Indian Statistical Institute
Back paper Examinations: 2018-19

Course Name: M. Math, 1st year
Subject Name : Linear Algebra
Maximum Marks: 100, Duration: Two hours

Answer all questions. There is no part marking.

1. **a.** Prove that an R-module M is decomposable if and only if the ring of endomorphims $\mathrm{End}_R(M)$ has a non trivial idempotent.

   b. Let R be a Principal Ideal Domain and p a prime element in R.

   i. Prove that $R/(p^n)$ is an indecomposable module over R.

   ii. Deduce the structure of a finitely generated indecomposable module over R.
   $\mathbf{10 + 10 + 10 = 30}$

2. Use the theory of Jordan canonical forms to prove that every complex matrix is similar to its transpose. **10**

3. Let $V = M_n(\mathbb{C})$ equipped with the inner product $\langle A, B\rangle = \mathrm{Tr}(AB^*)$. Find the orthogonal complement of the subspace of diagonal matrices. **10**

4. Suppose V is a finite dimensional complex inner product space and W is a subspace of V. Prove that if E is an idempotent with range W such that $\|E(v)\| \leq v$, then E is the orthogonal projection with range W. **10**

5. **a.** Prove that if two complex matrices A and B of the same size are similar, then the corresponding $k[x]$ modules are isomorphic.

   b. Use part a. to prove that if two idempotent complex matrices of the same size are similar, then they have the same rank. $\mathbf{5 + 5 = 10}$

6. **a.** Let A be a 4×4 matrix whose minimal polynomial is $x(x-2)(x+2)$ and whose rank is 2. What is the characteristic polynomial of A?

   b. Classify upto similarity all matrices whose characteristic polynomial is $(x-1)(x-2)^2$ and minimal polynomial is $(x-1)(x-2)$. **10 + 10 = 20**

7. Let T be a linear transformation from a vector space V to V. Prove that if every vector of V is a cyclic vector, then the characteristic polynomial of T is irreducible. **10**

INDIAN STATISTICAL INSTITUTE
Back-paper Semester Examination: 2018-19 (First Semester)

M. Math. I YEAR
Algebra

Date : 07.01.2019 Maximum Marks : 100 Duration : 3 Hours

ANSWER ANY QUESTIONS.

1. Let a be a nonzero element of an integral domain R. Prove that:

   (i) a is an irreducible element of R if and only if aR is maximal among all proper principal ideals.

   (ii) a is a prime element of R if and only if aR is a prime ideal of R. [12]

2. Prove that if R is an integral domain which is not a field then $R[X]$ is not a PID. [10]

3. (i) Let f be an endomorphism of a Noetherian module M. Prove that there exists a positive integer n such that $Ker(f^n) \cap Im(f^n) = (0)$.
   (ii) Deduce that any surjective endomorphism of a Noetherian module is an automorphism. [10+6]

4. Define tensor product of two R-modules M and N over R. Prove the existence uniqueness of tensor product. Compute $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. [20]

5. Let G be an Abelian group of order n. Prove that for every factor m of n, G has a subgroup of order m. Give an example with justification to show that this is not true in case G is non-Abelian. [10+10]

6. Prove that a simple group of order 60 cannot have a subgroup of index 4. [10]

7. Prove that Δ_n^2 is a symmetric function in $A := \mathbb{C}[X_1, \ldots, X_n]$, where $\Delta_n = \prod_{i<j}(X_i - X_j)$. Express Δ_3^2 in elementary symmetric functions. [12]

Indian Statistical Institute
Second Semester Examinations: 2017-18

Course Name: M. Math, 2nd year
Subject Name : Lie groups and Lie algebras
Maximum Marks: 50, Duration: Two and a half hours
Date: 19.4.2018, 5 PM

Marks will be deducted for indirect, incomplete, unnecessary long and imprecise answers. Consultation of OWN notes are permitted.

Question number 1 is compulsory. Answer any three questions among the rest

1. **a.** Let π_1 and π_2 are representations of a Lie group G on finite dimensional vector spaces V_1 and V_2. Prove that the representation $D(\pi_1 \otimes \pi_2)$ of $\mathfrak{g}$ on $V_1 \otimes V_2$ is given by:

$$D(\pi_1 \otimes \pi_2)(X) = D(\pi_1)(X) \otimes I + I \otimes D(\pi_2)(X), \text{ where } X \in \mathfrak{g}.$$

b. Let $L = \mathrm{sl}(3, \mathbb{C})$. Prove that each dominant integral element occurs as the highest weight of an irreducible representation of L.

c. Let $\{x_1, x_2, x_3, y_1, y_2, y_3, h_1, h_2\}$ be the canonical generators of $L = \mathrm{sl}(3, \mathbb{C})$. Thus, $\{x_1, y_1, h_1\}$, $\{x_2, y_2, h_2\}$, $\{h_1 + h_2, x_3, y_3\}$ are the canonical copies of $\mathrm{sl}(2, \mathbb{C})$ inside L. The other Lie-algebra relations are as follows:

$$[h_2, x_1] = -x_1,\ [h_2, y_1] = y_1, [h_1, x_2] = -x_2,\ [h_1, y_2] = y_2, [h_1, x_3] = x_3, [h_1, y_3] = -y_3,$$

$$[h_2, x_3] = -x_3,\ [h_2, y_3] = -y_3, [x_1, x_2] = x_3,\ [y_1, y_2] = -y_3, [x_1, y_2] = 0, [x_2, y_1] = 0,$$

$$[x_1, x_3] = 0,\ [y_1, y_3] = 0, [x_2, x_3] = 0,\ [y_2, y_3] = 0, [x_2, y_3] = y_1, [x_3, y_2] = x_1,$$

$$[x_1, y_3] = -y_2,\ [x_3, y_1] = -x_2.$$

Moreover, the root spaces for the Lie algebra L corresponding to roots $(2,-1), (-1,2), (1,1), (-2,1), (1,-2), (-1,-1)$ are $\mathbb{C}x_1, \mathbb{C}x_2, \mathbb{C}x_3, \mathbb{C}y_1, \mathbb{C}y_2, \mathbb{C}y_3$ respectively.

i. Consider the simple root system $\{\alpha_1 = (2,-1),\ \alpha_2 = (-1,2)\}$. Find a highest weight vector and the corresponding weight of the adjoint representation.

P.T.O

ii. Show that the adjoint representation of L is irreducible.

iii. Deduce that L is simple. $\mathbf{3 + 5 + (8 + 2 + 2) = 20}$

2. **a.** Suppose $\phi, \psi : G \to H$ be Lie group homomorphisms such that $D\phi = D\psi$. If G is connected, prove that $\phi = \psi$.

   b. Can there exist a compact Lie group G and a non-compact Lie group H such that $\mathfrak{g}$ and $\mathfrak{h}$ are not isomorphic but their complexifications are isomorphic? Prove or disprove. $\mathbf{5 + 5 = 10}$

3. Answer the following questions:

   a. Suppose L is a complex Lie algebra such that $[L, L]$ is nilpotent. Is L nilpotent?

   b. Prove that a nilpotent Lie algebra has an ideal of codimension 1.

   c. Give an example of a semi-simple Lie group which is not simple.

   d. Prove that the Lie group $SL(2, \mathbb{R})$ is simple.

   e. Prove that the Lie group $\mathrm{GL}(n, \mathbb{C})$ is reductive. $2 \times 5 = 10$

4. **a.** Prove that the Killing form is identically zero for a nilpotent Lie algebra.

   b. Prove that a Lie algebra L is solvable if and only if $[L, L]$ lies in the radical of the Killing form.

   c. Prove that the 2-dimensional matrix Lie group $G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^*, b \in \mathbb{R}\}$ is solvable but not nilpotent. $\mathbf{3 + 3 + 4 = 10}$

5. Attendance **10**

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2018-19 (First Semester)

M. Math. I YEAR
Algebra I

Date : 03.09.18 Maximum Marks : 40 Duration : 2 Hours

ANSWER ANY THREE QUESTIONS.
k will denote a field, R a commutative ring with unity and $\mathbb{Z}$ the ring of integers.

1. Let S be a multiplicatively closed subset of a ring R.

   (i) Define (with proofs) an equivalence relation $\sim$ on the set $R \times S$ and well-defined binary operations on the equivalence classes of $\sim$ so that the equivalence classes form a ring $S^{-1}R$ with a ring homomorphism f from R to $S^{-1}R$ such that that $f(s)$ is a unit for every s in S.

   (ii) Show that if S contains a nilpotent element, then $S^{-1}R = 0$.

   (iii) Prove that $\mathbb{Z}[X]/(2X-3) \cong \mathbb{Z}[\frac{1}{2}]$ as rings. [6+2+7=15]

2. (i) Suppose that A is an integral domain in which every nonzero nonunit of A can be expressed as a product of irreducible elements and any two elements of A have a GCD. Prove that A is a UFD.

   (ii) Examine whether $1+2i$ is prime in $\mathbb{Z}[i]$. [7+8=15]

3. Let A be the polynomial ring $k[T]$ and B be the k-subalgebra $k[T^2, T^3]$.

   (i) Prove that A is a UFD with infinitely many maximal ideals.

   (ii) Examine whether B is (a) a UFD (b) a Noetherian ring. [9+6=15]

4. Let $R = k[T]/(T^3 - T)$ and t denote the image of T in R.

   (i) Prove that there exists an isomorphism ϕ from R to the product ring $k \times k \times k$.

   (ii) Explicitly describe $\phi(t)$ as an element (a, b, c) of $k \times k \times k$.

   (iii) For which element f in R, is $\phi(f) = (0, 0, 1)$? [7+4+4=15]

5. Let $A = k[X, Y, Z]/(XY - Z^2)$.

   (i) Show that the images of Y and $Z - 1$ are co-maximal in A.

   (ii) Describe two maximal ideals of A.

   (iii) Find an element f in the ring A such that $A/fA \cong k[T]$.

