Indian Statistical Institute

Analysis of Discrete Data
Mid-Semestral Examination
M.Stat. II Year, 2013
Full Marks - 70
Time - 2 hrs. 30 mins.

Attempt all questions

1. Initially an experiment is set up with one toxic (*Dinophysis sp.*) and two non-toxic phytoplankton (*Chaetocerous gracilis* and *Biddulphia regia*) species. All the three phytoplankton growth profiles are monitored with regular recorded biomass of 16 samples on each of the 8 experimental days. The phytoplankton are collected from the deltaic region of river Subarnarekha ($87^o31^{'}$E and $21^o37^{'}$N) and the isolation is done in the laboratory. Species culture are maintained at optimal conditions in the laboratory in 30 separate bikers although the species might exhibits a negligible amount of genetic variation.

(a) Let us assume multivariate normality on empirically estimated Relative Growth Rate Vector evaluated from recorded biomass for each of the sample with exponential decay mean structure and suitably chosen covariance matrix. Unfortunately due to initial irregularities of the laboratory setup the recorded biomass are available only for the toxic species at final time point with grouped frequency distribution format. Treating these groups as cells and assuming suitable distribution of cell counts find the asymptotic distributions of estimators of model parameters and cell probabilities.

(b) For cell i define standardised residuals and comments on its asymptotic properties.

[15 + 5 = 20]

2. How a combination of several risk factors can be transformed to individual risk in epidemiological study ? Define odds and odds ratio in this context. How do you interpret the intercept term and risk coefficient from combined risk ? Estimation of individual risk is more appropriate in follow up study than case control study Justify. Define risk ratio for 2×2 contingency table and suggest an estimate of log relative risk. Modify this estimate with a less biased estimate and derived its asymptotic confidence interval. Show that ML estimates obtained through Newton-Raphson and Fishers scoring methods for logistic regression model with K covariates are equivalent.

[4 + 4 + 4 + 3 + 3 + 7 + 10 = 35]

3. Let p denote the sample proportion of successes in n Bernoulli trials. Find the asymptotic distribution of the sample logit. From 2×2 contingency table show that conditional ML estimate of odds ratio satisfies the equation $E(n_{11}) = n_{11}$, where n_{11} is the cell frequency for $(1,1)$ cell. In problem (1) if the number of cells N is fixed and all cell probabilities are positive then show that when the model holds, the likelihood - ratio statistic is asymptotically equivalent to χ^2 statistic as the total cell counts is infinite.

[4 + 5 + 6 = 15]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2013-2014, First Semester

M-Stat II

Set Theory and Topology

Date: 02/09/13 **Max. Marks 60** **Duration: 3 Hours**

Note: Answer all questions. Your score will be reduced to 30.

In the following, **R** and **N** will denote set of real numbers and set of natural numbers respectively.

1. a) Show that a linearly ordered set $(W, \leq)$ is well-ordered if and only if there is no infinite strictly descending sequence $w_1 > w_2 > w_3 \cdots$ in W.

   b) Let $\mathbf{N}' = \mathbf{N} - \{1\}$ be the set of positive integers greater than 1. Define $m \leq n$ if n divides m. Show that this is a partial ordering, that every chain has an upper bound and determine the set of maximal elements.

   [7+8]

2. a) Let X be a topological space and $x \in X$ be a cluster point of $A \subset X$. Suppose x has a countable basis, i.e. a countable collection $\{U_n^x\}$ of open neighbourhoods of x such that every open neighbourhood of x contains at least one U_n^x. If $x \notin A$ then show that there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$.

   b) Let $\{(X_i, d_i)\}_{i \in J}$ be an uncountable family of metric spaces, X_i containing more than two points for all $i \in J$. Show that $X := \prod_{i \in J} X_i$ with product topology is not first countable, i.e. there exists $x \in X$ which does not admit a countable basis.

   [7+8]

3. Let $g : X \to Z$ be a surjective continuous map. Define an equivalence relation $\sim$ on X as $x \sim x'$ if and only if $g(x) = g(x')$. Let X^* be the set of corresponding equivalence classes with quotient topology. Show that X^* and Z are homeomorphic if and only if g is a quotient map.

b)Let $X := \mathbf{R}^2$ with usual topology. Let $(x, y) \sim (x', y')$ if and only if $x + y^2 = x' + y'^2$. Let X^* be the set of corresponding equivalence classes with quotient topology. Identify X^* with a well-known space—-show that this space is homeomorphic to X^*.

c) Let X be a topological space. Let R_1 and R_2 be two equivalence relations on X such that $xR_1x' \Rightarrow xR_2x'$ for every pair x, x'. Let $X^*_{R_1}$ and $X^*_{R_2}$ be the sets of equivalence classes with respect to R_1 and R_2 respectively with quotient topologies. Show that $X^*_{R_1}$ is a quotient space of $X^*_{R_2}$.

[5+5+5]

4. a) Let $X := \mathbf{R}^2$ with usual topology. Let $A \subset X$ be countable. Show that $X - A$ is path-connected.

b) Consider $X := \mathbf{R}^{\mathbf{N}}$ with box topology. Show that $\tilde{x}$ and $\tilde{y}$ in X lie in the same connected component if $\tilde{x} - \tilde{y}$ is eventually zero , that is zero except for finitely many coordinates.

[7+8]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : Semester I (2013-14)

M. Stat. II Year

Actuarial Methods

ıte: 02.09.2013 Maximum marks: 50 Time: 2 hours 15 minutes

ılculator and Actuarial table can be used. Answer as many as you can. Total mark is 54.

1. (a) Describe domination of strategies in the context of zero-sum two-players game.

   (b) An entrepreneur is trying to decide whether to expand his business. He can either go ahead and *expand*, or *wait* until a more suitable time. He believes that there are two possible economic scenarios for the current year: *good* and *poor*. If he waits, he expects to make a profit of Rs 70,000 in the coming year if the economy is good, and Rs 30,000 if the economy is poor. If he expands, he expects to make a profit of Rs 150,000 if the economy is good, and a profit of Rs 6,000 if the economy is poor. The entrepreneur wishes to minimize his expected loss in the coming year. Let p be the probability of a recession (poor economy) in the current year. What values of p will make him wait?

[2+6=8]

2. An actuarial student observes that the size of claims follows a Gamma distribution with parameters a and l having density

$$f(x; a, l) = \frac{l^a}{\Gamma(a)} e^{-lx} x^{a-1}.$$

Past experience suggests that while a is known, l is unknown and has a prior distribution that can be modeled as exponential with mean m. The actuarial student has obtained recent claims data $x_1, \cdots, x_n$, where x_i is the size of the ith claim.

   (a) Derive the posterior distribution of l.

   (b) Determine the Bayesian estimate of l under zero-one loss and quadratic loss.

   (c) If $a = 100$, $m = 10$ and the student observes that the last 10 claims total 250, calculate the Bayesian estimate of l under both zero-one and quadratic loss.

[4+(2+2)+2=10]

3. The last ten claims (in rupees) under a particular class of insurance policy were: 1330 , 201 , 111 , 2368 , 617 , 309 , 35 , 4685 , 442 , 843.

   (a) Assuming that the claims come from a log-normal distribution with parameters μ and σ^2, find the maximum likelihood estimates of these parameters using the observed data.

P.T.O.

(b) Assuming that the claims come from a Pareto distribution with parameters λ and α, use the method of moments to estimate these parameters.

(c) If the insurance company takes out reinsurance cover with an individual excess of loss of Rs. 3,000, estimate the percentage of claims that will involve the re-insurer under each of the two models above.

[4+4+4=12]

4. Describe compound Poisson distribution in the context of general insurance and derive its moment generating function using that of the claim distribution. [2+2=4]

5. (a) Mention two characteristics of an insurable risk.

(b) Describe employers' liability.

(c) Mention two differences between collective and individual risk models.

(d) Consider an XOL arrangement with retention limit M. The re-insurer, having extensive experience in this line of business, suggests that 70% of the claims are exponentially distributed with mean 4 and 30% of claims are exponentially distributed with mean 10. Determine the probability that a randomly selected claim will need to go to the re-insurer.

(e) Consider an XOL reinsurance arrangement with retention limit M (fixed) and annual inflation factor k on the claim amount. Will the mean insurer's pay-off be inflated by the factor k? Justify your answer.

[2+2+2+2+2=10]

6. The ISI employees are covered by a group life insurance which pays a specific benefit amount (in Rs.) if an employee dies while in service. There are two categories of employees who are entitled to the following benefit amounts with corresponding probability of dying during a year:

Category	No. of employees	Benefit amount	Prob. of dying
Active	1250	50,000	0.008
Affiliated	250	20,000	0.012

Using individual risks model, calculate the mean and variance of the aggregate claim amount during a year. Find the probability that the aggregate claim amount in a given a year will exceed Rs 1000,000. What loading factor should be used to fix the premium to be 99% sure of making a profit in this portfolio? Assume a normal approximation whenever necessary.

[2+3+2+3=10]

INDIAN STATISTICAL INSTITUTE

MID-SEMESTRAL EXAMINATION

FIRST SEMESTER, 2013-14

TIME: 3 HOURS | STATISTICAL COMPUTING | FULL MARKS: 60

1. Give suitable algorithms for generating random variables from the following distributions.

   (a) $F(x) \propto (x + x^2 + x^4 + x^7), \ \ 0 \le x \le 1.$ [4]

   (b) $F(x) \propto \int_0^1 x^y e^{-y}\, dy, \ \ 0 \le x \le 1.$ [6]

2. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$ and $D = (x_4, y_4)$ be the four vertices of a quadrilateral $ABCD$.

   (a) How will you check whether the quadrilateral is convex ? [4]

   (b) If a new point $P = (x, y)$ is given to you, describe how you will check whether it lies inside the quadrilateral. [6]

3. Suppose that $\binom{X_1}{Y_1}, \binom{X_2}{Y_2}, \ldots, \binom{X_n}{Y_n}$ are i.i.d. observations from a bivariate distribution with the dispersion matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, where ρ is unknown.

   (a) Check whether $\frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})$ is an unbiased estimator of ρ. [3]

   (b) Find the corresponding jackknife estimator and check its unbiasedness. [7]

4. Consider a linear model $Y_i = \beta_0 + \beta x_i + \epsilon_i$; $i = 1, 2, \ldots, n$, where the x_i's are non-stochastic and the ϵ_i's are independent and identically distributed with the mean 0 and the unknown variance σ^2. Describe how you will construct a 95% confidence interval for β using the residual bootstrap method. [5]

5. If $\mathbf{X}$ follows an elliptically symmetric distribution with the mean vector $\boldsymbol{\mu}$ and the dispersion matrix $\boldsymbol{\Sigma}$, prove the following results.

   (a) The density function of $\mathbf{X}$ is of the form $f(\mathbf{x}) = C|\boldsymbol{\Sigma}|^{-1/2} g\left\{(\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\}$, where C is a normalizing constant, and $g : [0, \infty) \to [0, \infty)$ is a scalar function. [5]

   (b) The distribution of $\boldsymbol{\alpha}'(\mathbf{X} - \boldsymbol{\mu})/\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha}}$ does not depend on the direction vector $\boldsymbol{\alpha}$. [5]

   (c) Half-space depth of an observation $\mathbf{x}_0$ with respect to this distribution is a decreasing function of $(\mathbf{x}_0 - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}_0 - \boldsymbol{\mu})$. [5]

6. In order to estimate the value of e, a person generated 500 observations from a Poisson distribution with the mean unity. The frequency distribution of the observations are given below.

Value (i)	0	1	2	3	4	5	6	Total
Frequency (f_i)	186	183	86	27	9	7	2	500

   (a) Find an estimate of e and comment on its unbiasedness. [4]

   (b) Find a minimizer of $\psi(\theta) = \left[\sum_{i=0}^6 |i - \theta|\ f_i \ + \ 0.82 \sum_{i=0}^6 (i - \theta)\ f_i\right]$ and comment its uniqueness. [6]

Indian Statistical Institute
Semester 1, Academic Year: 2013-14
Mid-Semester Examination
Course: M. Stat 2nd Year
Subject: Advanced Probability 1

Total Points: $8 \times 4 = 32$

Date: 6.9.2013 **Time: 2 Hours 30 minutes**

The maximum you can score is 30. Answers must be justified with clear and precise arguments. If you refer to a theorem/result proved in class, state it explicitly. More than one answer to a question will not be entertained and only the first uncrossed answer will be graded.

1. (a) Suppose F is a distribution function on $\mathbb{R}$ satisfying the following condition: $4 + 4 = 8$ pts.

   For every $\epsilon > 0$ one can find $\delta > 0$ such that $\sum_{i=1}^{\infty}(F(b_i) - F(a_i)) < \epsilon$ for non overlapping intervals $(a_i, b_i), a_i < b_i, i = 1, 2, \ldots$ with $\sum_{i=1}^{\infty}(b_i - a_i) < \delta$.

   Show that under the above condition $\lambda(A) = 0$ implies $\mu_F(A) = 0$ where $A \in \mathcal{B}(\mathbb{R})$, λ is Lebesgue measure and, μ_F is the measure corresponding to F.

   (b) Suppose μ and ν are finite measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The Lebesgue decomposition theorem says $\nu = \nu_1 + \nu_2$ where $\nu_1 \perp \mu$ and $\nu_2 \ll \mu$. Prove the uniqueness of the Lebesgue decomposition.

2. (a) Suppose X and Y are nonnegative random variables in $L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a sub sigma-algebra of $\mathcal{F}$. Suppose also that Y is $\mathcal{G}$-measurable. Is the following true (if true prove it) or can you give a counterexample

   $$E(XY|\mathcal{G}) = YE(X|\mathcal{G}), a.s.?$$

   4 pts.

   (b) Suppose $\{X_n\}_{n\geq 1}, X$ are random variables in $L^p(\Omega, \mathcal{F}, P)$ for some fixed $p \geq 1$ and $\mathcal{G}$ is a sub sigma-algebra of $\mathcal{F}$. Show that

   $$X_n \xrightarrow{L^p} X \Rightarrow E(X_n|\mathcal{G}) \xrightarrow{L^p} E(X|\mathcal{G}).$$

   4 pts.

3. According to the notation in Kakutani's theorem, on $(\Omega_i, \mathcal{F}_i)$ one has two probability measures Q_i and P_i, $i = 1, 2, \cdots$. Assume that $Q_i \sim P_i$, and denote $\rho_i = dQ_i/dP_i, \forall i$. Let Q^n, P^n denote the n-fold product measures, $\rho^n = \prod_{i=1}^n \rho_i$ the Radon-Nikodym derivative of Q^n w.r.t. P^n (remember our assumption implies $Q^n \sim P^n$). Let Q, P denote the infinite product measures. $4 + 4 = 8$ pts.

   (a) Assume Kakutani's theorem and show that if $\prod_{i=1}^\infty \int_{\Omega_i} \sqrt{\rho_i} dP_i > 0$ then ρ^∞, the L^1-limit of ρ^n, satisfies $0 < \rho^\infty < \infty$ a.s. P.

   (b) Under the assumption of (a) show that $\rho^n \to \rho^\infty$ a.s. P. (This part can be done with your previous knowledge of probability and reference to the Martingale Convergence Theorem will not be awarded points.)

4. (a) Consider $\mathbb{R}^{[0,1]}$ with the Borel sigma-algebra that makes each projection measurable. On the other hand $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ denotes the product space with countably many components with the corresponding Borel sigma-algebra. In $\mathbb{R}^{[0,1]}$ with its Borel sigma-algebra consider two cylinder sets

$$A_1 = \pi^{-1}_{t_{11}, t_{12}, \dots}(B_1), A_2 = \pi^{-1}_{t_{21}, t_{22}, \dots}(B_2),$$

   where $B_1, B_2 \in \mathcal{B}(\mathbb{R}^\infty)$. Describe the base of the cylinder set $A_1 \cap A_2$. (The point is that the coordinate sets $\{t_{11}, t_{12}, \dots\}$ and $\{t_{21}, t_{22}, \dots\}$ may intersect.)

   2 pts.

   (b) Write down the statement of Kolmogorov's consistence theorem for probability measures on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ and the **proof** of countable additivity (measure on an algebra condition) of the probability defined on the cylinder sets. You need to write only one of the proofs, due to Ionescu-Tulcea or due to Kolmogorov.

   6 pts.

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination (2013–2014)

M. STAT. II (MSP)

Functional Analysis

Date : 09.09.2013 Maximum Marks : 80 Time : $2\frac{1}{2}$ hrs.

The paper carries 85 marks. Maximum you can score is 80. Precisely justify all your steps. Carefully state all the results you are using.

1. Let $\mathcal{P}$ be the vector space of all polynomials on $[a, b]$. If $p(x) = a_0 + a_1 x + \ldots + a_n x^n$, define $\|p\| = |a_0| + |a_1| + \ldots + |a_n|$.

   (i) Show that $\|\cdot\|$ is a norm on $\mathcal{P}$. [5]

   (ii) Show that $(\mathcal{P}, \|\cdot\|)$ is not a Banach space. [10]

   [Indeed, no norm on $\mathcal{P}$ is complete!]

2. Let X and Y be normed linear spaces and $T : X \to Y$ be a linear map. Define

$$\|x\|_1 = \|x\| + \|T(x)\|, \quad x \in X.$$

   (a) Show that $\|\cdot\|_1$ is a norm on X. [5]

   (b) Show that $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ if and only if T is continuous. [5]

3. Show that if any two norms on a vector space X are equivalent, then X must be finite dimensional. [15]

   [The converse was proved in class!]

4. A subset $A \subseteq X^*$ is said to separate points of X if whenever $x_1 \neq x_2 \in X$, then there exists $f \in A$ such that $f(x_1) \neq f(x_2)$.

   Let X, Y be Banach spaces. Let $A \subseteq Y^*$ separate points of Y. Let $T : X \to Y$ be a linear map. Show that T is continuous if and only if $g \circ T$ is continuous for all $g \in A$. [10]

5. Let X, Y be Banach spaces. Let $T \in B(X, Y)$. Show that there is a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$ if and only if T is 1-1 and $T(X)$ is closed in Y. [15]

[PTO]

6. Give an example of closed subspaces M and N of a Hilbert space $\mathcal{H}$ such that

$$M + N = \{x + y : x \in M,\ y \in N\}$$

is not closed. [10]

7. Let $\{e_n\}$ be an orthonormal basis in a Hilbert space $\mathcal{H}$. Show that any orthonormal set $\{f_n\}$ that satisfies

$$\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$$

is also an orthonormal basis. [10]

Advanced Design of Experiments
M. Stat II Yr. 2013-2014
Full Marks. 60
Time: 2 hours

Date: 10.09.13

(You can assume the form of the C- matrix of a BIBD, if you need to use it at any place)

1. (a) Define A-, D- and E-optimality criteria briefly stating(no proof) their statistical significance for a nonsingular inferential problem $\Pi : \eta = L\tau$.
   (b) State(proof is not required) a set of sufficient conditions to characterize an A-, D- and E- optimal design for estimating a full set of orthonormal contrasts.
   (c) Obtain the expression of the average variance of all elementary treatment contrasts for a connected block design. Show that a BIBD if exists, is A- optimal for the estimation of the full set of elementary treatment contrasts.
   (d) Obtain the value of the generalised variance of the BLUE of a full set of orthonormal contrasts using the following block design d_0 with 10 treatments in 5 blocks of size 6 each:

$d_0 =$

1	2	3	4	5	10
1	2	3	6	7	8
1	4	5	6	7	9
2	4	6	8	9	10
3	5	7	8	9	10

   Will you recommend this design for use? Briefly justify your answer.

[(9+2)+(7+8)+(8+3)=37]

2. Construct a Universally optimal row column design for 6 treatments in a 6x5 array. [7]

3. Prove or disprove the following claims with proper justification.
   (a) A Hadamard matrix of order 13 always exists.
   (b) If a design is E- optimal for the estimation of a set of parametric functions η, then it also remains E- optimal for the estimation of a nonsingular transform of η.
   (c) The following row-column design d^* is a treatment connected design.

$d^* =$

1, 4, 1	2	5, 5	6, 4, 1
3, 3	1, 1	6, 7, 1	8, 8
8, 8	6, 4, 4	1, 4, 4	3, 3
4, 7	2, 2	1, 6	5

[5+3+8= 16]

Date: 10.9.2013
Time: 2 hours

Statistical Methods in Genetics – I
M-Stat (2nd Year) 2013-14
Mid-Semester Examination

The paper carries 40 marks. This is an open notes examination.
Answer all questions.

1. Consider a disorder controlled by an autosomal biallelic locus. Suppose 100 nuclear families each comprising two unaffected parents and one offspring and 60 nuclear families each comprising exactly one parent affected and one offspring are randomly selected from the population. It was found that in the first group of families, there were 10 affected offspring while in the second group of families, there were 15 affected offspring. Do the above data support a recessive mode of inheritance? [16]

2. In every generation, a certain proportion α of a population practices self-mating, while the remaining $(1-\alpha)$ proportion of the population practices random mating. Will the population be in Hardy-Weinberg Equilibrium at a triallelic autosomal locus? Will the allele frequencies at such a locus remain constant over generations? [6 + 6]

3. Given the genotype of one parent at an autosomal locus, are the genotypes of a pair of sibs independent? Given the genotypes of a pair of sibs, are the genotypes of the two parents independent? [6 + 6]

Theory of Finance I

Midsem. Exam. / Semester I 2013-14

Time - 2 hours/ Maximum Score - 30

10.09.13

1. (3+3+6=12 marks)

   (a) Let $\{R_i\}$ be a sequence of i.i.d. $N(\mu, \sigma^2)$ random variables and let $X_n = R_1 + \cdots + R_n$. Define a filtration $\{\mathcal{F}_n\} = \sigma\{R_1, \ldots, R_n\}$. Are the sequences below follow martingale property w.r.t. $\{\mathcal{F}_n\}$ under some conditions on μ and σ^2? Justify.

   (i) $Z_n = X_n^2 - n$; (ii) $W_n = \exp\{X_n - (\mu n + \sigma^2 n/2\}$,

   (b) Let $\{W_1(t)\}_{t\geq 0}$ and $\{W_2(t)\}_{t\geq 0}$ be two independent standard Brownian motion. For any non-zero constants σ_1 and σ_2 show that $W(t) = \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}} W_1(t) + \frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}} W_2(t)$ is a standard Brownian motion. Can you modify the definition of $W(t)$, when the above two Brownian motions are not independent but jointly Normal non-zero covariance as ρt, so that it becomes a standard Brownian motion?

2. (4+4+2=10 marks) Assume a binary tree model for the price of a security, i.e., each time it goes up by a percentage u or it goes down by a percentage d, where $d = 1/u$. Let r be the risk-free interest rate per annum. Someone buys a product that pays one unit of money if the stock goes above a level K at the maturity time T.

   (a) Assuming one step model, Find the value of the Δ in terms of future share prices (S_u and S_d) and future payoffs (f_u, f_d), so that the portfolio would be riskless.

   Let r be the risk-free interest rate per annum. Find the risk-neutral probability p, (in terms of S_0, u, d and r). Show that, (assuming $S_d < K < S_u$), it satisfies the equation,

   $$f_0(1 + rT) = [pf_u + (1 - p)f_d]$$

   (b) Now, using the limiting form of the distribution of S_T (i.e., Log-Normal distribution), find the expression for f_0, the price of the derivative.

   (c) Calculate the price of this derivative at time zero, taking, $S_0 = 60$, $r = 6\%$, the strike price $K = 63$, $\sigma = 30\%$ and the maturity time $T = 6$ months.

3. (6+6=12 marks)

   (a) For B-S-M equation, $\frac{\partial f}{\partial t} + (r - \frac{\sigma^2}{2})\frac{\partial f}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial z^2} = rf$, derive the expressions for the parameters α, β, γ corresponding to Explicit difference method. Write the conditions, with arguement, for which this will give a trinomial model.

   (b) Let $S_0 = 60$, $K = 63$, $T = 3$ month , $r = 6\%$ and $\sigma = 25\%$. Show that for some appropriate Δt, there exist a range of $C = \Delta t/(\Delta z)^2$ for which S_t is a discounted martingale for the trinomial model (coming out of Explicit difference method). Write down the probabilities (of the trinomial model) and for two such C's find the Put option price at time 0. Which of the C's would you take?

All the best.

LIFE CONTINGENCIES

Date: 10 September, 2013 Time: 2:30 pm

Duration: 3 hours Maximum Marks: 100

Note: (i) Desk calculators are allowed; (ii) Actuarial tables are allowed; (iii) Symbols and notations have their usual meaning. The entire question paper is for 115 marks.

1. If $P[X > x] = [1 - (x/100)]^{1/2}$, $0 \leq x \leq 100$, evaluate

   (a) ${}_{15}p_{36}$; [2]

   (b) ${}_{17}q_{19}$; [2]

   (c) ${}_{17|15}q_{19}$; [3]

   (d) μ_{50}. [3]

2. Calculate the following quantities from the AM92 tables ($i = 0.04$):

   (a) ${}_{3|3}q_{45}$; [2]

   (b) ${}_{4|}q_{[36]+1}$; [2]

   (c) Variance of the present value of an insurance benefit, where the mean of that random variable is $A_{[41]}$; [2]

   (d) ${}_{5|}\ddot{a}_{45}$; [2]

   (e) $(IA)_{50:\overline{10|}}$. [2]

3. Give an expression for μ_{x+t} in terms of t and p_x for $0 \leq t \leq 1$ and $x = 1, 2, 3, \ldots$, when interpolation between integer ages is made

   (a) by assuming constant force of mortality; [3]

   (b) by assuming uniform distribution of death. [3]

4. Show algebraically that, under the assumption of a uniform distribution of death over the insurance year of age, $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$. [6]

5. If $q_{70} = 0.04$ and $q_{71} = 0.05$, calculate the probability that (70) will die between ages 70.5 and 71.5, assuming that deaths are uniformly distributed within each integer year of age. [5]

6. If $A_x = 0.25$, $A_{x+20} = 0.40$ and $A_{x:\overline{20|}} = 0.55$, calculate $A^{1}_{x:\overline{20|}}$. [8]

[P.T.O.]

7. Prove and interpret the following relations:

(a) $a_{x:\overline{n}|} = {}_1E_x \times \ddot{a}_{x:\overline{n}|}$; [6]

(b) ${}_{n|}a_x = \dfrac{A_{x:\overline{n}|} - A_x}{d} - {}_nE_x$. [6]

8. A life insurance company issues identical deferred annuities to each of 100 women aged 63 exact. The benefit is Rs. 5,000 per annum payable continuously from a woman's 65th birthday, if still alive at that time, and for life thereafter.

