4 = 10)$

3. (a) Prove that the neccesary and sufficient condition for the existence of an unbiased estimator for a survey population total Y is that $\pi_i > 0 \ \forall i \in U$.

   (b) Derive the variance and an unbiased estimator for variance of Horvitz-Thompson estimator for Y, stating clearly the required conditions.

   $(4 + 6 = 10)$

Indian Statistical Institute
Semestral Examination First Semester (2013-2014)
M.Stat. First Year
Large Sample Statistical Methods

Maximum Marks: 60 Date : 11.11.2013 Duration :- 4 hours

This question paper carries 66 marks. Answer as many questions as you can. The maximum you can score in this exam is 60.

1. Let $X_1, \ldots, X_n$ be iid with density $f(x) = \alpha x^{-(\alpha+1)} I_{\{x>1\}}$, where $\alpha > 0$. Find the values of α such that $\frac{X_n}{n}$ converges to 0 almost surely as $n \to \infty$. [5]

2. Suppose $X_1, X_2, \ldots$ are independent random variables such that $X_n = \sqrt{n}$ with probability $\frac{1}{2}$ and $X_n = -\sqrt{n}$ with probability $\frac{1}{2}$, for $n = 1, 2, \ldots$. Does $\bar{X}_n$ converge in distribution as $n \to \infty$ where $\bar{X}_n = \sum_{i=1}^{n} X_i/n$? Justify your answer. [5]

3. Let $X_1, \ldots, X_n$ be iid with a Uniform distribution on $[0, \theta]$ where $\theta > 0$ is unknown. Can you suggest a function $h(.)$ of $X_1, \ldots, X_n$ and real constants a_n and b_n (possibly dependent on θ) such that $a_n(h(X_1, \ldots, X_n) - b_n)$ converges in distribution to a non-degenerate random variable whose distribution is free of the parameter θ ? [5]

4. Let $X_1, \ldots, X_n$ be iid with standard Cauchy distribution with density $f(x) = \frac{1}{\pi(1+x^2)}$ on the real line. Prove that $\frac{\pi X_{(n)}}{n}$ converges in distribution to a non-degenerate limit as $n \to \infty$, where $X_{(n)}$ is the maximum of $X_1, \ldots, X_n$. [8]

5. Suppose $X_1, \ldots, X_n$ are iid with common density $f(x, \theta)$, where $\theta \in \Theta$, Θ consisting of *only finitely many real numbers*. Assume also that density under each θ has the same support and that the distributions under different θ's are different. If θ_0 is the true value of θ, prove that with probability tending to 1 (under θ_0) as $n \to \infty$, the likelihood will be maximized at the value $\theta = \theta_0$. [6]

6. Suppose $X_1, \ldots, X_n$ are iid with density $f(x, \theta)$ where $\theta \in \Theta$, Θ being an open interval on $\mathcal{R}$. Assume the regularity conditions needed to prove asymptotic normality of a consistent sequence of roots of the likelihood equation. Suppose now that $\hat{\theta}_n$ is a consistent sequence of roots of the likelihood equation and θ_0 is the true value of the parameter θ. Prove that, under θ_0, with probability tending to 1 as $n \to \infty$, $\hat{\theta}_n$ is a local maximum of the likelihood equation. [8]

7. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be iid each having a Bivariate Normal distribution with $\mathrm{E}(X_1) = 0 = \mathrm{E}(Y_1)$, $\mathrm{Var}(X_1) = 1 = \mathrm{Var}(Y_1)$ and $\mathrm{Cov}(X_1, Y_1) = \rho$, where $\rho \in (-1, 1)$ is unknown. Find an estimator $\hat{\rho}_n$ of ρ based on the given observations such that $\sqrt{n}(\hat{\rho}_n - \rho)$ converges in distribution to $N(0, \frac{1}{I(\rho)})$ as $n \to \infty$. Prove your assertion. Here $I(.)$ refers to the Fisher Information based on one observation from the above bivariate normal distribution. [6]

8. Let $X_1, \ldots, X_n$ be iid $N(\theta, 1)$ where $\theta \in \mathbf{R}$. Let T_n denote the Hodges' estimator that estimates θ by 0 if the sample mean lies in $[-n^{-1/4}, n^{-1/4}]$ and by the sample mean otherwise. Can you propose a criterion with respect to which the asymptotic performance of the sample mean is better than that of the Hodges' estimator? Justify your answer. [3]

9. Suppose $X_1, \ldots, X_n$ are iid having density $f(x, \theta)$, where $\theta \in \mathbf{R}$. Invoking appropriate regularity assumptions, derive the asymptotic null distribution of the likelihood ratio test statistic for testing $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. [7]

10. (a) Stating appropriate assumptions, prove that the delete-d Jackknife estimator of variance is consistent for estimating the variance of certain statistics. [8]

    (b) Can you given an example where the Bootstrap is inconsistent for the purpose of estimating the distribution function ? Justify your answer. [5]

INDIAN STATISTICAL INSTITUTE

M. Stat 1st Year – B-Stream 2013-2014

Semestral Examination

Subject – Applied Stochastic Processes.

Full Marks: 70 **Date: 13/11/2013** **Duration: 3 hours**

The paper contains questions of 75 marks. Attempt all questions. Maximum marks you can score is 70.

1. a) What is meant by genes identical by decent? What are the coefficient of parentage and inbreeding coefficient?
   b) Consider a bisexual population with alleles A, a at some autosomal loci. Let, at generation n, the inbreeding coefficient be F_n. Under the mating type of 'Sibmating' through generations show that panmictic index $P_n = 1 - F_n \to 0$ as $n \to \infty$. Find the rate of convergence. [6+12]

2. Consider a bisexual population with alleles A, a at some autosomal loci. Let at generation n, in both males and females proportion of A = p_n and proportion of a = $1 - p_n$. Under the random mating and the ratio of relative advantage in zygotic selection AA: Aa: aa = $\sigma_1 : \sigma_2 : \sigma_3$, find p_{n+1}.
   Let $\sigma_1 < \sigma_2$ and $\sigma_3 < \sigma_2$. Considering the sequence $\{p_n\}$ find the equilibrium values.
   Which of them are stable equilibria and which are unstable equilibria? Justify your answer.
   Explain this with the example of the case of sickle cell anaemia. [15]

3. Write down the General Stochastic Epidemic Model with related assumptions. Deduce the forward equation corresponding to the related Markov Pure Jump Process.
   Find the parameters of the process. What are the absorbing states? [12]

4. Write down the Simple Deterministic Epidemic Model with related assumptions. Solve the corresponding differential equation and find the time point at which the rate of infection is the highest. [12]

5. In a city, let there be k families with the same family name. Under the assumption that the family name comes from the father in a family and that the probability of male birth and that of the female birth are equal and they occur independently, find the probability that the family name eventually survives.
   It is given that families adopt (throughout generations) the policy to have two children except when both the children are girls, when they have one more child. [7]

6. Consider a bisexual population with two alleles A, a at some autosomal loci. In both males and females let the ratio AA: Aa: aa = $p^2 : 2pq : q^2$ where $p + q = 1$, $0 < p < 1$. Under the assumption of random mating, find the conditional probability that a grandson has genotype AA when it is given that the grandfather has genotype AA. [11]

INDIAN STATISTICAL INSTITUTE

Semestral Examination: (2013–2014)

M. Stat First Year

Statistical Inference I

Date: 18.11.13 Full Marks: 100 Duration: $3\frac{1}{2}$ hours

Attempt all questions

1. Suppose that for every $m = 1, 2, \ldots,$

$$f_{X_1,\ldots,X_m}(x_1,\ldots,x_m) = \frac{2}{(m+1)c_m(x_1,\ldots,x_m)^{m+1}}, \quad \text{if all } \quad x_i \geq 0$$

where $c_m(x_1,\ldots,x_m) = \max\{2, x_1, \ldots, x_m\}$.

