A Class of Exact Solutions of the Combined Gravitational and Electro-Magnetic Field Equations of General Relativity.

R. L. BRAHMACHARY

Research and Training School, Indian Statistical Institute - Calcutta

(ricevuto il 23 Luglio 1957)

1. – In our generalization of the Reissner-Nordström-Jeffery solution (1), we attempted a solution of the field-equations

$$(1)\quad \begin{cases} (a) & -8\pi T_1^1 - 8\pi t_1^1 = e^{-\lambda}\left(\dfrac{\nu'}{r}+\dfrac{1}{r^2}\right)-\dfrac{1}{r^2}+\Lambda \\ (b) & 8\pi T_2^2 - 8\pi t_2^2 = e^{-\lambda}\left(\dfrac{\nu''}{2}-\dfrac{\lambda'\nu'}{4}+\dfrac{(\nu')^2}{4}+\dfrac{\nu'-\lambda'}{2r}\right)+\Lambda\,; \\ (c) & T_2^2 = T_3^3 \\ (d) & 8\pi T_4^4 - \dfrac{C}{r^4} = -e^{-\lambda}\left(\dfrac{\lambda'}{r}-\dfrac{1}{r^2}\right)-\dfrac{1}{r^2}+\Lambda\,. \end{cases}$$

We will now attempt the interior solution of a space containing matter which is distributed with a spherical symmetry around a charged particle. We will later see that the charged particle should be something like an electron having a finite radius.

In the space surrounding the charged particle or sphere, we have

$$-8\pi T_1^1 - 8\pi t_1^1 = 8\pi p - C/p^4\,,$$

where (2) C depends on the charge of the particle (sphere) and the density is

$$8\pi\varrho + C/r^4\,. \qquad (8\pi t_1^1 = -8\pi t_2^2 = 8\pi t_4^4 = C/r^4)\,.$$

(1) R. L. BRAHMACHARY: *Nuovo Cimento*, **4**, 1216 (1956).

(2) A. S. EDDINGTON: *The Mathematical Theory of Relativity* (1929), p. 185.

Following a method of TOLMAN [3], we combine the first and second equation (1) thus

$$\left\{(8\pi T_1^1 - 8\pi t_1^1) - (8\pi T_2^2 + 8\pi t_1^1)\right\}\frac{2}{r} = \frac{2}{r}\,[e^{-\lambda}\{...\}],$$

and obtain

$$\frac{4C}{r^5} = \frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{(e^{-\lambda}-1)}{r^2}\right\} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{-\lambda}\nu'}{2r}\right) + e^{-\lambda-\nu}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{\nu}\nu'}{2r}\right). \tag{2}$$

We thus see that it is necessary and sufficient to solve the three equations, (1*a*), (1*d*) and (2). Thus we avoid the complicated equation (1*b*).

Assumption: $\nu' = 0$, $\nu'' = 0$.

Our equations reduce to

$$(3)\quad\begin{cases} (a) & \dfrac{(e^{-\lambda}-1)}{r^2} = 4C\displaystyle\int\frac{\mathrm{d}r}{r^5}\,, \\[2ex] (b) & 8\pi p - \dfrac{C}{r^4} = \dfrac{e^{-\lambda}}{r^2} - \dfrac{1}{r^2} + \Lambda\,. \\[2ex] (c) & 8\pi\varrho + \dfrac{C}{r^4} = e^{-\lambda}\left(\dfrac{\lambda'}{r} - \dfrac{1}{r^2}\right) + \dfrac{1}{r^2} - \Lambda\,. \end{cases}$$

From (3*a*) we immediately obtain

$$e^{-\lambda} = 4C\alpha r^2 - C/r^2 + 1\,,$$

where α is a constant of integration. Putting this value in 3 (*b*) we obtain

$$8\pi p = 4C\alpha + \Lambda\,.$$

We further obtain from 3 (*c*)

$$8\pi\varrho = -\Lambda - 12C\alpha - 2C/r^4\,.$$

Discussion: We find that for $r \ll 1$, $e^{-\lambda}$ becomes negative. As $e^{-\lambda}$ must always be positive, we thus find that even outside the origin $r=0$, there is a singular domain, unless we assume a shell like distribution of mass around the charged particle with a gap between the particle and the inner shell.

If however, a charged sphere of finite radius whose gravitational part of the energy tensor is negligible in comparison with its electromagnetic part (i.e. an electron) is assumed, our solution is to remain valid only outside the electron.