(a) Write down an expression for the random variable for the present value of future benefits for one policy at outset. [3]

(b) Calculate the total expected present value at outset of these annuities. [3]

Basis:

Mortality : PFA92C20
Interest : 4% per annum

(c) Calculate the total variance of the present value at outset of these annuities, using the same basis as in part (b). [8]

9. If $P^{1\ (12)}_{x:\overline{20}|} = 1.032 P^{1}_{x:\overline{20}|}$ and $P_{x:\overline{20}|} = 0.040$, what is the value of $P^{(12)}_{x:\overline{20}|}$? [8]

10. Write the prospective formula for the benefit reserve required at the end of 5 years for a unit benefit 10-year term insurance issued to (45) on a single premium basis. [6]

11. A 20-year special endowment assurance policy is issued to a group of lives aged 45 exact. Each policy provides a sum assured of Rs. 10,000 payable at the end of the year of death or Rs. 20,000 payable if the life survives until the maturity date. Premiums on the policy are payable annually in advance for 15 years or until earlier death.

You are given the following information:

Number of deaths during the 13th policy year : 4
Number of policies in force at the end of the 13th policy year : 195

Calculate the profit or loss arising from mortality in the 13th policy year. [8]

Basis:

Mortality : AM92 Ultimate;
Interest : 4% per annum;
Expenses : none.

12. The expense-loaded annual premium for a Rs. 1,000 endowment-at-age-65 life insurance with level annual premiums issued at age 40 is calculated using the following assumptions:

- selling commission is 40% of the expense-loaded premium in the first year;
- renewal commissions are 5% of the expense-loaded premium for policy years 2 through 10;
- premium tax is 2% of the expense-loaded premium each year;
- maintenance expense is 1.25% of insurance in the first year and 0.4% of insurance thereafter;
- the benefit premium is to provide for the immediate payment of death claims with no premium adjustment on death;
- a 15-year select-and-ultimate mortality table is to be used.

Write an expression for the expense-loaded premium. [8]

13. A life insurance company is considering selling with-profit endowment policies with a term of twenty years and initial sum assured of Rs. 100,000. Death benefits are payable at the end of the policy year of death. Bonuses will vest at the end of each policy year. The company is considering three different bonus structures:

I. Simple reversionary bonuses of 4.5% per annum.

II. Compound reversionary bonuses of 3.84615% per annum.

III. Super compound bonuses where the original sum assured receives a bonus of 3% each year and all previous bonuses receive an additional bonus of 6% each year.

(a) Calculate the amount payable at maturity under the three structures. [4]

(b) Calculate the expected value of benefits under structure II for an individual aged 45 exact at the start. [4]

Basis:

Mortality	:	AM92 Select;
Interest	:	8% per annum;
Expenses	:	ignore.

[P.T.O.]

(c) Calculate the expected value of benefits, using the same policy and basis as in (b) but reflecting the following changes, considering one of these at a time:

   i. Bonuses vest at the start of each policy year, including the first policy year (the death benefit is still payable at the end of the policy year of death). [2]

   ii. The death benefit is payable immediately on death (while bonuses vest at the end of each policy year). [2]

   iii. The death benefit is payable immediately on death, and bonuses vest continuously. [2]

INDIAN STATISTICAL INSTITUTE

Midterm Examination : 2013-14(Third Semester)

M. Stat. II year

Graph Theory and Combinatorics

Date: 11.09.13 Maximum Marks : 80 Duration : 3 hrs

Note: Answer as many as you can. The maximum you can score is 80. Notation is as used in the class.

1. (i) Show that the sequence $(6, 6, 5, 4, 3, 3, 1)$ is not graphic.

   (ii) Show that if $\mathbf{d} = (d_1, \ldots, d_n)$ is graphic and $d_1 \geq d_2 \geq \ldots \geq d_n$, then Σd_i is even and

$$\Sigma_{i=1}^{k} d_i \leq k(k-1) + \Sigma_{i=k+1}^{n} min\{k, d_i\}, 1 \leq k \leq n.$$

[3+7=10].

2. (i) Show that for any graph $G = (V, E)$ and any subset X of V

$$|\partial(X)| = \Sigma_{v \in X} d(v) - 2e(X),$$

   where $e(X)$ denotes the number of edges in the subgraph induced by X.

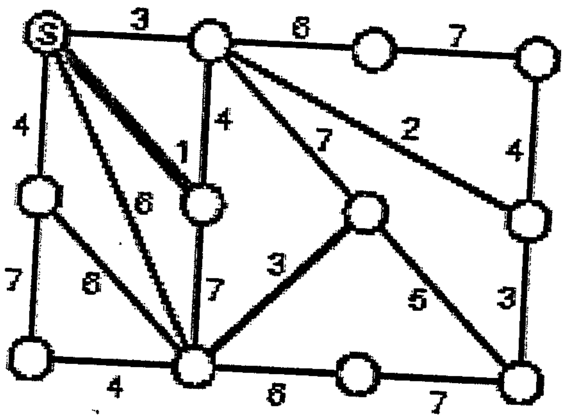
   (ii) Show that a connected graph is Eulerian iff it is even. [5+10]

3. (i) Denote the number of spanning trees of a graph G by $t(G)$. With usual notation, show that, for a link e of G,

$$t(G) = t(G-e) + t(E/e).$$

   (ii) Write down the Jarnik-Prim's algorithm for finding a minimum weight spanning tree of a weighted connected graph.

Use the algorithm for the following weighted graph starting from S.

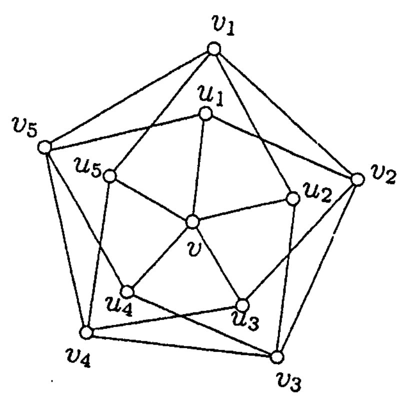


[5+10=15]

4. (i) In a k-colouring of a k-chromatic graph, show that there is a vertex of each colour which is adjacent to vertices of every other colour.

(ii) Show that there is no graph with chromatic polynomial $x^4 - 3x^3 + 3x^2$.

(iii) Find the chromatic number of the following graph.



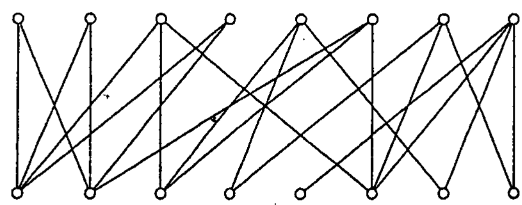
(iv) Let G be a triangle-free graph with chromatic number k. Form a graph G' from G which is triangle-free and has chromatic number $k+1$. [5+5+5+5=20]

5. (i) State Hall's Matching Theorem.
Let $\mathcal{A} = \{A_i; i \in I\}$ be a finite family of finite sets. Deduce that

$\mathcal{A}$ has a system of distinct representatives iff $|\bigcup_{i\in J} A_i| \geq |J|$ for all subsets J of I.

(ii) Let M be a matching and K a covering in a graph G such that $|M| = |K|$. Show that M is a maximum matching and K is a minimum covering.

(iii) Apply the Hungarian Algorithm to find a maximum matching in the following bipartite graph.



(iv) Let $G = (V, E)$ be a graph. Define an $n \times n$ matrix $A = (a_{ij})$ as follows.

$$a_{ij} = x_{ij} \text{ if } \{v_i, v_j\} \in E$$

$$a_{ij} = 0 \text{ if } \{v_i, v_j\} \notin E.$$

Define $Q(x_{11}, \ldots, x_{nn}) = det(A)$. Show that G has a perfect matching iff $Q \not\equiv 0$ [7+5+7+6=25]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination : Semester I (2013-2014)

M. Stat 2nd Year

Pattern Recognition and Image Processing

Date: 12. 9. 13 Maximum marks: 30 Time: 2 hours.

Note: Answer all questions. This paper carries 36 points. Maximum you can score is 30. You may use any result proved in class but the result has to be stated clearly.

1. For the two-class classification problem with multivariate normal class-conditional probability density functions with known parameters $\boldsymbol{\mu}_i, i = 1, 2$ and common Σ, compute the Bayes error assuming equal costs of misclassification where the prior class probabilities $\pi_1 = 0.6$, $\pi_2 = 0.4$ and

$$\boldsymbol{\mu}_1 = (2, 4), \quad \boldsymbol{\mu}_2 = (6, 8), \quad \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 4 \\ 4 & 9 \end{bmatrix}$$

[10]

2. Based on an iid sample of size n from the uniform[0,1] density, calculate the bias of the kernel density estimator $\hat{f}(x)$ for all real values of x using bandwidth h and

   (a) a gaussian kernel function. [6]

   (b) an exponential kernel function, $K(x) = e^{-x}$ for $x > 0$ and 0 for $x < 0$. [8]

3. Consider a J-class problem with prior class probabilities $\pi_i, i = 1, \ldots, J$ based on a two-dimensional feature vector $\mathbf{x}$. The density of $\mathbf{x}$ for the j-th class is uniform over the union of the unit ball centered at 0 and the square $[j, j+1] \times [j, j+1]$. Find out the overall Bayes error probability for this problem and the overall asymptotic error probability of the 1-NN rule when

   (a) $\pi_i, i = 1, \ldots, J$ are all equal. [6]

   (b) $\pi_1 = 1/2$, and $\pi_i, i = 2, \ldots, J$ are all equal. [6]

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination : (2013-2014)

M.Stat. 2nd Year

TOPICS IN BAYESIAN INFERENCE

Date: 14 September, 2013 Max. Marks: 90 Duration: $2\frac{1}{2}$ Hours

Answer as many questions as you can. Maximum you can score is 90.

1. What is the difference between the Bayesian paradigm and classical inference with respect to evaluation of performance of a decision rule.

Give an example showing the paradoxical behaviour of an inference procedure based on averaging over the sample space. [4+5=9]

2. Consider i.i.d. observations $X_1, \ldots, X_n \sim f(x|\theta) = \exp\{c(\theta) + \theta t(x)\}h(x)$ (one parameter exponential family). It is given that the usual regularity conditions hold and $c(\theta)$ is sufficiently smooth. A statistically natural parameter is $\mu = E_\theta t(X)$. Show that μ is a one-one function of θ. Find MLE of μ. Show that for a conjugate prior $\pi(\theta) = c\exp\{mc(\theta) + \theta s\}$, the posterior mean of μ is a weighted average of prior estimate and classical estimate (MLE). (assume that π is supported on $[a, b]$ where $\pi(a) = \pi(b) = 0$.)

[16]

3. Let $X_1, \ldots, X_n$ be i.i.d. $N(0, \sigma^2)$ where $\sigma(> 0)$ is unknown. Consider a conjugate prior for σ and find the posterior distribution. Also find the Bayes estimate of σ for this prior and the loss function $L(a, \sigma) = (a - \sigma)^2/\sigma^2$. (It will be enough to find the Bayes estimate as a one-dimensional integral.)

[12]

4. Let $X_1, \ldots, X_n$ be a random sample from a $U(\theta - 1/2, \theta + 1/2)$ distribution, $-\infty < \theta < \infty$. Consider the noninformative prior $\pi(\theta) = 1$. What will be your 95% credible interval for θ?

[9]

5. (a) Let $X_1, \ldots, X_n$ be i.i.d. with a common density $f(x|\theta)$ where $\theta \in R$. State the result on asymptotic normality of posterior distribution of suitably normalized and centered θ under suitable conditions on the density $f(\cdot|\theta)$ and the prior distribution.

(b) Suppose that the above result is proved for a proper prior. Consider an improper prior for which there is an n_0 such that the posterior distribution of θ given $x_1, \ldots, x_{n_0}$ is proper for *a.e.* $(x_1, \ldots, x_{n_0})$. How will you modify the proof?

(c) Use a stronger version of the result on asymptotic normality of posterior (to be stated by you) to prove that $\sqrt{n}(\tilde{\theta}_n - \hat{\theta}_n) \to 0$ with probability one, where $\tilde{\theta}_n$ and $\hat{\theta}_n$ denote respectively the posterior mean and MLE.

[6+5+8=19]

6. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two independent random samples from two normal populations with distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ respectively. Assume that the prior distribution of $(\mu_1, \mu_2, \log \sigma^2)$ is improper uniform where μ_1, μ_2, and σ^2 are independent. Find the posterior distribution of $\mu_1 - \mu_2$.

[20]

7. Let $X_1, \ldots, X_n$ be i.i.d. observations with a common density $f(x|\theta)$, $\theta \in \Theta = \{\theta_1, \theta_2\}$. Consider a prior distribution (π_1, π_2) for θ.

(a) Find the posterior distribution of θ.

(b) Assume that

$$E_{\theta_i} \log \frac{f(X_1|\theta_i)}{f(X_1|\theta_j)} > 0 \text{ for } i \neq j.$$

Show that the posterior distribution is consistent at each θ_i.

[3+9=12]

Semestral Examination

Advanced Design of Experi ments

M.Stat- 2nd Year : 2013-14

Full Marks: 100 | Time : 3.5 hours | Date: 11·11·13

Answer any four questions.

Q1. a). Define Universal Optimality criterion. State and prove a set of sufficient conditions for a design d^* to be universally optimal in a specified class and for a given inference problem.

b) Define a Balanced Block design (BBD) d^* and show that d^* is universally optimal for estimating a full set of orthonormal treatment contrasts in the class of appropriate block designs for given b, v and k.

c) Construct a BBD with b=10, v=5, k=8.

(3+2+6) +(3+6)+5=25

Q2. a) Prove the non existence and existence of the Hadamard matrices of order 7 and 24 respectively.

b) Indicate how an orthogonal array of appropriate parameters can be constructed from a Hadamard matrix and vice versa.

(6+15)+4=25

Q3) a)Construct a 2_{IV}^{6-2} design, clearly indicating your method of construction and justifying the Resolution.

b) Prove that the design constructed in part (a) above is an OA[16, 6, 2, 2].

c) Define a uniform strongly balanced crossover design with t treatments, p periods and n units. Construct a strongly balanced crossover design with 7 treatments and 14 units.

(2+8)+6+(3+6)=25

Q4)a) State the necessary and sufficient conditions for existence of a second order rotatable response surface design (SORD) and prove the sufficiency of these conditions.

b) Give an example of a useful SORD in two variables.

(5+15)+5=25

Q5) a)Construct a Generalised Youden Design (GYD) with 13 treatments in 13 rows and 4 columns.

b) Obtain the C- matrix (for treatments) of the GYD constructed in part(a) above and prove that it is universally optimal in the class of treatment connected 13x4 row-column designs with v=13.

c) Obtain with full justification the rank of the C- matrix(for treatments) of the following 3x3 row column design in three treatments A, B and C.

A B C

B C A

C A A

10+(4+6)+5=25

Date: November 11, 2013
Time: 3 hours

Statistical Methods in Genetics – I
M-Stat (2nd Year)
First Semester Examination 2013-14

The paper carries 60 marks. This is an open notes examination.
Answer all questions.

Question 1

The genotype distribution of a random set of 200 individuals at an autosomal biallelic locus in an inbred population is as follows:

Genotype	Frequency
AA	105
Aa	65
aa	30

Obtain an approximate 95% confidence interval for the allele frequency of *A* based on its m.l.e. [10]

Question 2

(a) Consider a dominant disorder controlled by an autosomal biallelic locus. If an individual is affected, who among the following relatives has the maximum risk of being affected: the mother, a brother or a daughter of the individual?

(b) Suppose that the initial genotype frequencies at an autosomal biallelic locus are according to Hardy-Weinberg equilibrium proportions. If the fitness coefficients corresponding to the genotypes are proportional to the initial genotype frequencies, explain whether the allele frequencies at the locus ever reach non-trivial equilibrium values. [6 + 6 = 12]

Question 3

(a) Explain what is meant by a LOD score of 2.7 in the context of genetic linkage analysis.

(b) Consider two autosomal biallelic loci with alleles (*A*,*a*) and (*B*,*b*), respectively. Consider a nuclear family where one parent is heterozygous at both loci, while the other parent has genotype *aaBb*. There are four offspring with genotypes *Aabb*, *aaBB*, *AaBb* and *AaBB*. Assuming that the possible phases of the double heterozygous parent are equally likely, which of them has a higher posterior probability?

(c) If the exact identity-by-descent scores at a marker locus are available for a random set of affected first cousin-pairs, develop a mean allele-sharing test for linkage between the marker locus and the disease locus. [3 + 6 + 7 = 16]

Question 4

(a) Consider two autosomal biallelic loci with alleles (*A*,*a*) and (*B*,*b*), respectively. If the two locus haplotype with the highest frequency is *aB*, is it possible that the alleles *a* and *B* are negatively associated?

(b) Given genotype data on a random set of individuals at two autosomal biallelic loci, explain how you would test whether there exists linkage disequilibrium between the two loci. [4 + 8 = 12]

Class presentation carries 10 marks.

INDIAN STATISTICAL INSTITUTE

Semestral Examination : Semester I (2013-14)

M. Stat. II Year

Actuarial Methods

ate: 11.11.2013 Maximum marks: 100 Time: $3\frac{1}{2}$ hours

Calculator and Actuarial table can be used.

1. (a) The number of patients treated by a hospital emergency unit is Poisson with mean parameter λ. The prior distribution of this parameter is gamma with mean 100 and variance 50. If 200 patients are treated by the unit on a single day, determine the posterior distribution of λ, based on this data. What is the Bayes estimate of λ based on the above data under squared error loss? What is the maximum likelihood estimate of λ?

   (b) Consider n independent and identically distributed observations on claims following a $N(\mu, 100)$ distribution. Assuming that μ is equally likely to be either 50 or 60, find the Bayes estimator of μ under squared error loss. Find this Bayes estimator of μ, when it is only given that the mean of the sample exceeds 57.

   (c) Describe a credibility estimate, specifying its characteristics.

   The Bayesian approach using quadratic loss always produces an estimate which can be readily expressed in the form of a credibility estimate. Prove or disprove the statement.

   [(3+1+1)+(3+4)+(2+5)=19]

2. The table below shows the cumulative claims (in Rs '000s) incurred on a particular class of insurance policies, divided by accident year and development year.

Accident Year	Development Year 0	1	2	3
1997	502	556	589	600
1998	487	565	593	
1999	608	640		
2000	551			

State the assumptions underlying the basic chain-ladder method. Using the method, estimate the outstanding claims reserve as on 31 December 2000. [2+6=8]

3. The profit per client-hour made by a privately owned health centre depends on the variable cost involved. Variable cost, over which the owner of the health centre has no control, takes one of the three levels θ_1 = high, θ_2 = medium and θ_3 = low. The owner has to decide at what level to set the number of client-hours that can be either $d_1 = 16,000$, $d_2 = 13,400$ or $d_3 = 10,000$. The profit (in Rs.) per client-hour is as follows:

	θ_1	θ_2	θ_3
d_1	85	95	110
d_2	105	115	130
d_3	125	135	150

Determine the minimax solution. Given the probability distribution $p(\theta_1) = 0.1$, $p(\theta_2) = 0.6$, $p(\theta_3) = 0.3$, determine the solution based on the Bayes criterion. [3+5=8]

4. Claim amounts from a portfolio have the distribution with pdf

$$f(x) = 2cxe^{-cx^2}, \ x \geq 0, \ c > 0.$$

An XOL reinsurance arrangement is in force with retention limit $M = 3$. A sample of reinsurer's payment amounts gives the following values: $n = 10$, $\sum y_i = 8.7$ and $\sum y_i^2 = 92.3$. Find maximum likelihood estimate of c. [8]

5. (a) Define the surplus process, in the context of ruin theory, and, then, the probability of ultimate ruin as a function of initial surplus. Prove that this ruin probability is a decreasing function of the initial surplus.

   (b) Aggregate claims from a policy in a year is assumed to follow a *Normal* distribution with mean $0.7P$ and variance $4.0P^2$, where $P =$ Rs 5,000 is the annual premium. An insurer with initial surplus Rs 100,000 expects to sell 100 policies with associated expense of $0.2P$ per policy. Calculate the probability of no ruin at the end of one year.

   (c) Describe the adjustment coefficient κ in the context of ruin theory. Calculate the adjustment coefficient κ when the loading factor $\theta = 0.32$ and the claim size distribution is Gamma with shape parameter 2 and scale parameter β.

[(2+2+2)+4+(2+4)=16]

6. (a) Describe product liability in the context of general insurance.

   (b) Describe policy excess and then give an expression for mean pay-off by the direct writer on a claim under policy excess.

   (c) Derive the expression for the moment generating function of a compound Binomial distribution.

   (d) Describe the difference between parameter variability and parameter uncertainty by means of example.

   (e) Describe the differences between a pseudo random number and a true random number.

[2+2+2+2+2=10]

7. The annual aggregate claim amount from a risk has a compound Poisson distribution with Poisson parameter 10. Individual claim amounts are uniformly distributed on $(0, 2000)$. The insurer of this risk has effected an excess of loss reinsurance with retention level 1600. Calculate the mean and variance of both the insurer's and the reinsurer's aggregate claim amounts under this reinsurance arrangement. Derive the same measures for the aggregate claim before reinsurance and check for their additive property with comments. [2+4+2=10]

8. The no claims discount system for an insurer has three categories with discount levels of 0%, 40% and 60%. If a policyholder makes any claims during the year he or she moves down a single category (or stays at the 0% discount level). If no claims are made, then the policyholder moves to the next higher category (or stays at the 60% discount level). The probability that a policyholder will make at least one claim in any one year is p in the 0% discount category and $0.8p$ and $0.6p$ in the 40% and the 60% discount category, respectively. The premium charged at the 0% discount category is c.

   (a) Write down the transition matrix for this system in terms of p.

   (b) Derive the steady state distribution of policyholders in each discount category in terms of p.

(c) Calculate the average premium paid in the steady state in terms of p and c. How does it behave with p?

[2+5+2=9]

9. (a) For Binomial regression model in the context of GLM, give expressions for the deviance residuals and explain their roles in model selection.

(b) Calculate the auto-correlation function of the process $X_n = 1 + e_n - 5e_{n-1} + 6e_{n-2}$.

(c) Consider

$$f(x) = \begin{cases} kx^{-0.5}, & \text{if } 0 \leq x \leq 1 \\ ke^{-x}, & \text{f } x > 1, \end{cases}$$

with $k > 0$ being the normalizing constant. Describe a method for simulating an observation from this distribution.

[(3+1)+4+4=12]

INDIAN STATISTICAL INSTITUTE

Semestral Examination (2013–2014)

M. STAT. II (MSP)

Functional Analysis

Date : 13.11.2013 Maximum Marks : 100 Time : $3\frac{1}{2}$ hrs.

This paper carries 125 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. (a) Let X be a normed linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $f : X \to \mathbb{K}$ be a linear map. Show that $\ker f = \{x \in X : f(x) = 0\}$ is either closed or dense in X. [5]

   (b) Let X be a Banach space and $f : X \to \mathbb{K}$ be a linear map. Show that f is continuous if and only if $\ker f$ is a G_δ-set in X. [15]

2. A closed subspace Y of a Banach space X is said to be *complemented in* X if there exists a continuous linear operator $P : X \to X$ such that $Py = y$ for all $y \in Y$ and $P(X) = Y$.

   (a) Show that a closed subspace $Y \subseteq X$ is complemented in X if there exists a closed subspace $Z \subseteq X$ such that $X = Y \oplus Z$. [12]

   (b) If Y is a closed subspace of X such that X/Y is finite dimensional, then Y is complemented in X. [8]

   (c) Let X be a Banach space containing a closed subspace Y isometric to ℓ_∞. Show that Y is complemented in X by a norm 1 projection. [15]

   [Hint : Hahn-Banach Theorem?]

3. Let (X, d) be a metric space and $f : X \to \mathbb{R}$. Show that the following are equivalent :

   (a) f is lower semi-continuous

   (b) For $x_n, x_0 \in X$, $x_n \to x_0$ implies $\liminf f(x_n) \geq f(x_0)$

   (c) for any $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(x_0, \delta)$, $f(x) > f(x_0) - \varepsilon$. [12]

[PTO]

4. The Volterra operator $V : L_2(0,1) \to L_2(0,1)$ is defined by

$$Vf(x) = \int_0^x f(y)\, dy, \qquad f \in L_2(0,1).$$

Show that $\sigma(V) = \{0\}$. [10]

5. Let $S : \ell_2 \to \ell_2$ be the unilateral shift, that is,

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \qquad (x_1, x_2, \ldots) \in \ell_2.$$

Show that the only reducing subspaces of S are $\{0\}$ and ℓ_2. [10]

6. Let $(\alpha_n) \in \ell^\infty$. Define a linear map $T : \ell_2 \to \ell_2$ by

$$T((x_n)) = (\alpha_n x_n), \qquad (x_n) \in \ell_2.$$

Show that

(a) T is continuous, normal and $\|T\| = \|(\alpha_n)\|_\infty$. [10]

(b) T is invertible if and only if $\{\alpha_n\}$ is bounded away from 0. [8]

(c) λ is an eigenvalue of T if and only if $\lambda = \alpha_n$ for some $n \geq 1$. [4]

(d) $\sigma(T) = \overline{\{\alpha_n : n \geq 1\}}$. [4]

(e) T is compact if and only if $(\alpha_n) \in c_0$. [12]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination : (2013-2014)

M.Stat. 2nd Year

TOPICS IN BAYESIAN INFERENCE

Date: 16 November, 2013 Max. Marks: 100 Duration: 3 Hours

This question paper carries 110 points.

Answer as many questions as you can. The maximum you can score is 100.

1. Consider the linear regression model $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{y} = (y_1, \ldots, y_n)'$ is the vector of observations on the "dependent" variable, $\boldsymbol{X} = ((x_{ij}))_{n\times p}$ is of full rank, x_{ij} being the values of the nonstochastic regressor variables, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is the vector of regression coefficients and the components of $\boldsymbol{\epsilon}$ are independent, each following $N(0, \sigma^2)$. Consider the noninformative prior $\pi(\boldsymbol{\beta}, \sigma^2) \propto \frac{1}{\sigma^2}$, $\boldsymbol{\beta} \in R^p$, $\sigma^2 > 0$. Find the following:

(a) The marginal posterior distribution of $\boldsymbol{\beta}$.
(b) The marginal posterior distribution of σ^2.
(c) The $100(1-\alpha)\%$ HPD credible set for $\boldsymbol{\beta}$.

[6+5+12=23]

2. (a) What are the difficulties with improper noninformative priors in Bayes testing? Describe the intrinsic Bayes factor (IBF) as a solution to this problem with improper priors.

(b) What is an intrinsic prior in the context of nonsubjective Bayes testing? Consider the general nested case and find the intrinsic prior determining equations corresponding to AIBF. Show that the solution suggested by Berger and Pericchi satisfies the intrinsic prior determining equations.

[(3+7)+15=25]

3. Consider p independent random samples, each of size n, from p normal populations $N(\theta_j, \sigma^2)$, $j = 1, \ldots, p$. Assume σ^2 to be known. Also assume that $\theta_1, \ldots, \theta_p$ are i.i.d. $N(\eta_1, \eta_2)$. Our problem is to estimate $\theta_1, \ldots, \theta_p$.

(a) A natural estimate of $(\theta_1, \ldots, \theta_p)$ is the vector of sample means. Why a suitable shrinkage estimate is expected to perform better than this estimate?