(a) Prove that X_i are exchangeable and that these distributions are consistent. [5]

(b) Find the distribution of $Y_n = c_n(X_1,\ldots,X_n)$ and the limit of this distribution as $n \to \infty$. [5]

(c) Find the conditional density of X_{n+1} given $X_1 = x_1, \ldots, X_n = x_n$, and assume that $\lim_{n\to\infty} c_n(x_1,\ldots,x_n) = \theta$. Find the limit of the conditional density as $n \to \infty$. [5]

(d) Use DeFinetti's representation theorem to show that the prior (the answer to part (b)) and the likelihood (the asnwer to part (c)) combine to give the original joint distribution. [5]

2. (a) Suppose that $\{Y_n\}_{n=1}^{\infty}$ are conditionally *iid* with Cauchy distribution $\mathcal{C}(\theta, 1)$, given $\Theta = \theta$. Here the parameter space Ω and the action space $\aleph$ are both equal to the real line $\mathbb{R}$. Let the loss function be $L(a,\theta) = (a-\theta)^2$ and the prior distribution of Θ be $\mathcal{C}(0,1)$. Now consider $X_i = \min\{t, Y_i\}$, where $t > 0$. Define $X = (X_1, X_2, X_3)$. Does a formal Bayes rule exist in this case? Justify. [10]

(b) Suppose that $X \sim Binomial(n,\theta)$. Let the parameter space be $\Omega = (0,1)$ and $\aleph = [0,1]$. Let $L(\theta, a) = (\theta - a)^2$ and

$$\delta(x) = \begin{cases} \frac{x}{n} & \text{with probability } \frac{1}{2}, \\ \frac{1}{2} & \text{with probability } \frac{1}{2}. \end{cases}$$

Find a non-randomized rule that dominates δ. [10]

3. (a) Suppose that for $i = 1, \ldots, m$, $X_i \sim N(\theta_i, 1)$ independently. Let the loss function be $L(\theta, a) = \sum_{i=1}^{m} (\theta_i - a_i)^2$. Obtain a minimax rule for this problem. Is your minimax rule also admissible? Justify. [10]

(b) Let $X \sim Geometric(\theta)$. Assume the loss function $L(\theta, a) = (\theta - a)^2/[\theta(1-\theta)]$. Consider the rule $\delta(x) = I_{\{0\}}(x)$. Is $\delta(x)$ minimax? Justify. [10]

4. (a) Suppose that an observation X is to be made and it is believed that X has one of the two densities:

$$f_0(x) = \frac{1}{2}\exp(-|x|), \quad f_1(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right).$$

Find all of the admissible procedures according to the Neyman-Pearson fundamental lemma. Express the rules in terms of intervals in which each decision is taken. [10]

(b) In the class of decision rules associated with the Neyman-Pearson lemma, show that the rules do not dominate each other, without assuming that the class is minimal complete. [10]

5. (a) Suppose that $X_1, \ldots, X_n$ are *iid* with density

$$f_{X_1|\Theta}(x|\theta) = \sqrt{\frac{2}{\pi}}\exp\left\{-\frac{1}{2}(x-\theta)^2\right\} I_{(\theta,\infty)}(x).$$

Let $L(\theta, a) = (\theta - a)^2$ and let both the parameter space Ω and the action space $\aleph$ be the real line $\mathbb{R}$. Let G be the one-dimensional location group, $g_c(x_1, \ldots, x_n) = (x_1 + c, \ldots, x_n + c)$. Obtain the MRE estimator. [10]

(b) A function $g : \mathbb{R}^n \mapsto \mathbb{R}$ is *even* if $g(-x_1, \ldots, -x_n) = g(x_1, \ldots, x_n)$. A function g is *odd* if $g(-x_1, \ldots, -x_n) = -g(x_1, \ldots, x_n)$. Suppose that S is odd and location equivariant and that T is even and location invariant. Suppose that $X_1, \ldots, X_n$ are *iid* with density f such that $f(c - x) = f(c + x)$ for some c and all x. Suppose that the variances of $S(X_1, \ldots, X_n)$ and $T(X_1, \ldots, X_n)$ are both finite. Prove that the covariance between S and T is zero. [10]

INDIAN STATISTICAL INSTITUTE

End-Semester Examination – Semester I : 2013-2014

M.Stat. I Year

Measure Theoretic Probability

Date : 22.11.13 **Maximum Score : 60** $3\frac{1}{2}$ Hours

Note : This paper carries questions worth a **total** of **75 MARKS**. Answer as much as you can. The **MAXIMUM** you can score is **60**.

1. (a) Show that if $\{f_n\}$ is a sequence of extended real-valued measurable functions on $(\Omega, \mathcal{A})$, then the set $A = \{\omega \in \Omega : \lim_n f_n(\omega) \text{ exists}\}$ belongs to $\mathcal{A}$.
(b) Consider the function f on $\mathbb{R}$ given by $f(x) = x - [x]$ where $[x]$ ='interger part' of x. Which of the following sets belong to the σ-field generated by f? (give reason): (i) the set of all integers, (ii) the set $\bigcup_{n=0}^{\infty}(n, n+\frac{1}{3})$, (iii) the set of all rationals. (5+(3×2))=**[11]**

2. (a)Let f be a real-valued funtion on $\mathbb{R}$. For $\alpha > 0$, let f_α denote the funtion defined by $f_\alpha(x) = f(\alpha x)$. Show that (i) if f is Borel-measurable then so is f_α, (ii) $\int f_\alpha \, d\lambda$ exists if and only if $\int f \, d\lambda$ exists and in that case $\int f_\alpha \, d\lambda = \alpha^{-1} \int f \, d\lambda$.
(b) Let X be a random variable on a probability space $(\Omega, \mathcal{A}, P)$ such that for all $A \in \mathcal{A}$ with $P(A) > 0$, one has $-P(A) < E[X 1_A] < 3P(A)$. Show then that $-1 < X < 3$, P-almost surely. ((4+5)+5)=**[14]**

3. Let f be an extended real-valued measurable function on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. Denote $\mathcal{B}^+$ to be the borel σ-field on $[0, \infty)$ and λ the Lebesgue measure.
(a) Show that the set $\{(\omega, t) \in \Omega \times [0, \infty) : |f(\omega)| > t\}$ belongs to the σ-field $\mathcal{A} \otimes \mathcal{B}^+$.
(b) Show that the non-negative $\mathcal{A} \otimes \mathcal{B}^+$-measurable function $g(\omega, t) = 4t^3 1_C(\omega, t)$ satisfies $\int_0^\infty g(\omega, t) \, d\lambda(t) = |f(\omega)|^4$, for each $\omega \in \Omega$, and using this (or otherwise), deduce that $\int_\Omega |f(\omega)|^4 \, d\mu(\omega) = \int_0^\infty 4t^3 \mu(\{|f| > t\}) \, d\lambda(t)$. (5+(3+3))=**[11]**

4. (a) Define what is meant by convergence in measure. Show that if f_n, $n \geq 1$ and g_n, $n \geq 1$ are measurable functions on a measure space converging in measure to f and g respectively, then the functions $f_n - g_n$, $n \geq 1$ converge in measure to $f - g$.
(b) Suppose X_n, Y_n, $n \geq 1$ and X, Y are real random variables on a probability space such that $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Show that, if $P(X_n < Y_n) \leq 0.25$ for all n, then $P(X < Y) \leq 0.25$. ((2+4)+5)=**[11]**

5. (a) State Kolmogorov's zero-one law.
Let $\{X_n\}$ be a sequence of random variables on a probability space and let $\{S_n\}$ be the sequence of partial sums. Which of the following events are "tail events"?
(i) $\{X_n \geq X_{n+1}$ for infinitely many $n\}$, (ii) $\{S_{2n} \geq S_n$ for all but finitely many $n\}$, (iii) $\{S_n/n$ converges $\}$, (iv) $\{\sqrt{n}\, S_n$ converges to $0\}$.
(b) Find the limit $\lim_n n^{-(1/2)} \int_1^2 \cdots \int_1^2 \sqrt{x_1 + \cdots + x_n} \, dx_1 \cdots dx_n$.
(c) For a sequence $\{X_n\}$ of i.i.d. random variables with density $f(x) = e^{-x}, x \in (0, \infty)$, show that $\limsup_n (X_n / \log n) = 2$, almost surely. ((2+4×1)+5+7)=**[18]**

6. Answer **ANY ONE** of the following. If you attempt more than one and do not strike off the one(s) you don't want to count, the one first attempted will be graded.
(a) Let f, f_α be as defined in 2(a) above. Show that if $f \in L_1$ and $\{\alpha_n\}$ is any sequence of positive reals with $\alpha_n \to 1$, then the sequence $\{f_{\alpha_n}\}$ converges to f in L_1.
(b) Show directly from definition that if $X_n \xrightarrow{P} X$ where X is a non-zero random variable, then $1/X_n \xrightarrow{P} 1/X$.
(c) Using the fact that for any $\alpha > 1$, $\sum_{n=j}^\infty n^{-\alpha} \leq C_\alpha \cdot n^{-(\alpha-1)}$, show that if $\{X_n\}$ is an i.i.d. sequence of non-negative random variables with $E[\sqrt{X_1}\,] < \infty$ and if for $n \geq 1$, $Y_n = X_n 1_{\{X_n \leq n^2\}}$, then both the series $\sum_1^\infty E[Y_n/n^2]$ and $\sum_1^\infty V[Y_n/n^2]$ converge. Hence (or otherwise) deduce that, as $n \to \infty$, $(X_1 + \cdots + X_n)/n^2 \longrightarrow 0$ a.s. **[10]**

INDIAN STATISTICAL INSTITUTE

First Semester Examination : 2013-14

Course Name: M.Stat. 1st Year (B-Stream)

Subject Name: Design of Experiments

Date: Total Marks: 30.