   (iv) Describe a nonzero prime ideal of A which is not maximal. [3+4+6+2=15]

Indian Statistical Institute, Kolkata
M. Math I
Fall Semester 2018
Mid-Term Examination
Course: Analysis of Several Variables
5 - 09 - 18
2 hours

- *Answer as many questions as you can.*
- *Maximum marks is 40.*

1. Let $1 < p < \infty$. Show that the L^1 and L^p norms on $C[0,1]$ are not equivalent. (6 marks)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Show that if p is a local maximum or local minimum of f then $Df_p = 0$. (6 marks)

3. Let U be an open set in $\mathbf{R}^n$ and let $f : U \to \mathbf{R}^n$ be a C^1, 1-to-1 map such that Df_p is invertible for all $p \in U$. Show that $V = f(U)$ is open, and $f^{-1} : V \to U$ is C^1. (6 marks)

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Suppose there are constants a_{ij} such that $\frac{\partial^2 f}{\partial x_i \partial x_j} \equiv a_{ij}$ for all i, j. Show that f is a quadratic polynomial of the form
$$f(x) = \frac{1}{2} \sum_{i,j} a_{ij} x_i x_j + \sum_i b_i x_i + c$$
(8 marks)

5. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a C^2 function such that $D^2 f_p(h,h) \geq 0$ for all $p, h \in \mathbf{R}^n$. Show that f is convex, i.e. for all $p, q \in \mathbf{R}^n$ and $t \in [0,1]$, $f((1-t)p + tq) \leq (1-t)f(p) + tf(q)$. (7 marks)

6. Let A be an $n \times n$ real matrix diagonalizable over $\mathbf{C}$ with all eigenvalues purely imaginary. Show that all integral curves of the vector field $X(x) = Ax$ are bounded. (5 marks)

7. Compute the flow (ϕ_t) of the vector field X on $\mathbb{R}^2$ given by $X(x, y) = (ax + by, -bx + ay)$, where a, b are constants. Describe the set of values of (a, b) such that $\phi_t(p) \to 0$ as $t \to +\infty$ for all p. (5+3 = 8 marks)

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination – Semester I : 2018-2019
M.Math. I Year
Measure Theoretic Probability

Date : 06.09.13 | Maximum Marks : 40 | Time : 2 Hours

Note : This paper carries **five** questions with **each worth 10 marks**. Answer as much as you can. The **MAXIMUM** you can score is **40**.

1. Let $\mathcal{F}$ and $\mathcal{G}$ be two fields on a non-empty set Ω. Show that the class of sets of the form $F \cap G$ with $F \in \mathcal{F}, G \in \mathcal{G}$, forms a semifield and hence describe the smallest field on Ω containing both $\mathcal{F}$ and $\mathcal{G}$.

2. Let $\{A_t, t \in T\}$ be an uncountable collection of subsets of a non-empty set Ω. For each countable subset $S \subset T$, let $\mathcal{A}_S$ denote the σ-field on Ω generated by the sets $\{A_t, t \in S\}$.
(a) Show that $\mathcal{A} = \{A : A \in \mathcal{A}_S \text{ for some countable } S \subset T\}$ is a σ-field.
(b) Deduce that $\mathcal{A}$ (as in (a)) equals the σ-field generated by the sets $\{A_t, t \in T\}$.

3. (a) Define clearly what is meant by a measure on a field $\mathcal{F}$.
(b) Suppose μ is a $[0, \infty]$-valued set function on a field $\mathcal{F}$ satisfying $\mu(\emptyset) = 0$ and $\mu(F \cup G) = \mu(F) + \mu(G)$ for every pair of disjoint sets $F, G \in \mathcal{F}$. Show that if for every sequence $\{F_n\}$ in $\mathcal{F}$ with $F_n \downarrow \emptyset$, one has $\mu(F_n) \to 0$, then μ is a measure on $\mathcal{F}$.

4. Let $\mathcal{A}$ denote the σ-field on $\mathbb{R}$, consisting of all countable subsets of $\mathbb{R}$ and their complements. Show that a function $f : (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable if and only if f is constant outside some countable subset $A \subset \mathbb{R}$ (A is allowed to depend on f).

5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and f a non-negative measurable function on Ω.
(a) Show that the set function $\nu(A) = \int_A f d\mu$, $A \in \mathcal{A}$, defines a measure on $\mathcal{A}$.
(b) Show that if g is a non-negative real-valued simple function on Ω, then $\int g d\nu = \int g f d\mu$.
(c) Show that for any extended-real-valued measurable function g on Ω, $\int g d\nu = \int g f d\mu$ in the sense that if one of the integrals exist, then so does the other and they are equal.

Indian Statistical Institute
First Semester Examinations: 2018-19

Course Name: M. Math, 1^{st} year
Subject Name : Topology I
Maximum Marks: 40, Duration: Two and a half hours
Date: 7.9.2018, 10:15 AM - 12:45 PM

- Answer as many questions as you can.
- Maximum marks is 40.

1. a) Let $f : X \to Y$ be a function where X is a set and Y a topological space. Consider the collection $\mathfrak{B} = \{f^{-1}(U) \mid U \textit{ is open in } Y\}$. Prove that $\mathfrak{B}$ is a basis for a topology. 5
b) Consider the function $f : \mathbb{R} \to S^1$ which sends t to $e^{2\pi it}$. Prove that the induced topology on $\mathbb{R}$ using the construction in a) is the same as the usual topology. 5

2. Prove that X is connected $\iff$ $X \times [0,1]$ is connected. 3

3. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers with the subspace topology induced from $\mathbb{R}$. Prove that the space $\mathbb{R}^2 - 0$ is homeomorphic to the product $\mathbb{R}_{>0} \times S^1$. 5

4. a) Define the terms : (i) T_1, (ii) Hausdorff. 2
b) Let X be an infinite set with finite complement topology and let Y be a Hausdorff space. Prove that any continuous function from $X \to Y$ is constant. 4

5. a) Let X be a Hausdorff space, $K \subset X$ be compact and $x \in X$. Prove that there exists open U, U' such that $x \in U$, $K \subset U'$ and $U \cap U' = \varnothing$. 4
b) Prove that any map from a compact space to a Hausdorff space is an open map. 4

6. a) Let $X = [0,1) \cup (3,4]$.
(i) Show that X is locally compact. 2
(ii) Prove that the one-point compactification of X is homeomorphic to the closed interval $[0,1]$. 4
b) Let $Y = (0,1) \cup (3,4)$. Prove that the one-point compactification of Y is not homeomorphic to $[0,1]$. 4

7. a) Define the terms : (i) Regular space, (ii) Normal space. 2
b) Show that a metric space is second countable $\iff$ there is a countable dense subset. 3
c) Does there exist a continuous function from $S^n \to \mathbb{R}$ which takes the value 0.5 on the equator and 0.75 on the north and south poles? 3

INDIAN STATISTICAL INSTITUTE

End-Semester Examination – Semester I : 2018-2019

M.Math. I Year

Measure Theoretic Probability

Date : 12.11.18 | **Maximum Marks : 60** | Time : 3 Hours

Note : This paper carries **SIX** questions worth **a TOTAL of 74 marks**. Answer as much as you can. The **MAXIMUM** you can score is **60**.

1. (a) For $t > 0$ and $B \subset \mathbb{R}$, let $t \cdot B = \{t \cdot x : x \in B\}$. Show that, for every $t > 0$ and $B \in \mathcal{B}$, the set $t \cdot B$ is borel and $\lambda(t \cdot B) = t \cdot \lambda(B)$.

   (b) For $t > 0$ and $f : \mathbb{R} \to \mathbb{R}$, let $S_t f$ be defined on $\mathbb{R}$ by $(S_t f)(x) = f(x/t)$. Show that if f is measurable, then so is $S_t f$ and that $\int_{\mathbb{R}} (S_t f)(x) dx = t \int_{\mathbb{R}} f(x) dx$, in the sense that if one of the sides is well-defined, then so is the other and they are equal. (6+6)=**[12]**

2. (a) State Hölder's inequality.

   (b) Let f be a real-valued measurable function on a measure space $(\Omega, \mathcal{A}, \mu)$. Show then that the set $I = \{p \in (0, \infty) : \int |f|^p d\mu < \infty\}$ is an interval (i.e., a convex set) and that the function ϕ defined on I by $\phi(p) = \log \left(\int |f|^p d\mu\right)$ is a convex function. (4+8)=**[12]**

3. Consider the function $f(x) = \sqrt{x}$ on $(0, \infty)$ and let μ be the measure λf^{-1} on $\mathcal{B}((0, \infty))$. Show that μ is absolutely continuous with respect to the Lebesgue measure λ on $(0, \infty)$. [Hint: Consider $\mu\left((a, b]\right)$ for $0 \leq a < b < \infty$, and identify a candidate for the R-N-derivative.] **[8]**

4. (a) State Fubini's Theorem.

   (b) Show that for a non-negative random variable X on a probability space, $Y = \tan^{-1}(X)$ is a bounded random variable and $\mathbf{E}[Y] = \int_0^\infty (1 + t^2)^{-1} \mathbf{P}(X > t)\, dt$. (4+6)=**[10]**

5. Let X be a real radom variable on some probability space.

   (a) Show that if G is any probability distribution function on $\mathbb{R}$, then $G(z - X)$ is a random variable for each $z \in \mathbb{R}$, and $H(z) = \mathbf{E}\left[G(z - X)\right]$ is a probability distribution function.

   (b) Show that if Y is a real random variable with distribution function G, which is independent of X, then H is the distribution function of $Z = X + Y$. (6+6)=**[12]**

6. (a) For real random variables $X_n, n \geq 1$ and X, define what is meant by $X_n \xrightarrow{P} X$.

   (b) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, with all the random variables defined on the same probability space, then show that $X_n Y_n \xrightarrow{P} XY$.

   (c) Suppose $\{X_n\}$ is a sequence of real random variables on the same probability space and $\{\alpha_n\}$ is a sequence of real numbers such that $\mathbf{P}(X_n < \alpha_n) \leq 0.3$, for all $n \geq 1$. Show that if $X_n \xrightarrow{P} X$ and $\alpha = \limsup_n \alpha_n$, then $\mathbf{P}(X < \alpha) \leq 0.3$. [Hint: Show that $\mathbf{P}(X < \beta) \leq 0.3$, for any $\beta < \alpha$.] (4+8+8)=**[20]**

Indian Statistical Institute, Kolkata
M. Math Ist year
Academic year 2018-2019
Final Examination
Course: Analysis of Several Variables
15 - 11 - 18
3 hours

- *Answer as many questions as you can.*
- *Maximum marks is 60.*

1. Let $M_n(\mathbb{R})$ denote the space of $n \times n$ real matrices equipped with the operator norm, and $GL_n(\mathbb{R})$ the subset of invertible $n \times n$ matrices.

   (a) Show that $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.