(b) Describe the Hierarchical Bayes and the parametric empirical Bayes (PEB) approaches in this context. Derive the James-Stein estimate as a PEB estimate.

[4+(15+7)=26]

4. Suppose we are studying the distribution of the number of defectives X in the daily production of a product. Consider the model $X|Y, \theta \sim \text{Bin}(Y, \theta)$ where Y, a day's production, follows Poisson (λ). The difficulty is that Y is not observable and inference has to be made on the basis of X only.

Consider a Beta(α, β) prior for θ, and show how Gibbs sampling can be used to sample from the posterior distribution of θ. Find all the required conditional distributions.

[17]

5. Write down the Laplace approximation for an integral in the multidimensional case. Describe how this approximation can be used to obtain the Bayesian information criterion (BIC) for model selection.

[8]

6. Suppose X has density $e^{-(x-\theta)} I(x > \theta)$ and the prior density of θ is $p(\theta) = [\pi(1 + \theta^2)]^{-1}$, $\theta \in R$. Find the Bayes estimate of θ for the loss function $L(\theta, a) = I(|\theta - a| > \delta)$ for some specified $\delta > 0$.

[11]

Indian Statistical Institute

Analysis of Discrete Data
Semestral Examination
M.Stat. II Year, 2013
Full Marks - 100
Time - 3 hrs.

Date : 18.11.2013

Attempt all questions

1. In a psychometric survey the IQ distribution of 1000 individuals is available in a grouped frequency distribution format. Treating these groups as cells and assuming suitable distribution of cell counts find the joint asymptotic distributions of the two vectors based on model-based maximum likelihood (ML) estimates and the simple sample proportion estimates of the cell probabilities. Prove that the ML estimate is more efficient than the estimate based on simple sample proportion. For the said fixed number of cells with non zero cell probabilities prove that when the model holds, the likelihood ratio statistic G^2 is asymptotically equivalent to χ^2 statistic as the total of cell counts tends to infinity. Show that $G^2 \geq 0$, with equality if and only if $p_i = \hat{\pi}_i$ for all i, p_i and $\hat{\pi}_i$ are respectively the sample proportion estimate and the maximum likelihood estimate of the ith cell probability.

[12 + 8 + 6 + 4 = 30]

2. (a) Show that the likelihood equations obtained for fitting of the logit model and least square equations of linear regression have similar form.

(b)Probabilities in a 2 × 2 table are hypothesized to follow the pattern $\pi_{11} = \theta^2$, $\pi_{12} = \pi_{21} = \theta(1-\theta)$, $\pi_{22} = (1-\theta)^2$, for some unknown θ. That is, the marginal distributions are identical and there is independence.

(i) For a multinomial sample, find the ML estimate of θ.

(ii) Construct a suitable test for the said hypothesis.

[10 + 4 + 4 = 18]

3. The following Table contains results of a study by Mendenhall *et al.* (1984) to compare radiation therapy with surgery in treating cancer of the larynx. Use appropriate test in testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$, where θ denotes odds ratio constructed from the above study. Explain how you will evaluate the P-value. Discuss how the testing procedure can be extended for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$, where θ_0 is a specified value of θ. Find the approximate confidence interval of the odds ratio for the extended case.

	Cancer Controlled	Cancer Not Controlled
Surgery	21	2
Radiation therapy	15	3

[5 + 2 + 6 + 3 = 16]

4. Discuss loglinear model of independence for two categorical variables. Extend the model in case of dependence. Establish the relationship between the odds ratio and association parameters in this extended loglinear model setup for two separate contingency tables viz. $I \times 2$ and 2×2.

[3 + 5 + 7 = 15]

5. (a) Define odds ratio for 2×2 contingency table and suggest an estimate of log odds ratio. Discuss the

Mantel-Haenszel estimate of the common odds ratio with a suitable example. Also state the logit estimat of the common odds ratio. Derive the asymptotic confidence interval for the common odds ratio considerin the logit estimate.

(b) Find the asymptotic standard error of the Kruskal Gamma as a measure of the ordinal association.

[(2 + 2 + 5 + 3 + 4) + 5 = 21]

or

Assuming the natural exponential family for the probability density function of the response variabl state and discuss different components of generalized linear model. Derive likelihood equations for the sa model. Under the same setup establish a relation between ML estimation using Fisher scoring and weighte least squares estimation.

[8 + 5 + 4 + 4 = 21]

INDIAN STATISTICAL INSTITUTE

Sementral Examination : 2013-14(Third Semester)

M. Stat. II year

Graph Theory and Combinatorics

Date: 19.11.13 Maximum Marks : 90 Duration : 3 hrs

Note: Answer as many as you can. The maximum you can score is 90. Notation is as used in the class.

1. (i) Show that every digraph D contains a directed path with χ vertices, where χ is the chromatic number of D.

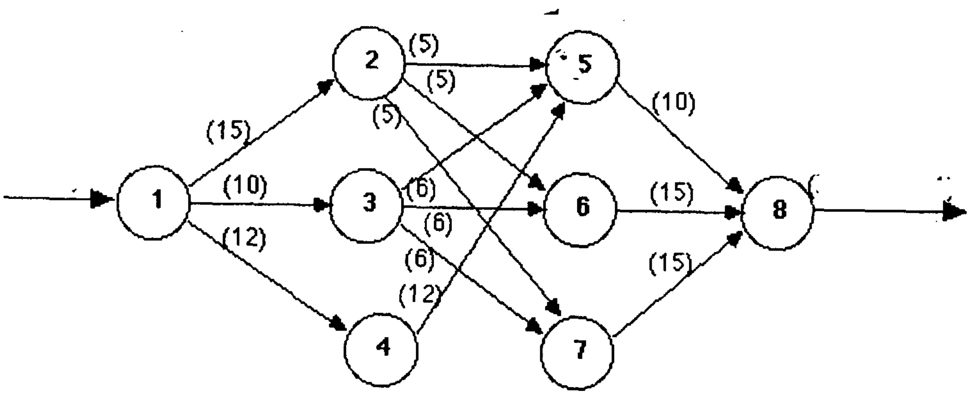
   (ii) Show that a matching M in a graph G is a maximum matching iff G contains no M-augmenting path. [8+7=15].

2. (i) Let G be a bipartite graph with edge-chromatic-number χ'. Show that $\chi' = \Delta$, where Δ is the maximum degree of G.

   (ii) Describe a polynomial-time algorithm for finding a proper Δ-edge colouring of a bipartite graph G. [8+7=15]

3. (i) Describe the Ford-Fulkerson Max-Flow-Min-Cut algorithm and discuss its correctness.

   (ii) Use the algorithm to find a maximum flow in the following network $N(1, 8)$, where 1 is the source and 8 is the sink.



[10+7]

4. (i) Show that in any finite bipartite graph, the number of edges in a maximal matching equals the number of vertices in a minimal vertex cover.

(ii) Let $N = N(X, Y)$ be a network with source set X and sink set Y. Construct a new network $N'(x, y)$ with a single source x and a single sink y such that for any flow f in $N(X, Y)$ there is a flow f' in $N'(x, y)$ with the same value as f. [8+7=15]

5. (i) State and prove Euler's formula for a connected plane graph. Hence, or otherwise, show that K_5 and $K_{3,3}$ are non-planar.

(ii) Let M be a minimally non-planar graph. Show that M must be 2-connected.

(iii) Let M be a graph with the smallest number of edges among all non-planar graphs which have no Kuratowski subgraph. Assume that M has no isolated vertices. Show that M must be 3-connected.

(iv) State Kuratowski's theorem . Deduce that a graph is planar iff it has no Kuratowski minor. [10+5+8+7=30]

INDIAN STATISTICAL INSTITUTE

SEMESTRAL EXAMINATION: (2013-2014)

MSQE I and M.Stat II

Microeconomic Theory I

Date: 20.11.2013 Maximum marks: 60 Duration: 3 Hours

Please answer Sections I and II on separate answer books. Each part is worth 30 marks.

Section I

Throughout this section, $\mathbb{R}^L$ is the L-dimensional Euclidean space. Let

$$\mathbb{R}^L_+ = \{x = (x_1, \cdots, x_L) \in \mathbb{R}^L : x_i \geq 0 \text{ for all } 1 \leq i \leq L\}$$

and

$$\mathbb{R}^L_{++} = \{x = (x_1, \cdots, x_L) \in \mathbb{R}^L : x_i > 0 \text{ for all } 1 \leq i \leq L\}.$$

Q1. Consider an economy with only one consumer and $\mathbb{R}^L$ as the commodity space. The demand function of the consumer is denoted by $x : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to X$, where $X \subset \mathbb{R}^L$ denotes the consumption set of the consumer.

(i) State the weak axiom of revealed preference ($WARP$) for x. Show that if x satisfies the $WARP$ then for any compensated price change from (p, w) to $(p', w') = (p',\ p' \cdot x(p, w))$, the following inequality holds:

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0. \qquad [1+5]$$

(ii) Take $L \geq 3$ and $X = \{(x, y, z, 0, \cdots, 0) \in \mathbb{R}^L : x, z \geq 0 \text{ and } y < 0\}$. Define $x : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to X$ by

$$x(p, w) = \left(\frac{p_2}{p_3}, -\frac{p_1}{p_3}, \frac{w}{p_3}, 0, \cdots, 0\right).$$

Prove that x is homogeneous of degree zero and satisfies Walras' law. Show further that x violates the $WARP$. [2+2]

Q2. Answer any **two** questions.

(i) For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_+$, define the lexicographic preference relation by letting $(x_1, y_1) \succeq (x_2, y_2)$ if and only if "$x_1 > x_2$ or ($x_1 = x_2$ and $y_1 \geq y_2$)". Show that it is not continuous. Prove that there is no utility function that can represent the lexicographic preference relation. [2+3]

(ii) Consider a preference relation over $\mathbb{R}^2_+$ represented by the utility function $U(x,y) = \sqrt{x} + \sqrt{y}$. Find the demand functions for the commodities 1 and 2 as they depend on prices and wealth. [5]

(iii) Given a $(p,w) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, the *utility maximization problem* (UMP) of the consumer is the following:

$$\sup\{U(x) : x \in B(p,w)\}$$

where $B(p,w) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the budget set for (p,w) and $U : \mathbb{R}^L_+ \to \mathbb{R}$ is a utility function representing a preference relation $\succsim$.

(a) Show that the solution of this problem exists if U is continuous. [2]

(b) Prove that the function $v : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}$, defined by

$$v(p,w) = \sup\{U(x) : x \in B(p,w)\},$$

is strictly increasing in w whenever $\succsim$ is locally non-satiated and U is continuous. [3]

Q3. Answer **all** questions.

(a) Suppose that f is the production function associated with a single-output technology, and let Y be the production set of this technology. Show that Y satisfies constant returns to scale if and only if f is homogeneous of degree one. [5]

(b) A firm has a production function $y = x_1 x_2$, where x_1 and x_2 denotes the amount of inputs. If the prices of inputs are equal to 1, then the minimum cost of production is 4. Find the value of y when the minimum occurs. [3]

(c) Recall that the *addition closure* of a production set Y is the smallest production set that is additive and contains Y. For a single-output technology with only one input, let $f(x) = x^2$ denote the production function. Find the addition closure of the production set of this technology. [2]

Section II

Q.1. Suppose we have a single good in a market with I buyers, each of whom wants at most one unit of the good. Buyer i is willing to pay up to v_i for his unit, with $v_1 > \cdots > v_I$. There are a total of $q < I$ units available. Suppose that buyers simultaneously submit bids, and that the units of the good go to the q highest bidders, who pay the amounts of their bids.

Show that every buyer making a bid of v_{q+1} and the good being assigned to buyers $1, \cdots, q$ is a Nash equilibrium of this game. [5]

Show also that in any pure strategy Nash equilibrium, buyers 1 through q receive a unit, while buyers $q+1$ through I do not. [5]

Q.2. Recall in the linear city model, we analysed equilibrium under the assumption $v > c + 3t$, where v is consumer valuation, c is cost of production per unit, and t is transport cost. Suppose instead we look at the case where $v \in (c + 2t, c + 3t)$.

Derive the best-response functions. Find the unique symmetric Nash equilibrium of this game. [5 +5]

Q.3. Consider an infinitely repeated Cournot duopoly with discount factor $\delta < 1$, unit costs of $c > 0$, and inverse demand function $p(q) = a - bq$, with $a > c$, and $b > 0$.

Under what conditions can the symmetric joint monopoly outputs $(q_1, q_2) = (\frac{q^m}{2}, \frac{q^m}{2})$ be sustained as an equilibrium of the infinitely repeated game with strategies that call for $(\frac{q^m}{2}, \frac{q^m}{2})$ to be played if no one has yet deviated and for the static Cournot equilibrium to be played otherwise? [10]

INDIAN STATISTICAL INSTITUTE

Semestral Examination, First Semester, 2013-14

M.Stat Second Year

20.11.2013

Time: $3\frac{1}{2}$ Hours

Statistical Computing

Full Marks: 100

1. Consider the following data set consisting of 12 pairs of observations on x and y.

i	1	2	3	4	5	6	7	8	9	10	11	12
x_i	-6	-5	-3	-2	0	1	1	2	2	3	3	4
y_i	5	4	2	0	0	-2	1	-2	-1	-1	-3	-3

$\left[\sum_{i=1}^{12} x_i = 0,\ \sum_{i=1}^{12} y_i = 0,\ \sum_{i=1}^{12} x_i^2 = 118,\ \sum_{i=1}^{12} y_i^2 = 74,\ \sum_{i=1}^{12} x_i y_i = -87 \right]$

(a) Show that the regression line $y = \alpha + \beta x$ that minimizes $\sum_{i=1}^{12}(y_i - \alpha - \beta x_i)^2$ passes through the origin. [2]

(b) Find the regression line $y = \beta x$ that minimizes $\sum_{i=1}^{12} |y_i - \beta x_i|$. [6]

(c) Show that both $S_1(\beta) = \sum_{i=1}^{12} |y_i - \beta x_i|$ and $S_2(\beta) = \sum_{i=1}^{12}(y_i - \beta x_i)^2$ are convex functions in β. [6]

(d) For any $a_1, a_2 \geq 0$, show that the minimizer of $a_1 S_1(\beta) + a_2 S_2(\beta)$ cannot lie outside the interval $[-0.8, -0.7]$. [6]

(e) Find the least median of squares (LMS) estimate for the location of y. [5]

(f) Find the regression depth of the fit $y = -(x + 0.5)$. [4]

(g) Define $z_i = I\{y_i \geq 0\}$ for $i = 1, 2 \ldots, 12$, where $I\{\cdot\}$ denotes the indicator function. Show that if we use a logistic regression model to predict Z, the maximum likelihood estimate of the model parameters will not exist. [5]

(h) If a regression tree with square error impurity function has only two leaf nodes, find the estimate of Y at $x = 0.5$. [6]

2. (a) Write down the basic principle of the MM (Minorization-Maximization or Majorization-Minimization) algorithm and give its mathematical justification. Show that the EM algorithm can be viewed as a special case of the MM algorithm. [2+2+4]

(b) Consider the data set given in Question no. 1. Show that the iteratively re-weighted least squares method can be used as an MM algorithm to find a minimizer of

$$S(\alpha, \beta) = 3 \sum_{i:\ y_i \geq \alpha + \beta x_i} (y_i - \alpha - \beta x_i) - \sum_{i:\ y_i < \alpha + \beta x_i} (y_i - \alpha - \beta x_i).$$

Describe your minimization algorithm (clearly mention about the initial choice of parameters, the updating scheme the and stopping rule). [4+4]

(c) Find a value of α that minimizes $S(\alpha, 0)$. Is this minimizer unique ? Justify your answer. [2+4]

3. (a) If you have n independent observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ from a four dimensional standard normal distribution, how will you use them to approximate

$$\int_A \exp -(x_1^2 + x_2^2/2 + x_3^2/3 + x_4^2/4)\, d\mathbf{x},$$

where $A = \{\mathbf{x} = (x_1, x_2, x_3, x_4) : x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 \geq 10\}$? [4]

(b) Describe the Metropolis-Hastings algorithm for generating observations from a target distribution π. Show that if an observation X_k follows the target distribution π, all subsequent observations $\{X_{k+i};\ i \geq 1\}$ generated by the algorithm will follow the same distribution. [2+4]

(c) Describe the Gibbs sampling algorithm for generating observations from the joint distribution with p.d.f. $f(x,y,z) \propto e^{-(y+2yz+4z)}x^y/(1+3x)^{y+z}, \quad x,y,z \geq 0.$ [4]

4. (a) Show that the Nadaraya Watson estimate can be viewed as a locally constant estimate of a regression function. Describe how this estimate can be used to construct a decision rule in a two-class classification problem. [3+4]

(b) Consider the data set given in Question no. 1. If Gaussian kernel is used to construct the Nadaraya Watson estimate $\hat{f}_h(\cdot)$ of the regression function $f(\cdot)$, find the limiting values of $\hat{f}_h(1.5)$ when the smoothing parameter h (i) tends to infinity (ii) shrinks to zero. [2+4]

(c) Describe how you will construct a locally linear estimate of a regression function when there are more than one regressor variables. [4]

(d) Consider a regression problem involving the response variable Y and two covariates X_1 and X_2. Briefly describe how you will fit an additive model using cubic splines. [4]

(e) Give an example of a regression problem where additive model leads to a poor result, but projection pursuit model works well. Justify your answer. [3]

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2013-2014, First Semester
M-Stat II
Subject: Set Theory and Topology

Date: 22.11.13 Total Marks 66 Duration: 3 Hours

Note: Answer all questions. The maximum you can score is 60. All the spaces considered are Hausdorff.

1. a) Let X be a locally compact space and Y any topological space.

   i) If $f : X \to Y$ is continuous, then is the space $f(X)$ necessarily locally compact as a subspace of Y?
   ii) Is $f(X)$ necessarily locally compact if f in (i) is both continuous and open?

   Justify your answer.

   b) Suppose G is a locally compact topological group and H is a subgroup. Show that G/H is a locally compact space in quotient topology.

   [(6+6)+8]

2. a) Show that a locally compact space is completely regular.

   b) If X is a normal topological space then show that

   i) if A and B are closed disjoint subsets of X then there exist open sets U_1 and U_2 such that $A \subset U_1$ and $B \subset U_2$ and closures of U_1 and U_2 are disjoint,

   ii) if A, B, C are closed subsets of X such that $A \cap B \cap C = \phi$ then there exist open sets U_1, U_2, U_3 such that $A \subset U_1$, $B \subset U_2$ and $C \subset U_3$ and $U_1 \cap U_2 \cap U_3 = \phi$.

   [6+(5+5)]

3. a) Show that every regular Lindelöf space is paracompact.

b) If X is the set of reals with lower limit topology, then is X paracompact? Justify your answer.

[8+8]

4. Let X be a completely regular space and $\beta(X)$ be its Stone-Čech compactification.

a) Show that X is connected if and only if $\beta(X)$ is connected.

b) If X has discrete topology, then show that for any $A \subset X$, $\overline{A}$ and $\overline{X - A}$ are disjoint, where $\overline{D}$ denotes the closure of D in $\beta(X)$.

[7+7]

Indian Statistical Institute
Semester 1, Academic Year: 2013-14
Semestral Examination
Course: M. Stat 2nd Year
Subject: Advanced Probability 1

Total Points: $14 \times 5 = 70$

Date: 25.11.2013 **Time: 3 Hours**

Answers must be justified with clear and precise arguments. If you refer to a theorem/result proved in class, state it explicitly. More than one answer to a question will not be entertained and only the first uncrossed answer will be graded.

1. (a) Let μ, ν and ρ be sigma-finite measures on $(\Omega, \mathcal{F})$. Suppose $\mu \ll \rho, \nu \ll \rho$ and let A be the set
$$\left\{\frac{d\nu}{d\rho} > 0 \text{ and } \frac{d\mu}{d\rho} = 0\right\}.$$
Show that $\nu \ll \mu$ iff $\rho(A) = 0$, in which case
$$\frac{d\nu}{d\mu} = 1_{A^c}\frac{d\nu/d\rho}{d\mu/d\rho}.$$
(b)Let $\mathcal{F}_0$ be the field consisting of the finite and the cofinite sets in an uncountable Ω. Define ϕ on $\mathcal{F}_0$ by taking $\phi(A)$ to be the number of points in A if A is finite, and the negative of the number of points in A^c if A is cofinite. Show that if $A_1, A_2, \cdots$ is a sequence of pairwise disjoint sets from $\mathcal{F}_0$ such that $\cup_n A_n \in \mathcal{F}_0$, then $\phi(\cup_n A_n) = \sum_n \phi(A_n)$.
$7 + 7 = 14$ pts.

2. (a) If Y is an integrable random variable and X, Z are random variables, show that if (X, Y) and Z are independent then $E(Y|X, Z) = E(Y|X)$.
(b) In Kakutani's theorem, if $P_j \sim Q_j$ for all j, then show that
$$\prod_1^\infty \int \sqrt{\frac{dQ_j}{dP_j}}dP_j > 0 \Leftrightarrow \prod_1^\infty \int \sqrt{\frac{dP_j}{dQ_j}}dQ_j > 0.$$
Also state the implication of the above condition for the product measures P and Q in the case $P_j \sim Q_j$, for all j.
$7 + 7 = 14$ pts.

3. Let $X_1, X_2, \ldots, X_n$ be a martingale relative to some filtration and τ be a finite stopping time for $X_1, \ldots, X_n$. $7 + 7 = 14$ pts.

(a) Let $X_n^+ = \max\{X_n, 0\}$. Show that $E|X_\tau| \leq 2EX_n^+ - EX_0$.

(b) Show by actual verification of the definition of a martingale that $\{X_{\tau\wedge n}\}$ is a martingale for $n \in \{1, 2, \ldots, k\}$. (You must work with this sequence and give a proof without referring to any result on two stopping times, one less than or equal to the other.)

4. (a) Let $X_0 = 1$ and for $n \geq 1$ given $X_{n-1} = x, X_n \sim U(0, x)$. If U_i are iid $U(0,1)$ then X_n can be written as $X_n = U_n.U_{n-1}.\ldots.U_1$. Show that $M_n = 2^n X_n$ is a martingale and that M_n converges a.s. to zero.

(b) Show that if X_n is a sequence of L^p-bounded random variables for some $p > 1$, then X_n is uniformly integrable.

(c) Suppose $\{(X_k, \mathcal{F}_k)\}, 1 \leq k \leq n$, is a martingale satisfying $E|X_k|^p < \infty, 1 \leq k \leq n$ where $p > 1$. State Doob's inequality between $E(\max_{1\leq k\leq n} |X_k|)^p$ and $E|X_n|^p$. $7 + 5 + 2 = 14$ pts.

5. Suppose $X_1, X_2, \ldots$ are iid random variables taking values ± 1 with equal probability. Let $S_0 = 0, S_n = X_1 + \cdots + X_n$. For r, s, positive integers define $\tau = \inf\{n : S_n = r \text{ or } S_n = -s\}$.

(a) Show that $P(\tau < \infty) = 1$, $P(S_\tau = r) = s/(r+s)$, and that $E\tau < \infty$. (For the last one, the martingale $S_n^2 - n$ may help.)

(b) In the case $r = s$ compute the value of $Ee^{-\lambda\tau}$ for $\lambda > 0$. (You may use the martingale $M_n = e^{tS_n}/(\phi(t))^n, n \geq 1, M_0 = 1$, where $\phi(t) = Ee^{tX_1}$.) $7 + 7 = 14$ pts.

MSTAT II / Advanced Probability II

Compensatory Exam. / Semester II 2012-13

Time - 3 hours

25 · 11 · 13

Maximum Score - 35

NOTE : SHOW ALL YOUR WORK TO GET THE FULL CREDIT. RESULTS USED MUST BE CLEARLY STATED.

1. (8+5+8+9=30 marks)

   Let $\{B_t\}_{t\geq 0}$ be the standard Brownian motion starting from 0, and $\{\mathcal{F}_t\}_{t\geq 0}$ be its canonical filtration given by $\mathcal{F}_t = \sigma\{B_u : 0 \leq u \leq t\}$.

   (a) Let $\mathcal{F}_{0+} = \bigcap_{s>0} \mathcal{F}_s$. Show that $\mathcal{F}_0 \subset \mathcal{F}_{0+}$, strictly. Also show that, $A \in \mathcal{F}_{0+} \Rightarrow P(A) =$ 0 or 1.

   (b) Let $\sigma = \inf\{t > 0 : B_t = 0\}$. Then show that $P(\sigma = 0) = 1$.

   (c) Let $Y_t = tB_{1/t}$, for $t > 0$, and $Y_0 = 0$. Show that $\{Y_t\}_{t\geq 0}$ is a standard Brownian motion.

   (d) Fix $a > 0$. Define $\tau = \sup\{t > 0 : B_t = at\}$. Find the distribution of τ.

2. (6+6+6+4+3=25 marks)

   Let $S = [0, \infty)$ and $v > 0$. Let $\Omega = C[0, \infty)$, that is, the set of all real continuous functions on $[0, \infty)$ and $\mathcal{F}$ the σ-field on Ω generated by the coordinate maps. If you started at $x > 0$ move with velocity v. If you started at $x = 0$ wait for $Exp(\lambda)$ amount of time and then move with velocity v. Thus, for $x > 0$, P^x, probability on $C[0, \infty)$, is the point mass at $w_x(t) = x + vt$. For $\xi \sim Exp(\lambda)$, define $w_0(t) = 0$, for $t < \xi$ and $w_0(t) = v(t - \xi)$ for $t \geq \xi$. Let P^0 be the induced probability. Then justify that, with the usual definition of filtration $\{\mathcal{F}_t\}$ and shift operartors $\{\theta_t\}$ on Ω, the family $\{P^x, x \in R\}$ is a Markov process. Find its Resolvent, infinitesimal Generator and its Range and Kernel.

3. ((2× 3)+9=15 marks)

   (a) Define what are meant by (i) local martingale, (ii) the predictable σ-field on $[0, \infty) \times \Omega$ and (iii) a predictable process on $(\Omega, \mathcal{F}, P)$.

   (b) Let L be a linear space of real-valued bounded measurable functions on $[0, \infty) \times \Omega$. Suppose L contains all bounded $\{F_t\}$-adapted processes with left-continuous trajectories. Suppose also that L satisfies the property: if $X_n \in L$, $n \geq 1$, $X_n \uparrow X$ and X is bounded, then $X \in L$. Show that L contains all bounded predictable processes.

P.T.O

4. (6× 5=30 marks)

Write TRUE or FALSE and justify your answer.

(a) Recall that $\mathcal{L}_2^{loc}$ denotes the set of all real functions f on $[0, \infty) \times \Omega$ such that $f(t, \omega)$ is jointly measurable in (t, ω), $f(t,)$ is $\mathcal{F}_t$-measurable for each t, and $\int_0^t f^2(s, \omega) ds < \infty$, P-almost surely, for each t.