Duration: $1\frac{1}{2}$ hours

Note: Answer Question 1 and any two questions from Questions 2, 3 and 4.

1. Consider a fraction d of a 2^3 factorial consisting of the four treatment combinations $000, 110, 101, 011$ and one more treatment combination, say $\alpha_1\alpha_2\alpha_3$, where each α_i is either 0 or 1. Each of these 5 treatment combinations is replicated once in d. Let $Y(i_1i_2i_3)$ be the observation arising from a typical treatment combination $i_1i_2i_3$ in the fraction. Suppose all interactions are absent and consider the linear model

$$E\{Y(i_1i_2i_3)\} = \beta_0 + \sum_{j=1}^{3}(2i_j - 1)\beta_j,$$

where β_0 is the general mean and β_j represents the main effect of the jth factor. As usual, assume that the errors are uncorrelated and homoscedastic. Write

$$\beta = (\beta_0, \beta_1, \beta_2, \beta_3)' \text{ and } Y = (Y(000), Y(110), Y(101), Y(011), Y(\alpha_1\alpha_2\alpha_3))'$$

for the parametric vector, and the 5×1 vector of observations arising from d, respectively.

(a) Write down the 5×4 design matrix X, where $E(Y) = X\beta$.

(b) Obtain $X'X$ and verify that it is nonsingular.

(c) Obtain the inverse of $X'X$ and hence find $\sum_{j=0}^{3} Var(\hat{\beta}_j)$, where $\hat{\beta}_j$ is the best linear unbiased estimator of β_j. Does the answer depend on the choice of the treatment combination $\alpha_1\alpha_2\alpha_3$?

(d) Obtain the determinant of the dispersion matrix of $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)'$. Does it depend on the choice of the treatment combination $\alpha_1\alpha_2\alpha_3$?

[3+1+4+4=12]

2. (a) Consider the set up of a 2^4 factorial experiment. In this set up, construct an 8-run resolution (1,2) plan.(You must give the actual treatment combinations in the plan but no proof needed).

   (b) Let d_0 be an 8-run plan for a 2^5 factorial such that d_0 minimizes the average variance of the best linear unbiased estimators of the general mean and the main effects parameters among all 8-run plans when all interactions are absent. Construct d_0 showing the actual treatment combinations in it and justify your claim. [4+5=9]

3. (a) Define a Balanced Incomplete Block (BIB) design. Write down the usual model for analysing these designs. Write down the C matrix for this design under this model,(no proof needed)

   (b) Consider 19 treatments labeled as $0, \ldots, 18$. Show that on developing the 3 initial blocks (0,6,10), (1,13,2), (3,8,5), one can construct a BIB design. Give the parameters of this design. (Actual construction of the design should not be done)

   (c) Hence indicate how you can construct a BIB design with $r = 48, k = 16, \lambda = 40$. [3+4+2=9]

4. (a) Define a system of distinct representatives.

   (b) Show that if we write the blocks of any symmetric BIB design $(v = b, r = k, \lambda)$ as columns, (each column being of size k), then we can always ensure that each of the v treatments appears exactly once in the first row.

   (c) Construct a Youden square design with 7 treatments arranged in 3 rows and 7 columns. [3+4+2=9]

INDIAN STATISTICAL INSTITUTE

First Semester Examination : 2013-14

Course Name : **M.Stat. 1st Year**

Subject Name : **Sample Survey and Design of Experiments** **Date** : 25.11, **2013**

Note: Use separate answer sheets for two groups.

Group – Sample Survey. (Total Marks = 30) Duration : 1 hrs. 30 mins.

Answer any three. Notations are as usual.

1. (a) State and prove the theorem on the existence/non-existence of the uniformly minimum variance estimator for Y within the class of all homogeneous linear unbiased estimators.

   (b) Prove that for any sampling design p with $\pi_i > 0, \forall i$, the Horvitz-Thompson estimator of population total Y of a variable y is admissible in the class of all homogeneous linear unbiased estimators.

   $(5 + 5 = 10)$

2. (a) Given any design p and an unbiased estimator t for Y depending on order and/or multiplicity of units in sample s, derive an improved estimator for Y through Rao-Blackwellization.

   (b) Let $P_i(0 < P_i < 1, \sum_{i=1}^{N} P_i = 1)$ be known numbers associated with the units i of a population U. Suppose on the first draw a unit i is chosen from U with probability P_i and given that the first unit is i on the second draw a unit $j(\neq i)$ is chosen with probability $\dfrac{P_j}{1 - P_i}$.
   For a sample of size 2 drawn under this sampling scheme, write down Des Raj's (1956) unbiased estimator for Y. Improve that estimator applying Rao-Blackwellization.

   $(6 + 4 = 10)$

3. (a) State and prove Rao (1979)'s theorem on the General Mean Square Error formula for a homogeneous linear estimator (HLE) for population total Y.

   (b) Use that theorem to obtain the variance of Horvitz and Thompson (1952)'s estimator by deriving the proper condition to be satisfied.

   $(6 + 4 = 10)$

4. Describe Rao, Hartley and Cochran (1962)'s sampling scheme to draw a sample of size n out of N population units with known positive size measures $x_i > 0 \ \ \forall i = 1, \ldots, N$. Derive for this scheme an unbiased estimator, variance and variance estimator for population total Y of a variable of interest y. Mention the optimality condition of this scheme and also the sign property of the variance estimator.

(10)

5. (a) Describe Politz and Simmon (1949, 1950)'s technique of using 'at-home-probabilities' to estimate the population mean $\bar{Y}$ by SRSWR scheme with an estimator for variance.

   (b) Describe a quantitative randomized response model to estimate a sensitive population mean by a sampling design with $p(s)$ being the selection probability of a sample s of respondents. Give an estimator of the variance.

$(5 + 5 = 10)$

INDIAN STATISTICAL INSTITUTE

End-Semester Examination – Semester I : 2013-2014

M.Stat. I Year

Measure Theoretic Probability

Date : 10.12.13 | **Maximum Score : 60** | 3 Hours

Note : This paper carries questions worth a **total** of **75 MARKS**. Answer as much as you can. The **MAXIMUM** you can score is **60**.

1. Let $\{f_n\}$ be a sequence of real-valued measurable functions on $(\Omega, \mathcal{A})$. Define τ on Ω as $\tau(\omega) = \sup\{n \geq 1 : f_n(\omega) \geq f_k(\omega) \text{ for } 1 \leq k \leq n\}$. Show that τ is measurable with respect to $\mathcal{A}$. **[8]**

2. For a measurable function f on a measure space $(\Omega, \mathcal{A}, \mu)$, let $I(f) = \{p \in (0, \infty) : \int |f|^p d\mu < \infty\}$.
(a) Show that $I(f)$ is an interval (possibly empty) and that φ defined on $I(f)$ by $\varphi(p) = \log\left[\int |f|^p d\mu\right]$ is a convex function.
(b) For the measure space $((0, \infty), \mathcal{B}, \lambda)$, find a measurable function f for which $I(f) = (1, 2]$. ((5+3)+7)=**[15]**

3. Let f be an integrable function on a measure space $(\Omega, \mathcal{A}, \mu)$. Show that if $\{A_n\}$ is any sequence of sets in $\mathcal{A}$ with $\mu(A_n) \to 0$, then $\int_{A_n} f d\mu \to 0$. **[8]**

4. Let f be a non-negative measurable function on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. Consider the set $A = \{(\omega, t) \in \Omega \times [0, \infty) : f(\omega) \geq t\}$. Show that the set A belongs to the product σ-field $\mathcal{A} \otimes \mathcal{B}$ and its $\mu \otimes \lambda$-measure equals $\int f d\mu$. [Here $\mathcal{B}$ denotes the Borel σ-field on $[0, \infty)$ and λ the lebesgue measure.] (6+4)=**[10]**