   (b) Show that the map $f : GL_n(\mathbb{R}) \to M_n(\mathbb{R}), A \mapsto A^{-1}$, is differentiable and compute its derivative $Df_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ at an $A \in GL_n(\mathbb{R})$. (3 + 7 = 10 marks)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. Show that $p \in \mathbb{R}$ is a local minimum of f if $Df_p = 0$ and $h \mapsto D^2 f_p(h, h)$ is a positive definite quadratic form. (5 marks)

3. Let f, g be C^1 functions on $\mathbb{R}^3$ and let $A = \{x \in \mathbb{R}^3 | f(x) = 0\}, B = \{x \in \mathbb{R}^3 | g(x) = 0\}$. Suppose that for some $p \in \mathbb{R}^3$, $f(p) = g(p) = 0$, and Df_p, Dg_p are linearly independent linear functionals on $\mathbb{R}^3$. Show that there is a neighbourhood U of p and a C^1 curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ such that $A \cap B \cap U = \{\gamma(t) | t \in I\}$ where I is an open interval in $\mathbb{R}$. (6 marks)

4. Calculate the flow of the linear vector field $f(x) = Ax$ on $\mathbb{R}^2$ where

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

   where a, b are constants. (5 marks)

5. Let $D = \mathbb{R}^2 - \{0\}$, and let ω be the 1-form on D defined by

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

(a) Show that ω is closed.

(b) Show that for any C^1 closed curve γ in D, the integral $\int_\gamma \omega$ is an integer multiple of 2π.

(c) Show that ω is not exact. (3+7+4 = 14 marks)

6. Let $c : [0,1]^k \to \mathbb{R}^n$ be a singular k-cube such that $Dc_x : \mathbb{R}^k \to \mathbb{R}^n$ has rank strictly less than k for all $x \in [0,1]^k$. Show that $\int_c \omega = 0$ for all k-forms ω on $\mathbb{R}^n$. (5 marks)

7. Let $X = (X_1, \ldots, X_n)$ be a smooth vector field on $\mathbb{R}^n$. Define the divergence of X to be the function

$$divX := \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}$$

Define an $(n-1)$-form ω on $\mathbb{R}^n$ by $\omega_p(v_1, \ldots, v_{n-1}) := det[X_p|v_1|\ldots|v_{n-1}]$ for $p \in \mathbb{R}^n$, $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$. Show that $d\omega = (divX)dx_1 \wedge \cdots \wedge dx_n$. (8 marks)

8. Let ω be a k-form on a domain $D \subset \mathbb{R}^n$ and let f be a smooth function on D. Show that

$$\int_\sigma f d\omega = \int_{\partial\sigma} f\omega - \int_\sigma df \wedge \omega$$

for all $(k+1)$-chains σ in D. (6 marks)

9. Let $D \subset \mathbb{R}^n$ be a domain and let $H : D \times [0,1] \to D$ be a smooth homotopy between two smooth maps $f, g : D \to D$, so that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in D$. Let $h_t : D \to D$ be defined by $h_t(x) = H(x,t)$, and let $P : C_{k-1}(D) \to C_k(D)$ be the corresponding prism operator from $(k-1)$-chains in D to k-chains in D, defined by

$$(Pc)(t, x_1, \ldots, x_{k-1}) := h_t(c(x_1, \ldots, x_{k-1}))$$

for any singular $(k-1)$-cube c.

(a) Show that for any singular $(k-1)$-cube c, $\partial Pc = (g \circ c - f \circ c) - P\partial c$.

(b) Show that for any k-form ω on D, there is a unique $(k-1)$-form $P^*\omega$ on D such that

$$\int_{Pc} \omega = \int_c P^*\omega$$

for all singular $(k-1)$-cubes c in D.

(c) Show that if ω is closed and $f^*\omega = g^*\omega$, then $P^*\omega$ is closed. (5 + 5 + 5 = 15 marks)

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2018-19 (First Semester)

M. Math. I YEAR
Algebra I

Date : 19.11.18 Maximum Marks : 60 Duration : 3 Hours

Throughout the paper R will denote a commutative ring with unity.
Answer any four questions from Group A and any three questions from group B

Group A
ANSWER ANY **FOUR** QUESTIONS.

1. (i) Let R be a Noetherian ring. Prove that the polynomial ring $R[X]$ is Noetherian.
(ii) Let D, A be commutative rings such that $D \subset A \subset D[X]$. Show that if A is Noetherian then so is D. [9+5]

2. (i) Let k be a field. Show that $k[X, 1/X]$ is a Euclidean domain by defining a suitable Euclidean function.
(ii) Deduce that the ring $\mathbb{C}[X,Y]/(X^2+Y^2-1)$ is a Euclidean domain. Justify all steps. [6+8]

3. (i) Let $\phi : M \to F$ be a surjective R-linear map from a finitely generated R-module M onto a free R-module F. Show that the kernel of ϕ is a finitely generated R-module.
(ii) Suppose there exists a faithful Noetherian R-module. Prove that R is a Noetherian ring. [7+7]

4. Let M, N be R-modules and I, J be ideals of R. Prove the following isomorphisms of R-modules.
(i) $M \otimes_R R/I \cong M/IM$.
(ii) $R/I \otimes_R R/J \cong R/(I+J)$.
(iii) $M/IM \otimes_{R/I} N/IN \cong (M \otimes_R N)/I(M \otimes_R N)$. [7+3+4]

5. Let $D := \mathbb{C}[X,Y,Z]/(X^2+Y^2+Z^2-1)$.
(i) Prove that D is a Noetherian integral domain.
(ii) Find an element f in the ring D such that $D/fD \cong \mathbb{C}[T]$. [5+9]

6. (i) Let R be an integral domain and $A = R[X_1, X_2, \ldots, X_n]$ be a polynomial ring in n-variables over R. For any permutation $\sigma \in S_n$ and any polynomial $f \in A$, let $\sigma.f = f(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)})$. Set
$$A^{S_n} := \{f \in A | \sigma(f) = f, \ \forall\, \sigma \in S_n\}.$$
Determine A^{S_n}.
(ii) Let $A = \mathbb{R}[X,Y]/(X^2+YX)$ and x denote the image of X in A. Let J denote the ideal generated by x^2+1 in A. Examine whether J is a prime ideal of A. [9+5]

PTO

7. (i) Let G be a finite group acting transitively on a finite set X with $|X| \geq 2$. Show that G necessarily has an element g which has no fixed point.
(ii) Suppose H is a proper subgroup of a finite group G. Prove that $G \neq \bigcup_{g \in G} gHg^{-1}$.
[9+5]

8. (i)Let p and q be distinct primes. Prove that a group of order p^2q cannot be simple.
(ii) Find the number of ways the edges of an equilateral triangle can be painted using 4 colours. Only one colour is used in a single edge and two different edges can have same colour. [9+5]

Group B

ANSWER ANY **THREE** QUESTIONS.

Each question carries 3-marks.

1. Examine whether the following statements are True or False.

   (i) Any two bases of a finitely generated free R-module have same cardinality.

   (ii) Any group of order 15 is cyclic.

   (iii) If $R = \mathbb{Z}/25\mathbb{Z}$ and $S = \{\bar{5}^n | n \geq 0\}$, then $S^{-1}R = 0$.

   (iv) The additive group of rational numbers and the multiplicative group of positive rational numbers are isomorphic to each other.

   (v) $\mathbb{Z}/25\mathbb{Z}$ is an indecomposable $\mathbb{Z}$-module.

Indian Statistical Institute
First Semester Examinations: 2018-19

Course Name: M. Math, 1st year
Subject Name : Linear Algebra
Maximum Marks: 50, Duration: Two and a half hours
Date: 22.11.2018, 2.30 PM

Answer all questions. In case you use any matrix corresponding to a linear transformation in your solutions, please indicate the basis w.r.t which the matrix is defined. If you do not specify the matrix, you will not get any marks. Marks will be deducted for indirect, incomplete, unnecessary long and imprecise answers. There is no part marking.

1. Suppose V is a finite dimensional complex inner product space and $T \in \mathcal{L}(V)$.

   (i) Prove that if T is unitary, then T is an isometry.

   (ii) Prove that the following statements are equivalent:

   (a) T preserves inner products.

   (b) T carries some orthonormal basis of V onto another orthonormal basis.
   $\mathbf{3 + 7 = 10}$

2. **(i)** Suppose V is a complex finite dimensional inner product space. If $P \in \mathcal{L}(V)$ is such that P is an idempotent with orthogonal range and kernel, then prove that P is self-adjoint.

   (ii) Suppose $T \in \mathcal{L}(V)$. Use the polar decomposition theorem to prove that there exists a positive operator N and a unitary operator U such that $T = NU$.

   (iii) Prove that if two normal operators on V commute, then their product is again a normal operator. $\mathbf{3 + 4 + 3 = 10}$

3. **(i)** Suppose V is a finite dimensional vector space. Classify all $T \in \mathcal{L}(V)$ such that

P.T.O

T commutes with every idempotent.

(**ii**) Classify all possible matrices up to similarity whose minimal polynomial is $(x-1)^5$ and characteristic polynomial is $(x-1)^7$.

(**iii**) Prove that if T is a normal operator on an inner product space. then $\mathrm{Ker}(T) = (\mathrm{Ran}(T))^{\perp}$.

(**iv**) Let T be a linear transformation from a vector space V to V. Prove that if the characteristic polynomial of T is irreducible, then every vector of V is a cyclic vector. $\mathbf{5 \times 4 = 20}$

4. Suppose A is an $n \times n$ matrix over a field $\mathbb{K}$ with characteristic polynomial

$$f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}.$$

Use the theory of Jordan canonical forms to compute the trace of A. **10**

Indian Statistical Institute
First Semester Examinations: 2018-19

Course Name: M. Math, 1^{st} year
Subject Name : Topology I
Maximum Marks: 60, Duration: Three hours
Date: 26.11.2017, 2 PM – 5 PM

- Answer as many questions as you can.
- Maximum marks is 60.
- You may use any results proved in class. Any other results (including those in homework problem sets) require proof.