Let $M_t = \int_0^t f(s, .) dB_s^m$, $N_t = \int_0^t g(s, .) dB_s^n$, $t \geq 0$ for $f, g \in \mathcal{L}_2^{loc}$, where $\{B^m\}$ and $\{B^n\}$ are two independent Brownian motion. Then $\{M_t N_t\}_{t \geq 0}$ is a continuous local martingale.

(b) Let $\{B_t\}$ be a standard Brownian Motion. We define $\{Z_n(t), t \geq 0\}$ recursively as follows: $Z_0(t) \equiv 1$; and for each $n \geq 1$, $Z_n(t) = \int_0^t Z_{n-1}(s) dB_s$, $t \geq 0$. Then, with probability 1, $Z_n(t)$ converges uniformly on compact set to the '0' random variable.

(c) Let $\{X\} \in \mathcal{L}_{2,T}$ for every $T > 0$ and satisfies the Stochastic Differential Equation $dX_t = X_t dB_t$, where $\{B\}_{t \geq 0}$ is a standard Brownian motion, $X_0 = 1$. Then for any α, $\lambda > 0$,

$$P\left(\sup_{0 \leq t \leq T} |\alpha X_t| \geq \lambda \right) \leq e^{-\alpha \lambda}.$$

(d) Let $\{M_t\}$ be a continuous square-integrable martingale. Then there is a P-null set N, such that, for all $\omega \notin N$, for some interval $[a, b]$, $< M >_t (\omega) = 0$, $\forall t \in [a, b]$, iff $M_t(\omega) = 0$, $\forall t \in [a, b]$.

(e) Let $\{X_t\}$ be a real-valued stochastic process which is continuous in t for all sample paths. Assume, for any reals $a < b$, $\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2 \to (f(b) - f(a))$ in L_2, as mesh of the partition, $a = t_0 < t_1 < \cdots < t_n = b$, is going to zero, where f is a strictly increasing continuous real valued function. Then, for almost all sample paths, $\{X_t\}$ has unbounded variation.

All the best.

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2013-14
Course Name: MSTAT II
Subject Name: Advanced Multivariate Analysis (Special Topics)
Date: November 2013 Maximum Marks: 100 Duration: 3 hrs.

27·11·2013

Attempt question numbers 1, 2 and either 3 or 4.
Show all your work. Marks are indicated in the margin.

Q. 1. (a)Characterize, with proof, the most general family of bivariate distributions, whose conditionals are specified to be members of p- and q- parameter exponential families ($p \neq q$).
(b) (i) Derive the p.d.f. of the bivariate normal conditionals distribution for which the components are uncorrelated but dependent. State any theorem you need to derive this distribution.
(ii) Obtain consistent estimators of the parameters of the distribution in (i)

(10 +10 +10) = [30]

Q. 2. (a) Based on the concept of a generalized test variable, derive an **exact** test for the specified non-null difference of the marginal means in the bivariate exponential distribution with independent components. Show how you will obtain the cut-off points of this test.
(b) Consider the "Small n, Large p" problem. Suggest a test for testing $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$, when n(« p) i.i.d. observations are available from $N_p(\mu, \Sigma)$, with Σ unknown. Derive the asymptotic null distribution of your test statistic.

(10 +15) = [25]

Q. 3. (a) Show that the MLE of μ in $N_p(\mu, \Sigma)$ is rendered inadmissible by the class of James-Stein estimators in the sense of Pitman closeness.
(b). Derive the likelihood ratio test for the homogeneity of Generalized Variances of several independent multivariate normal distributions of possibly different dimensions.

(10 + 10) = [20]

Q. 4. (a) State and prove the theorem on the form of the Locally Most Mean Powerful Unbiased (LMMPU) test in the multivariate multiparameter case.
(b). Derive the LMMPU test for testing $H_0 : \mu = 0$ for $N_p(\mu, \Sigma)$, with Σ known. Show how you will obtain the cut-off point for this test.

(10 + 10) = [20]

Q. 5. TAKE HOME PROBLEM [25]

INDIAN STATISTICAL INSTITUTE
Second-semestral Examination : (2013-2014)
M. Stat 2nd Year
Pattern Recognition and Image Processing

Date: November 28, 2013 Maximum marks: 100 Time: $3\frac{1}{2}$ hours.

Note: Attempt all questions. Maximum you can score is 100. Answer Group A and Group B questions in separate answerscripts.

Group A

1. In a two-class classification problem, there were 500 samples from each class in a training sample. A binary classification tree T was grown using equal prior probabilities and costs of misclassifications. Let $n_j(t), j = 1, 2$ denote the number of observations present in a node $t \in T$ for the two classes respectively. The stopping rule for splitting a node was that either the resubstitution error of the node was less than 2% or $n_j(t) \leq 5$ for any $j = 1, 2$. It was observed that whenever a node was split, $\lfloor 0.9 * n_1(t) \rfloor$ and $\lfloor 0.1 * n_2(t) \rfloor$ observations went to the left descendant node, and the remaining observations went to the right descendant node, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

   (a) Calculate the number of terminal nodes of the tree. [5]

   (b) Calculate the resubstitution error of T. [3]

   (c) Generate the α-sequence for the cost-complexity pruning, and the corresponding minimizing subtrees where α is the cost-complexity parameter. [10]

2. In a two-class classification problem with two features, the density of the observations from the first class is uniform over the circle centered at 0 and radius 2 while the density of the observations from the other class is a standard bivariate normal distribution. Assuming equal prior probabilities and costs of misclassifications,

   (a) find the Bayes optimal error for this problem. [7]

   (b) find the required architecture of a multilayer perceptron with proper weights which can produce the optimal boundary of classification for this problem. [10]

Group B

1. Consider the following run-length coding scheme for a binary $2^m \times 2^n$ image, where m, $n > 2$:

 - The image is scanned row-wise from top to bottom.
 - Only the runs of 0's in rows are coded by code pairs of the type (x, r), where x denotes the starting position of the run in the row and r denotes the length of the run.
 - The start of each row is coded by $(0, 0)$.

 (a) Is the resulting data compression lossless? Justify your answer.

 (b) Deduce the maximum average number of runs per row required to guarantee data compression under this coding scheme.

 (c) Will this coding scheme always result in greater compression than the coding scheme in which all runs of 0's and 1's in a row are assigned code pairs of the type $(0, r)$ and $(1, s)$ respectively? Justify your answer. (r and s are variables which represent respectively the length of a run of 0's and a length of 1's.)

[3+3+4=10]

2. *Unsharp masking* is a spatial domain technique in which a blurred version of an image is subtracted from the image itself.

 (a) If the blurring operation applied is 8-neighbourhood averaging, write down the corresponding 3×3 spatial filter that implements unsharp masking.

 (b) What, in your opinion, is the objective of this procedure from the viewpoint of image enhancement? Justify your answer with the help of the filter in part (a) of this problem.

 (c) What will be the enhancement effect of applying a filter obtained by adding a positive integer A to the $(2, 2)$-th element of the mask in part (a) of this problem, to the original image? Explain.

[4+3+3=10]

3. (a) Describe briefly how the Hough transform is used to detect subsets of dark pixels which lie on circles of the type $(x-a)^2 + (y-b)^2 = c^2$ on a light background in a binary image.

 (b) What is the effect of applying the following 5×5 spatial filter on a graylevel image? Explain.

0	0	-1	0	0
0	-1	-2	-1	0
-1	-2	16	-2	-1
0	-1	-2	-1	0
0	0	-1	0	0

[7+3=10]

4. (a) Describe a frequency domain filter that can be used to sharpen a graylevel image without introducing an undesirable "ringing" effect.

   (b) Define the Laplacian of a graylevel image. Briefly discuss its role in image sharpening. Describe how the Laplacian can be implemented in the frequency domain.

[3+(2+2+3)=10]

5. (a) Explain how it is possible to infer that a graylevel image is corrupted by sinusoidal noise. Describe a frequency domain filter that can restore such an image.

   (b) Consider a graylevel image that has been captured under non-uniform illumination. Describe an effective method for segmenting it into internally homogeneous subregions.

[(3+3)+4=10]

6. (a) Define *Hue*, *Saturation* and *Intensity* of colour.

   (b) Describe the HSI colour model diagrammatically providing adequate labels, and highlight its relationship with the RGB model of colour.

   (c) Discuss briefly any method for improving contrast in a colour image.

[(3+(3+2)+2=10]

7. (a) Consider the problem of clustering a set $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ of n observations on $\boldsymbol{X}$, p-variate random vector of binary values. Assume that $\boldsymbol{X}$ is distributed as a mixture of c multivariate Bernoulli distributions

$$p(\boldsymbol{x}|\boldsymbol{\theta}) = \sum_{i=1}^{c} \pi(\omega_i) p_i(\boldsymbol{x}|\omega_i, \boldsymbol{\theta}_i), \text{ where } p_i(\boldsymbol{x}|\omega_i, \boldsymbol{\theta}_i) = \prod_{j=1}^{p} {\theta_{ij}}^{x_{ij}} (1-\theta_{ij})^{(1-x_{ij})}.$$

Here, $\boldsymbol{\theta}_i' = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{ip})$, $\boldsymbol{\theta}' = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2', \ldots, \boldsymbol{\theta}_c')$, $\theta_{ij} \in (0,1)\ \forall i, j$, and the mixing proportions $\pi(\omega_i)$ $(\in (0,1))$ satisfy $\sum_{i=1}^{c} \pi(\omega_i) = 1$.

Show that, for $i = 1, 2, \ldots, c$, the maximum likelihood (ML) estimates $\hat{\boldsymbol{\theta}}_i$ of $\boldsymbol{\theta}_i$ based on $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ must satisfy

$$\hat{\boldsymbol{\theta}}_i = \frac{\sum_{k=1}^{n} \hat{\pi}(\omega_i|\boldsymbol{x}_i, \hat{\boldsymbol{\theta}}_i)\ \boldsymbol{x}_i}{\sum_{k=1}^{n} \hat{\pi}(\omega_i|\boldsymbol{x}_i, \hat{\boldsymbol{\theta}}_i)},$$

where $\hat{\pi}(\omega_i|\boldsymbol{x}_i, \hat{\boldsymbol{\theta}}_i)$ is the ML estimate of

$$\pi(\omega_i|\boldsymbol{x}_i, \boldsymbol{\theta}_i) = \frac{\pi(\boldsymbol{x}_i|\omega_i, \boldsymbol{\theta}_i)}{p(\boldsymbol{x}|\boldsymbol{\theta})},$$

obtained by substituting the MLE $\hat{\boldsymbol{\theta}}_i$ of $\boldsymbol{\theta}_i$.

(Please Turn Over)

(b) Suppose observations are available on a D-dimensional measurement vector $\boldsymbol{y}$ for a c-class pattern recognition problem, where the classes are labeled by $\omega_1, \omega_2, \ldots, \omega_c$. Let $p(\boldsymbol{z}|\omega_i)$ denote the probability density function of an arbitrary $d \times 1$ subvector $\boldsymbol{z}$ of $\boldsymbol{y}$ in class ω_i, $i = 1, 2, \ldots, c$, and let π_i be the *a priori* probability for ω_i. Consider the problem of selecting an optimal $d \times 1$ feature vector from the D measurements in $\boldsymbol{y}$ where $d << D$.

i. Write down an optimization criterion in terms of the Bhattacharya distance

$$d_B(\boldsymbol{z}) = -\ln \int \sqrt{p_1(\boldsymbol{z})p_2(\boldsymbol{z})}d\boldsymbol{z}$$

between two probability distribution functions $p_1(\cdot)$ and $p_2(\cdot)$.

ii. Show that if $p(\boldsymbol{z}|\omega_i) = N_d(\boldsymbol{\mu}_i, \Sigma)$, $i = 1, 2, \ldots, c$, then this criterion reduces to a function of the Mahalanobis distances between pairs of class-conditional densities $p(\boldsymbol{z}|\omega_i)$ and $p(\boldsymbol{z}|\omega_j)$.

[(4+(2+4)=10]

INDIAN STATISTICAL INSTITUTE
Semestral Examination, First Semester: 2013-14
M.Stat. II Year (AS)
Life Contingencies

ate: November 28, 2013 | Maximum marks: 100 | Duration: $3\frac{1}{2}$ hours

Students are permitted to use non-programmable calculators and Actuarial Formulae and Tables.

1. A certain population is subject to two independent modes of of decrement: α and β. The following relation between multiple and single decrement rates are proposed.

$$(aq)_x^\alpha = q_x^\alpha\left(1-\frac{1}{2}q_x^\beta\right);$$
$$(aq)_x^\alpha = \left(q_x^\alpha + q_x^\beta - q_x^\alpha q_x^\beta\right) \times \frac{\log(p_x^\alpha)}{\log\left[(1-q_x^\alpha)(1-q_x^\beta)\right]}.$$

Show that these two expressions follow from the assumption of uniform distribution of death within an integer year, for either *each of the multiple decrements* or *each of the single decrements.* Which expression corresponds to the first assumption, and which one to the second? [9]

2. An insurer is about to sell a life annuity to a life age 40 exact. The annuity at the annual rate of Rs. 1 lakh would be paid monthly in advance. It can be for a fixed term of 25 years or for whole life. Obtain an expression for the difference between the present values of the two potential annuities. Calculate the expected value of this random variable, by using AM92 Select table and 4% rate of interest. [5]

3. Ten years ago, an insurer had sold pure endowments of Rs. 50,000 for 15 years to a number of lives aged 45 exact. The premiums were payable as annuity-due. Nine years after policy issue, there were still 400 policies in force. Only two policyholders died last year. Calculate the mortality profit or loss for the last year.

   Basis

   Mortality: AM92 Ultimate
   Interest: 4% per annum

   [7]

4. A whole life 'with profits' policy for a life aged 33 exact provides for a sum assured of Rs. 50,000 payable at the end of the year of death. A compound bonus of 3% vests at the start of each policy year, together with an additional bonus of 5% on all *previously* existing bonuses. Premiums are paid annually in advance. Calculate

   (a) the present value of the benefit in terms of the curtate future life;

   (b) the net premium payable annually in advance. [2+6]

   Basis

   Mortality : Exponentially distributed with mean 70
   Rate of interest : 6% per annum

   [Total 8]

5. Calculate ${}_{0.75}p_{40.75}$ from the ELT15 table for males, under the assumption of constant force of mortality in between integer years. **[3]**

6. For a female life aged 60 and a male life aged 65, an annuity of Rs. 500,000 is paid annually in advance from the year after the first death till the year of the second death. Level annual premiums are paid in advance till the year of the first death. Calculate the gross premium.

Basis

Expenses :	Initial	Rs. 10,000
:	Renewal	Rs. 1,000 plus 2% of gross premium
:		(incurred from policy inception till second death)
Mortality :	Male	PMA92C20
:	Female	PFA92C20
Interest :		4% per annum

[9]

7. Consider two independent lives, (x) and (y) having continuous life distributions.

   (a) Prove the following statements

   i. $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$.

   ii. ${}_nq_{\overline{xy}} = {}_nq^{2}_{xy} + {}_nq^{\;2}_{x y}$. [2+2]

   (b) A 20 year term assurance issued to (x) and (y) provides benefit of Rs. 2 lakhs at the instant of the *second* death, if it happens during the term. Level premium is paid continuously as long as the last survivor status continues. What should be the net premium if (x) and (y) have mortalities 0.01 and 0.02, and the rate of interest is 4%?[7]

[Total 11]

8. A pension scheme assumes a continuous time Markov model for its members with three states: *Active*, *Retired* and *Dead.* The transition rate from either *Active* or *Retired* state to *Dead* state is 0.02 per year. Transition from *Active* to *Retired* state occurs at the rate of 0.04 per year.

   (a) Calculate (i) the dependent probability of retirement in one year and (ii) the independent probability of death from active service in one year. [3+2]

   (b) A 30 year term assurance policy provides for a benefit of Rs. 100,000 for a healthy life at the instant of death from the *Active* state only. Calculate the expected present value of this policy at the outset, assuming interest rate $i = 4\%$. [4]

[Total 9]

9. (a) Explain the difference between *adverse selection* and *spurious selection.* [4]

   (b) What is *temporary initial selection*? Explain how it can be handled. [2]

   (c) The mortality experience of workers in an organization tasked with maintenance of hazardous mountain roads gives rise to the following summary.

Age	Population	Number of deaths
28	2203	8
29	2238	10
30	2221	11

   Calculate the crude mortality rate and the mortality ratio standardized with respect to AM92 select, in respect of workers in the age group 28-30. [4]

[Total 10]

10. (a) Explain the terms *bid price* and the *offer price* of a unit. What does *bid-offer spread* mean? [3]

(b) An insurer issues a 4-year endowment assurance policy to a male life aged 56 exact. The sum assured is Rs. 100,000 payable at maturity or at the end of the year of earlier death. Annual premium of Rs. 22,000 is paid in advance. The insurer holds net premium reserves calculated prospectively by equivalence principle, using the AM92 Ultimate table with 4% interest rate, at the beginning of the second, third and fourth policy year. Calculate (i) the reserves held in the three years, (ii) the net present value of the profits and (iii) the profit margin for this contract. [3+9+2]

Basis

Interest rate on cash flows and reserves:	5% per annum
Mortality:	AM92 Select
Initial expenses:	10% of annual premium
Renewal expenses:	Rs. 50 plus 2% of annual premium
Risk discount rate:	6% per annum

[Total 17]

11. End-of-the year cash-flows computed by an insurer (assuming no reserve) in respect of a ten-year unit-linked policy for a female aged 55 exact are as under.

(–72.14, –82.34, –23.56, 62.84, 129.42, 31.40, 83.12, 1.87, 9.29, 41.90)

Determine the minimum reserves needed to be held at the beginning of renewal years in order to avoid negative cash-flows. You can assume an interest of 6% in respect of reserves and mortality as per PFAbase. **[3]**

12. A benefit scheme provides pension on age retirement or retirement due to ill health. The annual rate of pension is one-sixtieth of the final pensionable salary for each year of service (fractions of years being counted proportionately). The final pensionable salary is the average salary over three years preceding retirement. A 45 year old male has been in service for 15 years and has become a member of the scheme for 10 years. Assuming that salary increases continuously and his current salary is Rs. 5 lakh, calculate the following.

(a) The expected present value of the member's aggregate benefits arising from his past services only. [3]

(b) The expected present value of the member's aggregate benefits arising from his future services only. [3]

(c) The expected present value of the member's future service contributions to the pension scheme, if $k\%$ of the salary is contributed. [2]

(d) The cost of providing future service benefits as a percentage of salary. [1]

Basis: Example Pension Scheme Table in *Formulae and Tables for Examinations*, 4% interest rate, retirement age 65

[Total 9]

Date: 05.02.14
Time: 3 hours

Statistical Methods in Genetics – I
M-Stat (2^{nd} Year)
Backpaper Examination 1^{st} Semester 2013-14

This paper carries 100 marks. Answer all questions.

1. Explain what is meant by each of the following:
Snyder's Ratio; heterozygote advantage; linkage phase; segregation ratio; population stratification. [2 x 5 = 10]

2(a) Consider a biallelic X-linked locus with alleles (A,a). If the initial frequencies of the allele A among males and females are p_1 and p_2, respectively, in how many generations will the frequency of A be p^* among males?

(b) Suppose that the mutation rate from allele A to allele a is 0.003 per generation and that from a to A is 0.002 per generation. In how many generations will the frequency of allele a increase from 0.1 to 0.2?

(c) Consider a disease controlled by an autosomal biallelic locus. If an individual is affected, show that irrespective of the disease model, it is equally likely for the grandmother and the nephew of the individual to be affected. [10 + 10 + 10 = 30]

3. Compute the expected identity-by-descent (i.b.d.) score at an autosomal biallelic locus for a pair of sibs, both of whom are heterozygous at that locus. If it is known that one of the parents is also heterozygous at that locus, what is the expected i.b.d. score for the sib-pair? [8 + 8 = 16]

4(a) Consider two autosomal biallelic linked loci with alleles (D,d) and (M,m) respectively. The following are genotype data on two nuclear families:

Family 1: father is DM/dm, mother is Ddmm, offspring are DdMm, Ddmm and DDmm

Family 2: both parents are Dm/dM, offspring are DDmm and DdMM

Compute the LOD score for testing the presence of linkage between the two loci based on these two families. [15]

(b) Consider the following data from an affected sib-pair study (both sibs are affected and affection status of parents are unknown) on schizophrenia . *CHRNA7* (cholinergic receptor nicotinic alpha polypeptide 7) on Chromosome 15 is believed to be a candidate gene for schizophrenia. Twelve sib-pairs comprising all affected siblings (along with their parents) were genotyped at a triallelic marker locus *D15S1012* near this candidate gene. Do these data provide evidence of linkage between *D15S1012* and a locus controlling schizophrenia? [15]

Sib-pair	Parental genotypes	Genotypes of affected sibs
1	AA,AB	AA,AB
2	AB,BC	AB,BC
3	AB,CC	AC,BC
4	BB,BC	BB,BB
5	AB,AB	AB,AB
6	AC,AC	AA,AA
7	AB, BB	AB,BB
8	AA,AB	AA,AB
9	BB ,BC	BC,BC
10	CC,CC	CC,CC
11	AB,AC	AC,BC
12	AC,AC	AA,AC

5(a) Explain why linkage disequilibrium exists over much smaller distances on the genome compared to linkage.

(b) In a case-control association study based on 200 cases and 300 controls, the following genotype data were observed at a marker locus:

Genotype	*AA*	*AB*	*BB*
Cases	100	85	15
Controls	110	140	50

Do the above data show significant evidence of association between the marker locus and the disease based on Odds Ratio? [4 + 10 = 14]

INDIAN STATISTICAL INSTITUTE

Second-semestral (Backpaper) Examination : (2013-2014)

M. Stat 2nd Year

Pattern Recognition and Image Processing

Date: 05.02.14 Maximum marks: 100 Time: 3 hours.

Note: Attempt all questions. Answer Group A and Group B questions in separate answerscripts.

Group A

1. For a two-class problem with equal prior probabilities and misclassification costs, let the class densities be $p_i(\mathbf{x}) = N(\boldsymbol{\mu}_i, \sigma^2 I)$, for $i = 1, 2$.

   (a) Express the optimal error rate P^* in terms of the c.d.f. of the standard normal density and $\frac{\|\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1\|}{2\sigma}$. [8]

   (b) Let, $\boldsymbol{\mu}_1 = 0$ and $\boldsymbol{\mu}_2 = \mu \mathbf{1}_M$, where $\mathbf{1}_M$ is the M-dimensional vector with all entries equal to one. Show that in this case, P^* goes to 0 as M goes to infinity. Give an interpretation of this result. [8]

2. Assuming equal prior probabilities and equal misclassification costs, find the overall Bayes error and the overall asymptotic probability of error for the 1-NN rule in a 2-class problem based on a single feature x where the class densities $p_1(x)$ and $p_2(x)$ are as given below: [6 + 8]

$$p_1(x) = \begin{cases} x & \text{for } 0 \le x \le 1, \\ 2 - x & \text{for } 1 \le x \le 2 \end{cases} \quad \text{and} \quad p_2(x) = \begin{cases} x - 1 & \text{for } 1 \le x \le 2, \\ 3 - x & \text{for } 2 \le x \le 3. \end{cases}$$

3. Show the required architecture (with appropriate weights) of a multilayer perceptron which can solve the two-dimensional two-class classification problem where the first class consists of all points in the first and the third quadrants of the measurement space and the second class is the complement of the first class. [10]

4. Describe briefly the minimal cost-complexity pruning algorithm for generating the α-sequence and the corresponding smallest minimizing subtree sequence in CART. [10]

(Please Turn Over)

Group B

1. (a) What is the outcome of applying histogram equalization to the histogram-equalized version of a low-contrast image? Explain.

   (b) What is salt-and-pepper noise in the context of a graylevel image? How can you infer whether a given image is corrupted by this kind of noise? Write down a spatial filter which is most successful in reducing such noise.

   [3+(2+3+2)=10]

2. (a) Proposing an appropriate model for image degradation, discuss how the Wiener filter is used to restore a graylevel image.

   (b) Explain how a graylevel image can be segmented into internally homogeneous regions, through region splitting and merging based on quadtrees. State clearly the underlying assumptions.

   [5+5=10]

3. (a) Use the Huffman coding technique to generate a compression scheme for a graylevel image in which there are 8 graylevels with probabilities of occurrence 0.1, 0.15, 0.25, 0.3, 0.05, 0.15, 0, and 0 respectively.

   (b) Discuss how first-order digital derivatives can be used to detect edges in a gray-level image.

   (c) To identify the boundaries of internally homogeneous regions in a gray-level image, is it enough just to apply an edge detection technique? If not, explain what additional operation(s) need to be performed, and how.

   [(4+3+3=10]

4. Consider a set of n observations on a D-dimensional measurement vector Y for a c-class pattern recognition problem. in which n_i of the observations are from class i. $i = 1, 2, \ldots, c$. Describe any suboptimal feature selection algorithm that can reduce the number of variables iteratively to d ($<< D$) so that the successive feature subsets obtained are not nested. [10]

5. State a method of splitting a dataset consisting of N observations into k clusters by iterative optimization of the sum-of-squared-errors criterion, explaining clearly the rationale behind the key steps. [10]

Indian Statistical Institute
Semester 1, Academic Year: 2013-14
Backpaper Examination
Course: M. Stat 2nd Year
Subject: Advanced Probability 1

Total Points: 5 × (10 + 10) = 100

Date: [redacted] 05.12.14 **Time: 3 Hours**

Answers must be justified with clear and precise arguments. If you refer to a theorem/result proved in class, state it explicitly. More than one answer to a question will not be entertained and only the first uncrossed answer will be graded.

1. (a) Consider $[0,1]$ with $\mu =$ counting measure and $\lambda =$ Lebesgue measure on $\mathcal{B}([0,1])$. Show that $\lambda \ll \mu$ but there is no Borel measurable g such that $\lambda(A) = \int_A g d\mu$ for all $A \in \mathcal{B}$. Does this contradict the Radon-Nikodym theorem?

   (b) Suppose on $(\Omega, \mathcal{F}, P)$ we have independent random variables X and Y. By $E(\cdot|X)$ we mean $E(\cdot|\sigma(X))$. Show that for any Borel set $P(Y \in B|X) = P(Y \in B)$ a.s. In addition if $E|Y| < \infty$ then show that $E(Y|X) = E(Y)$ a.s.

2. Let $\forall n$, $\Omega_n = \{0,1\}, \alpha_n, \beta_n \in (0,1)$, and μ_{α_n} be the measure given by $\mu_{\alpha_n}(\{0\}) = \alpha_n, \mu_{\alpha_n}(\{1\}) = 1 - \alpha_n$. Similarly define μ_{β_n}. Denote $\alpha = (\alpha_1, \alpha_2, \cdots)$ and similarly β. Let μ_α and μ_β be the product measures constructed from μ_{α_n}'s and μ_{β_n}'s.