5. Let $X_n, n \geq 1; X$ be random variables on a probability space such that $X_n \xrightarrow{P} X$. For each $n \geq 1$, let θ_n be such that $P[X_n \leq \theta_n] \geq 0.7$ and $P[X_n < \theta_n] \leq 0.7$. Show that if $\theta_n \to \theta$, then $P[X \leq \theta] \geq 0.7$ and $P[X < \theta] \leq 0.7$. **[10]**

6. (a) Let $\{X_n\}$ be an i.i.d. sequence of random variables with density $f(x) = e^{-x}, x \in (0, \infty)$. Show that $\min_{1 \leq k \leq n} X_k \to 0$ almost surely.
(b) Let $\{X_n\}$ be an i.i.d. sequence of random variables on a probability space. Show that if $(\max_{1 \leq k \leq n} X_k)/n \to 0$ almost surely, then the X_n must have a finite mean. (7+7)=**[14]**

7. Find the value of $\lim_{n \to \infty} \int_0^1 \cdots \int_0^1 \left[x_1(1 - x_2)x_3(1 - x_4) \cdots x_{2n-1}(1 - x_{2n})\right]^{1/n} dx_1 \cdots dx_{2n}$.
You need to provide complete argument for your answer. **[10]**

INDIAN STATISTICAL INSTITUTE

Back-Paper Examination – Semester I : 2013-2014

M.Stat.(B-Stream) I Year

Measure Theoretic Probability

Date: 30.01.14   Total Marks : 100   Time : 3 Hours

Note : This paper carries a total of **100** marks. You may answer as many as you want. The **maximum** you can score is **45**.

1. Let $\mathcal{A}$ denote the class of all those sets $A \subset \mathbb{R}$ such that either A or A^c is countable.
(a) Show that $\mathcal{A}$ is the smallest σ-field on $\mathbb{R}$ containing all the singleton sets.
(b) Show that a function $f : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable if and only if there exists a set $A \subset \mathbb{R}$ with A^c countable such that f is constant on A. (8+8)=**[16]**

2. Let F be a probability distribution function on $\mathbb{R}$.
(a) Show that if X is *any* non-negative real random variable on some probability space, then for any real number a, the function $F(aX)$ is a random variable with finite expectation and the function $G(a) = E[F(aX)]$ is a probability distribution function on $\mathbb{R}$.
(b) Show that if F has a density, then so does G. ((6+6)+6)=**[18]**

3. Let $(\Omega, \mathcal{A}, P)$ be a probability space. (a) Show that if $\mathcal{S}_1 \subset \mathcal{A}$ and $\mathcal{S}_2 \subset \mathcal{A}$ are two semi-fields that are independent, then the σ-fields $\sigma(\mathcal{S}_1)$ and $\sigma(\mathcal{S}_2)$ are also independent.
(b) Show that if $X_1, \ldots, X_n$ are independent random variables and $1 < k < n$, then for any borel functions $h : \mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R}^{n-k} \to \mathbb{R}$, the random variables $Y = h(X_1, \ldots, X_k)$ and $Z = g(X_{k+1}, \ldots, X_n)$ are independent. (8+8)=**[16]**

4. (a) Show, directly from definitions, that $X_n \xrightarrow{\text{a.s.}} X$ if and only if $\sup_{k \geq n} |X_k - X| \xrightarrow{P} 0$.
(b) Show that for random variables X_n, $n \geq 1$ and X defined on a discrete probability space $(\Omega, \mathcal{A}, P)$, one has $X_n \xrightarrow{P} X$ if and only if $X_n \xrightarrow{\text{a.s.}} X$. (8+(4+6))=**[18]**

5. Show that if $X_n \xrightarrow{P} X$, then for any bounded continuous function $h : \mathbb{R} \to \mathbb{R}$, one has $E[h(X_n)] \longrightarrow E[h(X)]$ as $n \to \infty$. **[16]**

6. Let $\{X_n\}$ be an independent sequence of random variables with $P(X_n = 2) = P(X_n = n^\alpha) = \theta_n$, $P(X_n = \theta_n) = 1 - 2\theta_n$, for some $\alpha \in \mathbf{R}$ and $\theta_n \in (0, \frac{1}{3})$. Show that the series $\sum_n X_n$ converges almost surely if and only if $\sum_n \theta_n < \infty$. **[16]**

INDIAN STATISTICAL INSTITUTE

Metric Topology and Complex Analysis
M. Stat 1st year
Mid Semester Examination: 2013-14
24 February , 2014.

Maximum Marks 40 Maximum Time 3 hrs.

(1) Prove that every connected metric space (X, d) with more than one point is uncountable. [5]

(2) Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $f : S^1 \to \mathbb{C} \setminus \{0\}$ be continuous with respect to the restricted metric of $\mathbb{R}^2$. If $f(z_1 z_2) = f(z_1)f(z_2)$ for all z_1 and z_2 in S^1 then prove that $f(S^1) \subset S^1$. [5]

(3) Does there exist a metric d on $\mathbb{Q}$ which is equivalent to the standard metric but $(\mathbb{Q}, d)$ is complete? Justify your answer. [5]

(4) Let $B[0,1]$ be the space of bounded real valued functions on $[0,1]$ with the metric $d(f,g) = \text{Sup}_{t\in[0,1]}|f(t) - g(t)|$. Is the set $S = \{f \in B[0,1] \mid d(f,0) = 1\}$ a compact subset of $B[0,1]$? Justify your answer. [5]

(5) Consider $\mathbb{R}^n$ with Euclidean norm.
  (a) Prove that an open vector subspace of $\mathbb{R}^n$ is necessarily $\mathbb{R}^n$ itself. [3]
  (b) Prove that every vector subspace of $\mathbb{R}^n$ is closed. [3]

(6) Let (X, d) be a metric space and $\delta(x,y) = \text{Min}\{1, d(x,y)\}$ for all $x \in X$ and $y \in X$. Prove that δ is a metric on X which is equivalent to d. [6]

(7) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and $g(z) = \overline{f(\bar{z})}$ for all $z \in \mathbb{C}$. Is g holomorphic? Justify your answer. [5]

(8) For $0 < b < 1$ prove that,

$$\int_{-\infty}^{\infty} \frac{1 - b + x^2}{(1 - b + x^2)^2 + 4bx^2} dx = \pi$$

by integrating the function $(1 + z^2)^{-1}$ around the rectangle with vertices $\pm R$ and $\pm R + i\sqrt{b}$, $R > 0$. [8]

Indian Statistical Institute
M.Stat I
Discrete Mathematics
Mid term Examination

e: February 25, 2014
e 2.5 hours

The question paper contains 7 questions. Total marks is 70. Maximum you can score is 60.

Prove that a simple graph with n vertices and k components can have atmost $(n-k)(n-k+1)/2$ edges. (10)

Prove that a set of edges in a connected planar graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*. (10)

A *king* is a vertex from which every vertex is reachable by a path of length at most 2. Prove that:

(a) Every tournament has a king.

(b) Every tournament having no vertex with in-degree 0 has atleast two kings.

$(5 + 5 = 10)$

Let T be a minimum spanning tree in G and T' is another spanning tree in G. Prove that T' can be transformed into T by a sequence of steps that exchange one edge of T' for one edge of T, such that the edge set if always a spanning tree and the total weight never increases. (10)

A graph is *Hamiltonian-connected* if for every pair of vertices u, v there is a Hamiltonian path from u to v. Prove that a simple graph G is Hamiltonian-connected if $e(G) \geq \binom{n(G)-1}{2} + 3$. (10)

Write an algorithm to test if a graph is bipartite? Prove the correctness of the algorithm and calculate the time complexity. $(4 + 3 + 3 = 10)$

Let G be a simple graph on $n > 3$ vertices.

(a) Prove that if G has more than $n^2/4$ edges, then G has a vertex whose deletion leaves a graph with more than $(n-1)^2/4$ edges.

(b) Use (a) to prove that G contains a triangle if $e(G) > n^2/4$. $(5+5 = 10)$

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination: (2013–2014)

M. Stat First Year

Regression Techniques

Date: 26/02/2014 Marks: 30 Duration: 2 hours

Attempt all questions

1. (a) Is the sum of the residuals always zero? Justify.

(b) Show that in the presence of pure error the square of the multiple correlation coefficient $R^2 < 1$.