1. Let X be a topological space and $\mathcal{B}$ be a basis for the topology of X. Let $f : [0,1] \to X$ be a continuous function. Show that there exists $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for every $i < n$, $\exists\, U \in \mathcal{B}$ for which $f([t_i, t_{i+1}]) \subset U$. 5

2. a) Prove that $S^1/[z \sim e^{\frac{2\pi i}{3}} z]$ is homeomorphic to S^1. 4
   b) Prove that $D^2/[x \sim -x]$ is homeomorphic to D^2. 4

3. Identify the space of complex polynomials of degree $\leq n$ with $\mathbb{C}^{n+1}$ via

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \mapsto (a_0, a_1, \cdots, a_n)$$

   a) Prove that polynomial multiplication induces a continuous function $\mu_{n,m}$ from $(\mathbb{C}^{n+1} - 0) \times (\mathbb{C}^{m+1} - 0)$ to $(\mathbb{C}^{n+m+1} - 0)$. 5
   b) Prove that the above continuous function induces a continuous function $\mathbb{C}P^n \times \mathbb{C}P^m \to \mathbb{C}P^{n+m}$. 5

4. a) Write down an action of C_2 (the group of order 2) on S^n such that the orbit space S^n/C_2 is homeomorphic to $S(\mathbb{R}P^{n-1})$. (Here, S stands for the suspension.) 5
   b) With respect to the usual inclusion of $\mathbb{R}P^{n-1}$ in $\mathbb{R}P^n$, prove that $\mathbb{R}P^n/\mathbb{R}P^{n-1}$ is homeomorphic to S^n. 5

5. Write the group C_2 as $\{1, \sigma\}$ such that $\sigma^2 = 1$. Consider the action of C_2 on $\mathbb{C}$ induced by $\sigma(z) = \bar{z}$.
   a) Prove that $\mathbb{C}/C_2$ is homeomorphic to the half plane H ($H = \{a + bi \in \mathbb{C} | b \geq 0\}$). 5
   b) Is the quotient map $q : \mathbb{C} \to H$ a covering space ? 5

6. Let $S^{n-1} \subset S^n$ be the equatorial sphere. Prove that there is an open set containing S^{n-1} in S^n that deformation retracts to S^{n-1}. 4

7. Suppose that $f : X \to Y$ is a covering space and $g : Y \to Z$ is a covering such that for every $z \in Z$, $g^{-1}(z)$ is finite. Prove that $g \circ f : X \to Z$ is a covering space. 5

8. Let p, q, r be three distinct points of S^2. Compute $\pi_1(S^2 - p - q, r)$. 8

9. Let X be a path-connected space. Let $S(X)$ denote the suspension of X.
   a) Prove that $S(X)$ is the union of two open sets U and V such that
   (i) $U \simeq V \simeq C(X)$, the cone on X. ($\simeq$ stands for homotopy equivalent).
   (ii) $U \cap V \simeq X$. 7
   b) Prove that $S(X)$ is simply connected. 3

INDIAN STATISTICAL INSTITUTE

Backpaper Examination – Semester I : 2018-2019

M.Math. I Year

Measure Theoretic Probability

Date : 04/01/19 **Maximum Marks : 45** Time : 3 Hours

Note : This paper carries **SIX** questions worth **a TOTAL of 100 marks**. Answer as much as you can. The **MAXIMUM** you can score is **45**.

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Denote $\mathcal{M} = \{M \in \mathcal{A} : \mu(M) = 0\}$.

 (a) Show that $\overline{\mathcal{A}} = \{A \triangle N : A \in \mathcal{A}, N \subset M \in \mathcal{M}\}$ is a σ-field on Ω that contains $\mathcal{A}$.

 (b) Show that, for $A \in \mathcal{A}$ and $N \subset M \in \mathcal{M}$, if we put $\mu(A \triangle N) = \mu(A)$, then μ is well-defined and defines a measure on $\overline{\mathcal{A}}$.

 (c) Show that $(\Omega, \overline{\mathcal{A}}, \mu)$ is the completion of $(\Omega, \mathcal{A}, \mu)$, that is, $\overline{\mathcal{A}}$ is a μ-complete σ-filed and is the smallest such containing $\mathcal{A}$. (6+6+6)=**[18]**

2. Let $f_n, n \geq 1$, f and h be extended real-valued measurable functions on $(\Omega, \mathcal{A})$.

 (a) Show, directly from definition of measurability, that each of the following four sets belongs to $\mathcal{A}$: (i) $\left\{\omega \in \Omega : \lim_n f_n(\omega) \text{ exists}\right\}$, (ii) $\left\{\omega \in \Omega : \lim_n f_n(\omega) \text{ exists and is finite}\right\}$, (iii) $\left\{\omega \in \Omega : \sum_n f_n(\omega) \text{ converges and is finite}\right\}$, (iv) $\left\{\omega \in \Omega : \lim_n f_n(\omega) = f(\omega)\right\}$.

 (b) Consider the function g defined on Ω by $g(\omega) = \begin{cases} \lim_n f_n(\omega) & \text{if } \lim_n f_n(\omega) \text{ exists,} \\ h(\omega) & \text{otherwise.} \end{cases}$

 Show that g is an extended real-valued measurable function. ((5×4)+6)=**[26]**)

3. Let $\{f_n, n \geq 1\}$ be a sequence of extended real valued measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$.

 (a) Show that if $f_n \uparrow f$ and if for some measurable g with $\int g^- \, d\mu < \infty$, one has $f_n \geq g$ for all n, then the integrals $\int f_n \, d\mu$, $n \geq 1$ and $\int f \, d\mu$ all exist, and, $\int f_n \, d\mu \uparrow \int f \, d\mu$.

 (b) Show that if for some measurable g with $\int g^- \, d\mu < \infty$, one has $f_n \geq g$ for all n, then the integrals $\int f_n \, d\mu$, $n \geq 1$, all exist and, $\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$. (6+6)=**[12]**

4. (a) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $a < b$ be any two real numbers. Suppose f is a real-valued function on $(a, b) \times \Omega$ such that (i) for each $t \in (a, b)$, the function $\omega \mapsto f(t, \omega)$ is $\mathcal{A}$-measurable and μ-integrable, (ii) $\frac{\partial f(t,\omega)}{\partial t}$ exists for all $t \in (a, b)$ and for each $\omega \in \Omega$. If there exists a μ-integrable measurable function g on Ω such that $\left|\frac{\partial f(t,\omega)}{\partial t}\right| \leq g(\omega)$ for all $t \in (a, b)$ and all $\omega \in \Omega$, then show that the function $\varphi(t) = \int f(t, \omega) d\mu(\omega)$, $t \in (a, b)$ is everywhere differentiable with $\varphi'(t) = \int \frac{\partial f(t,\omega)}{\partial t} d\mu(\omega)$.

 (b) Let μ be a measure on the Borel σ-field on $[0, \infty)$. Show that, if $\int e^{-\lambda x} d\mu(x) < \infty$ for all $\lambda > 0$, then the function $\phi(\lambda) = \int e^{-\lambda x} d\mu(x)$ is infinitely differentiable on $(0, \infty)$ with $\phi^{(n)}(\lambda) = (-1)^n \int x^n e^{-\lambda x} d\mu(x)$ for all $n \geq 1$, $\lambda > 0$. (6+6)=**[12]**

5. Let X and Y be independent random variables on some probability space and let h be a real-valued measurable function on $\mathbb{R}^2$.

 (a) Show that for each $x \in \mathbb{R}$, $h(x, Y)$ is a real random variable.

 (b) Show that, if h is non-negative, then $\phi(x) = E[h(x, Y)]$ is a non-negative measurable function and $E[\phi(X)] = E[h(X, Y)]$.

 (c) Show that, if the real random variable $h(X, Y)$ has finite expectation, then $E[h(x, Y)]$ is finite for P_X - a.e. $x \in \mathbb{R}$. Putting $\phi(x) = E[h(x, Y)]$, if $E[h(x, Y)]$ is finite, and $\phi(x) = 0$, otherwise, show that ϕ is measurable and $E[\phi(X)] = E[h(X, Y)]$. (6+6+6)=**[18]**

6. Suppose $\{X_n\}$ is a sequence of real random variables converging in probability to a real random variable X.

 (a) Show that there is a subsequence $\{X_{n_k}\}$ that converges almost surely to X.

 (b) Show that $\{X_n\}$ converges in distribution to X. (7+7)=**[14]**

Indian Statistical Institute, Kolkata
M. Math Ist year
Academic year 2018-2019
Backpaper Examination
Course: Analysis of Several Variables
3 hours

DATE: 07.01.1

- *Answer as many questions as you can.*
- *Maximum marks is 60.*

1. Let $M_n(\mathbb{R})$ denote the space of $n \times n$ real matrices equipped with the operator norm. Show that the map $f : M_n(\mathbb{R}) \to \mathbb{R}, A \mapsto det\ A$, is differentiable and compute its derivative $Df_A : M_n(\mathbb{R}) \to \mathbb{R}$ at an $A \in M_n(\mathbb{R})$. (8 marks)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a C^2 curve. Show that

$$(f \circ \gamma)''(t) = D^2 f_{\gamma(t)}(\gamma'(t), \gamma'(t)) + Df_{\gamma(t)}(\gamma''(t))$$

for all $t \in \mathbb{R}$. (8 marks)

3. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (e^x \cos y, e^x \sin y)$ is a C^1 diffeomorphism from a neighbourhood of $(0,0)$ onto a neighbourhood of $(1,0)$. (6 marks)

4. Let X be a linear vector field $X(x) = Ax$ on $\mathbb{R}^n$ where $A \in M_n(\mathbb{R})$. Show that if A is skew-symmetric then all elements of the flow of X are orthogonal linear maps. (6 marks)

5. Let ω be an alternating k-tensor on a finite dimensional vector space V. Show that $Alt(\omega) = \omega$. (5 marks)

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6. Let $D \subset \mathbb{R}^n$ be a domain and let $\gamma_0, \gamma : [0,1] \to D$ be C^1 closed curves in D. We say that γ_0, γ_1 are homotopic in D if there is a continuous map $\sigma : [0,1] \times [0,1] \to D$ such that $\sigma(s,0) = \gamma_0(s), \sigma(s,1) = \gamma_1(s)$ for all $s \in [0,1]$, and $\sigma(0,t) = \sigma(1,t)$ for all $t \in [0,1]$. Suppose there is a C^1 homotopy of closed curves $\sigma : [0,1]^2 \to D$ between γ_0 and γ_1.

   (a) Show that the boundary of the singular 2-cube σ is given by $\partial\sigma = \gamma_0 - \gamma_1$.

   (b) Show that if ω is a closed 1-form in D then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.

   (6+5 = 11 marks)

7. Let ω be a k-form on $\mathbb{R}^n$ such that $\int_c \omega = 0$ for all singular k-cubes $c : [0,1]^k \to \mathbb{R}^n$. Show that $\omega = 0$. (8 marks)

8. Let $\omega = Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$ be a 2-form on $\mathbb{R}^3$. Show that there exists a vector field X on $\mathbb{R}^3$ such that $\omega_p(v,w) = det[X_p|v|w]$ for all $p \in \mathbb{R}^3$, $v, w \in \mathbb{R}^3$. (7 marks)

9. Let $\omega = hdx_1 \wedge \cdots \wedge dx_n$ be an n-form on a domain $D \subset \mathbb{R}^n$. Let $\sigma : [0,1]^n \to D$ be a singular n-cube such that σ is a C^1 diffeomorphism onto its image $V = \sigma([0,1]^n) \subset D$, and such that $detD\sigma_x > 0$ for all $x \in [0,1]^n$. Show that

$$\int_\sigma \omega = \int_V hdx_1 \ldots dx_n$$

   (7 marks)

10. Let ω be a k-form on $\mathbb{R}^n$ and let X be a smooth complete vector field on $\mathbb{R}^n$ with flow (ϕ_t). Suppose $\frac{d}{dt}_{|t=0} \phi_t^* \omega = 0$. Show that for any singular k-cube c, the function $t \mapsto \int_{\phi_t \circ c} \omega$ is constant. (14 marks)

Indian Statistical Institute
Second Semester Examinations: 2018-19

Course Name: M. Math, 1^{st} year
Subject Name : Topology II
Maximum Marks: 40, Duration: Two hours
Date: 18.2.2019, 2:30 PM - 4:30 PM

- Answer as many questions as you can.
- Maximum marks is 40.
- You may consult any book or course notes during the exam.
- You may use any results proved in class. Any other results (including those in homework problem sets) require proof.