   (a) Prove that either ($\mu_\alpha \ll \mu_\beta$ and $\mu_\beta \ll \mu_\alpha$) or $\mu_\alpha \perp \mu_\beta$. Show that ($\mu_\alpha \ll \mu_\beta$ and $\mu_\beta \ll \mu_\alpha$) holds iff $\sum_1^\infty (1 - \sqrt{\alpha_n\beta_n} - \sqrt{(1-\alpha_n)(1-\beta_n)}) < \infty$.

   (b) Show that if $\alpha_n, \beta_n \in (\delta, 1-\delta), 0 < \delta < 1/2$, then $\sum_1^\infty (1 - \sqrt{\alpha_n\beta_n} - \sqrt{(1-\alpha_n)(1-\beta_n)}) < \infty$ iff $\sum_1^\infty (\alpha_n - \beta_n)^2 < \infty$.

3. Consider the following Markov chain: $X_0 = 0$ and for all $n \in \{0,1,2,\ldots\}$, and for all $k \in \{1,2,\ldots\}$

$$\begin{aligned} P(X_{n+1} = n+1 | X_n = n) &= p_{n+1} \\ P(X_{n+1} = -(n+1) | X_n = n) &= 1 - p_{n+1} \\ P(X_{n+1} = -k | X_n = -k) &= 1. \end{aligned}$$

(a) Show that if $p_{n+1} = (2n+1)/(2n+2)$ for all n, then $\{X_n\}$ is a martingale.

(b) If the p_n are chosen as in (a), show that the martingale converges a.e. to a finite limit although $E|X_n| \to \infty$ (for the expectation, note that $|X_{n+1}| = (|X_n|+1).1_{(X_n \geq 0)} + |X_n|.1_{(X_n<0)})$.

4. Suppose $Y_1, Y_2, \cdots$ are independent, positive random variables and that $EY_n = 1$. Put $X_n = Y_1.Y_2, \cdots Y_n$.

   (a) Show that $\{X_n\}$ is a martingale and converges with probability 1 to an integrable X.

   (b) Assume further that Y_n assumes values $1/2$ and $3/2$ with probability $1/2$ each. Then show that $X = 0$ with probability 1.

5. Let $\{X_n\}$ be a Markov chain with state space $\{0, 1, \cdots, N\}$ and suppose that $\{X_n\}$ is a martingale.

   (a) Show that 0 and N must be absorbing states.

   (b) Let $\tau = V_0 \wedge V_N$ where $V_a = \inf\{n \geq 0 : X_n = a\}$. Suppose $P_x(\tau < \infty) > 0$ for $1 < x < N$. Show that $P_x(\tau < \infty) = 1$ and $P_x(V_N < V_0) = x/N$.

INDIAN STATISTICAL INSTITUTE

First Semester Backpaper Examination (2013–2014)

M. STAT. II (MSP)

Functional Analysis

Date : 07.02.2014 | Maximum Marks : 100 | Time : 3 hrs.

Precisely justify all your steps. Carefully state all the results you are using.

1. Let $Y = C[0,1]$ with sup norm. Let $X = \{f \in Y : f' \in Y\}$ with the sup norm. Define the mapping $D : X \to Y$ by $D(f) = f'$.

   (a) Show that D is an unbounded operator.

   (b) Show that D has a closed graph.

   (c) Why doesn't (b) contradicts the closed graph theorem? [5+5+5 = 15]

2. Let X be a Banach space and $P : X \to X$ be a linear map such that $P^2 = P$ and both $P(X)$ and $\ker(P)$ are closed. Show that P is continuous. [10]

3. Show that if X and Y are normed spaces and the space $B(X,Y)$ of bounded linear operators from X to Y is a Banach space, then Y is a Banach space. [10]

4. (a) Let X be a normed linear space over $\mathbb{R}$ and $f : X \to \mathbb{R}$ be a linear map. Show that $\ker f = \{x \in X : f(x) = 0\}$ is either closed or dense in X. [5]

   (b) Let X be a Banach space and $f : X \to \mathbb{R}$ be a linear map. Show that f is continuous if and only if $\ker f$ is a G_δ-set in X. [15]

5. A subset $A \subseteq X^*$ is said to separate points of X if whenever $x_1 \neq x_2 \in X$, there exists $f \in A$ such that $f(x_1) \neq f(x_2)$.

   Let X, Y be Banach spaces. Let $A \subseteq Y^*$ separate points of Y. Let $T : X \to Y$ be a linear map. Show that T is continuous if and only if $g \circ T$ is continuous for all $g \in A$. [10]

6. Let $\mathcal{H} = L^2[0,1]$. For $\phi \in L^\infty$, let $M_\phi : \mathcal{H} \to \mathcal{H}$ denote the multiplication operator defined by $M_\phi f = \phi f$ for $f \in \mathcal{H}$. Show that $\|M_\phi\| = \|\phi\|_\infty$. [15]

7. Let $N : \mathcal{H} \to \mathcal{H}$ be a normal operator on a Hilbert space $\mathcal{H}$. Show that λ is an eigenvalue of N if and only if $\bar{\lambda}$ is an eigenvalue of N^*. [10]

8. If T is a compact, self adjoint operator on a Hilbert space, then there exist an eigenvalue λ of T such that $|\lambda| = \| T \|$. [10]

(b) What is a projection pursuit regression model? How will you fit this model to a data set consisting of n observations on Y, X_1 and X_2, where Y is the response variable and the other two are regressor variables? [2+4]

6. (a) Suppose that $X_1, X_2, \ldots, X_n \overset{i.i.d}{\sim} F$, where F is a univariate continuous distribution with finite second moments (i.e., $E_F(|X^2|) < \infty$). Show that $T_0(F_n)$ is a biased estimators of $T_0(F) = E_F(X^2) - [E_F(X)]^2$. Find the corresponding jackknife estimator and check whether it is unbiased. [2+4+2]

   (b) Study the behavior of the bias of the jackknife estimator when the bias of $T_n = T(F_n)$ is of the order (i) $O(n^{-2})$ (ii) $O(n^{-\delta})$ for some $\delta \in (0,1)$. [2+3]

   (c) If n observations are chosen at random with replacement from set of n distinct observations, how many different samples are possible? Let R be the number of distinct observations in the sample. Find the expectation and the variance of R.. Show that $R/n \xrightarrow{P} 1 - e^{-1}$ as $n \to \infty$. [3+4+2]

7. (a) Consider a regression problem where X takes values in the interval $[-1, 1]$ and the true regression function f is monotonically increasing. Show that the Nadaraya Watson estimate of $f(1)$ will have negative bias. [3]

   (b) Describe how you will construct a locally cubic estimate of a regression function. [3]

   (c) Briefly describe how cost-complexity pruning can be used to select the size of a regression tree. [4]

8. (a) Let f be a non-negative continuous function defined on $[a, b]$ and $f_0 = \max_{a \le x \le b} f(x) < \infty$. In order to approximate $\theta = \int_a^b f(x)dx$, a person generates some independent observations $u_1, u_2, \ldots, u_n$ from $U(a, b)$ and uses $T_1 = (b-a)\sum_{i=1} f(u_i)/n$ as an estimate. Another person generates some independent observations $(z_1, w_1), (z_2, w_2), \ldots, (z_n, w_n)$ from $U(a, b) \times U(0, 1)$ and uses $T_2 = \sum_{i=1}^{n} I(f_0 w_i \le f(z_i))/n$ as an estimate. Check whether these estimates are unbiased. Which of these two estimates will you prefer and why ? [2+4]

   (b) Give an appropriate sampling algorithm for generating observations from the joint distribution with probability density function $f(x, y, z) \propto e^{-(xy+2yz+4zx)} x^{y+1} yz/(1+x)^{y+z+1}, \quad x, y, z \ge 0$. [4]

INDIAN STATISTICAL INSTITUTE

Semestral (Backpaper) Examination, First Semester, 2013-14

M.Stat Second Year

7.2.14

Time: $3\frac{1}{2}$ Hours | Statistical Computing | Full Marks: 100

1. A set of n independent observations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are generated from a mixture of three bivariate normal distributions differing only in their locations.

   (a) Describe how EM algorithm can be used to estimate (i) the mixing proportions (ii) locations of the three distributions and (iii) the common dispersion matrix. [6]

   (b) If these mixing proportions are of the form p^2, $2pq$ and q^2 $(p > q)$, how will you modify your algorithm to estimate p ? [4]

2. Suppose that the distribution of a p-dimensional random vector $\mathbf{X}$ is symmetric about $\boldsymbol{\mu}$.

   (a) Show that $\boldsymbol{\mu}$ is the half-space median of the distribution. [4]

   (b) If $\mathbf{H}(\mathbf{X}-\boldsymbol{\mu})$ has the same distribution for all orthogonal matrices $\mathbf{H}$, for any $\mathbf{x} \in R^p$, show that spatial depth of $\mathbf{x}$ is a decreasing function of $\|\mathbf{x}-\boldsymbol{\mu}\|$. [8]

   (c) If $\mathbf{H}(\mathbf{X}-\boldsymbol{\mu})$ has the same distribution for all orthogonal matrices $\mathbf{H}$, and $\mathbf{X}$ has finite second moments, show that $E[(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})']$ is a constant multiple of the identity matrix. [6]

   (d) Give an example of a p-dimensional symmetric distribution, which is not elliptically symmetric. [2]

3. (a) Show that in the case of simple linear regression, the least squares estimate of the slope can be viewed as a weighted average of the slopes between the pairs of observations. What is the Theil-Sen estimator of the slope ? [4+2]

   (b) Define regression depth of a fit. Give an example of a nonfit that yields some positive as well as some negative residuals. [2+2]

4. Following observations are generated from a univariate distribution.

12.7	14.6	13.9	10.8	10.1	13.2	12.7	15.8	13.3	11.8
15.2	14.1	13.6	12.4	13.0	11.4	10.5	15.2	14.0	12.1

   Find a value of θ that minimizes

   (a) $\text{Median}\{|y_1-\theta|, |y_2-\theta|, \ldots, |y_{20}-\theta|\}$. [4]

   (b) $4\sum_{i=1}^{20}|y_i-\theta| + \sum_{i=1}^{20}(y_i-\theta)$. [4]

5. (a) Suppose that the regression function f is to be estimated from the data using linear splines based on power basis functions. Let $t_1 < t_2 < \ldots < t_{10}$ be 10 equi-spaced knots placed on the range of the X-variable. If there are no observations in the interval $[t_4, t_6]$, show that the design matrix cannot have full column rank. [4]

INDIAN STATISTICAL INSTITUTE

Back-paper Examination, First Semester: 2013-14

M.Stat. II Year (AS)

Life Contingencies

Date: 7.4.14 Maximum marks: 100 Duration: 3 hours

Students are permitted to use non-programmable calculators and Actuarial Formulae and Tables.

1. Find l_x, if $\mu_x = 1/(100 - x)$. [5]

2. A life office issues an annuity to a woman aged 65 exact and a man aged 68 exact. The annuity of Rs. 20,000 per annum is payable annually in arrears as long as either of the lives is alive.

   (a) Calculate the expected present value of this benefit. [2]

   (b) Calculate the probability that the life office makes a profit in this case if it charges a single premium of Rs. 320,000. [4]

   Basis: Interest: 4% per annum; Mortality: Female: PFA92C20, Male: PMA92C20.

   [Total 6]

3. (a) List the cash flows involved in profit testing a regular premium term assurance contract by giving the sign of the cash flow ("+" for income and "–" for outgo to the Company). [3]

   (b) Define and explain the following terms: (i) Profit Vector, (ii) Profit Signature, (iii) Net Present Value of Profit and (iv) Profit Margin. [4]

   [Total 7]

4. (a) Under what circumstances the non-unit reserve is required to be set up in a unit linked contract. Explain how reserving bases affects the profitability of a life insurance contract. [4]

   (b) In-force expected net cash flows of a single premium 5 year life insurance contract are (-300, -200, 1000, -400, 200). Calculate the reserves required at the beginning of each of the second, third, fourth and fifth policy year and give the revised profit vector allowing for the reserves.

   Assume that reserves earn interest at a rate of 6% per annum and the regulations do not allow negative reserves. You may ignore mortality. [4]

   [Total 8]

5. A 2-year term assurance policy is issued to a life aged x. The benefit amount is Rs. 100,000 if the life dies in the first year, and Rs. 200,000 if the life dies in the second year. Benefits are payable at the end of the year of death.

   (a) Give an expression for the present value random variable for this benefit. [2]

   (b) Calculate the standard deviation of the present value random variable assuming that $q_x = 0.025$, $q_{x+1} = 0.030$ and $i = 0.06$. [4]

   [Total 6]

6. As a pricing actuary, you are profit testing a unit linked product with the following features:

   - The sum assured plus unit account value is payable on death. Sum assured is Rs. 500,000 in the first year which reduces by Rs. 100, 000 at the end of each year.

- The unit account value is payable on maturity.
- The unit account value less surrender charge is payable on surrender.
- The policy term and premium payable term is 3 years and all premiums are payable on annual basis i.e., at the beginning of every year.

The Company will offer the customers to choose either of the following two charging structures:

Charge	Charge Structure A	Charge Structure B
Allocation rate (% of premiums invested into unit account)	80% of first year premium and 101% of each subsequent premium of the year	94% of all premiums
Bid offer spread	5%	5%
Surrender Charge	Nil	15% of outstanding premiums
Fund Management Charge	1% per annum	1% per annum
Mortality Charge	Nil	Nil

The following profit test assumptions are given to you.

- Expenses: Rs. 2,000 at the start of the first policy year and Rs. 500 at the start of subsequent policy years.
- Commission: 10% of first year premium.
- Dependant Rate of Mortality: 0.001 per annum for all ages.
- Dependant Rate of Surrender: 0.15 for first year and 0.10 for second policy year.
- Unit Growth Rate: 10% per annum.
- Interest rate on non unit cash flows: 6% per annum.

You may further assume the following.

- Expenses (excludes commission), death claims and surrenders occur at the end of the year.
- The Fund Management Charge for a year is deducted at the end of the year.
- The investment return on the unit account is credited at the end of the year.
- The interest income on non unit cash flows is earned at the end of the year.
- Non unit reserve is zero at the end of each year.

(a) Calculate the unit account value payable to the policyholder in respect of both the charge structures at the end of the three year period if the annual premium is Rs. 100,000. [6]

(b) Which charge structure is more profitable to the company assuming risk discount rate of 10% per annum? [3]

(c) Comment on the capital need of the two charge structures for this product. [3]

[Total 12]

7. Explain how the level of education or the type of occupation influences the mortality and morbidity experience of a group of insured lives. [5]

8. List the advantages and disadvantages for using single figure indices to measure the mortality rates. [3]

9. A male life aged 63 years exact and his spouse aged 60 exact buys a joint life annuity. The contract provides following three benefits.

   - Benefit A: an annuity certain of Rs. 20,000 per annum payable monthly in advance during first 10 year of the contract.
   - Benefit B: A deferred annuity of Rs. 20,000 per annum payable monthly in advance. This benefit commences immediately after the 10 year expiry period of Benefit A. This benefit is payable as long as both spouse are alive on the due date of the annuity installment.
   - Benefit C: A deferred reversionary annuity of Rs. 15,000 per annum payable monthly in advance. The first annuity installment is payable on the 'monthiversary' following the later of *the expiry of the 10 year benefit A period* and *the death of any one of the two annuitants.*

   Calculate the single premium for this contract. You should use the following assumptions to calculate the premium.

   Mortality: PA92C20 mortality tables (as applicable for male and females).

   Interest Rate: 4% per annum.

   Expenses: Policy acquisition expenses of Rs. 500 per contract; No Maintenance expense.

   Commission: 2% of the single premium.

   [10]

10. A pension scheme provides retirement benefit payable on ill health retirement only. The benefit is 1/80th of final pensionable salary for each complete year of service including the prospective service until 'age retirement' at age 60.

    Final pensionable salary is defined as the average of the earnings in the three years immediately preceding the date of retirement. Derive formulae for the evaluation of the present value of the benefit for a new member aged exactly x.

    Define all functions you use and state clearly all assumptions you make. [7]

11. Discuss the "Selection Process" in underwriting of cases in life insurance business. How can selective withdrawals affect the mortality experience of a portfolio of life insurance and annuity polices? [5]

12. A life insurance company issues a 10-year decreasing term assurance benefit to a man aged 50 exact. The death benefit is Rs. 100,000 in the first year, 90,000 in the second year and decreases by Rs. 10,000 each year so that the benefit in the 10th year is Rs. 10,000. The death benefit is payable at the end of the year of death.

    Level premiums are payable monthly in advance for the term of the policy, ceasing at earlier death.

    Calculate the annual premium using the following basis.

Interest:	6% pa.
Mortality:	AM92 select.
Initial expenses:	Rs. 200 and 25% of total annual premium (all incurred on policy commencement).
Renewal expenses:	2% of each premium from the start of the 2nd year and Rs. 50 pa, inflating at 1.923% pa, at the start of the second and subsequent policy years.
Claim expenses:	Rs. 200 inflating at 1.923% pa.
Inflation:	For the renewal and claim expenses, the amounts quoted are at outset, and the increases due to inflation start immediately.

[8]

13. (a) If P_x is the annual premium under a whole life assurance policy to a life aged x, then show that

$$({}_tV_x + P_x)(1+i) = {}_{t+1}V_x + q_{x+t}(1 - {}_{t+1}V_x).$$ [4]

(b) Explain the above equation by general reasoning. [2]

(c) Explain what is meant by "actual death strain" and "expected death strain". [2]

(d) Calculate the profit or loss from mortality for the year to 31st December, 2007. The deferred annuities are payable yearly from the 60th birthday and Sums Assured under the Whole Life policies are paid at the end of the year.

	As at 31-12-07			Year 1-1-07 to 31-12-07
Class	Age at entry	No. of years in force	Sum Assured or annuity pa (Rs.)	Contracts ceasing by deaths: Sum Assured or annuity pa (Rs.)
Whole life annual premium	30	10	500,000	5,000
Deferred Annuity single premium without return	40	20	60,000	1,200

Basis: Mortality: AM92 ultimate; Interest: 4% pa. [10]

[Total 18]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2013-2014
M. Stat. II Year: Semester-II

Survival Analysis

Date: 25 February, 2014 Full Marks: 80 Duration: 3 hours

Note: Answer all questions

1. (a) What is the difference between censoring and truncation?
   (b) Describe left truncated right-censored data. Write down likelihood function corresponding to a continuous lifetime distribution based on left truncated right-censored data.

[2+2+2 =6]

2. Consider a lifetime T with the hazard function

$$\lambda(t) = \begin{cases} \lambda_1 & \text{for } 0 \le t \le t_1 \\ \lambda_2 & \text{for } t_1 \le t < t_2 \\ \lambda_3 & \text{for } t_2 \le t < \infty \end{cases}$$

Find the survival function and mean lifetime.

[5+3=8]

3. Consider a mixture of two exponential distributions with hazard rates λ_1 and λ_2. Check whether the mixture distribution is IFR or DFR.

[10]

4. Suppose we have ten individuals on study with the following times on study (+ denotes a right-censored observation): 3, 4, 5+, 6, 6+, 8+, 11, 14, 15, 16+. Compute the Kaplan-Meier estimate of $S(15)$. Discuss a graphical method to check whether the data are from Weibull distribution or not.

[7+3=10]

5. Consider for a Type-I censoring scheme, with censoring time t_0 =0.4, observed failure times are 0.067, 0.090, 0.116, 0.140, 0.148, 0.174, 0.179, 0.190, 0.253, 0.384 out of $30(=n)$ samples. Perform the modified Kolmogrov Smirnov test to check whether the data are from $U(0,1)$ or not. For truncation proportion 0.4 upper alpha value $U_{0.05} = 1.1975$.

[10]

P.T.O.

6. What is the Mann-Whitney form of the Wilcoxon test for two sample problems ? Describe Gehan's Generalization for the censored data.

[4+6=10]

7. Suppose the survival prospect of two groups of patients are to be compared using a covariate Z. Group 0, indicated by $Z=0$, has survival function $S_0(t) = \exp(-\lambda t)$. Group 1, indicated by $Z=1$, has survival function $S_1(t) = S_0(t)^{\psi}$.

   (a) Show that the two groups follow the proportional hazards model.

   (b) Consider random right censored data (x_i, δ_i, z_i), $i = 1, \ldots, n$. Find the maximum likelihood estimate of λ and ψ.

[2+8=10]

8. Suppose a continuous lifetime random variable T given Z satisfying proportional hazard model

$$\lambda(t; Z) = \lambda_0(t) \exp(\beta' Z)$$

is grouped into intervals $I_1, I_2, \ldots, I_{k+1}$, where $I_i = [a_{i-1}, a_i]$ and $0 = a_0 < a_1 < \ldots < a_k < a_{k+1} = \infty$. Define $T_d = i$, if $a_{i-1} \leq T < a_i$. Find the discrete hazard of T_d for given Z.

[6]

9. Let T be a continuous survival time. Given a covariate Z, suppose that log survival time Y follows a linear model with logistic error distribution

$$Y = \ln T = \mu + \beta Z + \sigma W,$$

where pdf of W is given by $f(w) = e^w/(1+e^w), -\infty < w < \infty$.

   (a) Check whether the regression model is proportional hazard model or not.

   (b) Find the survival function of T for given Z.

   (c) Consider two individuals with different covariate values. Show that, for any time t, the ratio fo their odds of the death is independent of t.

[4+2+4=10]

Indian Statistical Institute
Midterm Examination
Asymptotic Theory of Inference
M.Stat. Second Year, Second Semester (2013-2014)

Date:- 26.02.14 Maximum Marks: 30 Duration :- 2 hours

Answer all questions

1. State and prove LeCam's Third Lemma. Explain with an example how this lemma might be useful in the study of asymptotic relative efficiency of nonparametric tests for location. [8]

2. For each $n \geq 1$, let us denote by P_n be the joint probability distribution of n iid observations $X_1, \ldots, X_n$ from $N(0,1)$ and by Q_n the joint probability distribution of n iid observations $X_1, \ldots, X_n$ from $N(\mu_n, 1)$. Show that Q_n is contiguous to P_n if and only if $\mu_n = O(n^{-1/2})$ as $n \to \infty$. [6]

3. Consider a sequence of statistical experiments $(\mathcal{X}^n, \mathcal{A}^n, P_\theta^n : \theta \in \Theta \subset \mathcal{R}^d)$. Define Local Asymptotic Normality at a certain parameter value $\theta_0 \in \Theta$. Show that under i.i.d. sampling from a common distribution $P_\theta, \theta \in \Theta \subset \mathcal{R}$, Θ open and usual Rao-Cramer type regularity conditions on the density of P_θ, the family $\{P_\theta^n : \theta \in \Theta\}$ satisfies the Local Asymptotic Normality condition at $\theta_0 \in \Theta$ with normalizer $\phi_n = \frac{1}{\sqrt{n}}$. [2+4]

4. Consider the model $\{P_\theta : \theta \in \Theta\}$, $\Theta \subset \mathcal{R}$ and $P_\theta << \gamma$, $\forall \theta \in \Theta$ and $\frac{dP_\theta}{d\gamma} = f(x, \theta)$. Assume Quadratic Mean Differentiability at $\theta_0 \in \Theta$. Show that for all $\epsilon > 0$, $P_{\theta_0}\{|h(X_1, u)| > \epsilon\} = o(u^2)$, where symbols have usual meanings. [6]

5. Consider the family $\{P_\theta, \theta \in (0, \infty)\}$ where $P_\theta = U[0, \theta]$ for each $\theta > 0$. Is the family QMD at any $\theta_0 > 0$? Justify your assertion. [4]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2013-14

M. Stat. II Year
Advanced Survey Sampling

Date: 28.02.14 **Maximum Marks: 50** **Duration: 3 Hours**

Answer any 4 of the following questions each carrying 10 marks.

Assignment Records to be submitted on the date of the examination carry 10 marks.

1 Find an admissible estimator for a finite population total among all of its unbiased estimators based on a general sampling design.

2 Derive Hartley & Ross's unbiased estimator for a finite population total. Give an unbiased estimator for its variance.

3 Given raw survey data based on a non-informative sampling design, derive a sufficient statistic. Show how this is useful in deriving a better estimator for a finite population total compared to one which is not based on a sufficient statistic.

4 Given stratified SRSWOR survey data show how you may assess if stratified SRSWOR may be better than SRSWOR from an entire population.

5 Appropriately denoting the

(i) Total number of residential buildings in a given city,
(ii) The number of households such a building may contain and
(iii) The number of members in such a household explain how (a) you may choose such a household with a probability proportional to the number of members in it and (b) also how you may randomly choose such a household.

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INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : Semester II (2013-14)

M. Stat. II Year

Actuarial Models

Date: 28.02.2014 Maximum marks: 60 Time: 2 hours 15 minutes

Calculator and Actuarial table can be used. Answer as many as you can. Total mark is 67.

1. Describe two benefits and two limitations of using models in actuarial work. [2+2=4]

2. (a) Define the following stochastic processes:
   i. a Poisson process
   ii. a compound Poisson process; and
   iii. a general random walk [3]

   (b) For each of the processes in part (a), state whether it operates in continuous or discrete time and whether it has a continuous or discrete state space. [3]

   (c) For each of the processes in part (a), describe one practical situation in which an actuary could use such a process to model a real world phenomenon. [3]

   [Total 9]

3. A die is rolled repeatedly. Consider the sequence B_n, which is the largest number found in the first n rolls.

   (a) Explain why B_n is a Markov chain. [1]

   (b) Determine the state space of the chain. [1]

   (c) Derive the transition probabilities. [1]

   (d) Explain whether the chain is irreducible and/or aperiodic. [1]

   (e) Describe the equilibrium distribution of the chain. [1]

   [Total 5]

4. A three state process with state space $\{A, B, C\}$ is believed to follow a Markov chain with the following possible transitions:
   A to A, A to C,
   B to B, B to A, B to C,
   C to C, C to B.
   An instrument was used to monitor this process, but it was set up incorrectly and only recorded the state occupied after every TWO time points. From these observations, the following two-step transition probabilities have been estimated:
   $P^2_{AA} = 0.5625$, $P^2_{AB} = 0.125$, $P^2_{BA} = 0.475$, $P^2_{CC} = 0.4$.
   Calculate the one-step transition matrix consistent with these estimates. [8]

5. (a) Consider continuously monitored data from a non-homogeneous Poisson process (NHPP) with rate $\lambda(t) = \lambda e^{-\beta t}$, $\lambda, \beta > 0$, and observed during $[0, \tau]$. Write down the corresponding likelihood function. [3]

(b) Suppose claims from a particular portfolio arise according to a Poisson process with rate λ_R per day during rainy season (16 June - 15 October) and λ_O per day during other time of a year. In the observation period of two years (1 January 2011 - 31 December 2012), there are 84 claims during rainy season and 102 claims during other time. Obtain the maximum likelihood estimates of λ_R and λ_O from this data. [5]

[Total 8]

6. Consider an illness model in which the instantaneous transition rate from the healthy state (H) to the ill state (I) is α and that from I to H is β. The instantaneous transition rate to the dead state (D) from H and I are μ and δ, respectively.