(c) Consider the four-run experiment involving three explanatory variables presented in Table 1 below, where X_1, X_2, X_3 are covariates and Y is the response variable. Assume that $Y \sim N\left(\mu, \sigma^2 \mathbf{I}\right)$, where μ is a linear function of the covariates and $\mathbf{I}$ denotes the identity matrix. Assume that σ is known. With reference to Table 1 discuss

Table 1: Factor setting for the hypothetical four-run experiment

X_1	X_2	X_3	Y
-1	-1	+1	Y_1
-1	+1	-1	Y_2
+1	-1	-1	Y_3
+1	+1	+1	Y_4

in details a geometric approach to formally testing

$$H_0 : \beta_0 \mathbf{1} + \beta_1 X_1 \text{ versus } H_1 : \mu = \beta_0 \mathbf{1} + \beta_1 X_1 + \beta_2 X_2,$$

where $\mathbf{1}$ denotes the vector with all elements equal to 1.

Marks: 6+2+2=10

2. The following question concerns a dynamic probit regression model. Here one observes $(y_1, x_1), \ldots, (y_n, x_n), \ldots$ sequentially in time, where $(y_n, x_n) \in \{0,1\} \times \mathbb{R}$. The proposed model is:

$$Y_n = \mathbf{I}_{(0,\infty)}(Z_n)$$
$$Z_n = \theta_n x_n + \epsilon_n$$
$$\theta_n = \theta_{n-1} + \nu_n$$

with $\theta_0 = 0$, $\epsilon_n \overset{iid}{\sim} N(0,1)$ and independently $\nu_n \overset{iid}{\sim} N(0,1)$.

(a) Show that

$$P(Y_n = 1|\theta_n) = \Phi(\theta_n x_n)$$

for any $n \geq 1$, where Φ the Gaussian CDF.

(b) Consider only the second two equations of the system of equations that define the model. Show that, for $n \geq 2$,

$$p(\theta_n|z_{1:n}) = \frac{p(z_n|\theta_n)p(\theta_n|z_{1:n-1})}{\int p(z_n|\theta_n)p(\theta_n|z_{1:n-1})d\theta_n}$$

where

$$p(\theta_n|z_{1:n-1}) = \int p(\theta_n|\theta_{n-1})p(\theta_{n-1}|z_{1:n-1})d\theta_{n-1}.$$

(c) Again, considering only the second two equations of the system of equations that define the model, show that:

$$[\theta_1|z_1] \sim N\left(\frac{z_1 x_1}{1+x_1^2}, \frac{1}{1+x_1^2}\right).$$

(d) Now assuming that:

$$[\theta_{n-1}|z_{1:n-1}] \sim N\left(\mu_{n-1}, \sigma_{n-1}^2\right)$$

show that:

$$[\theta_n|z_{1:n-1}] \sim N\left(\mu_{n|n-1}, \sigma_{n|n-1}^2\right).$$

(e) Hence show that

$$[\theta_n|z_{1:n}] \sim N\left(\mu_n, \sigma_n^2\right)$$

for some $\mu_{n|n-1}$, $\sigma_{n|n-1}^2$, μ_n, σ_n^2 to be determined.

Marks: 2+2+2+2+2=10

3. Suppose that data $\{y_1, \dots, y_n\}$ is available where $E(y_i) = \mu(t_i)$; $t_i = (2i-1)/2n$. The function $\mu(\cdot)$ is assumed to be unknown. Now consider the following extension of the regressogram estimator. Define the partitions $P_j = [\frac{j-1}{\lambda}, \frac{j}{\lambda})$, $j = 1, \dots, \lambda - 1$, and $P_\lambda = [\frac{\lambda-1}{\lambda}, 1]$. For $t \in P_j$ let $\mu_\lambda(t) = b_{0j} + b_{1j}(t - \bar{t}_j)$ for $b_{1j} = \sum_{t_i \in P_j} y_i(t_i - \bar{t}_j) / \sum_{t_i \in P_j}(t_i - \bar{t}_j)^2$, $b_{0j} = \bar{y}_j = n_j^{-1} \sum_{t_i \in P_j} y_i$, $\bar{t}_j = n_j^{-1} \sum_{t_i \in P_j} t_i$ and $n_j = \sum_{i=1}^n I_{P_j}(t_i)$. Show that if μ has two continuous derivatives and the number of partitions is allowed to grow at the rate $n^{1/5}$, then the squared error risk for μ_λ decays to zero at the rate $n^{-4/5}$.

Marks: 10

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2013-14
M. STAT. I YEAR
Optimization Techniques

Date: 27/02/2014 Maximum Marks: 50 Duration: 2 hours

Notation have usual meaning.

This paper carries 60 marks. However, maximum you can score is 50.

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1 Consider the constants: a_i an n-vector and b_i a scalar for $i = 1, 2, \ldots, m$. With $x \in R^n$ as the vector of decision variables, we are given the following mathematical programming problem:

$$\min \max_{1 \leq i \leq m} |a_i^{\mathrm{T}} x - b_i|.$$

Formulate the above as an equivalent linear programming problem. [10]

2 Consider the nonempty polyhedral set $P = \{x : Ax \leq b, x \geq 0\}$, where A is $m \times n$ matrix and b is m-vector. Let d be a direction of P. Prove that d is an extreme direction of P if and only if d is an extreme point of the set $D = \{d : Ad \leq 0, d \geq 0, e^{\mathrm{T}} d = 1\}$. [10]

3 Find all the extreme points and extreme directions of the polyhedral set given by:

$$\begin{aligned} x_1 + 2x_2 + \ x_3 &\leq 10 \\ -x_1 + 3x_2 \qquad &= \ 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

[10]

4 Let A be an $m \times n$ matrix, b an m-vector and c an n-vector. Show that the following problem has an optimal solution:

$$\begin{aligned} &\min \quad c^{\mathrm{T}} x \\ &\text{subject to} \\ &\qquad Ax \leq b \\ &\qquad x \geq 0, \end{aligned}$$

where every element of both A and b are non-negative. [10]

[P.T.O.]

5 For the linear program:

$$\max \quad x_1 - 2x_3$$
subject to
$$x_1 - x_2 \leq 1$$
$$2x_2 - x_3 \leq 1$$
$$x_1, x_2, x_3 \geq 0,$$

prove that $(3/2, 1/2, 0)$ is an optimal solution by using complementary slackness theorem. [10]

6 Let A be an $m \times n$ matrix. Prove that exactly one of the following two systems has a solution:

System 1. $Ax < 0$ for some $x \in R^n$.

System 2. $A^{\mathrm{T}}y = 0$ and $y \geq 0$ for some nonzero $y \in R^m$.

[10]

____*** xXx ***____

INDIAN STATISTICAL INSTITUTE
MID-SEMESTRAL EXAMINATION, 2013-2014
M.Stat. I & II years and M.S.(Q.E.) I year
Time Series Analysis/ Time Series Analysis and Forecasting

Date: 28.02.2014 Maximum Marks: 100 Time: **2 hours**

[*Note: Answer any* ***five*** *questions. Marks allotted to each question are given within parentheses*]

1. (a) Discuss what are meant by trend and seasonality of a time series. What types of differencing would remove these components and why?

   (b) Suppose that a time series contains trend, seasonality and noise components. Describe an appropriate procedure for 'obtaining' the trend and seasonal components present in the series.

[10+10 = 20]

2. (a) Distinguish between weak stationarity and strong stationarity of a time series.

   (b) Obtain the (weak) stationarity conditions, in terms of its parameters, of an ARMA(2, 1) process.

   (c) What are the statistical consequences if a stationary time series is further differenced?

[6+10+4 = 20]

3. (a) Obtain the autocorrelation function (ACF) of a special ARMA (1, 2) process given by

$$x_t - \Phi x_{t-1} = a_t - \theta a_{t-2}, |\Phi| < 1 \text{ and } |\theta| < 1.$$

   Discuss also the nature of the ACF of this special ARMA process.

   (b) Let $\{X_t\}$ be the monthly time series given by

$$X_t = (1+0.3B)(1-0.6B_{12})\, a_t; a_t \sim WN(0,\sigma^2).$$

   Determine the coefficients $\Pi_j, j = 0,1,2,\ldots$ in the representation

   $a_t = \sum_{j=0}^{\infty} \Pi_j X_{t-j}$. Further, is $\{X_t\}$ stationary and invertible? Justify your answer.

[10+10 = 20]

4. (a) Given two time series models, discuss how you would choose between them based on their forecast performances. What are the time series issues that are important in this choice? Give explanations in support of your answer.

   (b) Show that for an ARIMA (0, 1, 1) model, the forecast for all future vales is an exponentially weighted moving average of current and past values. Would a similar result hold for an ARIMA (1, 1, 1) process? Explain you answer.