1. Let
$$Y = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1, z = 0\} \subset \mathbb{R}^3,$$
$$X = Y \cup \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$
a) Write down a CW complex structure on X such that Y is a subcomplex. 5
b) Prove that $X \simeq S^2 \vee S^2$. 5

2. Let f be the map $S^1 \to S^1$ which is given by $f(z) = z^4$ on the upper semi-circle ($Im(z) \geq 0$) and $f(z) = \bar{z}^2$ on the lower semi-circle ($Im(z) \leq 0$). Prove that $Cone(f)$ (the mapping cone of f) is contractible. 5

3. Let C_2 (the cyclic group of order 2 generated by σ) act on $S^n \times S^n$ according to the formula $\sigma(x, y) = (-x, -y)$. Let $P = (S^n \times S^n)/C_2$.
a) Prove that the quotient map $S^n \times S^n \to P$ is a covering space. 3
b) Write down a map $q : P \to \mathbb{R}P^n \times \mathbb{R}P^n$ which is a covering space. 3
c) Prove that every map from P to S^1 is homotopic to a constant map. 4

4. Find all path connected covering spaces $E \to \mathbb{R}P^n \times \mathbb{R}P^n$. In each case, compute $G(E)$, the group of deck transformations. 6

5. Let Y be the combinatorial Δ-complex given by
$$Y_0 = \{v_0, v_1, v_2\},\ Y_1 = \{a, b, c\},\ Y_2 = \{\Delta_1, \Delta_2\},\ Y_n = \varnothing \text{ if } n \geq 3,$$
$$d_0(a) = d_1(b) = v_1,\ d_0(b) = d_0(c) = v_2,\ d_1(c) = d_1(a) = v_0,$$
$$d_0(\Delta_1) = d_0(\Delta_2) = b,\ d_1(\Delta_1) = d_1(\Delta_2) = c,\ d_2(\Delta_1) = d_2(\Delta_2) = a.$$
a) Prove that $|Y| \cong S^2$. 5
b) Calculate the homology groups $H_*^{\Delta}(Y)$. 5

6. Consider the sequence of Abelian groups C_n with $C_n = 0$ if $n \geq 3$, $C_2 = \mathbb{Z}^2$, $C_1 = \mathbb{Z}^4$, $C_0 = \mathbb{Z}^2$, together with maps between them as in the diagram below
$$\cdots 0 \to 0 \to \cdots 0 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} -3 & 3 \\ 1 & 1 \\ 1 & -2 \\ -2 & 1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix}} \mathbb{Z}^2$$
a) Prove that C is a chain complex. 2
b) Compute $H_*(C)$ 5

Indian Statistical Institute, Kolkata
M. Math I
Spring Semester 2019
Mid-Term Examination
Course: Complex Analysis
19 - 02 - 19
2 hours

- *Answer as many questions as you can.*
- *Maximum marks is 40.*

1. Let f be an entire function such that f maps horizontal lines into vertical lines. Show that f is of the form $f(z) = az + b$ where $a \in i\mathbb{R}, b \in \mathbb{C}$. (7 marks)

2. Let $D \subset \mathbb{C}$ be an open convex set, and let f be a holomorphic function on D such that $f(z) \neq 0$ for all $z \in D$. Show that there is a holomorphic function g on D such that $f(z) = e^{g(z)}$ for all $z \in D$. (8 marks)

3. Suppose f is holomorphic on $\{1/2 < |z| < 1\}$ and $f(z) = f(z^2)$ for all $z \in S^1$. Show that f is constant. (6 marks)

4. Let f be an entire function such that $\Im f(z) \geq \Re f(z)$ for all $z \in \mathbb{C}$. Show that f is constant. (6 marks)

5. Let T be a Moebius transformation with a unique fixed point z_0 in $\hat{\mathbb{C}}$. Show that if $z_0 \in \mathbb{C}$, then $T'(z_0) = 1$. (7 marks)

6. Let $N \geq 1$ be an integer. Let f_n be a sequence of holomorphic functions on the domain $D = \{0 < |z| < 1\}$ such that each f_n has either a removable singularity or a pole of order at most N at $z = 0$. Suppose f_n converges uniformly on compacts of D to a holomorphic function f. Show that f has either a removable singularity or a pole of order at most N at $z = 0$. (7 marks)

7. Let f be a meromorphic function on a domain $D \subset \mathbb{C}$ such that the function $g = e^f$ is also meromorphic on D. Show that f is holomorphic on D. (7 marks)

Indian Statistical Institute

Mid-semestral Examination: 2018-2019

Programme: Master of Mathematics

Course: **Algebra II**

Maximum Marks: 60

Duration: 2 Hours *Date:* 21·02·2019 *2019*

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1. Indicate which of the following are pairs of isomorphic fields. Do **not** write any explanation. (5 × 3=15 marks)

(a) $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt{2}+\sqrt{5})$
(b) $\mathbb{Q}(\pi)$, $\mathbb{Q}(\pi^2)$
(c) $\mathbb{Q}(3^{1/7})$, $\mathbb{Q}(3^{2/7})$
(d) $\mathbb{Q}[X]/(X^2+X+1)$, $\mathbb{Q}(\sqrt{-3})$
(e) $\mathbb{Q}(2^{1/4})$, $\mathbb{Q}(2^{1/4}\sqrt{-1})$.

2. Is it true that every field extension of degree 2 is a normal extension? Prove your assertion. (6 marks)

3. Show that $[A_4, A_4] = V_4$, where V_4 is the subgroup $\{(1), (12)(34), (13)(24), (14)(23)\}$ of A_4. (8 marks)

4. Let K be the splitting field for the polynomial $X^7 - 2$ over $\mathbb{Q}$ in $\mathbb{C}$? Describe K and determine $[K : \mathbb{Q}]$. (10 marks)

5. Show that a polynomial $f \in \mathbb{Q}[X]$ that is irreducible over $\mathbb{Q}$ remains irreducible over $\mathbb{Q}(\pi)$. (11 marks)

6. Suppose $K \subset \mathbb{C}$ is an algebraic extension of $\mathbb{Q}$ of degree 3. Show that $K \subset \mathbb{R}$ if it is given that K is the splitting field of a cubic polynomial. Give an example to show that not every extension of $\mathbb{Q}$ of degree 3 is contained in $\mathbb{R}$.(10+2=12 marks)

INDIAN STATISTICAL INSTITUTE, KOLKATA
Mid Semester Examination, Second Semester 2018-2019
Differential Geometry, M.Math First Year
Total Marks -30, Time : 3 hours , Date : 22. 02. 2019

1. Let $\alpha : I \to \mathbb{R}^3$ be a unit speed regular curve. Write down the Frenet-Serret equations of α. (2)

2. Consider the curve parametrized by

$$\alpha(t) = (t^3 + t, 2t^3 + 2t, 3t^3 + 3t).$$

   (a) Find out the speed (i.e. $|\alpha'(t)|$) and curvature of $\alpha(t)$. (1+2)

   (b) Find out a unit speed reparametrization $\beta(s)$ of this curve. (2)

   (c) Find out an orientable surface S such that $\beta(s) \subset S$ and it is a line of curvature of S. Here a curve C on a surface S is called a line of curvature of S, if for all $p \in C$, the tangent line of C at p is a principal direction at p. (3)

3. Let $C \subset \mathbb{R}^3$ be the helix given by $\alpha(t) = (r.cos(t), r.sin(t), h.t)$ for some $r, h > 0$. Find out a unit speed reparametrization of C. Show that the curvature and the torsion of C are nonzero constants. (2 + 2 +3)

4. Let $\alpha : I \to \mathbb{R}^3$ be a non-planar unit speed parametrized curve and let $n(s)$ be the normal to the curve at s. Assume $k(s) \neq 0$ and $n(s) \neq 0$ for all $s \in I$.

   (a) Suppose $n(s) = usin(rs) + vcos(rs)$ for $r > 0$ and $u, v \in \mathbb{R}^{\cancel{2}\,3}$. Show that curvature k and torsion τ of α are constants and $r^2 = k^2 + \tau^2$. Prove that $|u| = |v| = 1$ and $u.v = 0$. (3+3)

   (b) If the torsion τ and the curvature k of α are constant, then show that $n''(s) = -(k^2 + \tau^2)n$. Show that $n(s) = usin(rs) + vcos(rs)$ satisfies the equation $n''(s) = -(k^2 + \tau^2)n$, where $u, v \in \mathbb{R}^3$ and $r = \sqrt{k^2 + \tau^2}$. (2+1)

5. Define a geodesic of a regular surface S to be any regular curve $\alpha(t)$ on S such that $\alpha''(t)$ is normal to S at $\alpha(t)$.

   (a) Prove that a geodesic has constant speed. Show that a unit speed geodesic on the standard unit sphere is planar. (2 + 4)

   (b) Suppose $\alpha(t)$ is a unit speed geodesic on S and is also contained in a plane $P \subset \mathbb{R}^3$. Prove that α is also a line of curvature of $\cancel{M}$ S. (3)

6. A regular surface S is called a ruled surface if there exists a parametrization or ruling of S given by $x(u, v) = \beta(u) + v\alpha(u)$, where β and α are regular smooth curve.

(a) Prove that for every point p on a ruled surface, there must be a tangent direction v such that the normal curvature along the direction v is zero. (3)

(b) Prove that at every point on ruled surface, the principle curvatures k_1 and k_2 must have $k_1 \geq 0$ and $k_2 \leq 0$. (2)

Indian Statistical Institute
First Semester Examinations: 2018-19

Course Name: M. Math, 1st year
Subject Name : Functional Analysis
Maximum Marks: 50, Duration: Three hours
Date: 22.4.2019, 2.30 PM

Answer all questions. Marks will be deducted for indirect, incomplete, unnecessary long and imprecise answers. There is no part marking.

1. Suppose $H = L^2[0,1]$. Answer the following questions:

   (a) Consider the operator $M_x : H \to H$ defined by the formula

   $$M_x(f)(t) = tf(t) \ \forall \ f \in L^2[0,1] \text{ and } \forall \ t \ \in \ [0,1].$$

   Is this operator compact?