(a) Write this as a continuous time Markov jump process by defining the process and specifying the matrix of instantaneous transition rates. [2]

(b) Derive an expression for the conditional probability that the subject is in state I at time t with current sojourn time C_t being greater that $w(< t)$ given that the subject is in state H at time 0. This expression may involve some transition probabilities. [3]

(c) Assuming $\beta = 0$, obtain an explicit expression for the probability in (b) above without involving any transition probability. [4]

(d) Given that the subject is in state H at time 0, describe an algorithm for simulating continuous monitoring data of this illness model for the subject till time τ. [3]

[Total 12]

7. Consider a portfolio of car insurance policies in which major claims arise according to a Poisson process with rate λ and minor claims arise, independent of the major claims, according to a Poisson process with rate α. What is the probability of n claims during $[0, t]$? Given that there are n claims during $[0, t]$, what is the probability that n_1 of them are major? Obtain the probability that the first claim is a major one. [2+3+4=9]

8. (a) A continuous time stochastic process with states A, B and C is modelled as a Markov jump process with the following generator matrix

$$\begin{pmatrix} -\sigma_{AB} - \sigma_{AC} & \sigma_{AB} & \sigma_{AC} \\ \sigma_{BA} & -\sigma_{BA} - \sigma_{BC} & \sigma_{BC} \\ \sigma_{CA} & \sigma_{CB} & -\sigma_{CA} - \sigma_{CB} \end{pmatrix}.$$

Write down the one-step transition matrix of the jump chain associated with the presumed jump process. [3]

(b) The following data have been collected from observation of the process.

Transition from state	*Transition to state*	*Number of transitions in one jump*
A	B	110
A	C	90
B	A	80
B	C	45
C	A	120
C	B	15

Estimate the transition matrix of the jump chain you have obtained in part (a), assuming that the underlying process is a Markov jump process. [3]

(c) The following additional data have been collected from observation of the process, in respect of successive transitions.

Sequence of states visited	*Number of occurrences*		*Sequence of states visited*	*Number of occurrences*
ABC	42		BCA	38
ABA	68		BCB	7
ACA	85		CAB	64
ACB	4		CAC	56
BAB	50		CBA	8
BAC	30		CBC	7

Test the goodness-of-fit of the Markov jump process model by considering whether triplets of successive transitions are consistent with the hypothesized transition probability matrix, as estimated in part (b). [6]

[Total 12]

Indian Statistical Institute
Semester 2, Academic Year: 2013-14
Mid-Semester Examination
Course: M. Stat 2nd Year
Subject: Advanced Probability 2

Total Points: $8 \times 4 = 32$

Date: 28.2.2014 Time: 2 Hours

The maximum you can score is 30. Answers must be justified with clear and precise arguments. If you refer to a theorem/result proved in class, state it explicitly. More than one answer to a question will not be entertained and only the first uncrossed answer will be graded.

1. Suppose μ_W is the standard Wiener measure on $C[0,1]$ and $f \in L^2[0,1]$ (nonrandom). Consider the probability measure on $C[0,1]$

$$d\widetilde{\mu} = e^{\int_0^1 f_s dW_s - \frac{1}{2}\int_0^1 f_s^2 ds} d\mu_W,$$

and the process $\widetilde{W}_t = W_t - \int_0^t f_s ds, 0 \le t \le 1$. Compute for $\alpha, \beta \in \mathbb{R}$,

$$\int_{C[0,1]} e^{\alpha \widetilde{W}_s + \beta(\widetilde{W}_t - \widetilde{W}_s)} d\widetilde{\mu}, 0 < s < t < 1.$$

2. Suppose $(M_t, \mathcal{F}_t), 0 \le t \le 1$, is a martingale and τ is a stopping time taking value in $[0,1]$. It can be shown that the stopped process $(M_{\tau\wedge t}, \mathcal{F}_{\tau\wedge t}), 0 \le t \le 1$ is a martingale. Assuming this Show that $(M_{\tau\wedge t}, \mathcal{F}_t), 0 \le t \le 1$, is also a martingale. (Hint: The following

$$\int_A M_{t\wedge\tau} = \int_A M_{s\wedge\tau}$$

holds for all $A \in \mathcal{F}_{s\wedge\tau}$, you have to prove it for all $A \in \mathcal{F}_s$. So, for $A \in \mathcal{F}_s$, consider

$$A_1 = A \cap \{\tau > s\},\ A_2 = A \cap \{\tau \le s\}.$$

Show that $A_1 \in \mathcal{F}_{s\wedge\tau}$ and establish the required equality appropriately justifying all the steps.)

3. Calculate explicitly $E(S_t e^{r(T-t)-\frac{T-t}{2}+X} - c)^+$ where $X \sim N(0, T-t)$, and S_t is given. The result should be expressed in terms of the standard normal cdf. For convenience you can put $S_t = x > 0$, and take r, c to be positive real numbers.

4. (a) Compute $E\Big(\int_0^s W_u^2 dW_u\Big).\Big(\int_0^t W_v^2 dW_v\Big)$, where $0 < s < t < 1$, are fixed.

   (b) Write down the Ito formula connecting W_t^3 and W_0^3 for fixed $t \in [0, 1]$.

Statistical Methods in Public Health
Mid-Semestral Examination
M.Stat. II Year, 2013-2014
Total Marks - 70
Time - 2 hrs. 30 mins.

Indian Statistical Institute
Kolkata 700 108, INDIA

Attempt all questions:

1. (a) Let X be an $(n \times q)$ data matrix in which each row corresponds to a q - variate size measurement available at q time points on one of n individuals. Write down the expressions for four estimated relative growth rate (RGR) vectors (each having dimension $(1 \times \overline{q-1})$) based on the empirical RGR estimates and all $(q-1)$ samples each of size n.

(b) Discuss Lyapunov and asymptotic stability. State and proof the stability theorem for single species growth dynamics.

[(4 + 2) + (2 + 2) + 5 = 15]

or

Find the analytical solution of the growth curve governed by the following growth equation

$$\frac{1}{x(t)}\frac{dx(t)}{dt} = bt^{c}exp(-at) \qquad (*),$$

where, $a, b > 0$ and c is an integer. Compare the point of inflexions for $c = 0$ and 1 with proper interpretations. Write down the step(s) for testing the equality of expected time points for which RGR is maximized based on the data as described in 1 (a) for two species. You have to assume that the data matrix has proximity to the growth curve as expressed in equation (*).

[7 + 4 + 4 = 15]

2. Let us consider the following growth equation

$$\frac{dx(t)}{dt} = rx(t)^{\gamma}\left[1 - \left(\frac{x(t)}{K}\right)^{\theta}\right]^{\beta}, \qquad (**)$$

where, $x(t)$ be the size variable describes the growth profile of a specific species and (γ, θ, β) are the non-negative growth curve parameters.

(a) Interpret the above equation and its parameters through (i) the basic and two opposite growth pulses and (ii) the birth - death rate expressions.

(b) Find the approximate analytical solution of the growth curve assuming size to be bounded by twice the carrying capacity. Comments on the stability aspect of this approximate growth curve without using the stability theorem.

(c) Evaluate the point of inflexion of the growth curve and compare it with the point of inflexion of the logistic growth using the definition of log, lag and stationary phase (you may use some limiting value of the paramter(s)).

(d) Define Allee effect and its critical size ? Equation (**) can be used (with / without changing the model structure) to model Allee effect phenomena with positive critical size / null critical size - discuss.

(e) Derive the limiting form of the growth curve (**) with $\beta = 1$, high intrinsic growth rate and low value of curvature parameter.

(f) The RGR of equation (**) can capture both monotonic and non-monotonic profiles while plotted against density - proof or disproof it.

[(2 + 2) + (8 + 5) + (4 + 4) + (3 + 3) + 5 + 4 = 40]

3. A system consists of prey and predator population. The following dynamics is governed by the population.

(i) The prey population grow in a logistic fashion with intrinsic growth rate parameter r and carrying capacity K.

(ii) The interaction among prey and predator follow mass action law.

(iii) The predator population dies at a rate ' d '.

(a) Write down the system of differential equation based on the above descriptions.

(b) Find out the ecological feasible equilibria of the system with proper ecological meaning.

(c) Find out the condition(s) for which the system is locally asymptotically stable or unstable along each equilibrium.

(d) How can you conclude about the global asymptotic stability of the system around the co-existence point.

[4 + 2 + 7 + 2] = 15]

INDIAN STATISTICAL INSTITUTE

Mid-semestar Examination: (2013-2014)

MS(QE) II

Social Choice and Political Economy

Date: 03.03.2014 Maximum Marks: 40 Duration: 3 hrs.

(1) Consider the social aggregation problem, where $A = \{x, y\}$ is the set of alternatives, $|A| = 2$, N is the finite set of agents and $|N| \geq 2$.

(a) Suppose that the social welfare function satisfies symmetry and neutrality and the number of agents preferring x over y is the same as the number of agents preferring y over x. Then show that the social welfare function necessarily prescribes indifference between x and y.

(b) Define non-triviality and positive responsiveness of a social welfare function.

(c) Can you find a social welfare function that satisfies non-triviality and positive responsiveness but fails to satisfy Pareto? Justify your answer.**(5+1+4=10)**

(2) State Arrow's impossibility theorem. Prove Arrow's impossibility theorem using the notion of an 'extremely pivotal' agent. **(2+18=20)**

(3) Define 'almost decisiveness' and 'decisiveness'. Show that if a social ordering satisfies positive responsiveness, then 'almost decisiveness' and 'decisiveness' are equivalent. **(2+8=10)**

INDIAN STATISTICAL INSTITUTE

MID-SEMESTER EXAMINATION 2013-14

Course name: MSQE I and M.Stat II
Subject name: Microeconomic Theory II
Date: 4th March 2014 Maximum marks: 40 Duration: 2 hours

Throughout this section, $\mathbb{R}^\ell$ is the ℓ-dimensional Euclidean space. Let

$$\mathbb{R}^\ell_+ = \{x = (x^1, \cdots, x^\ell) \in \mathbb{R}^\ell : x^h \geq 0 \text{ for all } 1 \leq h \leq \ell\}$$

and

$$\mathbb{R}^\ell_{++} = \{x = (x^1, \cdots, x^\ell) \in \mathbb{R}^\ell : x^h > 0 \text{ for all } 1 \leq h \leq \ell\}.$$

Q1. Answer **all** questions.

(i) Let $\mathcal{O} = \{A, B, C, D\}$ and $\succeq$ be a *preference relation* on the set of compound lotteries over $\mathcal{O}$. Suppose also that $\succ$ and $\sim$ are the *strict preference relation* and the *indifference relation* respectively associating with $\succeq$. Assume that $\succeq$ satisfies the von Neumann-Morgenstern axioms and

$$C \sim \left[\frac{3}{5}(A), \frac{2}{5}(D)\right], \; B \sim \left[\frac{3}{4}(A), \frac{1}{4}(D)\right], \; A \succ D.$$

If

$$L_1 = \left[\frac{2}{5}(A), \frac{1}{5}(B), \frac{1}{5}(C), \frac{1}{5}(D)\right] \text{ and } L_2 = \left[\frac{2}{5}(B), \frac{3}{5}(C)\right],$$

Determine which of the following is true: (a) $L_1 \succ L_2$; (b) $L_2 \succ L_1$; and (c) $L_1 \sim L_2$. [6]

(ii) Let $\mathbb{R}_+$ denote the set of wealth and $N = \{1, 2\}$ denote the set of agents. Suppose that $U_i : \mathbb{R}_+ \to \mathbb{R}$ be a strictly monotonically increasing and twice differentiable function for $i \in N$. If agent 1 is more risk averse than agent 2 then show that there is a strictly increasing and concave function $V : \mathbb{R} \to \mathbb{R}$ such that $U_1 = V \circ U_2$. [8]

(iii) Consider the utility function $U(x) = \sqrt{x}$. Find the certainty equivalence and the probability premium for lotteries

$$L_1 = \left[\frac{1}{2}(16), \frac{1}{2}(4)\right] \text{ and } L_2 = \left[\frac{1}{2}(36), \frac{1}{2}(16)\right].$$

Compare the probability premium of L_1 and L_2. [6]

Q2. Suppose that an agent has a preference relation $\succeq$ on $\mathbb{R}^\ell_+$ represented by a utility function $U : \mathbb{R}^\ell_+ \to \mathbb{R}$ and an initial endowment $w \in \mathbb{R}^\ell_+ \setminus \{0\}$.

(i) Let $\ell = 3$ and

$$U(x, y, z) = \sqrt{x} + \sqrt{y} + y + \frac{z}{1+z}.$$

If $z > 0$, then verify that $(x, y+z, 0) \succ (x, y, z)$. If a price $p = (p^1, p^2, p^3) \in \mathbb{R}^3_{++}$ satisfies $p^2 = p^3$, then show that the demand set $D(p) \subseteq \{(x, y, 0) : x, y \in \mathbb{R}_+\}$. [6]

(ii) Let $\ell = 2$ and $U(x, y) = x^\alpha y^\beta$, where $\alpha, \beta \geq 0$. Find the demand set $D(p)$ for all $p \in \mathbb{R}^2_{++}$ when (i) $\alpha \neq 0$ and $\beta \neq 0$; and (ii) $\alpha = 0$ and $\beta \neq 0$. [8]

(iii) If $\{p_n : n \geq 1\} \subseteq \mathbb{R}^\ell_{++}$ satisfies $p_n \to p \in \mathbb{R}^\ell_{++}$, then there exists a bounded subset B of $\mathbb{R}^\ell_+$ such that the demand set $D(p_n) \subseteq B$ holds for each $n \geq 1$. [6]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination
Semester II : 2013-2014
M.Stat. II Year
Stochastic Processes I

Date : 04.03.14 | **Maximum Score : 40** | Time : 2 Hours

Note : This paper carries questions worth a **total** of **48 marks**. Answer as much as you can. The **maximum** you can score is **40**.

Throughout, (S, ρ) denotes a metric space and $\mathcal{S}$ denotes the Borel σ-field on S. For any function g on S, the set of discontinuity points of g is denoted by D_g. For $A \subset S$, the boundary of A is denoted by ∂A.

1. Show that if S is complete and separable, then any probability measure P on $(S, \mathcal{S})$ is tight. [8]

2. (a) Show that for any real-valued function f on S and any real number t, one has $\partial\{f > t\} \subset D_f \cup \{f = t\}$.
(b) Using (a) and Portmanteau, show that if $P_n \xrightarrow{w} P$ on $(S, \mathcal{S})$, then for any bounded measurable function $f : S \to \mathbb{R}$ with $P(D_f) = 0$, $\int f\, dP_n \to \int f\, dP$. [Enough (?) to do it when $0 \leq f \leq 1$.]
(c) Use (b) to prove: If $P_n \xrightarrow{w} P$ on $(S, \mathcal{S})$, then for any measurable $h : S \to S'$ with $P(D_h) = 0$, one has $P_n h^{-1} \xrightarrow{w} Ph^{-1}$ on $(S', \mathcal{S}')$. (6+6+6)=**[18]**

3. Let $\{x_n\}$ be a sequence in S. Show that if $\delta_{x_n} \xrightarrow{w} P$ on $(S, \mathcal{S})$, then $P = \delta_x$ for some $x \in S$. [Enough (?) to do it for separable S.] **[10]**

4. (a) Let $\{B_t, 0 \leq t \leq 1\}$ be a standard Brownian motion on some probabilituy space. Define $X_t = B_t - tB_1$ for $0 \leq t \leq 1$. Show that $\{X_t\}$ is a 0-mean Gaussian process with continuous paths and $X_0 = X_1 \equiv 0$.
(b) With $\{X_t\}$ as defined in (a), define $Y_t = (1+t)X_{t/(1+t)}$ for $0 \leq t < \infty$. Show that $\{Y_t, 0 \leq t < \infty\}$ is a standard Brownian motion. (6+6)=**[12]**

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2013–14(Second Semester)

M. Stat. II Year

APPLIED MULTIVARIATE ANALYSIS

Date: March 5, 2014 Maximum Marks: 50 Duration: 2 hr

1. Let $\boldsymbol{X} = (X_1, X_2, X_3)'$ be a trivariate random vector with mean vector $\mathbf{0}$ and covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & 0 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 \\ 0 & \rho\sigma^2 & \sigma^2 \end{bmatrix},$$

where $\rho \in (-1/\sqrt{2}, 1/\sqrt{2})$.

(a) Find the principal components of $\boldsymbol{X}$.

(b) Determine the proportion of the total population variance that is explained by each principal component.

(c) Will the principal components remain the same if they are determined from the correlation matrix

$$\boldsymbol{R} = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix},$$

rather than from $\mathbf{\Sigma}$? Justify your answer.

[15+6+4=25]

2. Let X_1, X_2, X_3, X_4, X_5 denote respectively the weekly rates of return of stocks (over a fixed period) of five companies A, B, C, D and E, listed on the New York Stock Exchange. Data on $n = 100$ weekly rates of return was used to obtain the sample correlation matrix

$$\boldsymbol{R} = \begin{bmatrix} 1 & .577 & .509 & .387 & .462 \\ & 1 & .599 & .389 & .322 \\ & & 1 & .436 & .426 \\ & & & 1 & .523 \\ & & & & 1 \end{bmatrix}$$

(Please Turn Over)

Based on this, the estimates of factor loadings for a 2-factor model were computed by the maximum likelihood method. These are given in the following table, together with the estimated varimax-rotated factor loadings:

Variable	MLEs of factor loadings		MLEs of rotated factor loadings	
	F_1	F_2	F^*_1	F^*_2
X_1	0.684	0.189	0.601	0.377
X_2	0.694	0.517	0.850	0.164
X_3	0.681	0.248	0.643	0.335
X_4	0.621	-0.073	0.365	0.507
X_5	0.792	-0.442	0.208	0.883

(a) Using the estimated unrotated factor loadings, obtain estimates of the specific variances, explaining briefly the underlying theory.

(b) In the unrotated factor model, suggest an interpretation of the two factors, given that the companies A, B and C are chemical-manufacturing companies, while D and E are petroleum companies.

(c) In what manner does the interpretation of the factors change after rotation? Explain.

(d) Describe how the Maximum Likelihood method directly provides a test for determining the optimal number (k) of factors.

(e) Discuss any method for estimating the factor scores.

[7+6+4+4+4=25]

Indian Statistical Institute

Second Mid-Semestral Examination

M. Stat. II year
Statistical Inference I

Date: March 6, 2014 | Maximum Marks: 60 | Duration: 2 hours

Answer all Questions. Paper carries 70 points

1 (a) Define sufficiency, weak conditionality and likelihood principles for statistical experiements with examples.

(b) Show that sufficiency and weak conditionality principles imply and implied by the likelihood principle.

[(5+5+5) + 15 =30]

2 (a) Let $(\mathcal{X}, \mathcal{A}, \{P_\theta\})$ be a statistical experimental model. For sub $\sigma-$ fields, $\mathcal{B} \subset \mathcal{C} \subseteq \mathcal{A}$ define $\mathcal{B}$ to be $\mathcal{C}$-*sufficient* if for any $\mathcal{C}$ measuable, bounded function ϕ there is a uniform version $E(\phi||\mathcal{B})$ independent of θ. Also, let any two sub $\sigma-$ fields, $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be P_θ-*independent* if $P_\theta(B \cap C) = P_\theta(B)\, P_\theta(C)$ for every θ, $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Finally, let us define sub $\sigma-$ field $\mathcal{B}$ to be *ancillary* if $P_\theta(B)$ is independent of θ for every $B \in \mathcal{B}$.

Suppose that a sub $\sigma-$ field $\mathcal{C}$ is P_θ-independent of an ancillary sub σ- field $\mathcal{B}$ and together with $\mathcal{B}$ generates $\mathcal{A}$, i. e., $\mathcal{A}$ is the smallest σ- field that contains both $\mathcal{C}$ and $\mathcal{B}$, $\sigma(\mathcal{C}, \mathcal{B}) = \mathcal{A}$. Then show that $\mathcal{C}$ is $\mathcal{A}$-sufficient. (restrict to discrete sample space and partition σ-fields only)

(b) Consider the experiement where $X_1, X_2, \ldots, X_k$ are iid samples from discrete $\mathrm{U}(0, \theta)$ for $\theta \in \mathbb{N}$. Compute the defifciency error minimizer between $\theta, \eta \in \mathbb{N}$ as discussed in the class (that is a a linear oprator $\tau : \ell_\infty(\mathbb{N}^k) \to \ell_\infty(\mathbb{N}^k)$ which minimizes for $\phi \in \ell_\infty(\mathbb{N}^k)$

$$\sup_{||\phi||_\infty \leq 1} |E_\theta(\phi) - E_\eta(\tau(\phi))|.$$

(You can assume $k = 1$ to begin with)

[20 + 20 =40]

INDIAN STATISTICAL INSTITUTE

MID-SEMESTRAL EXAMINATION

M. Stat, 2nd year 2013-14

Subject: Theory of Games and Statistical Decisions

Date: 07/03/2014 Duration: 1hr 30 minutes

Full Marks: 30

The paper contains questions of 35 marks. Attempt all questions. Maximum you can score is 30.

1. Let A be an n x n positive definite matrix .Consider the mixed extension of the matrix game A. Show that value (A) ≥ (β_{min})/n , where β_{min} =minimum eigen-value of A .Give example of real n x n matrices A where in the above inequality the following are attained.

   i) equality

   ii) strict inequality [8+2+2]

2. Consider matrix A = $((a_{ij}))_{n \times n}$

   Let a_{ij} = 1 if i-j =1 or 1-n

   = -1 if i-j = -1 or n-1

   =0 otherwise

   i) Drive all equilibrium situations and optimal strategies in context of the mixed extension of the game A, with n=4.

   (P.T.O.)

ii) In the context of the above matrix A (in ques. 2i)), give example of strategies X_1, Y_1 of players I and II respectively such that

$X_1 A Y_1^T$ = value (A), but (X_1, Y_1) is not an equilibrium situation. [8+4]

3. Let $A_{m \times n}$ be a real matrix. Connsider mixed extension of the matrix game A . Show that for all such A , $\max_X \min_Y X A Y^T$ and $\min_Y \max_X X A Y^T$ both exist ,where X varies over the set of mixed strategies of player I and Y varies over the set of mixed strategies of player II.

[11]

XXXXXXXXXXXXXXXXXXXXXXXX

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2013-2014
M. Stat. II Year: Semester-II

Survival Analysis

Date: 29 April, 2014 Maximum Marks: 60 Duration: 3 hours

Note: Answer as many questions as you can but the maximum you can score is 60. Total mark is 68.

1. Let T be a discrete survival random variable which takes values $t_1 < t_2 < \ldots$. Show that the mean residual life at age t for $t_i \le t < t_{i+1}$ can be expressed as

$$MRL(t) = \frac{(t_{i+1} - t)S(t_{i+1}) + \sum_{j \ge i+1}(t_{j+1} - t_j)S(t_{j+1})}{S(t)}.$$

[5]

2. Consider the following table to compare two treatments A and B,

Treatment →	A	B	Total events
Event(Yes)	d_{Ah}	d_{Bh}	d_h
Event(No)	$n_{Ah} - d_{Ah}$	$n_{Bh} - d_{Bh}$	$n_h - d_h$
Total subjects	n_{Ah}	n_{Bh}	n_h

(a) Derive the Cochran-Mantel-Haenszel test statistics and its large sample distribution in hypergeometric set up with proper statement of hypothesis and assumptions.

(b) Extend the test for multiple tables where $h = 1, 2, ..., k$.

(c) How can you use this idea to compare two survival curves?

[4+4+2=10]

3. Suppose that the joint survival function of the latent failure times T_1 and T_2 for two competing risks is

$$S(t_1, t_2) = \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \lambda t_1 t_2), \quad 0 < t_1 < \infty, \quad 0 < t_2 < \infty.$$

(a) Find the marginal hazard function and cumulative incidence function of T_1.

(b) Describe censored data corresponding to this model. Discuss why it would not be possible to assess the adequacy of the joint model for (T_1, T_2) on the basis of data as you have described.

[(2+5)+(2+3)=12]

4. Suppose the continuous lifetimes T given z following Cox proportional hazard model $\lambda(t; Z) = \lambda_0(t)\exp(\beta^T Z)$ ($\lambda_0(t)$ is unspecified) are grouped into $k+1$ intervals $I_i = [a_{i-1}, a_i)$, $i = 1, \ldots, k+1$, with $a_0 = 0$ and $a_{k+1} = \infty$. Assume that all censoring in I_i takes place just prior to time a_i, after all deaths in I_i have occurred. Write down the likelihood function for estimating β and $S_0(a_1)$, $S_0(a_2)$, $\ldots$, where $S_0(t)$ is the baseline survival function of T.

[4]

P.T.O.

5. Consider the Cox proportional hazard model $\lambda(t; Z) = \lambda_0(t)\exp(\beta^T Z)$, where $\lambda_0(t)$, the baseline hazard function, is arbitrary and unspecified.

(a) Suppose you have complete data. The observed failure times are $t_1 < t_2 < \cdots < t_n$ with corresponding covariate vectors $Z_1, \ldots, Z_n$. Suppose $U(\beta)$ is the score function corresponding to β. Show that the score test statistic for the hypothesis $H_0 : \beta = 0$ can be written as

$$U(0) = \sum_{i=1}^{n} Z_i(1 - a_i),$$

where a_i is the expectation of ith order statistic in a sample of size n from an exponential distribution with mean 1.

(b) Consider a two sample problem using censored data with possible tie. Write down the score statistic using Breslow's approximate partial likelihood function. Argue whether it is the log-rank test or not.

[10+(4+2)=16]

6. Let T be a continuous survival time. Given a covariate Z, suppose that log survival time Y follows a linear model

$$Y = \ln T = \mu + \beta^T Z + \sigma e,$$

where e is error random variable.

(a) Derive a linear rank statistic for testing $H_0 : \beta = \beta_0$ based on complete data.

(b) Show that the expectation (under permutation distribution) of the test statistic under the null hypothesis is zero.

(c) Derive the scores if error follows standard extreme value distribution.

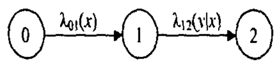
[5+4+4=13]

7. Consider a simple illness-death model as shown in the figure, where 0, 1 and 2 represents healthy, illness and death states, respectively. Let T_{01} be the sojourn time in state 0 and T_{12} the same in state 1 before death. Let C be the censoring random variable independent of both T_0 and T_{12}. Suppose that there are n individuals in a study with state 0.

(a) Describe the data with appropriate notation when time to occurrence of illness state is observed for all the individuals. Write down the likelihood function for estimating the parameters, when $\lambda_{01}(x) = \lambda_{01}$ and $\lambda_{12}(y|x) = \lambda_{12}e^{\beta x}$.

(b) Suppose that the time to occurrence of illness state is unknown for some of the individuals. Describe the data with appropriate notation. Construct the likelihood function when $\lambda_{01}(x) = \lambda_{01}$ and $\lambda_{12}(y|x) = \lambda_{12}$.