[8 + 12 = 20]

5. (a) Describe the full information maximum method of estimation of an AR(1) model. Is this method better than the other methods available for estimating this model? Justify your answer.

   (b) Find the process of $\{Z_t\}$ where Z_t is defined as $Z_t = X_t + Y_t$, $\{X_t\}$ and $\{Y_t\}$ follow two different AR (1) processes and they are independent. Is it possible that $\{Z_t\}$ follows an AR (2) process? If so, under what conditions on the parameters ? If not, why? Also discuss the special case where the root of one process is reciprocal of the other.

[8+12 = 20]

6. (a) Let $\{x_t\}$ be defined by

   $x_t - \Phi x_{t-1} + a_t, |\Phi| < 1$

   where a_t, $t \in (0, \pm 1, \pm 2, \ldots)$ is obtained from the sequence $\{u_t\}$ of independent Normal (0, 1) random variables as follows:

$$a_t = \begin{cases} u_t \quad , & t = 0, \pm 2, \pm 4, \ldots \\ \frac{1}{\sqrt{2}}(u_t^2 - 1), & t = \pm 1, \pm 3, \ldots \end{cases}$$

   Is $\{x_t\}$ a weakly stationary series? Is it also strongly stationary? Justify your answer with necessary derivations/arguments.

   (b) Suppose that $\{x_t\}$ and $\{y_t\}$ are stationary time series satisfying $x_t - \phi x_t = a_t$ and $y_t - \phi y_{t-1} = x_t + b_t$ where $|\phi| < 1$, each of $\{a_t\}$and $\{b_t\}$ follows $WN(0, \sigma^2)$, and a_t and b_t are uncorrelated. Find the autocorrelation function of $\{y_t\}$ in terms of the parameters involved. Is it possible to find the partial ACF (PACF) of this process? Give explanations for your answer.

[8+12 = 20]

MULTIVARIATE ANALYSIS

M Stat 1st Year (2013-14)

Mid Semester Examination

Date: 03.03.2014 Time: 2 hours

Total Marks 40

*Answer **Question no. 5** and any 3 (THREE) questions of the following.*

1. Explain union – intersection principle for testing $H_0 : \underset{\sim}{\theta} = \underset{\sim}{0}$ against $H_1 : \underset{\sim}{\theta} \neq \underset{\sim}{0}$ based on a random sample {$\mathbf{X}_1, \ldots, \mathbf{X}_n$}, where $\mathbf{X} \sim N(\underset{\sim}{\theta}, \Sigma)$ and Σ is positive definite. Find the distribution of the test statistic thus obtained and discuss how you will perform the test. Check whether this test is unbiased? [You have to prove any result that you want to use]
(3+2+1+4=10)

2. Find the equation of ellipses or ellipsoids on which a multivariate normal distribution in p dimensions has constant density. Draw ellipses of equal concentration for the bivariate normal distribution with suitably chosen constant in the equation (that you obtained) for the following cases
(a) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ (c) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
(2+2+3+3=10)

3. (a) If $\mathbf{X} \sim N_p(\underset{\sim}{0}, \sigma^2 I)$, show that $\mathbf{AX}$ and $(\mathbf{I}-\mathbf{A}^-\mathbf{A})\mathbf{X}$, where $\mathbf{A}^-$ is a generalized inverse satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A}=\mathbf{A}$, are independent and each has a normal distribution. [You have to prove any result that you want to use]
(b) If $\mathbf{X} \sim N_3(\mathbf{0}, \mathbf{I})$, then
$$\frac{X_1 e^{X_3} + X_2 \log|X_3|}{\{e^{2X_3} + (\log|X_3|)^2\}^{1/2}} \sim N(0,1).$$
(5+5=10)

4. Let $A \sim W_p(n, \Sigma)$ and $B \sim W_p(m, \Sigma)$ and A and B are independently distributed.
(a) Show that $\dfrac{|A|}{|A+B|} \sim \Lambda_{p,m,n}$, where $\Lambda_{p,m,n}$ has the same distribution as that of $\prod_{i=1}^{p} U_i$ where $U_i \sim B\left(\frac{n-i+1}{2}, \frac{m}{2}\right)$ $i = 1, \ldots, p$ and U_i's are independently distributed.
(b) Also show that the distributions of $\Lambda_{p,m,n}$ and $\Lambda_{m,p,m+n-p}$ are same for any choice of m, n, and p.
(5+5=10)

5. Derive likelihood ratio test for testing $H_0 : \underset{\sim}{\mu_1} = \underset{\sim}{\mu_2}$ against $H_1 : \underset{\sim}{\mu_1} \neq \underset{\sim}{\mu_2}$ based on two random samples of size n and m from $N_p(\underset{\sim}{\mu_1}, \Sigma_1)$ and $N_p(\underset{\sim}{\mu_2}, \Sigma_2)$ respectively, under the assumption of $\Sigma_1 = \Sigma_2 = \Sigma$, where Σ is unknown. Is there any condition that should be imposed on the sample sizes so that you can carry out your test? If so, explain it. Also derive union-intersection test for the same null hypothesis and check whether the two critical regions are same.
(4+1+5=10)

INDIAN STATISTICAL INSTITUTE

Metric topology and Complex Analysis
M. Stat 1st year
End Semester Examination: 2013-14
22.04. 2014.

Maximum Marks 60 | Maximum Time 3:30 hrs.

Answer all questions but maximum you can score is 60.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$, $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$.

(1) Find the residue of the function $f(z) = \frac{\log(1+z)}{z^2}$ at $z = 0$, where $\log z$ stands for the principal branch of the logarithm [5]

(2) If $f : \mathbb{D} \to \overline{\mathbb{D}}$ is holomorphic then prove that $|f'(0)| \leq 1$. [5]

(3) Suppose f is an entire function such that $|f(z) + e^z| > |f(z)e^z|$ for all $z \in \mathbb{C}$. Prove that $f(z) = 0$ for all $z \in \mathbb{C}$. [5]

(4) If f is a continuous function on $\partial\mathbb{D}$ then prove that

$$F(z) = \int_{\partial\mathbb{D}} \frac{f(w)}{w - z} dw$$

is a holomorphic function on $\mathbb{D}$. [5]

(5) Evaluate the integral

$$\int_\gamma \frac{e^{-z^2}}{z^2} dz,$$

where γ is the square with vertices $1 + i$, $-1 + i$, $-(1 + i)$, $1 - i$. [5]

(6) Find the number of zeroes of the polynomial $P(z) = z^7 - 4z^3 + z - 1$ in $A = \{z \in \mathbb{C} \mid |z| < 1\}$, $B = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$ and $C = \{z \in \mathbb{C} \mid |z| > 2\}$. [6]

(7) Suppose f is a nonconstant holomorphic function on an open set Ω containing $\overline{\mathbb{D}}$. If $|f(z)| = 1$ for all $z \in \partial\mathbb{D}$ then prove that f must have a zero. Using this prove that the image of f contains $\mathbb{D}$. [6]

(8) Prove that a nonconstant entire function f is a polynomial if and only if $\lim_{|z|\to\infty} |f(z)| = \infty$. [8]

(9) Describe all conformal maps $f : \mathbb{C} \to \mathbb{C}$. [8]

(10) Suppose f is holomorphic on the set $\mathbb{D} \setminus \{0\}$ and

$$|f(z)| \leq A|z|^{-\frac{1}{2}}, \quad \text{for all, } 0 < |z| < \frac{1}{2}.$$

Prove that f has a removable singularity at $z = 0$. [8]

(11) If $a > 0$ then prove that

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

[8]

INDIAN STATISTICAL INSTITUTE

Semestral Examination: (2013–2014)

M. Stat First Year

Regression Techniques

Date 25/4/14 Full Marks: 100 Duration: 3 hours

Attempt all questions

[You may use the following result if necessary, without proof: If $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$, then $\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} \end{pmatrix}$ where $\mathbf{A}^{11} = \left(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1} = \mathbf{A}_{11}^{-1} - \mathbf{A}^{12}\mathbf{A}_{21}\mathbf{A}_{11}^{-1}$, $\mathbf{A}^{12} = -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}^{22}$, $\mathbf{A}^{22} = \left(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\right)^{-1}$, $\mathbf{A}^{21} = -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}^{11}$.]