   (b) Suppose T is a compact operator on H and suppose $T = UA$ be the polar decomposition of T. If U commutes with A, then prove that T is normal.

   (c) Let us denote the spectrum of an element $A \in B(H)$ by the symbol $\sigma(A)$. Suppose T is a compact self adjoint operator on H. Suppose f is a real-valued continuous bounded function on $\mathbb{R}$ such that $f(0) = 0$. Prove that

   $$\sigma(f(T)) = f(\sigma(T)).$$

   (d) Suppose T is a compact normal operator on H. Prove that the set

   $$\{\phi(T) : \ \phi \text{ is a bounded function from } \mathbb{C} \text{ to } \mathbb{C}\}$$

   is closed in the weak operator topology. **5 + 5 + 7 + 8 = 25**

2. Let X denote a Banach space. Prove the following statements:

   (a) Suppose Y is a norm-closed subspace of X. Prove that $Y = \cap\{\mathrm{Ker}(f) : f \in X^*, \ f(y) = 0 \ \forall \ y\}$.

   (b) Suppose X is reflexive and Z is a subspace of X^*. Prove that Z is norm-closed if and only if Z is closed in the weak-$*$-topology.

   (c) Suppose Y is a Banach spaces and T is a linear map from X to Y. If $\{f_i\}_{i \in I}$ is a set of functionals in Y^* which separates points of Y and $f_i \circ T$ is continuous for all $i \in I$, prove that T is continuous.

(d) Suppose $g \in L^1(\mathbb{R})$ and $X = L^3(\mathbb{R})$. Prove that the operator

$$T : X \to X, f \mapsto g * f$$

is a bounded linear operator on X.

(e) Consider the Banach space $L^3[a,b]$ and let μ denote the Lebesgue measure on $[a,b]$. Prove that a sequence $\{f_n\}_n$ in $L^3[a,b]$ converges weakly to a function f in $L^3[a,b]$ if and only if $\{f_n\}_n$ is bounded in the L^3-norm and

$$\lim_{n\to\infty} \int_{[c,d]} f_n d\mu = \int_{[c,d]} f d\mu \ \forall [c,d] \subseteq [a,b].$$ **2 + 3 + 5 + 5 + 10 = 25**

Indian Statistical Institute

Semestral Examination: 2018-2019
Programme: Master of Mathematics
Course: **Algebra II**

Maximum Marks: 100
Duration: 3 Hours Date: April 24, 2019

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1. Show that if a field K is perfect and $char(K) = p > 0$ then (8)
$$K = \{a^p : a \in K\}.$$

2. Suppose F is a field of characteristic 0. Let ζ_n denote a primitive n-th root of unity in some extension of F. Show that $F(\zeta_n)/F$ is a Galois extension and $Gal(F(\zeta_n)/F)$ is isomorphic to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$. (8)

3. Suppose K/F is a finite purely inseparable extension and $char(F) = p$. Show that $[K : F]$ is a power of p. (10)

4. (a) Show that $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ if and only if $(m, n) = 1$. You may assume that the Euler ϕ function is multiplicative. Here ζ_n denotes a primitive n-th root of unity.
(b) Determine the Galois group of $x^{10} + x^9 + \cdots + x + 1$ over $\mathbb{Q}(\zeta_7)$. (10+5=15)

5. Let K be an infinite field and let $a_1, a_2, \cdots, a_n$ be pairwise distinct elements in K. Show that there is a natural number r such that (10)
$$\sum_{i=1}^{n} a_i{}^r \neq 0.$$

6. (a) Show that there is a subfield K of $\mathbb{C}$ such that $K/\mathbb{Q}$ is a Galois extension and $[K : \mathbb{Q}] = 15$. Hint: Use the theory of cyclotomic fields.
(b) Can such a field K contain $2^{1/3}$? (8+4=12)

7. Let K/F be a Galois extension with $Gal(K/F) \cong S_3$. Show that there is an irreducible cubic polynomial $f \in F[x]$ such that K is a splitting field of f over F. Hint: Consider the subfield of K fixed by a subgroup of $Gal(K/F)$ of order 2. (15)

8. For the following polynomials $f \in F[x]$, determine the Galois group of f over F:
(a) $f(x) = (x^p - 1)(x^p - 2)$ over $F = \mathbb{F}_p$, where $p > 2$ is a prime,
(b) $f(x) = (x^2 - 2)(x^2 - 3)$ over $F = \mathbb{Q}$,
(c) $f(x) = (x^2 - 2)(x^2 - 3)$ over $F = \mathbb{F}_5$,
(d) $f(x) = (x + 2000)^3 + 2x + 2$ over $F = \mathbb{Q}$. (5+5+10+10=30)

Indian Statistical Institute, Kolkata
M. Math Ist year
Academic year 2018-2019
Final Examination
Second semester
Course: Complex Analysis
26 - 04 - 19
3 hours

- *Answer as many questions as you can.*
- *Maximum marks is 60.*

1. Let f be a holomorphic function on the unit disk such that $\Re f(z) > 0$ for all z. Show that $|f'(0)| \leq 2\Re f(0)$ with equality if and only if f is a Moebius transformation mapping the unit disk onto the half-plane $\{\Re z > 0\}$. (5+3=8 marks)

2. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with a zero of order N at the origin. Show that $|f(z)| \leq |z|^N$ for all $z \in \mathbb{D}$ (here $\mathbb{D}$ denotes the unit disk). (6 marks)

3. Find the number of zeroes (counted with multiplicity) of $f(z) = 2z^2 + 5z^4 + \sin(z)$ in the unit disk $\mathbb{D}$. (6 marks)

4. Let f be a meromorphic function on a simply connected domain D with finitely many poles in D. Suppose all residues of f at all of its poles are zero. Show that there is a meromorphic function F on D such that $F' = f$. (8 marks)

5. (i) Let f be a holomorphic function on the unit disk $\mathbb{D}$ such that $|f(z)| \to 1$ as $|z| \to 1$, and suppose $f(z) \neq 0$ for all z. Show that f is constant.

   (ii) For $a \in \mathbb{D}$, let ϕ_a denote the Moebius transformation $\phi_a(z) = (z-a)/(1-\bar{a}z)$. Let $f : \mathbb{D} \to \mathbb{D}$ be a proper holomorphic function (here proper means that the inverse image of a compact set is compact). Show that there are points $a_1, \ldots, a_n \in \mathbb{D}$ and $\lambda \in S^1$ such that $f(z) = \lambda\phi_{a_1}(z) \ldots \phi_{a_n}(z)$ for all $z \in \mathbb{D}$. (5+8 = 13 marks)

6. Let $\mathcal{F}$ be a family of one-to-one holomorphic functions on a domain D such that $\mathcal{F}$ omits 0. Show that $\mathcal{F}$ is a normal family (when considered as a family of meromorphic functions). (8 marks)

7. Let $f_n : \mathbb{D} \to D_n$ be a sequence of conformal mappings onto an increasing sequence of simply connected domains D_n such that $B = \cup_n D_n$ is a simply connected proper subdomain of $\mathbb{C}$ and $f_n(0) = z_0 \in D_1, f_n'(0) > 0$ for all n.

   (i) Show that the family $\{f_n\}$ is a normal family of holomorphic functions.

   (ii) Show that if a subsequence $f_{n_k} \to g$ uniformly on compact sets in $\mathbb{D}$, then g is one-to-one and holomorphic.

   (iii) Show that if a subsequence $f_{n_k} \to g$ uniformly on compact sets in $\mathbb{D}$, then $g(\mathbb{D}) = B$.

   (iv) Let $f : \mathbb{D} \to B$ be the unique conformal map such that $f(0) = z_0, f'(0) > 0$. Show that $f_n \to f$ uniformly on compact sets in $\mathbb{D}$. (5+5+6+6 = 22 marks)

Indian Statistical Institute
First Semester Examinations: 2018-19

Course Name: M. Math, 1^{st} year
Subject Name : Topology II
Maximum Marks: 60, Duration: Three hours
Date: 29.04.2019, 2:30 PM - 5: 30 PM

- Answer as many questions as you can.
- Maximum marks is 60.
- You may consult any book or course notes during the exam.
- You may use any results proved in class. Any other results (including those in homework problem sets) require proof.

1. Let G be a topological group with multiplication $m : G \times G \to G$. Let ω denote the composite
$$G \vee G \subset G \times G \xrightarrow{m} G.$$
Calculate the effect of ω on homology groups using the isomorphism $\tilde{H}_*(G \vee G) \cong \tilde{H}_*(G) \oplus \tilde{H}_*(G)$. 6

2. a) Consider $TS^2 = \{(x,v) | x \in S^2, v \in \mathbb{R}^3, \langle v, x \rangle = 0\} \subset S^2 \times \mathbb{R}^3$. Show that TS^2 deformation retracts to $A = \{(x,0) | x \in S^2\} \subset TS^2$, which is homeomorphic to S^2. 3
b) Let $X = \{(x,v) \in TS^2 | |v| = 1\}$. Show that $TS^2 - A$ deformation retracts to X. 3
c) Consider $\pi_1 : TS^2 \to S^2$ the projection onto the first factor. Show that any $s : S^2 \to TS^2$ such that $\pi_1 \circ s = id$, satisfies $Im(s) \cap A \neq \phi$. 4

3. Let $\Delta : S^n \to S^n \times S^n$ be the diagonal map given by $\Delta(x) = (x,x)$. Calculate $H_*(\text{Cone}(\Delta))$, the homology of the mapping cone of Δ. 8

4. Let $q : S^n \to \mathbb{R}P^n$ be the usual quotient map.
a) Compute $q_* : H_n(S^n) \to H_n(\mathbb{R}P^n)$. 5
b) Compute $q_* : H_n(S^n; \mathbb{Z}/2) \to H_n(\mathbb{R}P^n; \mathbb{Z}/2)$. 5

5. Let X be a CW complex, and Σ_2 be the surface of genus 2. Prove that $H_k(X \times \Sigma_2) \cong H_k(X) \oplus H_{k-1}(X)^4 \oplus H_{k-2}(X)$. 7

6. Let X be a path connected space such that $\pi_1(X)$ is a finite group. Prove that for any map $X \to S^1 \times S^1$, the induced map $f_* : \tilde{H}_*(X) \to \tilde{H}_*(S^1 \times S^1)$ is 0. 7

7. a) For any map $f : S^{2n} \to S^{2n}$, prove that $\exists\, x \in S^{2n}$ with $f(x) = x$, that is, f has a fixed point. 2
b) Prove that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. 5
c) Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points. 3

8. Let $L_3(k)$ be the Lens space S^3/C_k.
a) Using the usual CW complex structure on $L_3(k)$ and $\mathbb{R}P^2$, and the induced CW complex structure on $L_3(k) \times \mathbb{R}P^2$, write down the cellular chain complex of $L_3(k) \times \mathbb{R}P^2$. 6
(b) Compute $H_*(L_3(k) \times \mathbb{R}P^2)$. 4
b) Prove that any CW complex homotopic to $L_3(k) \times \mathbb{R}P^2$ must have dimension ≥ 5. 3

INDIAN STATISTICAL INSTITUTE, KOLKATA
SEMESTER EXAMINATION, SECOND SEMESTER 2018-2019
Differential Geometry I, M. Math I year
Total Marks -50, Time : 3 hours , Date : 03. 05. 2019

1. Suppose a unit speed curve $\alpha(s)$ has constant curvature $\kappa > 0$ and zero torsion τ.

    (a) Show that $\gamma(s) = \alpha(s) + (1/\kappa)(n_{\alpha(s)})$ is a constant curve. [3]

    (b) Using the previous part, prove that the curve $\alpha(s)$ is part of a circle centered at $\gamma(s)$. What is the radius of the circle ? [2]

2. Compute the geodesic curvature of any circle on the unit sphere. [4]

3. Let $S \subset \mathbb{R}^3$ be an orientable surface and $p \in S$. Let $h(q) := (q-p).N(p)$ for all $q \in S$. Let $\alpha : (-a, a) \to S$ be a differentiable curve with $\alpha(0) = p$ and $\alpha'(0) = w$. Let $(H_p h)(w) := \frac{d^2(h \circ \alpha)}{dt^2}|_{t=0}$.