[3+5=8]



Indian Statistical Institute
Second Semestral Examination
2013-2014 Academic Year
Asymptotic Theory of Inference
M. Stat. Second Year

Date: 29.04.2014 Maximum Marks: 55 Duration : $3\frac{1}{2}$ hours

Answer all questions

1. Suppose X_i, $i \geq 1$ are independent with $X_i \sim N(\theta, \sigma_i^2)$, where $\theta \in \mathcal{R}$ is unknown and σ_i's are known. Consider the problem of estimating θ based on $(X_1, \ldots, X_n)$. Is it possible to construct a sequence $\{T_n\}_{n\geq 1}$ of estimators based on $(X_1, \ldots, X_n)$ which is consistent for θ when $\sigma_i = i$ for $i \geq 1$? What happens if $\sigma_i = \sqrt{i}$ for $i \geq 1$? Justify your answers. [5]

2. Let $X_1, \ldots, X_n$ be iid with common density $f(x, \theta)$ (with respect to the Lebesgue measure on R) and $\theta \in \Theta$, Θ being a closed subset of $\mathcal{R}$. Let $\theta_0 \in \Theta$ be the true value of the parameter θ. Assume further that,
(i) for any $\theta \neq \theta_0$, $E_{\theta_0}(\log f(X, \theta)) < E_{\theta_0}(\log f(X, \theta_0))$,
(ii) $E_{\theta_0}(|\log f(X, \theta_0|) < \infty$,
(iii) $\lim_{\rho\to 0} E_{\theta_0}(\log f(X, \theta, \rho)) = E_{\theta_0}(\log f(X, \theta))$, $\forall \theta \in \Theta$, and,
(iv) $\lim_{r\to\infty} E_{\theta_0} \log \phi(X, r) = -\infty$, where
$f(x, \theta, \rho) = \sup\{f(x, \theta') : \theta' \in \Theta \cap [\theta-\rho, \theta+\rho]\}$ and $\phi(X, r) = \sup\{f(x, \theta) : |\theta| > r\}$.
Prove that
$$\lim_{n\to\infty} \sup_{\theta\in F} \frac{\prod_{i=1}^{n} f(X_i, \theta)}{\prod_{i=1}^{n} f(X_i, \theta_0)} = 0 \text{ a.s. } [P_{\theta_0}^{\infty}],$$
where F is any closed subset of Θ not containing θ_0. [8]

3. Let $\{P_{0n}\}$ and $\{P_{1n}\}$ be two sequences of probabilities on some sequence of measurable spaces $(\mathcal{X}_n, \mathcal{A}_n)$ such that $\{P_{1n}\}$ is contiguous with respect to $\{P_{0n}\}$. Let $f_{in} = \frac{dP_{in}}{dS_n}$, $i = 0, 1$ where $S_n = P_{0n} + P_{1n}$. Let $M_n = S_n\left(\log \frac{f_{1n}}{f_{0n}}\right)^{-1}$. Show that $M_n([n, \infty]) \to 0$ as $n \to \infty$. [8]

4. (a) Consider the family of probabilities $\{P_\theta : \theta \in \mathcal{R}\}$ where P_θ has a density $f(x - \theta)$ with respect to the Lebesgue measure on $\mathcal{R}$ and
$$f(x) = \frac{1}{2}e^{-|x|}, \; x \in \mathcal{R}.$$
Is the family QMD at $\theta = 0$? Justify your answer. [6]
(b) Define Local Asymptotic Mixed Normality. Give an example (without proof) where this property is satisfied. [4]

5. (a) Define regularity of a sequence of estimators under the LAN setup. [2]
(b) Suppose $X_1, \ldots, X_n$ are iid $N(\theta, 1)$ where $\theta \in \mathcal{R}$ is unknown. Consider the Hodges' estimator of θ that estimates θ by $\bar{X}_n$ if $|\bar{X}_n| > n^{-1/4}$ and by 0 if $|\bar{X}_n| \leq n^{-1/4}$, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Is this estimator regular at $\theta = 0$? Justfy your answer. [5].
(c) Suppose now that $\{T_n\}$ is a regular (at θ_0) sequence of estimators under the LAN setup with normalizing matrtix ϕ_n. Show that
$$\liminf_{n\to\infty} E_{\theta_0} L[\phi_n^{-1}(T_n - \theta_0)] \geq EL(Z),$$
where $Z \sim N(0, I^{-1}(\theta_0))$ and L is any subconvex loss. [6]

(d) State the Hajek-LeCam Local Asymptotic Minimax Theorem. State a necessary condition for a sequence of estimators to be Locally Asymptotically Minimax. [4]

6. Suppose $\{P_\theta^n : \theta \in \mathcal{R}\}$ satisfies the LAN condition at θ_0 with normalizer $\frac{1}{\sqrt{n}}$ and non-singular $I(\theta_0)$. Show that the sequence of experiments $\{P^n_{\theta_0+\frac{h}{\sqrt{n}}} : h \in \mathcal{R}\}$ converges to the experiment $\{N(h, I^{-1}(\theta_0)) : h \in \mathcal{R}\}$. [7]

INDIAN STATISTICAL INSTITUTE

203, B.T. ROAD, KOLKATA – 700108

SEMESTER II EXAMINATION (2013–14)

M. STAT. I and II Years & M.S. (Q.E.) I Year

Time Series Analysis / Time Series Analysis & Forecasting

Date: 02 May 2014 Maximum Marks: 100 Time: 3 hours

*The question paper carries a total of **120** marks. You can answer any part of any question. But the maximum that you can score is **100**. Marks allotted to each question are given within parentheses.*

1. (a) State what you understand by PACF of a time series. Show that for an AR (2) process, $\Phi_{11} \neq 0, \Phi_{22} \neq 0$, but $\Phi_{kk} = 0$ for $k \geq 3$ where Φ_{kk} stands for the k^{th} partial autocorrelation coefficient of the time series.

   (b) Find the 2-step ahead minimum MSE forecast at origin n of the following time series

   $$(1-0.8B)(1-B)^2 X_t = (1+0.4B)\, a_t,\ a_t \sim WN(0,1).$$

   Further show that $V(e_{n,1}) < V(e_{n,2})$ where $e_{n,h}$, $h = 1,2$, is the forecast error associated with the h-step ahead forecast.

   $[\overline{4+11}+9=24]$

2. (a) Distinguish between deterministic trend and stochastic trend.

   (b) Discuss what, in your opinion, is the most important problem in case the ADF test is carried out without any consideration to structural break(s) in the time series.

   (c) Describe the Quandt-Andrews test for detecting the presence of a single structural break in a time series.

   [5+8+8 = 21]

3. (a) Discuss about the nature of unit roots in a quarterly time series.
   (b) Describe the HEGY test for detecting the presence of seasonal and non-seasonal unit roots in a quarterly time series data.

   [6+12 = 18]

4. (a) Discuss briefly the problem of missing observations and its consequences in time series analysis.

(b) Consider an AR (1) model with multiple missing observations, but no two missing observations being consecutive. Describe, with derivations wherever necessary, the forecasting replacement method for dealing with such missing observations. State also all the underlying assumptions.

[6+13 = 19]

5. State a simple state-space model along with all the assumptions. Show how prediction and then updating are carried out in this model by applying the Kalman filter.

[13]

6. (a) State a representation, along with all assumptions, of the stationary time series in the frequency domain approach, and then prove that under these assumptions, the time series is indeed stationary.

(b) Given a finite realization $\{x_1, x_2, \ldots, x_n\}$ of a stationary time series, obtain its periodogram, and then show that this can be regarded as a sample analogue of $2\pi f(\lambda)$ where $f(\lambda)$ is the spectral density function.

(c) The spectral density function of a real-valued stationary time series $\{x_t\}$ is defined on $[0, \pi]$ by

$$f(\lambda) = \begin{cases} 150, & \pi/3 - 0.02 < \lambda < \pi/3 + 0.02 \\ 0, & \text{otherwise} \end{cases}$$

and

on $[-\pi, 0]$ by $f(\lambda) = f(-\lambda)$.

(i) Evaluate the autocovariance function of $\{x_t\}$ at lags 0 and 1.

(b) Find the spectral density function of $\{y_t\}$ where $y_t = x_t - x_{t-4}$.

[8+8+9 = 25]

INDIAN STATISTICAL INSTITUTE
Second Semester Examination: 2013-14
M. Stat. II Year
Advanced Sample Survey

Date: 02/05/2014 **Maximum Marks: 50** **Duration: 3 Hours**

Four questions each carrying 10 marks are required to be answered. Records of Assignments carrying 10 marks required to be submitted on or before the last date of the semestral examination.

1 Considering a homogeneous linear estimator for a finite population total establish general results to show how uniformly non-negative unbaised estimators for the Mean Square Error in estimation may be derived.

2 Derive Brewer's Asymptotically Design unbiased predictor for a finite population total. Give a formula to estimate its measure of error.

3 Give an estimator for the finite population total correlation coefficient based on a varying probability sampling scheme. Show that it lies in [-1, +1]. How will you estimate its measure of error?

4 Derive Murthy's almost unbiased ratio type estimator for the ratio of two totals, explaining its rationale.

5 Discuss how you may employ an RR technique to suitably estimate the proportion of people loyal to their office bosses along with an estimated variance of the estimator employed.

*****

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2013-2014, Second Semester
M-Stat II and M-Math II
Ergodic Theory

Date: 02/5/14 Max. Marks 50 Duration: 3 Hours

Note: 1. Answer all questions.
2. All the measures considered are probability measures
3. Total Marks: 58. Maximum you can score: 50.

1. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Let f be a μ-integrable function. Then

   a) show that $a_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$ is uniformly integrable.

   b) using (a) show that $\lim_n a_n(x) = E(f|\mathcal{I})(x)$ a.s.$[\mu]$ where $\mathcal{I} := \{A \in \mathcal{B} : A = T^{-1}A\}$.

   (You may assume Birkoff's Ergodic Theorem: $\lim_n a_n(x)$ exists a.s. and is invariant under T.)

   [9 + 9]

2. First prove (a) then using (a) prove (b).

   a) Let $\{c_{i,j} : i, j \geq 1\}$ be a double indexed bounded sequence of real numbers such that $\lim_{|i-j|\to\infty} c_{i,j} = 0$. Then show that

   $$\lim_n \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{i,j} = 0$$

   b) Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be invertible measure preserving strong mixing. Show that for every strictly increasing sequence of integers $\{k_n\}$,

   $$\lim_n \int_X |\frac{1}{n}\sum_{i=1}^{n} I_A(T^{k_i}x) - \mu(A)|^2 d\mu = 0.$$

   [6+12]

P. T. O.

$f \in L_2(\mu)$

3. Let $(X, \mathcal{B}, \mu)$ be a probability space such that $\mathcal{B}$ is countably generated and let $T : X \to X$ be an invertible measure preserving transformation which has countable Lebesgue spectrum. Suppose that $\mathcal{P}$ is the spectral measure corresponding to U_T, the Koopman operator associated with T. If f is an L_2-function on $(S^1, \mathcal{B}_{S^1}, \lambda)$, where S^1 is the unit circle and λ Lebesgue measure and $\langle f, 1\rangle = 0$ and $\langle f, f\rangle = 1$ then show that $\langle \mathcal{P}(A)f, f\rangle = \lambda(A)$ for any Borel $A \subset S^1$.

[9]

4. Suppose $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $i = 1, 2$ are two measure preserving dynamical systems which are spectrally isomorphic. Then show that if one of them is ergodic, the other is also so.

[5]

5. a) If two measure preserving dynamical systems $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $i = 1, 2$ are conjugate then show that they have the same entropy.

   b) Give an example to show that the converse is not true.

[5+3]

Indian Statistical Institute
Second Semestral Examination 2013-14
M. Stst. II (MSP)
Statistical Inference II

Date: May 5, 2014 | Maximum marks: 100 | Duration: 3 hrs.

Answer all Questions. *Paper carries 115 points.*

1 (a) Let $(\mathcal{X}, \mathcal{A}, Q)$ be a probability space and P be another measure defined on $(\mathcal{X}, \mathcal{A})$. Suppose $\mathcal{G} \subseteq \mathcal{A}$, be a sub-sigma field. Further let (Q_0, ν, N) be a Lebesgue decomposition of Q with respect to P in the sense that $Q_0 << P$, $\nu \perp P$ and $P(N^c) = 0, \nu(N) = 0$ respectively. If $N \in \mathcal{G}$ and $dQ_0 \,/\, dP$ is $\mathcal{G}$-measurable show that for any bounded $\mathcal{A}$-measurable function f, $E_{Q_0}(f||\mathcal{G})\, I_N$ $+E_\nu(f||\mathcal{G})\,(1 - I_N)$ is a version of conditional expectation $E_Q(f||\mathcal{G})$

(b) Let the experiment $\mathcal{E}$ be described by iid observations $X_1, X_2, \ldots, X_n$ where $P(X_1 = \theta) = P(X_1 = \theta - \sigma) = P(X_1 = \theta + \sigma) = 1/3$, $\theta \in \mathbb{Z}$ and $\sigma \in \mathbb{N}$. Discuss differential nature of $Ev(\mathcal{E}, x)$ for different configurations of the sampled data, where $x = (x_1, x_2, \ldots, x_n)$ in terms of sufficiency and likelihood principles.

[15 + 10 = 25]

2. Consider a multinomial model with n observations and k cell probabilities given by $\pi(\theta) = (\pi_1(\theta), \pi_2(\theta), \ldots, \pi_k(\theta))$ where θ is a real valued parameter taking values in an open interval $\Theta \subseteq \mathbb{R}$. Let the observation vector be denoted by $x = (x_1, x_2, \ldots, x_k)$. Asume that π is a smooth function of θ and $0 < \pi_i(\theta) < 1$ for all i and $\theta \in \Theta$.

(a) Define local unbiasedness at $\theta = \theta_0$ and obtain the locally unbiased minimum variance estimator at θ_0.

(b) Define the notion of local sufficiency at $\theta = \theta_0$ and show that the score function $\ell_1(x, \theta_0)$ is locally suficient.

(c) Define the statistical curvature $\kappa(\theta_0)$ at θ_0 and show that locally unbiased linear estimator T at θ_0 (explicitly derive it) based on $\ell_1(x, \theta_0 + t/\sqrt{I(\theta_0)})$ satisfies

$$\text{Var}_{\,\theta_0}(T) = \frac{1}{I(\theta_0)} + \kappa^2(\theta_0)\, t^2 + o(t^2)$$

as $t \to 0$ where $I(\theta)$ is the Fisher information of the model.

[10 + 10 + 20 = 40]

... P. T. O.

3 (a) Describe the notion of a parametric testing problem remaining invariant under a group of transformations G with one nontrivial example (explicitly compute G, $\bar{G}$, maximal invarinat and invariance of hypotheses) .

(b) Suppose the problem of testing Ω_0 against Ω_1 remains invariant under a finite group $G = \{g_1, g_2, \ldots, g_N\}$ and that $\bar{G}$ is transitive over Ω_0 and Ω_1 repectively. Show that there exists a UMP invariant test of Ω_0 against Ω_1 and derive the test statistic describing critical regions at different levels of significance.

[10+15 =25]

4 (a) Define the notion of indifference zone and maximin tests with one nontrivial example.

(b) Let θ be a location parameter, so that $f_\theta(x) = g(x - \theta)$, and suppose for simplicity that $g(x) > 0$ for all x. Show that the location family has monotone likelihood ratio in x if and only if $-\log g$ is convex.

[10+ 15 =25]

INDIAN STATISTICAL INSTITUTE

Final Examination : Semester II (2013-2014)

Course Name : BSDA (M. Stat. 2nd year)

Subject Name : Statistical Methods in Biomedical Research

Date : 05·05·14 , Maximum Marks : 65. Duration : 3 hrs.
[Answer as many as you can.]

1. What is the difference between clinical trials and bioassays? What are the differences between direct bioassay and indirect bioassay? What is slope-ratio assay? Derive the relative potency, ρ in terms of the linear regression parameters in slope ratio assay. [2+2+2+5=11]

2. Discuss the $A + B$ design for dose finding in toxicity. [5]

3. Construct an example to illustrate how the imbalance with respect to prognostic factors induces allocation bias. How Pocock's minimization rule can be implemented to get rid of this difficulty? [4+4=8]

4. Suppose patients are entering in a system in a sequential way. If the response time of any patient is constant ($= c$), and interarrival times are exponential with expectation b, find the probability that the response of the 4-th patient will be obtained before the entrance of the 10-th patient. If, instead, interarrival times are normally distributed with mean μ and variance σ^2, find the required probability. [5+2=7]

5. Discuss how type I error spending functions can be constructed by accumulating boundary crossing probabilities. Suppose $\alpha_1^*(t) = \alpha t$, $0 \leq t \leq 1$, be one type I error spending function. Another type I error spending function α_2^* spends exactly one-third of the type I error of α_1^* at the half way mark. Derive one functional form of $\alpha_2^*(t)$. [4+5=9]

6. Describe a Truncated Binomial Design (TBD) for assigning subjects in two treatment clinical trial. How is TBD affected by the selection bias? Show that for such a design, the unconditional allocation probability to either treatment for any subject remains 0.5. Also calculate the expected number of subjects in the tail. [2+2+3+3=10]

7. Let there be four treatments (A, B, C & D) under comparison in a clinical trial where the treatment responses are binary. We start with an urn for allocating the entering patients. Initially the urn contains 2 balls of each type. For any entering patient, we treat him/her by drawing a ball from the urn and replace it immediately to the urn. If the response is a success, we add an additional 3 balls

of the same kind in the urn. On the other hand, if the response is a failure, we add an additional 1 balls of each of the other three types. Assuming the responses to be instantaneous, develop appropriate notations and write down the conditional probability that the nth patient is treated by treatment A, given all the previous allocation and response history. Find the unconditional probability for the second patient. [4+4=8]

8. Consider a two-treatment trial with binary responses as well as binary surrogates. Suppose both true and surrogate responses are available for m_1 and m_2 patients for the two treatments, while only surrogate responses are available for $(n_1 - m_1)$ and $(n_2 - m_2)$ patients. Let p_1 and p_2 be the true success probabilities by the two treatments. Discuss the probability set up and illustrate that the surrogate-augmented estimate of treatment difference is 'biased'. [3+5=8]

9. Consider a two-treatment response-adaptive allocation with binary responses. Let p_1 and p_2 be the success probabilities by the two treatments. Suppose both p_1 and p_2 can take only two possible values: a and b, where $a < b$. The prior distribution is as follows: all the four possible combinations of (p_1, p_2) have equal probability. State a sensible Bayesian response-adaptive design and find the allocation probability for the 26-th patient given that out of the first 25 patients 15 are treated by the first treatment and the number of failures are 6 for both the treatments so far. [2+8=10]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2013-14

M. Stat. II Year

APPLIED MULTIVARIATE ANALYSIS

Date: May 6, 2014 Maximum Marks: 100 Duration: 3 hr

Note: All questions carry equal marks. Answer any five.

1. The correlation matrix computed from the scores $\boldsymbol{X} = (X_1, X_2, X_3)'$ in three different subjects obtained by a group of schoolchildren was found to be

$$\begin{pmatrix} 1 & 0.83 & 0.78 \\ & 1 & 0.67 \\ & & 1 \end{pmatrix}.$$

   a. Fit a 1-factor model to the data, with appropriate justification.
   b. Describe the Maximum Likelihood approach to factor analysis.
   c. Hence deduce a likelihood ratio test (LRT) for the hypothesis that exactly $k\ (< p)$ common factors are sufficient to describe the data on a p-variate random variable $\boldsymbol{X}$, and state what form the asymptotic distribution of the LRT statistic has.

[6+6+(6+2)=20]

2. Consider two bivariate random variables $\boldsymbol{X} = (X_1, X_2)'$ and $\boldsymbol{Y} = (Y_1, Y_2)'$. For some $\boldsymbol{\alpha}$ and γ, where $|\alpha|<1$ and $|\gamma|<1$, the correlation matrix of $(X_1, X_2, Y_1, Y_2)'$ is

$$\begin{pmatrix} 1 & \alpha & \beta & \beta \\ \alpha & 1 & \beta & \beta \\ \beta & \beta & 1 & \gamma \\ \beta & \beta & \gamma & 1 \end{pmatrix}.$$

   a. Show that the first canonical correlation coefficient ρ_1 of $\boldsymbol{X}$ and $\boldsymbol{Y}$ is

$$\rho_1 = \frac{2\beta}{\sqrt{(1+\alpha)(1+\gamma)}}.$$

   b. Hence deduce the first canonical correlation vectors for $\boldsymbol{X}$ and $\boldsymbol{Y}$.

[10+(5+5)=20]

(Please Turn Over)

3. The following is a Euclidean distance matrix based on observations on 7 random variables:

$$
\mathbf{D} = \begin{pmatrix}
0 & 1 & \sqrt{3} & 2 & \sqrt{3} & 1 & 1 \\
 & 0 & 1 & \sqrt{3} & 2 & \sqrt{3} & 1 \\
 & & 0 & 1 & \sqrt{3} & 2 & 1 \\
 & & & 0 & 1 & \sqrt{3} & 1 \\
 & & & & 0 & 1 & 1 \\
 & & & & & 0 & 1 \\
 & & & & & & 0
\end{pmatrix}
$$

Apply the multidimensional scaling technique to this matrix to arrive at a configuration of points in the two-dimensional space $\mathbb{R}^2$ whose (Euclidean) distance matrix is equal to $\mathbf{D}$.

[20]

4.

a. Consider a dataset consisting of 6 observations, for which the matrix of Euclidean distances between pairs of observations is as follows:

$$
\mathbf{D} = \begin{pmatrix}
0 & 5 & 7 & 6 & 2 & 1 \\
5 & 0 & 1 & 2 & 3 & 4 \\
7 & 1 & 0 & 1 & 5 & 6 \\
6 & 2 & 1 & 0 & 4 & 5 \\
2 & 3 & 5 & 4 & 0 & 2 \\
1 & 4 & 6 & 5 & 2 & 0
\end{pmatrix}
$$

Apply single linkage clustering to divide the dataset into three clusters and sketch the dendrogram.

b. Discuss how a dataset consisting of N observations can be split into K clusters by iterative optimization of the sum-of-squared-errors criterion, explaining clearly the rationale behind the key steps.

[(7+3)+10=20]

5. Consider the problem of discrimination between two populations Π_1, Π_2 on the basis of observations on a random variable X. It is assumed that in Π_i, i=1,2, X is distributed as

$$
p_i(x) = \begin{cases} 1 - |x - a_i| & \text{if } |x - a_i| < 1, \\ 0 & \text{otherwise,} \end{cases}
$$

where a_1, a_2 are constants satisfying the condition $a_1 + 1 < a_2 < a_1 + 2$. Also, $\pi_1 = \pi_2 = \frac{1}{2}$, where π_i denotes the prior probability for the population Π_i, $i = 1,2$.

a. Write down the explicit form of the Bayes rule for discriminating between Π_i and Π_i under zero-one loss.

b. Determine the overall error probability for this rule.

c. Write down the modified Bayes discriminant rule if, instead of zero-one loss, the following loss function is used:

$$l(i,j) = \begin{cases} 0 & \text{if } i = j, \\ 2 & \text{if } i = 1, j = 2, \\ 1 & \text{if } i = 2, j = 1. \end{cases}$$

Here $l(i,j)$ denotes the loss incurred when an observation from Π_i is allocated to $\Pi_j, i,j = 1,2$. How does the overall error probability change under this loss? Explain.

[5+5+(6+4)=20]

6. Consider the problem of testing the null hypothesis that exactly $p-k$ of the p eigenvalues of the covariance matrix $\boldsymbol{\Sigma}$ of a p-variate random variable are equal, where $0 \le k < p$.

a. Write down the likelihood ratio test statistic λ for this problem, in terms of a_0 and g_0, which are respectively the arithmetic and geometric means of the sample estimates of the $p-k$ repeated eigenvalues of $\boldsymbol{\Sigma}$.

b. How many degrees of freedom does the asymptotic χ^2-distribution of $-2\log_e \lambda$ have? Justify your answer.

c. A 86 × 4 data matrix led to the sample covariance matrix

$$\mathbf{S} = 10^{-3} \times \begin{bmatrix} 29.004 & -8.545 & 1.143 & -6.594 \\ & 3.318 & 0.533 & 3.248 \\ & & 4.898 & 5.231 \\ & & & 8.463 \end{bmatrix},$$

whose eigenvalues are $\ell_1 = 0.0337, \ell_2 = 0.0111, \ell_3 = 0.0006$, and $\ell_4 = 0.0002$.

By testing null hypotheses of the type mentioned above, for $k = 0,1,2$, infer that the eigenvalues of $\boldsymbol{\Sigma}$ are all distinct.

[5+5+10=20]

7. Consider a two-category discrimination problem based on four binary variables X_1, X_2, X_3 and X_4. Suppose the following observations (in the form of 4-tuples of bits) are available from the two categories, where the i-th bit in each 4-tuple represents the observation on $X_i, i = 1,2,3,4$:

Category 1	0110	1010	0011	1111
Category 2	1011	0000	0100	1110

Use the entropy impurity measure to create an unpruned binary classification tree from this data.

[20]

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination: (2013-2014)
MS (Q.E.) I Year
Macroeconomics I

Date: 07.05.2014 Maximum Marks 60 Duration 3 hours

1. Show that in an OLG model, introducing a 'Pay as you go' pension scheme will reduce the per capita capital stock, in the steady state equilibrium.

Analyse the effect of introducing such a pension scheme on the steady state level of welfare?

[5+10

2. a) Show that in a flex price, monopolistically competitive equilibrium of the Blanchard-Kiyotaki kind, money is neutral.

Also show that in such a model the monopolistically competitive output is smaller than the competitive output.

b) Consider an economy with the representative agent having the utility function given by

$U = [C^\alpha(1-L)^{1-\alpha}]^\gamma \left[\frac{M}{P}\right]^{1-\gamma}$ with $0 < \alpha, \gamma < 1$

Where $C = n\left[\frac{1}{n}\sum_{1=1}^{n} {c_i}^{\rho}\right]^{1/\rho}$ with $0 < \rho < 1$; and c_i is the consumption of the i^{th} variety

L is the labour supply, P is the price index of varieties. Each agent is endowed with one unit of labour, thereby $(1-L)$ is the leisure enjoyed. M is the money balances (and suppose M_0 is the initial endowment of money). The household budget constraint is given by

$PC + w(1-L) + M = M_0 + w + \pi - T$ where w is the money wage rate and π is the economy wide profits and T is the taxes. Production of varieties is given by:

$$Y_i = 0 \; if \; L_i \le F$$

$$= \frac{L_i - F}{k} \; if \; L_i > F \text{ where } k>0$$

Y_i is the output of i^{th} variety and L_i is the labour employed in the production of the i^{th} variety.

Assume that there are no costs in adjusting prices (i.e. prices are fully flexible) and that there is no entry/exit of firms (fixed n).

(i) Derive the multiplier of a balanced budget ($PG=T$) increase in government expenditure where G takes the form:

$G = n\left[\frac{1}{n}\sum_{1=1}^{n} g_i{}^{\rho}\right]^{1/\rho}$; and g_i is the government consumption of i^{th} variety.