1. Let the full model $\mathbf{y}^{(n\times 1)} = \mathbf{X}_1^{(n\times \overline{k+1})}\boldsymbol{\beta}_1^{(\overline{k+1}\times 1)} + \mathbf{X}_2^{(n\times \overline{m-k})}\boldsymbol{\beta}_2^{(\overline{m-k}\times 1)} + \boldsymbol{\epsilon}^{(n\times 1)}$ be the true regression model. The components of $\boldsymbol{\epsilon}$ are distributed independently as $Normal(0, \sigma^2)$.

   Also, consider the reduced model: $\mathbf{y}^{(n\times 1)} = \mathbf{X}_1^{(n\times \overline{k+1})}\boldsymbol{\theta}_1^{(\overline{k+1}\times 1)} + \boldsymbol{\epsilon}^{(n\times 1)}$.

   Let $\hat{\boldsymbol{\beta}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$ and, $\hat{\boldsymbol{\theta}}_1 = \left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\mathbf{X}_1'\mathbf{y}$, where $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$.

   Further, let $\mathbf{x}^*$ and $\mathbf{x}_1^*$ denote two vectors of input having $m+1$ and $k+1$ components respectively. Let $\mathbf{A} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2$.

   Prove the following:

   (a) $\hat{\boldsymbol{\theta}}_1$ is biased for $\boldsymbol{\beta}_1$ and ${\mathbf{x}^*}_1'\hat{\boldsymbol{\theta}}_1$ is biased for ${\mathbf{x}^*}_1'\boldsymbol{\beta}_1$ unless $\boldsymbol{\beta}_2 = \mathbf{0}$ or $\mathbf{A} = \mathbf{0}$. Also prove that ${\mathbf{x}^*}_1'\hat{\boldsymbol{\theta}}_1$ is biased for $\mathbf{x}^{*\prime}\boldsymbol{\beta}$ unless $\boldsymbol{\beta}_2 = \mathbf{0}$ or $\mathbf{x}_2^* = \mathbf{A}'\mathbf{x}_1^*$.

   (b) Prove that $Cov(\hat{\boldsymbol{\beta}}_1) - Cov(\hat{\boldsymbol{\theta}}_1)$ is positive semi-definite, and $Var(\mathbf{x}^{*\prime}\hat{\boldsymbol{\beta}}) \geq Var({\mathbf{x}^*}_1'\hat{\boldsymbol{\theta}}_1)$.

   (c) Let RSS_1 and RSS be the residual sums of squares for the reduced and the full model respectively. Also, let $s_1^2 = RSS_1/(n-k-1)$. Then show that $RSS_1 \geq RSS$ and that s_1^2 is a biased estimator of σ^2 in general.

   (d) Let $H_1 = \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'$. Then, under the condition that

   $$\frac{1}{\sigma^2}\boldsymbol{\beta}_2'\mathbf{X}_2'(\mathbf{I} - \mathbf{H}_1)\mathbf{X}_2\boldsymbol{\beta}_2 \leq 1,$$

   show that $Var\left(\hat{\boldsymbol{\beta}}_1\right) - MSE\left(\hat{\boldsymbol{\theta}}_1\right)$ is positive semi-definite and $Var\left(\mathbf{x}^{*\prime}\hat{\boldsymbol{\beta}}\right) \geq MSE\left({\mathbf{x}^*}_1'\hat{\boldsymbol{\theta}}_1\right)$. Here MSE stands for mean square error.

Marks: 6+6+6+7=25

2. (a) In the ridge regression set up with $\boldsymbol{\mu}_\lambda = \mathbf{S}_\lambda \mathbf{y}$ $(\lambda \geq 0)$, $\mathbf{y} = (y_1, \ldots, y_n)'$ the vector of observations, and $\mathbf{S}_\lambda = \mathbf{X}\left(\mathbf{X}'\mathbf{X} + \lambda \mathbf{I}\right)^{-1}\mathbf{X}'$, where $\mathbf{X}$ is the design matrix and $\mathbf{I}$ is the identity matrix, prove that

$$|E\{GCV(\lambda)\} - P(\lambda)|/R(\lambda) \leq 3\frac{p}{n} + O\left(\left(\frac{p}{n}\right)^2\right)$$

In the above, p denotes the number of predictors, n is the number of observations, $GCV(\lambda) = n^{-1}RSS(\lambda)/\left(n^{-1}tr[I - \mathbf{S}_\lambda]\right)^2$, with $RSS(\lambda) = \sum_{i=1}^n (y_i - \mu_{\lambda i})^2$, $P(\lambda) = n^{-1}\sum_{i=1}^n E(y_i^* - \mu_{\lambda i})^2$, with y_i^* standing for the i-th future observation, and with $\mu_i = E(y_i)$, $R(\lambda) = E\left\{n^{-1}\sum_{i=1}^n (\mu_i - \mu_{\lambda i})^2\right\}$.

(b) Assume that there are infinitely many non-zero β_j in the model $y_i = \sum_{j=1}^\infty \beta_j x_{ij} + \epsilon_i$, $i = 1, \ldots, n$, where $\epsilon_i; i = 1, \ldots, n$ are normal random errors, $\sum_{j=1}^\infty \beta_j^2 < \infty$ and, for each n, for $(j,k) = 1, \ldots, n$, $\sum_{i=1}^n x_{ij}x_{ik} = n\delta_{jk}$, where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. Let $\hat{\lambda}$ be the minimizer of $\hat{P}(\lambda) = n^{-1}RSS(\lambda) + 2n^{-1}\sigma^2 tr(S_\lambda)$ over $\Lambda_n = \{1, \ldots, a_n\}$ with $a_n \to \infty$ as $n \to \infty$. Show that, in this case, $P(\hat{\lambda} \leq x) \to 0$ for any finite x.

Marks: 15+10=25

3. (a) Suppose that we observe data $(y_1, x_1), \ldots, (y_n, x_n)$, with $y_i \in \{0, 1, 2, \ldots\}$ The following regression model is adopted; for $i = 1, \ldots, n$,

$$\log(\mu_i) = \eta_i + \log(\phi_i).$$

Y_i given ϕ_i are *iid* Poisson random variables, with mean $E(Y_i|\phi_i) = \mu_i$ and $\phi_i \overset{iid}{\sim} Gamma(\nu^{-1}, \nu^{-1})$, with $\nu > 0$. Show that

$$E(Y_i) = e^{\eta_i}$$
$$V(Y_i) = e^{\eta_i} + \nu e^{2\eta_i}.$$

(b) Denote by V the collection of functions f with $f'' \in L^2[0,1]$ and consider the subspace

$$W_2^0 = \{f(x) \in V : f, f' \text{ absolutely continuous and } f(0) = f'(0) = 0\}.$$

Define an inner product on W_2^0 as

$$< f, g >= \int_0^1 f''(t)g''(t)dt.$$

(i) Show that for $f \in W_2^0$, and for any s, $f(s)$ can be written as

$$f(s) = \int_0^1 (s-u)_+ f''(u)du,$$

where $(a)_+$ is a for $a > 0$ and 0 for $a \leq 0$.

(ii) Hence, obtain the reproducing kernel of W_2^0.

Marks: 10+(5+10)=25

4. (a) In the general multiple linear regression set-up derive the formula for the leverage values p_{ii}.

(b) Show that the variance of the regression coefficient $\hat{\beta}_j$ $(j = 1, \ldots, p)$ in the usual linear regression set up can be written in terms of the variance inflation factor VIF_j.

(c) In the linear regression framework: $\mathbf{y}^{(n\times 1)} = \mathbf{X}^{(n\times p)}\boldsymbol{\beta}^{(p\times 1)} + \boldsymbol{\epsilon}^{(n\times 1)}$, with $\mathbf{X}$ having full rank, consider testing

$$H_0 : \mathbf{R}^{(k\times p)}\boldsymbol{\beta}^{(p\times 1)} = \mathbf{q}^{(k\times 1)} \quad vs \quad H_1 : \mathbf{R}^{(k\times p)}\boldsymbol{\beta}^{(p\times 1)} \neq \mathbf{q}^{(k\times 1)},$$

where $\mathbf{R}$ has rank k. Let

$$F = \frac{(\mathbf{e}_R'\mathbf{e}_R - \mathbf{e}'\mathbf{e})/k}{\mathbf{e}'\mathbf{e}/(n-p)},$$

where $\mathbf{e}_R$ and $\mathbf{e}$ are the restricted and unrestricted least square residuals. Show that F is distributed as the $F_{k,n-p}$ under any spherically contoured distribution of $\boldsymbol{\epsilon}$.