    (a) Show that $dh_p = 0$. [3]

    (b) Show that $H_p h : T_p(S) \to \mathbb{R}$ is a quadratic form. [5]

    (c) Let $w \in T_p(S)$, $|w| = 1$. Show that $(H_p h)(w)$ is the normal curvature at p in the direction of w. [5]

    (d) Conclude that $H_p(h)$ is the second fundamental form. [2]

4. Show that a curve α with positive curvature is asymptotic if and only if its binormal is parallel to the unit normal of S at all points of α. [5]

5. Show that

$$x(u,v) := (u(cos(v)), u(sin(v)), log(u))$$

and

$$\bar{x}(u,v) = (u(cos(v)), u(sin(v)), v)$$

have equal Gaussian curvature for all (u,v) but $\bar{x} \circ x^{-1}$ is not an isometry. [5]

6. Let S be an orientable regular surface such that for all $p, q \in S$ parallel transport joining p and q is independent of the curve α joining p and q, then the Gaussian curvature K is identically zero on S. [6]

7. Let S be a compact regular surface and let $W(S) := \iint_S H^2 d\sigma$, where H is the mean curvature.

    (a) Compute $W(S)$ when S is a sphere of radius R. [2]

    (b) Suppose S is homeomorphic to a sphere, show that $W(S) \geq 4\pi$. [5]

(c) Suppose S is homeomorphic to a sphere and $W(S) = 4\pi$.. Prove that S is a sphere. [5]

8. Let S be a compact surface with positive Gaussian curvature. Then any two simple closed geodesics intersects. Show that there does not exists any simple closed geodesic on a cone minus its vertex. [5 +5]

9. Let S be a connected compact regular surface and $p \in S$. Show that the map $exp_p : T_p(S) \to S$ is surjective. [6]

Indian Statistical Institute, Bangalore
B. Math (II) Second Semester 2018-19
Backpaper Examination : Statistics (II)

Date: 13-06-2019 Maximum Score 100 Duration: 3 Hours

1. Let $X_1, X_2, \cdots, X_n$ be a random sample from $Poisson(\lambda)$, $\lambda > 0$. Find *method of moments estimator* as well as *maximum likelihood estimator* for λ. What happens if $\sum_{i=1}^{n} X_i = 0$?

[5 + 7 + 2 = 14]

2. Suppose there are 12 reservation counters for state run buses in the city of Bangalore. A counter is open on any particular day with probability θ, $0 < \theta < 1$. The counters function independently of each other. For reasons of proximity and convenience, Shilpak uses either counter 1 or counter 2. Shilpak is interested in knowing *i)* $\tau_1(\theta)$, the probability that either counter 1 is open or counter 2 is open on any given day and *ii)* $\tau_2(\theta)$, the probability that exactly one of the counters 1 and 2 is open on a given day. Let $X_i = 1$ if the *ith* counter is open and $X_i = 0$ if the *ith* counter is closed on a given day, $1 \leq i \leq 12$. Let $X_1, X_2, \cdots, X_{12}$ be the random sample taken on some day indicating whether various of the reservation counters are open or not.

   (a) Find $\tau_1(\theta)$ and $\tau_2(\theta)$.

   (b) Show that $T = \sum_{i=1}^{12} X_i$ is a minimal sufficient statistic for θ.

   (c) Is $T = \sum_{i=1}^{12} X_i$ complete as well? Substantiate.

   (d) Find Fisher information $I(\theta)$ contained in the sample $X_1, X_2, \cdots, X_{12}$ about θ.

   (e) Find an unbiased estimator for $\tau_2(\theta)$. Hence or otherwise obtain *UMVUE* for $\tau_2(\theta)$.

[4 + 4 + 5 + 5 + 8 = 26]

3. Suppose that an electronic system contains n similar components which function independently of each other and which are connected in series, so that the system fails as soon as one of the components fails. Suppose $X_1, X_2, \cdots, X_n$ denote the lifetimes of the n components. Suppose also that the lifetime of each component, measured in hours, has *exponential distribution* with *pdf* $\frac{1}{\lambda}e^{-\frac{x}{\lambda}}I_{(0,\infty)}(x)$, $\lambda > 0$. The system user has reasons to believe that λ has a *prior distribution* given by $Gamma(a, b)$, $a > 0$, $b > 0$ known.

   (a) Find the distribution of Y, the lifetime of the system. Determine $E(Y)$ $(= \theta$ say), the expected lifetime of the system.

   (b) Obtain *posterior distribution* of θ given the observation Y. Obtain mean and variance of the *posterior distribution* of θ.

   (c) Suggest *Bayes estimator* for θ.

[(6 + 3) + 10 + 3 = 22]

[PTO]

4. Let $X_1, X_2, \cdots, X_n$ be the random sample from $N(\theta, \sigma^2)$, $\theta \in I\!R$ and $\sigma^2 > 0$ are both unknown. Consider the testing problem

$$H_0 : \theta = \theta_0 \; versus \; H_1 : \theta \neq \theta_0.$$

where $\theta_0 \in I\!R$ is a specified value.

(a) Derive *size* α *likelihood ratio test.*

(b) Find p-value.

(c) Find 90% *confidence interval for* θ.

[11 + 3 + 6 = 20]

5. An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ square fluid ounces. If σ^2, the variance of fill volume, exceeds 0.01 square fluid ounces, an unacceptable proportion of bottles will be under or overfilled.

(a) Is there evidence in the sample data to suggest that the manufacturer has a problem with under and overfilled bottles? Use $\alpha = 0.05$.

(b) Report the $p - value$.

(c) Obtain 90% one sided confidence interval for σ^2.

[12 + 4 + 6 = 22]

Indian Statistical Institute
Back paper Examinations: 2018-19

Course Name: M. Math, 1st year
Subject Name : Functional Analysis
Maximum Marks: 100, Duration: Two hours

Answer all questions. There is no part marking.

1. Suppose A, B are closed subspaces of a normed linear space N. Prove or disprove: $A+B$ is a closed subspace.

2. Suppose (X, μ) is a measure space and $k \in L^2(X \times X, \mu \otimes \mu)$ and let K be the integral operator with kernel k. Is K a compact operator from $L^2(X, \mu)$ to $L^2(X, \mu)$?

3. For $z \in \mathbb{C}$ such that $|z| = 1$, define $\phi_z : L^1(S^1) \to L^1(S^1)$ by $\phi_z(f)(z') = f(z^{-1}z')$. Suppose W is a closed subspace of $L^1(S^1)$. Is the closed subspace generated by $\{\phi_z(W) : |z| = 1\}$ always a proper subspace of $L^1(S^1)$?

4. Suppose X, Y are Banach spaces and $A \in K(X, Y)$. Suppose $\{x_n\}_n$ is a sequence in X which converges to x weakly. Does the sequence $A(x_n)$ converge to Ax in norm?

5. Suppose H is a Hilbert space and T be a self-adjoint operator on H. Prove that $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$.

6. Suppose X, Y are Banach spaces and $T : X \to Y$ be a linear map such that $x_n \to x$ in norm implies that $T(x_n)$ converges to $T(x)$ weakly. Is T bounded?

7. Suppose $\{f_n\}_n$ be a sequence of positive functions on S^1 such that $\|f_n\|_{L^1} = 1$ for all n and for all $y \in (0, \pi)$, $\lim_{n\to\infty} \int_{|\theta|\geq y} |f_n(x)| = 0$. Suppose $1 < p < \infty$. Prove or disprove:

   for all $g \in L^p(S^1)$, $g * f_n \to g$ in $L^p(S^1)$.

8. Answer the following questions:

   (a) Let H be a Hilbert space. Is the map $B(H) \to B(H), T \mapsto T^*$, continuous in the strong operator topology?

   (b) Suppose T is a compact normal operator on a Hilbert space. Prove that the spectrum of T consists of only the eigenvalues.

9. Suppose f is a function on $\mathbb{R}^n$ which is C^∞ and $g \in C_c(\mathbb{R}^n)$. Is the function $f * g$ C^∞?

10. Let H be a Hilbert space. Prove that the closed unit ball of $B(H)$ is compact in the weak operator topology.

8/7/19

Indian Statistical Institute
Back paper Examination: 2018-19

Course Name: M. Math, 1^{st} year
Subject Name : Topology II
Maximum Marks: 100, Duration: Three hours

- Answer as many questions as you can.
- Maximum marks is 100.
- You may use any results proved in class. Any other results (including those in homework problem sets) require proof.