(ii) What would be the effect of such an increase in government expenditure on the price index of varieties?

[Hint: Try to write down the goods market equilibrium ($Y=C+G$) in a form which does not involve money balances. That would require a look into the money market equilibrium ($M = M_0$).]

[10+5]

3.(a) Distinguish between short-run equilibrium and long-run equilibrium in an aggregate supply demand model.

(b) Explain how does an increase in money supply generate a short-run fluctuation while making a shift from one long-run equilibrium to another. **[15]**

OR

3. Under rational expectation hypothesis, aggregate demand management policies designed to raise the level of output are ineffective - Examine the validity of this statement in the light of an aggregate supply demand model [15]

4. (a) Show how divergence in the expectations regarding the future rate of depreciation of domestic currency on the part of the domestic wealth holders can give rise to imperfect capital mobility in a small-country open economy macro model.

(b) Examine the effectiveness of fiscal policy under a flexible exchange rate regime in this model. Specify an index to measure the degree of capital mobility in this framework. Show how an increase in the value of this index makes the result closer to that under perfect capital mobility. Explain your answer. [8+7]

Indian Statistical Institute
Semester 2, Academic Year: 2013-14
Final Examination
Course: M. Stat 2nd Year
Subject: Advanced Probability II

Total Points: $5 \times 14 = 70$

Date: 9.5.14 **Time: 3 Hours**

Answers must be justified with clear and precise arguments. If you refer to a theorem/result proved in class, state it explicitly. More than one answer to a question will not be entertained and only the first uncrossed answer will be graded.

1. (a) Let M be a continuous L^2 martingale on $[0,T]$ with quadratic variation process $< M >$. Suppose f is predictable and such that $\int_0^T f_s^2 d < M >_s < \infty$ a.s. Show that for all $\lambda > 0$ and $\epsilon > 0$,

$$P\Big(\sup_{0 \le t \le T} | \int_0^t f_s dM_s| > \lambda\Big) \le P\Big(\int_0^T f_s^2 d < M >_s \ge \epsilon\Big) + 4\frac{\epsilon}{\lambda^2}.$$

(b) Suppose f is continuous on $[0,T]$. Use the above to prove that for any sequence of partitions $0 = t_0 < t_1 < \ldots < t_n = T$ such that the norm of the partition goes to zero, one has

$$\sup_{0 \le t \le T} | \int_0^t f_n dM - \int_0^t f dM| \to 0 \text{ in probability,}$$

where $f_n = \sum_{i=1}^n f(t_{i-1}) 1_{(t_{i-1}, t_i]}$. $7 \times 2 = 14$ pts.

2. Consider the following SDE:

$$dX_t = -\beta X_t \, dt + dW_t, \; t \ge 0; \; X_0 = \xi,$$

where $\beta > 0$ and $\xi \sim N(0, \sigma^2)$ is independent of W which is a standard Wiener process. Find σ such that the distribution of X_s is same for all $s > 0$. Here we are talking of the one dimensional distribution, not the finite dimensional distribution of the process. (Hint: consider the Ito differential of the process $Y_s = e^{\beta s} X_s$.)

3. Suppose $F(t, x)$ is twice continuously differentiable in both time and space variable. Write down the Ito formula for $F(t, X_t)$ where X_t satisfies $dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s$, $X_0 = x$. Here it can be assumed that b, σ satisfy the Lipschitz and usual growth conditions so that the SDE has a unique strong solution.

4. Suppose $(M_t, \mathcal{F}_t), t \in [0,T]$, is a continuous local martingale with $M_0 = 0$ and

$$Y_t = e^{M_t - \frac{1}{2} < M >_t}.$$

(a) Taking $\tau_n = \inf\{t : |M_t| \geq n \text{ or } < M >_t \geq n\}$ write down the Ito formula for $Y_{t\wedge\tau_n}$ and explain why $Y_{t\wedge\tau_n}$ is a martingale.

(b) Show that Y is a supermartingale, which is a martingale iff $EY_T = 1$. $7 \times 2 = 14$ pts.

5. Consider the standard Wiener space $(\Omega, \mathcal{F}, \mu_W)$ on $[0,T]$ and suppose Y with $EY = 0$ is in $L^2(\Omega, \mathcal{F}, \mu_W)$ i.e. a square integrable real valued random variable with zero mean on the Wiener space. Then prove that $Y = \int_0^T \Phi(s)dW_s$ for some $\Phi(\cdot)$ which is adapted and satisfies $\int_0^T E\Phi^2(s)ds < \infty$, using the steps below:

Suppose $\mathcal{M}$ is the class of all random variables M_T that can be represented as above. Notice that $EM_T = 0$.

(a) For $m \in L^2[0,T]$ (nonrandom) define

$$\beta_m(t) = e^{\int_0^t m(u)dW_u - \frac{1}{2}\int_0^t m^2(u)du}.$$

Write down the Ito formula for $\beta_m(T)$. Your main task is to show that $-1 + \beta_m(T) \in \mathcal{M}$.

(b) Consider $L^2(\Omega, \mathcal{A}, \mu_W)$, i.e. square integrable random variables on the Wiener space on $[0,T]$. If $Y \in L^2(\Omega, \mathcal{A}, \mu_W)$ with $EY = 0, Y \perp \mathcal{M}$, then $E(Y\beta_m(T)) = 0$, $\forall m \in L^2[0,T]$. With suitable choice of m, show that $E(Y|W_{t_1}, \ldots, W_{t_k}) = 0, \forall k$ (do it for t_1, then for $t_1 < t_2$ etc.). Explain why this implies $Y = 0$. $7 \times 2 = 14$ pts.

(Hint: The (normalized) Hermite polynomials $H_n(x_1)$ obtained from

$$e^{x_1\alpha_1 - \alpha_1^2/2} = \sum_{n=0}^{\infty} H_n(x_1)\alpha_1^n,$$

(here α_1 can vary over real numbers) form a complete orthonormal set wrt $e^{-x_1^2/2}dx_1/\sqrt{2\pi}$. This can be extended to a few i.i.d standard normal random variables as needed.)

Statistical Methods in Public Health
Semestral Examination
M.Stat. II Year, 2013-2014
Total Marks - 100
Time - 3 hrs. 30 min.

Indian Statistical Institute
Kolkata 700 108. INDIA

D.o.E - 09.05.14

Attempt all questions:

1. Write down a simple epidemic model. Show that the solution arises from simple deterministic model does not agree with the solution thus obtained from its stochastic analogue.

[2 + 8 = 10]

2. Write down a simple SIR epidemic model with proper assumptions. Find out the basic reproduction number of the model by next generation matrix method. How this number is useful for public health authority to eradicate the disease? Prove that from this model, at the end of the epidemic, everyone will still be susceptible.

[2 + 6 + 2 + 5 = 15]

3. Consider the following SIS model with standard incidence:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \frac{\beta SI}{(S+I)} - \mu S + \phi I \\ \frac{dI}{dt} &= \frac{\beta SI}{(S+I)} - (\alpha + \mu + \phi)I \end{aligned} \tag{1}$$

where, S and I are respectively susceptible and infected population. Λ is the rate of susceptible individuals recruited into the population (either by birth or immigration); the standard incidence is $\frac{\beta SI}{(S+I)}$. μ is the natural death rate; ϕ is the recovery rate; and α is the disease related death rate. The parameters are positive constants and variables are non-negative.

(a) Show that all solutions of the system are eventually confined in the compact subset

$$\Gamma = \left\{(S, I) \in \Re^2 : S \geq 0, I \geq 0, S + I \leq \frac{\Lambda}{\mu}\right\}.$$

(b) Find out disease free steady state, a unique endemic steady state and the basic reproduction number of the system.

(c) Show that the endemic steady state is locally and globally asymptotically stable if the basic reproduction number is greater than unity.

[3 + 3 + (3 + 3) = 12]

4. Write down a simple mathematical model on dengue fever based on the following assumptions:

(i) Divide total human population in three mutually exclusive sub-compartments susceptible, infected and recovered respectively.

(ii) Divide mosquito population in two mutually exclusive sub-compartments susceptible mosquito and infected respectively.

(iii) Total human population size is constant over time.

(iv) Total mosquito population size is constant over time.

Derive the expression of the basic reproduction number (R_0) from the formulated model. Derive the relation between the basic reproduction number (R_0) and the force of infection (Λ). Estimate R_0 from the following dengue incidence data:

Time (Month)	Number of dengue cases
January	224
February	296
March	700
April	1530
May	881
June	312
July	273
August	96

[3+4+6 = 13]

5. (a) Define Fisher's Relative Growth Rate (RGR) based on the size measurements at two specific time points. Comments on its extension and growth law non-invariant form. Derive the expression for this extended RGR metric for Gompertz growth law based on the size measurements at three consecutive time points. How this extended metric is affected through reading / measurement error under Gompertz law ? The RGR with time covariate structure can also be interpreted as the "Average rate of relative growth rate" - Explain.

(b) Let X be an $(n \times q)$ longitudinal data matrix in which any row corresponds to a q - variate size measurements available at q time points on one of the n individuals. Suggest two commonly use empirical estimates of RGR based on the data and comment on the consistency property of these estimates.

(c) Derive the expressions for bias and MSE for Fisher's RGR under both first and second order of approximations (where order is defined by the power of "difference of relative errors" at two time points). Write down the expressions for estimated bias and MSE based on the data structure given in 5(b).

[(1 + 3 + 3 + 3 + 3) + (2 + 5) + (8 + 2) = 30]

6. Let us assume that $(X_1, \ldots, X_q)' \sim N_q(\theta, \Sigma)$, where $X(t)$ = size at time point t, and $E(X(t) = \theta(t) = f(\phi, t)$, be a suitable growth curve, $t = 1, \ldots, q$. Suppose we are interested in testing the hypothesis of exponential quadratic growth curve model (EPQGM), i.e., to test

$$H_0 : \theta(t) = e^{b_0 + b_1 t + b_2 t^2} \text{ ag. } H_1 : \text{not } H_0$$

Using the approximate expression for expectation and variance of the logarithm of ratio of size variables for two consecutive time points describe two testing procedures and critical regions in testing the null hypothesis of EPQGM. Also suggest required modifications of test statistics when the time spacings are unequal and the errors are non-normal.

[11 + 5 + 4 = 20]

or

7. (a) Define quasi-equilibrium probabilities of a general birth death process.

(b) Consider the Von Bertalanffy growth equation for a single species population dynamics as follows:

$$\frac{dx}{dt} = ax^{\frac{2}{3}} - bx,$$

where parameters have their usual interpretations.

Derive the quasi equilibrium probabilities and determine the expression for the approximate mean and variance incorporating random perturbation in the above model. You may assume that the growth variable x to be bounded twice the carrying capacity.

(c) Consider the data structure as described in 5(b) and let us assume that for a given time point the distribution of absolute growth rate of n individuals is Gaussian with Von Bertalanffy growth as mean function (the right hand side of the equation stated in 7(b)) and homoscedastic variance (known). Discuss the profile likelihood method to estimate the most vital parameter a for this specific time point assuming b to be the nuisance parameter.

[3 + (6 + 6) + 5 = 20]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2013-14

Course name: MSQE II

Subject name: Incentives and Organisations

Date:

Maximum marks: 50

Duration: 3 hours

13-05-14

Answer any **two** of the following three questions. Each question carries 25 marks.

Q1. A principal (P) and an agent (A) play the following game.

1. P announces a contract (K, S).

2. A accepts or rejects the contract. Rejection yields utility 0.

3. If acceptance, A chooses effort e, at private cost e. This action is observable by the principal but not verifiable by third parties. Revenue from the project is $R(e)$, where $R(.)$ is increasing and concave.

4. P chooses whether to keep the project or sell it to the agent.

If he keeps it, he pays the agent K. Payoffs are then

$U_P = R(e) - K$, $U_A = K - e$

Alternatively, P sells the project to the agent for price S. Payoffs are then

$U_P = S$, $U_A = R(e) - S - e$

Let e^* maximise $R(e) - e$. We say a contract is *first-best* if it implements e^* and yields the agent utility $U_A = 0$.

Construct a contract which implements the first-best. [25]

Q2. Two agents, $i \in \{1, 2\}$, simultaneously choose efforts $e_i \in \{0, 1\}$ on a project. 0 effort costs 0, while effort 1 costs c_i, where $c_1 + c_2 < 1$ and $1 - x > max(c_1, c_2)$, $x \in (0, 1)$.

The production function is as follows: If both agents choose effort 1, the output is 1. If both choose 0 effort, the output is 0. If one agent chooses effort 1, while the other chooses effort 0, the output is x.

Suppose agent i gets share β_i of the output, where $\beta_1 + \beta_2 < 1$. We say the efficient outcome can be implemented if there exists an equilibrium where both agents exert high effort.

(a) For which values of x does there exist (β_1, β_2) such that the efficient outcome can be implemented? [20]

(b) Show there exist sharing rules (β_1, β_2) depending only on (c_1, c_2) and implementing the efficient outcome whenever it is implementable. [5]

Q3. A risk neutral agent chooses probabilities of success of two projects: p_1 and p_2. The private cost of effort is $\frac{p_1^2+p_2^2}{2} + cp_1p_2$, where $c \in (-1, 1)$. Agent choices are not observable or verifiable by third parties. Project i gives the principal (also risk neutral) x_i if it succeeds, and 0 if it fails.

Project outcome is also not verifiable by third parties. However, for each project there is a verifiable signal, correlated with true outcome, $s_i \in \{0, 1\}$, with

$Pr(s_i = 1|\text{project } i \text{ is successful}) = Pr(s_i = 0|\text{project } i \text{ is a failure}) = r_i \in (0.5, 1)$.

Suppose the principal offers the agent a contract of the form $w_1s_1 + w_2s_2$, with $w_i > 0$.

(a) What is the optimal contract? [20]

(b) What would have been the optimal contract (still of the form $w_1s_1 + w_2s_2$, with $w_i > 0$) if the probabilities chosen by the agent had been observable and verifiable? [5]

INDIAN STATISTICAL INSTITUTE

Semestral Examination Semester II : 2013-2014
M.Stat. II Year Stochastic Processes I

Date : 13.05.14 **Maximum Score : 60** Time : 3 Hours

Note : This paper carries a **total** of **78 marks**. The **maximum** you can score is **60**.

1. Let $\{B_t, t \in [0,\infty)\}$ be a SBM (standard Brownian motion). Consider disjoint intervals $I = [a,b]$ and $J = [c,d]$ where $0 < a < b < c < d < \infty$. Show that if $X = \sup_{t\in I} B_t$ and $Y = \sup_{t\in J} B_t$, then, P-almost surely, $X \neq Y$. [6]

2. Let $\{B_t, t \in [0,\infty)\}$ be a SBM. Fix $t \geq 0, T > 0$. Consider the process $\{X_u, u \in [0,T]\}$ defined by $X_u = B_{t+u} - \frac{u}{T}B_{t+T} - (1-\frac{u}{T})B_t$, for $u \in [0,T]$.
(a) Show that $\{X_u, u \in [0,T]\}$ is a zero-mean gaussian process with continuous paths and find its covariance kernel.
(b) Show that the process $\{X_u, u \in [0,T]\}$ is independent of the vector (B_t, B_{t+T}).
(c) Define $\{Y_u, u \in [0,T]\}$ by $Y_u = B_{t+u}$. Denoting ρ to be the metric on $C([0,T])$ given by $\rho(x,y) = \sup_{0\leq u\leq T} |x(u) - y(u)|$, show that $\rho(X(\cdot), Y(\cdot)) = \max\{|B_t|, |B_{t+T}|\}$.
(d) Denote P_ϵ to be the conditional distribution of the process $\{Y_u, u \in [0,T]\}$, given $(B_t, B_{t+T}) \in [-\epsilon,\epsilon] \times [-\epsilon,\epsilon]$. Show that, as $\epsilon \downarrow 0$, P_ϵ converges weakly to (the law of) the process $\{X_u, u \in [0,T]\}$. (6+6+2+6) = [20]

3. Let $\{B_t, t \in [0,\infty)\}$ be a SBM and let $\tau(\omega) = \inf\{t \in [0,\infty) : B(t,\omega) = 1\}$.
(a) Use properties of Brownian motion to show that $\tau < \infty$ almost surely.
(b) Show that τ is a stopping time with respect to the filtration $\mathcal{F}_t = \sigma\{B_s, s \leq t\}$.
(b) Show that if $\eta(\omega) = \inf\{t \geq \tau(\omega) : B(t,\omega) = 0\}$, then τ and $\eta - \tau$ are independent and identically distributed. (4+4+6)= [14]

4. Let $(S, \mathcal{S})$ denoting a Polish space equipped with its Borel σ-field.
(a) Define clearly what is meant by a continuous-path markov process with state space S.
(b) What is meant by the Feller property for a Markov process as in (a).
(c) For a continuous path Markov process with the Feller property, define what is meant by its resolvent operators R_λ on $C_b(S)$.
(d) State and prove the resolvent identity and hence show that (i) the R_λ for all λ have the same range and trivial null space; and (ii) the operator $\lambda I - R_\lambda^{-1}$ defined on the common range of the R_λ does not depend on λ. (2+2+2+(6+4+4))= [20]

5. Let $\{B_t, t \in [0,\infty)\}$ be a SBM. Let $\alpha > \frac{1}{2}$ and m an integer with $m > (\alpha - \frac{1}{2})^{-1}$.
(a) Denote D to be the set of all those ω, for which there exists $t \in [0,1)$ such that $|[B(t+h,\omega) - B(t,\omega)]/h^\alpha|$ remains bounded for all sufficiently small $h > 0$. Show that $D \subset \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n>(m+1)k}\bigcup_{i=1}^{n-m-1}\bigcap_{l=1}^{m}\{\omega : |B(\frac{l+1}{n},\omega) - B(\frac{l}{n},\omega)| \leq j[(l+1)^\alpha + l^\alpha]/n^\alpha\}$.
(b) Denoting the set on the righthand-side in (a) by N_0, show that $P(N_0) = 0$.
(c) Deduce that there is a P-null set N such that for all $\omega \notin N$, the trajectory $t \mapsto B(t,\omega)$ is *nowhere* Hölder-continuous with exponent α. (8+6+4)= [18]

INDIAN STATISTICAL INSTITUTE

Semestral Examination, Second Semester: 2013-14

M.Stat. II Year (AS)

Actuarial Models

Date: May 13, 2014 Maximum marks: 100 Duration: 3 hours

Answer all questions. Standard actuarial notations are followed.

1. Indicate whether the state space and the time set of the following stochastic processes are continuous or discrete.

   (i) A thinned Poisson process. [1]

   (ii) A Markov jump chain. [1]

   (iii) A compound Poisson process. [1]

   [Total 3]

2. Identify two major differences between stochastic and deterministic models. Mention two benefits of modelling in actuarial work. [4]

3. (i) Let $N(t)$ be a homogeneous Poisson process with rate λ. Derive the conditional distribution of n unordered occurrence times $t_1, \cdots, t_n$ before time t, given $N(t) = n$. [3]

   (ii) Suppose the claims from a portfolio occur according to a homogeneous Poisson process with rate λ, but the payments on the claims occurring before the end of a year are made at the end of the year. Define the withholding time for a claim as the time duration since the time of claim till the payment is made. Obtain the mean and variance of the total withholding time in a year. [3+4]

   [Total 10]

4. A no claims discount (NCD) system with discount levels 0%, 20% and 50%, modelled as a Markov chain, has run for a number of years, and can be reasonably assumed to have reached the steady state. The probabilities of transition are as follows.
   From 0% to 20% : α; from 20% to 50% : β; from 50% to 20% : $1-\gamma$;
   from 0% to 50% : 0; from 20% to 20% : 0; from 50% to 0% : 0,
   where α, β and γ take values in the unit interval.

   (i) Draw the transition diagram of this chain. [1]

   (ii) Write down the transition probability matrix. [1]

   (iii) Explain whether this Markov chain is irreducible and/or aperiodic. [2]

   (iv) Calculate the stationary probability distribution of the chain in terms of the parameters α, β and γ. [4]

   (v) The chain is observed in the steady state for two consecutive years, and the data are summarized as n_{AB}, the number of policyholders in state A in the first year and in state B in the second year, for A, B = 0%, 20%, 50%. Write down the likelihood of the parameters α, β and γ. [3]

   (vi) An actuary claims that the MLE of α is $n_{0\%,20\%}/(n_{0\%,0\%} + n_{0\%,20\%})$. Explain why this is not correct. [2]

   (vii) Describe precisely how you would simulate a single sample path of the Markov chain *in steady state*, by using the answers to parts (ii) and (iv). [3]

   [Total 16]

5. (i) Describe a marital status model by specifying all possible transitions and the corresponding rates (possibly time-dependent) with the states being bachelor (B), married (M), widow/widower (W), separated/divorcee (S) and dead (D). Give an expression for the conditional probability of being in state M at time t with the current holding time being greater than w, given that the state occupied at time s, for $s < t$, is M. This expression may involve some transition probabilities. [(1+3)]

(ii) The following data gives observations on ten individuals from the underlying Markov jump chain. Test whether the probability of separation/divorce depends on the status prior to marriage.
BMD, $BMSMSD$, $BMWMD$, $BMSMSMD$, $BMWMD$, $BMSMSD$, $BMWD$, $BMSMSD$, BMD, $BMWMSMSMD$. [6]

[Total 10]

6. The human mortality rate is known to follow a bathtub pattern, with a decreasing trend over the first decade of life and an increasing trend over the last few decades.

(i) Indicate whether any of the families of survival functions, $\overline{F}_1(t) = \exp(-\alpha t^\beta)$, $\overline{F}_2(t) = \exp(\alpha t + \beta\gamma^t)$, $\overline{F}_3(t) = \exp(-\alpha t - \beta t^2 - \gamma t^3)$, has a force of mortality satisfying the above property. [3]

(ii) By examining the increasing/decreasing nature of the force of mortality implied by the three common interpolation rules for life tables, indicate which of these would be most similar to the trend in human mortality for the later decades of life. [3]

[Total 6]

7. A graduation of the mortality experience of the population of North Eastern India has been carried out by using a standard table. The following is an extract from the results.

Age x	Actual number of deaths, θ_x	Graduated force of mortality, $\mathring{\mu}_x$	Central exposed to risk, E_x^c	Expected deaths $E_x^c\,\mathring{\mu}_x$
50	52	0.00549	10,850	59.57
51	54	0.00610	9,812	59.85
52	60	0.00679	10,054	68.27
53	65	0.00757	9,650	73.05
54	64	0.00845	8,563	72.36
55	87	0.00945	10,656	100.70
56	88	0.01057	9,667	102.18
57	97	0.01182	9,560	113.00
58	103	0.01323	8,968	118.65
59	105	0.01483	8,455	125.39

(i) Use the Chi-squared test to test the adherence of the graduated rates to the data. State clearly the null hypothesis you are testing and comment on the result. [5]

(ii) Perform two other tests which detect different aspects of the adherence of the graduation to the data. For each test state clearly the features of the graduation which the test is able to detect, and comment on your results. [6]

[Total 11]

8. In a mortality study of the Indian armed forces, the three services are asked to provide the following summary information:

(a) the number of soldiers on 1st January 2005, 2006 and 2007, classified by age as per the records;

(b) the number of soldiers died during the calendar years 2005 and 2006, classified by age at nearest birthday at the time of death.

An extract of the data provided by the three services is given below.

Army					
Age x	No. of lives aged x last birthday on			No. of deaths to persons aged x nearest at death during the year	
	1-1-2005	1-1-2006	1-1-2007	2005	2006
39	4529	4620	4707	22	20
40	4381	4439	4492	20	25
41	4467	4577	4616	25	21

Navy					
Age x	No. of lives aged x nearest birthday on			No. of deaths to persons aged x nearest at death during the year	
	1-1-2005	1-1-2006	1-1-2007	2005	2006
39	452	512	489	3	1
40	528	602	573	0	5
41	444	498	541	2	1

Air Force					
Age x	No. of lives aged x next birthday on			No. of deaths to persons aged x nearest at death during the year	
	1-1-2005	1-1-2006	1-1-2007	2005	2006
39	330	309	373	2	2
40	406	352	356	1	4
41	327	381	358	2	0

(i) State the type of rate interval. [1]

(ii) State the assumptions which you may need to obtain a single estimate of μ_{40}, by pooling together all the information. [3]

(iii) Give an expression for the central exposed to risk. [6]

(iv) Estimate μ_{40} from the available data. [3]

(v) Give an example of why the assumptions you have made may not be appropriate in this particular investigation. [2]

[Total 15]

9. Consider life table data in k age intervals and assume that the hazard rate is constant within an interval. In addition to the life table data, we also have information on the failure and censoring times within each age interval along with their failure/censoring status.

(i) Derive the maximum likelihood estimate (MLE) of the constant hazards in each interval. [6]

(i) Derive the MLE of the survival function at a time t in the jth interval. [3]

[Total 9]

INDIAN STATISTICAL INSTITUTE

SECOND SEMESTRAL EXAMINATION

M.STAT. (2 ND YEAR) 2013-14

Subject: THEORY OF GAMES AND STATISTICAL DECISIONS

DATE: 16.04.2014 FULL MARKS: 70 DURATION: 3 HOURS

ATTEMPT ALL QUESTIONS

1. a) Considering two person zero- sum games in statistical decision theory , describe least favorable prior ,minimax rule and minimax point.

b) Let a coin be tossed with probability of head, θ. $\Theta \in \{⅓, ⅔\}$. Let an action $\delta = (x, y)$ be given by : predict by $\Theta = x$ if head appears and $\Theta = y$ if tail appears. $0 \leq x, y \leq 1$.

Considering square error loss, find minimax rule and value of the game.

c) Let set of strategies of 1st player be $\{\Theta_1, \Theta_2\}$ in a two player zero - sum game.
Let $A = \{ (r(\Theta_1,d), r(\Theta_2,d)): d \in$ set of all possible actions of second player$\}$, where $r(\Theta_i, d)$ is the pay off of first player for the situation (Θ_i,d), $i= 1,2$. If $A = \{(x, y): (x-3)^2+(y-4)^2 \leq 5\}$, find the value of the game and optimal mixed strategy of player 1 ,(in mixed extension).

[3+11+12]

2 a) State and prove Nash's theorem in the context of finite non-cooperative games.

b) Give example of a 2 X 3 X 4 non-cooperative game with more than one equilibrium situations, but with different pay off at those situations for the same player. [(2+14) +4]

(P.T.O.)

Consider mixed extension of a matrix game with matrix $A_{n \times n}$ and A is skew- symmetric. Show sets of optimal strategies of the two players are identical subsets of R^n .

Consider mixed extension of a matrix game with matrix $A_{n \times n}$ and $A = ((a_{ij}))_{n \times n}$ be given by:

u_i if i= j

0 if i≠ j

> 0 for all i, find v(A) and set of all optimal strategies.

onsider a 2 x 2 x 2 non-cooperative game. Derive a method describing how to get the set of issible situations in mixed extension for the 1st player in terms of the pay off functions.

[6+10+8]

XXXXXXXXXXXXX