(3) R. C. TOLMAN: *Phys. Rev.*, 55, 364 (1939).

At $r=r_e$, the radius of the electron, the singular region commences, or in other words we put

$$(4) \qquad 4C\alpha r_e^2 - C/r_e^2 + 1 = 0\,.$$

Our solution is:

$$e^{\nu} = \text{const}\,, \qquad e^{-\lambda} = 4C\alpha r^2 - C/r^2 + 1\,,$$

$$8\pi p = 4C\alpha + \Lambda\,, \qquad 8\pi\varrho = -\Lambda - 12C\alpha - 2C/r^4\,.$$

As p is positive and ϱ must be positive up to $r=r_e$, we have α=negative, Λ=positive

$$|\Lambda| > |4C\alpha|$$

and

$$|12C\alpha| > |\Lambda + 2C/r^4|\,.$$

With the help of these conditions, we can find from equation (4), the value of α. If

$$C = 10^{-10} \text{ units} \quad \text{or} \quad 10^{-38}\ \text{cm},$$

(in relativistic units) and $r_e \sim 10^{-13}$ cm. We find $\alpha \sim -10^{64}$.

Instead of an electron we may also take a charged sphere, the solution for which has already been found [4]. Outside the sphere, the Reissner-Nordström solution would be valid (if the space were empty) with

$$T_1^1 = 0\,, \quad t_1^1 = C/r^4\,.$$

Thus our equations (3) correctly represent the energy-tensor of matter-distribution having a spherical symmetry around the charged sphere.

2. – We attempted so far a solution of spherically symmetric matter-distribution around a charged sphere of finite radius. Another example of this type of solutions is now being obtained by the simple expedient of putting $p=0$, namely, considering the case of dust like particles, offering no pressure.

Assumption: $p=0$, $\nu'=0$, $\nu''=0$.

The field-equations now reduce to

$$(5) \qquad \begin{cases} (a) & -8\pi t_1^1 = e^{-\lambda}/r^2 - 1/r^2 + \Lambda \\ (b) & -8\pi t_2^2 = e^{-\lambda}(-\lambda'/2r) + \Lambda \\ (c) & T_2^2 = T_3^3 \\ (d) & -8\pi\varrho - 8\pi t_4^4 = -e^{-\lambda}(\lambda'/r - 1/r^2) - 1/r^2 + \Lambda\,. \end{cases}$$

Outside the charged sphere,

$$8\pi t_1^1 = -8\pi t_2^2 = 8\pi t_4^4 = C/r^4\,.$$

(4) R. L. BRAHMACHARY: *Nuovo Cimento*, **5**, 1520 (1957).

We therefore obtain from (5)*a*,

$$e^{-\lambda} = 1 - \Lambda r^2 - C/r^2 .$$

Equation (5)*b* is seen to be identically satisfied for this value of $e^{-\lambda}$.
We can then easily find from (5)*d*, the value for ϱ:

$$8\pi\varrho = 2\Lambda - 2C/r^4 .$$

As $8\pi\varrho$ and $e^{-\lambda}$ must always be positive the range of validity of our solution is restricted.

Let the lower and upper boundaries of the range of validity be r_b and r_B, respectively.

We then have

$$e^{-\lambda} = 1 - \Lambda r_b^2 - C/r_b^2 = \text{positive} ,$$

$$8\pi\varrho = 2\Lambda - 2C/r_b^4 \qquad = \text{positive} .$$

If $r_b = 10^{-13}$ cm and $C = 10^{-38}$ cm, we have

$$e^{-\lambda} = 1 - \Lambda \cdot 10^{-26} - 10^{-12} = \text{positive} .$$

Thus

$$\Lambda < (10^{26} - 10^{14}) .$$

Again from the expression for $8\pi\varrho$ we have

$$8\pi\varrho = 2\Lambda - 2\cdot 10^{14} > 0 .$$

Thus,

$$\Lambda > 10^{14} .$$

But we can now easily see that the maximum allowable value for the range of validity of the solution is $r = r_B < 10^{-7}$ that is, $e^{-\lambda}$ attains the value zero at $r_B < 10^{-7}$ if we choose the minimum (possible) value of Λ.

By taking a greater value of r_b, we can increase the range of validity. With $r_b = 2$ cm, for example, we have

$$e^{-\lambda} = 1 - 4\Lambda - C/4$$

and

$$8\pi\varrho = 2\Lambda - 2C/(2)^4 .$$

Then we have

$$\Lambda > 10^{-38}/(2)^4$$

3365