Marks: 8+7+10=25

MULTIVARIATE ANALYSIS

M Stat 1st Year (2013-14)

Semester Examination

Date: 28.04.2014 Time: 2 hours 30 minutes

Total Marks 60

*Answer **Question 5** and any **3 (THREE)** questions of the following.*

1. Let U_i be the i-th principal component of $\mathbf{X}$, $i = 1,2,\ldots,p$. Then show that
(a) U_1 has maximum variance among all normed linear combination of $X_1, \ldots, X_p$
(b) U_i has maximum variance among all normed linear combination of $X_1, \ldots, X_p$ uncorrelated with $U_1,\ldots,U_{i-1}$; $i = 1,\ldots,p$.
(c) In principal component analysis explain how do you decide on the number of principal components.
(5+5+5=15)

2. (a) Let $A \sim W_p(n,\Sigma)$ and $B \sim W_p(m,\Sigma)$ and A and B are independently distributed. Then we know that $\frac{|A|}{|A+B|} \sim \Lambda_{p,m,n}$, where $\Lambda_{p,m,n}$ has the same distribution as that of $\prod_{i=1}^{p} U_i$ where $U_i \sim B\left(\frac{n-i+1}{2}, \frac{m}{2}\right)$ $i = 1,\ldots,p$ and U_i's are independently distributed.
(b) Assuming $n = r + 2\alpha$ where α is fixed, derive the asymptotic distribution of Wilk's lambda statistic $\Lambda_{p,m,n}$ and determine α suitably.
(7+8=15)

3. Let a random sample of size N_i be drawn from $N_p(\mu_i,\Sigma)$, $i = 1,2,\ldots,k$; Σ is an unknown positive definite matrix. Based on these samples, we want to test
$H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$ against H_1: *at least one equality in* H_0 *is false*
(a) Derive the likelihood ratio test statistic for the above test and find its distribution.
(b) Also construct a test for H_0 using the union-intersection principle. Check whether the test statistic thus obtained has the same distribution as that in (a). Justify your answer.
(6+9=15)

4. (a) Derive explicitly the characteristic function of A where $A \sim W_p(n,\Sigma)$.
Hence find the distribution of $A = \sum_{j=1}^{k} A_j$ where $A_j \sim W_p(n_j,\Sigma)$, $j = 1,2,\ldots,k$ and A_j's are independently distributed.
(b) Let $A \sim W_p(n,\Sigma)$. Also let $B = HAH'$ where H is any $p \times p$ orthogonal matrix, the elements of which are random variables distributed independently of A. Show that the distribution of B is distributed independently of H and also find the distribution of B.
(7+8=15)

5. What is the basic idea behind doing factor analysis? How does it differ from regression analysis? A set of $p = 10$ psychological variables were measured on $n = 810$ normal children with a certain correlation matrix. A maximum likelihood factor analysis was carried out with $k = 4$ factors yielding the following estimate of factor loadings (rotated by varimax):

$$\begin{pmatrix} -0.03 & 0.59 & -0.31 & 0.41 \\ -0.04 & 0.09 & -0.12 & 0.59 \\ -0.06 & 0.42 & -0.20 & 0.69 \\ -0.11 & 0.33 & -0.08 & 0.48 \\ -0.06 & 0.76 & -0.07 & 0.24 \\ 0.23 & -0.11 & 0.36 & -0.17 \\ 0.15 & -0.24 & 0.78 & -0.16 \\ 0.93 & -0.04 & 0.37 & -0.06 \\ 0.27 & -0.11 & 0.44 & -0.18 \\ 0.09 & -0.00 & 0.63 & -0.04 \end{pmatrix}$$

Give interpretation of the four factor using factor loadings. Justify your answer by using appropriate assumption / theory / idea relating to factor analysis. (15)

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination : 2013-14
M. STAT. I YEAR
Optimization Techniques

Date: 30 April 2014 | Maximum Marks: 60 | Duration: $2\frac{1}{2}$ hours

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This paper carries 70 marks. However, maximum you can score is 60.

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1 (a) Consider the transportation problem with m origins and n destinations. Let $\bar{A}$ be the coefficient matrix after one of the constraints (say, the last one) is dropped, i.e., $\bar{A}$ is an $(m+n-1)\times mn$ matrix. Show that every basis of $\bar{A}$ is lower triangular (possibly after appropriate permutation of its columns and rows).

(b) Prove that the North-West Corner rule gives a basic feasible solution of the transportation problem.

[10+10 = 20]

2 (a) Consider the Hungarian method for solving assignment problem. Develop a linear programming formulation of the problem to determine the maximum number of independent zero cells in a reduced cost matrix.

(b) Let a_i and b_j (for $i, j = 1, 2, \ldots, n$) be arbitrary given numbers. Solve the assignment problem having cost matrix $C = ((c_{ij}))$, where $c_{ij} = a_i + b_j$ for all $i, j = 1, 2, \ldots, n$.

[10+10 = 20]

3 (a) Let A be an $m \times n$ matrix, b an m-vector and c an n-vector. Suppose that the problem: $\min c^{\mathrm{T}}x$ subject to $Ax = b$ and $x \geq 0$ has an optimal solution. Show by duality theory that the new problem: $\min c^{\mathrm{T}}x$ subject to $Ax = d$ and $x \geq 0$ cannot be unbounded, no matter what value the vector d might take.

(b) Formulate the following as a dynamic programming problem:

$$\max\ z = (x_1+2)^2 + x_2x_3 + (x_4-5)^2$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 5$$
$$x_1, x_2, x_3, x_4 \geq 0, \text{ are integers.}$$

[P.T.O.]

(c) Formulate the following optimization problem as an equivalent mixed integer linear programming problem:

$$\begin{aligned} \min\ z &= 3x_1 + g(x_2) \\ \text{subject to} \\ x_1 + x_2 &\geq 15 \\ |x_1 - x_2| &= 0, \text{ or } 5, \text{ or } 10 \\ x_1, x_2 &\geq 0, \end{aligned}$$

where

$$g(x_2) = \begin{cases} 20 + 5x_2 & \text{if } x_2 > 0 \\ 0 & \text{if } x_2 = 0. \end{cases}$$

[10+10+10 = 30]

——*** xXx ***——

Indian Statistical Institute

M.Stat I

Discrete Mathematics

Semestral Examination

Maximum Marks: 80

Date: May 3, 2014,
Time 3 hours

The question paper contains 9 questions. Total marks is 90. Maximum you can score is 80. Unless otherwise mentioned, all notations are the same as presented in class.

1. Write an algorithm to find a maximum matching in a bipartite graph. Analyze the time complexity of your algorithm. (5+ 5 = 10)

2. Suppose G is a bipartite graph. Prove that,
   (a) $\alpha(G) \geq n(G)/2$.
   (b) $\alpha(G) = n(G)/2$ if and only if G has a perfect matching. (3 + 7 = 10)

3. Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersection points of the lines, with two vertices adjacent if they appear consecutively on one of the lines. Prove that $\chi(G) \leq 3$. (10)

4. Suppose that G is a simple graph with diameter 2 and that $[S, \bar{S}]$ is the minimum edge cut with $|S| \leq |\bar{S}|$.
   - Prove that every vertex of S has a neighbor in $\bar{S}$.
   - Use (a) to prove that $\kappa'(G) = \delta(G)$. (6+4 = 10)

5. If $\chi(H) < \chi(G)$, for all subgraphs H of G, then G is called *color-critical*. Prove that if G be a color critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical. (10)

6. Consider a graph $G(n, p)$ with $p = 1/n$. Let X be the number of triangles in the graph.
   - Show that, $Pr(X \geq 1) \leq 1/6$.
   - Using conditional expectation inequality to show that $\lim_{n\to\infty} Pr(X \geq 1) \geq 1/7$.

   (4 + 6 = 10)

7. Prove that the disappearance of isolated vertices in $G(n,p)$ has a sharp threshold of $\frac{\ln n}{n}$. (10)

8. Let $X_1, X_2, \cdots, X_n$ be independent random variables and $a > 0$, such that,
$$Pr(X_i = 1 - p_i) = p_i \text{ and } Pr(X_i = -p_i) = 1 - p_i.$$
   Let $X = \sum_{i=1}^{n} X_i$. Prove that,
$$Pr(|X| \geq a) \leq 2e^{-2a^2/n}.$$
   Hint: You might assume the inequality $p_i e^{\lambda(1-p_i)} + (1 - p_i)e^{-\lambda p_i} \leq e^{\lambda^2/8}$. (10)

9. Deduce the number of distinct cubes that can be obtained by coloring the vertices with two distinct colors. (10)