1. a) Prove that with respect to the natural inclusion of $\mathbb{R}P^i \subset \mathbb{R}P^n$, $\mathbb{R}P^n - \mathbb{R}P^i \simeq \mathbb{R}P^{n-i-1}$. 4
   b) Compute $H_*(\mathbb{R}P^n, \mathbb{R}P^n - \mathbb{R}P^i; \mathbb{Z}/2)$. 4

2. Write the group C_2 as $\{1, \sigma\}$ such that $\sigma^2 = 1$. Consider the C_2 action on $\mathbb{C}P^2 \times S^2$ given by $\sigma([z_0, z_1, z_2], x) = ([\bar{z_0}, \bar{z_1}, \bar{z_2}], -x)$. Define P as the space $\mathbb{C}P^2 \times S^2/C_2$.
   a) Show that P is path-connected. 2
   b) Prove that the quotient map $\mathbb{C}P^2 \times S^2 \to P$ is a covering space. 5
   c) Compute $\pi_1(P, p)$ for any point p. 3

3. Let $L_{2n+1}(k)$ be the Lens space S^{2n+1}/C_k.
   a) Prove that the quotient map $S^{2n+1} \to L_{2n+1}(k)$ is a covering space. 5
   b) Calculate $\chi(L_{2n+1}(k))$. 4
   c) Calculate $\pi_1(\mathbb{R}P^n \times L_{2n+1}(k))$. 4
   d) Calculate all the possible covering spaces of $\mathbb{R}P^n \times L_{2n+1}(k)$ in the case k is odd. 6

4. Let $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$ be the usual inclusion. Let X be the space $\mathbb{R}P^n \cup_{f:\mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}} \mathbb{R}P^n$ with $f =$ the identity map. Compute the homology groups of X. 8

5. Let $X = S(\mathbb{C}P^2)$ be the suspension of $\mathbb{C}P^2$. Compute the homology groups $H_*(X, X - p)$ for points $p \in X$. 10

6. Let f be a map $S^n \to S^k$, $n > k$.
   a) Calculate $H_*(C(f))$ where $C(f)$ is the mapping cone of f. 4
   b) Calculate $H_*(M(f))$ where $M(f)$ is the mapping cylinder of f. 3
   c) Consider the quotient map q from $M(f)$ to $C(f)$. Calculate the effect of q_* on homology groups. 3

7. a) Suppose X is an n-dimensional CW complex with exactly one n-cell. We say that the top cell e^n splits off from X if the attaching map $\partial e^n \to X^{(n-1)}$ is null homotopic. Show that in this case $X \simeq X^{(n-1)} \vee S^n$. 5
   b) Consider the CW complex structure on $\mathbb{R}P^{2n+1}$ with one i-cell for every $0 \leq i \leq 2n+1$. Show that the top cell does not split off from $\mathbb{R}P^{2n+1}$. 5

8. For $p, q \in \mathbb{Z}$ such that $(p, q) \neq (0, 0)$, let $C_{p,q}$ be the circle on the Torus $T \cong \mathbb{R}^2/\mathbb{Z}^2$ which is the image of the line $px = qy$ under the quotient map $\mathbb{R}^2 \to T$.
   a) Compute the image of $\pi_1(C_{p,q})$ in $\pi_1(T)$. 5
   b) Compute $\tilde{H}_*(T/C_{p,q})$. 5

9. Recall that $M(\mathbb{Z}/n, k) = \text{Cone}(f_n)$ where $f_n : S^k \to S^k$ is a map of degree n.
   a) Show that $\mathbb{R}P^2 \simeq M(\mathbb{Z}/2, 1)$. 4
   b) Compute $H_*(M(\mathbb{Z}/2, 5); \mathbb{Z})$ and $H_*(M(\mathbb{Z}/2, 5); \mathbb{Z}/2)$. 3
   c) Consider the composite $S^5 \xrightarrow{f} S^5 \xrightarrow{g} S^5$ where f, g are maps of degree 2. Show that we obtain a map ψ as the composite
   $$M(\mathbb{Z}/4, 5) \simeq C(g \circ f) \to C(g) \simeq M(\mathbb{Z}/2, 5)$$
   (i.e, verify the existence of the arrows and the homotopy equivalences) 5
   d) Compute the effect of ψ_* on homology groups. 3

Indian Statistical Institute, Kolkata
M. Math Ist year
Academic year 2018-2019
Backpaper Examination
Second semester
Course: Complex Analysis
3 hours

- *Answer as many questions as you can.*
- *Maximum marks is 100.*

1. Let T be a Moebius transformation mapping the unit disk $\mathbb{D}$ onto itself such that T has no fixed points in $\mathbb{D}$. Show that $|T^n(z)| \to 1$ as $n \to \infty$ for all $z \in \mathbb{D}$. (12 marks)

2. Suppose f is holomorphic on the unit disk $\mathbb{D}$, and $|f(z)| \to M$ as $|z| \to 1$. Show that $|f^{(n)}(0)| \leq Mn!$ for all n. (12 marks)

3. Let $n \geq 1$ and let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + e^{z-(n+2)}$ be an entire function, where $|a_j| \leq 1$ for all j. Show that f has n zeroes (counted with multiplicity) in the disc of radius $(n+1)$ around the origin. (10 marks)

4. Let $D \subset \mathbb{C}^*$ be a domain such that there is a piecewise C^1 closed curve γ in D such that $n(\gamma, 0) = 1$. Show that there is no holomorphic function f on D such that $f(z)^2 = z$ for all z in D. (10 marks)

5. Let f_1, f_2 be one-to-one holomorphic functions on the unit disk $\mathbb{D}$ such that $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$. Show that $|f_1'(0)| \leq |f_2'(0)|$ with equality if and only if $f_1(\mathbb{D}) = f_2(\mathbb{D})$. (6+6 = 12 marks)

6. Let $R_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a sequence of rational functions converging uniformly on $\hat{\mathbb{C}}$ (with respect to the chordal metric) to a nonconstant function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Show that f is a rational function and the degree

of R_n is equal to the degree of f for all n large enough (here the degree of a rational function $P(z)/Q(z)$ is the maximum of the degrees of the polynomials P, Q). (7+9 = 16 marks)

7. Let g be a nonconstant analytic function on a domain D and let $\mathcal{F}$ be a family of holomorphic functions on $g(D)$. Show that $\mathcal{F}$ is normal on $g(D)$ if and only if the family $\tilde{\mathcal{F}} = \{f \circ g : f \in \mathcal{F}\}$ is normal on D. (12 marks)

8. Let $\lambda > 1$, let $\mathbb{H}$ denote the upper half-plane, and let $T : \mathbb{H} \to \mathbb{H}$ be given by $T(z) = \lambda z$.

   (i) Find a conformal map from the domain $\{z \in \mathbb{H} : 1 < |z| < \lambda\}$ to the domain $\{0 < \Re z < 1, 0 < \Im z < \pi/\log\lambda\}$.

   (ii) Show that there is $R > 1$ and a holomorphic covering map $p : \mathbb{H} \to \{1 < |w| < R\}$ such that the group of deck transformations of the covering p is given by the cyclic group generated by T. (6+10 = 16 marks)

Indian Statistical Institute
Backpaper Examination: 2018-2019
Programme: Master of Mathematics
Course: **Algebra II**

Maximum Marks: 100
Duration: 3 Hours

Date: 10.07.2019

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1. Suppose G is a group and H, K are normal subgroups of G. It is given that both H and K are solvable. Show that HK is solvable. (10)

2. Let G be a finite solvable group and let

$$\{1\} \trianglelefteq H_n \trianglelefteq H_{n-1} \cdots \trianglelefteq H_1 = G$$

be its composition series. Show that each factor H_i/H_{i+1} is a cyclic group of prime order. (8)

3. Let L/K be an extension of fields such that $[L : K] = q$, a prime. Let $f \in K[x]$ be an irreducible polynomial of degree p, a prime. If f is reducible in $L[x]$ then show that $p = q$. (10)

4. Suppose L/K is an abelian extension of fields and $\alpha \in L$ is a root of a polynomial $f \in F[x]$ which is irreducible over F. Is it necessarily true that f splits in $F(\alpha)$? Explain. (8)

5. Let $K = \mathbb{Q}\left(\cos\frac{2\pi}{37}\right)$. Determine if the extension $K/\mathbb{Q}$ Galois. Determine the group $Gal(K/\mathbb{Q})$. (4+8=12)

6. Determine $Gal\left(\mathbb{Q}(2^{1/11}, \zeta_7)/\mathbb{Q}(2^{1/11})\right)$. Here ζ_n denotes a primitive n-th root of unity in $\mathbb{C}$. (12)

7. Suppose $F \subset K \subset L$ are fields with L/F Galois. Let

$$H = \{\sigma \in Gal(L/F) : \sigma(K) = K\}.$$

Show that H is the normalizer of $Gal(L/K)$ in $Gal(L/F)$. (12)

8. Suppose $K/\mathbb{F}_q$ is a finite extension, where $\mathbb{F}_q$ denotes a finite field of order q. Show that (a) $K/\mathbb{F}_q$ is a Galois extension and (b) $Gal(K/\mathbb{F}_q)$ is cyclic. (c) Determine the Galois group of $x^{30} - 1$ over $\mathbb{F}_2$ (3+5+10=18)

9. Suppose F is a field of characteristic p and let K/F be an extension. Let $\alpha \in K$ be algebraic over F. Show that there is an integer $m \geq 0$ such that α^m is separable over F. (10)

INDIAN STATISTICAL INSTITUTE, KOLKATA
BACK PAPER, SECOND SEMESTER 2018-2019
Differential Geometry I, M. Math I year
Total Marks -100, Time : 3 hours , Date : dd. mm. yyyy   12.07.20

Consider the parametrized curve $\alpha(t) = (cos(3t), sin(3t), 4t)$.

(a) Find the speed of α. [5]

(b) Find a unit speed reparametrization of α. [5]

(c) Compute the curvature. [5]

(d) Compute all three vectors of the Frenet frame (t, n, b). [5]

(e) Compute torsion τ. [5]

(f) Give the equation for a surface on which this curve is a geodesic. [10]

Let S be a regular surface $p \in S$, $x(u, v) : U \to S$ parametrization around p with $N = \frac{x_u \times x_v}{|x_u \times x_v|}$. Let E, F, G (resp. e, f, g) be the coefficients of the first fundamental form (resp. coefficients of the second fundamental form). Assume that $F = 0$. Prove that $x_{uv} = \frac{E_v}{2E} x_u + \frac{G_u}{2G} x_v + f.N$. [7]

Compute the first fundamental form of the following parametrized surfaces

(a) $x(u, v) := (u - v, u + v, u^2 + v^2)$.[4]

(b) $x(u, v) = (u, v, u^2 + v^2)$. [4]

Show that the mean curvature H at $p \in S$ for regular orientable surface S is given by $H = \frac{1}{\pi} \int_0^\pi k_n(\theta) d\theta$. Here $k_n(\theta)$ is the normla curvature at p along a direction making an angle θ with a fixed direction. [10]

Show that there exists no surface S such that $E = G = 1$, $F = 0$, $e = 1, g = -1, f = 0$. [10]

Show that if all geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere. [10]

Prove that on every compact regular surface S, there is at least one point of S with positive Gaussian curvature. A surface is called minimal if its mean curvature H is identically zero. Show that there does not exist a compact minimal surface. [5+5]

Let S be a regular compact connected surface with Gaussian curvature $K > 0$ and with constant mean curvature H. Show that S is a sphere. [10]