INDIAN STATISTICAL INSTITUTE

Semestral Examination: 2019-20

Course Name: M. Tech. I Year

Subject Name: Introduction to Programming

Date: 26/11/2019 **Maximum Marks: 100** **Duration: 3 hours**

Note: Answer all questions

1. The monkey-and-bananas problem is faced by a monkey in a laboratory with some bananas hanging out of reach from the ceiling. A box is available that will enable the monkey to reach the bananas if he climbs on it. Initially, the monkey is at A, the bananas at B, and the box at C. The monkey and the box have a height ***Low***, but if the monkey climbs on to the box he will have height ***High***, the same as the bananas. The actions available to the monkey include ***Go*** from one place to another, ***Push*** an object from one place to another, ***ClimbUp*** onto or ***ClimbDown*** from an object, and ***Grasp*** or ***Ungrasp*** an object. The result of a ***Grasp*** is that the monkey holds the object if the monkey and object are in the same place at the same height.

   a. Write down the initial state description.

   b. Write the six action schemas.

   c. Suppose the monkey wants to fool the scientists, who are off to tea, by grabbing the bananas, but leaving the box in its original place. Write this as a general goal (i.e., not assuming that the box is necessarily at C).

   20

2. Write a C function

   a. To find the size in byte(s) of a given text file without reading the content of the file.

   b. mycalloc(int iN, int iSize) which returns a pointer to iN objects of size iSize. with storage initialized to zero. If storage is not available it will return NULL. You are not allowed to use the calloc( ) function.

   c. Which takes variable number of integer parameter(s) and returns the maximum among them.

   20

3. What is Polymorphism in Object Oriented Programming? Describe each type of them.

   20

4. Design and implement myatoi() C function which takes at least a string (expected to represent integer number) and converts to its equivalent integer. If no valid conversion takes place, the function should have some mechanism to report to the calling function.

   20

5. What is a functional programming? Explain Recursion, Lazy Evaluation and Referential Transparency with examples.

   20

Endterm
MTech CS Discrete Mathematics, 2019

25 November, 2018

Time: 3 hours, Maximum Marks: 100
There are two sections: Section A and Section B. Answer as many questions as possible but the maximum possible marks one can obtain from Section A is 40 and the maximum marks obtained from Section B is 60.
Your answers should be well-written and you should explain your arguments properly.
Note that this is a closed book exam.

Part A

1. (5 Marks) If G is graph with even number of vertices and each vertex has even degree (that is, even number of neighbors) then G has even number of edges. Either prove the statement or disprove it by demonstrating a counter example.

2. (10 Marks) Prove: if the graph G has no paths of length k then G is k colorable.

3. (5+10 Marks) Consider an $n \times m$ rectangular grid, with the co-ordinates of the corners being $(0,0)$, $(n,0)$, (n,m) and $(0,m)$. How many paths are there along the rectangular grid from $(0,0)$ to (n,m) such that

   (a) The paths are the shortest among all paths from $(0,0)$ to (n,m).

   (b) Every horizontal move of unit length is followed by at least one vertical move and the paths are the shortest.

4. (5 Marks) Out of n candidates, an association elects a president, two vice presidents, and a treasurer. Count the number of possible outcomes of the election.

5. (10 Marks) How many ways can you color n chairs in a circle with three colors (red, blue and white) such that no two consecutive chairs have the same color? (Hint: Obtain a recurrence relation and then try to solve it).

6. (5 Marks) Prove or disprove: in a directed graph the relation "vertex u is related to vertex v if there is a directed path from u to v **and** there is a directed path from v to u" is an equivalence relation.

Part B

1. (10 marks) Prove that a planar graph (with $|V| \geq 4$) has at least 4 vertices of degree ≤ 5.

2. (3 + 7 Marks) If A_G is the adjacency matrix of a simple graph (with diagonal entries 0) then what is the

   (a) Sum of all the entries in A_G^3.

   (b) If G is bipartite then prove that if $(i,j)th$ entry of A_G^4 is non-zero then the $(i,j)th$ entry for A_G^5 must be 0.

3. (10 Marks) Prove: every tournament on 2^{k-1} vertices contains a subtournament on k vertices which is a Directed Acyclic Graph (DAG). (A tournament is a directed complete graph: every pair of vertices is directed in exactly one direction.)

4. (10 Marks) Girth of a undirected graph is the size of the smallest cycle in the graph. That is, if a graph has girth k then the graph has no cycles of length $k-1$. If G be a d-regular undirected graph with girth 5 then prove that the number of vertices in G is more than d^2+1.

5. (10 + 10 Marks) For a simple graph G an independent set is a subset S of vertices such that no two vertices in the set S are adjacent. Prove that

   (a) If a graph G on n vertices has a matching of size k then G cannot have an independent set of size $> (n-k)$.

   (b) If the graph G on n vertices has a maximal matching of size k then G has an independent set of size $\geq (n-2k)$.

6. (10 Marks) A tree where every vertex has degree 3 or 1 (except at most one vertex. called the root, that has degree 2) is called a 2-ary tree. Prove that a 2-ary tree on n vertices has at least $\lfloor n/2 \rfloor + 1$ number of leaves. (Hint: induction pivoting at the root. Also consider the case when no vertex has degree 2).

7. (10 Marks) Let G be a bipartite graph with bipartitions A and B such that $|A| = |B| = n$. If the minimum degree of G is $n/2$ then prove that G has a perfect matching.

Endterm - back paper
MTech CS Discrete Mathematics, 2019

15 January, 2020
~~15 November,~~ 2019

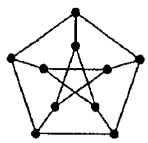
Time: 3 hours, Maximum Marks: 100
Answer as many questions as possible but the maximum possible marks one can obtain is 100.
Your answers should be well-written and you should explain your arguments properly.
Note that this is a closed book exam.

1. (10 marks) Let $a_n, b_n \to \infty$. Show if $a_n = \Theta(b_n)$ then $\log a_n \sim \log b_n$.

2. (10 marks) Prove that a tournament always has a Hamiltonian path.

3. (10 marks) In how many ways can five distinct books be tied up in at most three bundles. (Here the order of books in the bundle does not matter and the bundles are not labeled. So the bundles are not distinguishable from one another.)

4. (10 marks) Prove or disprove: The following graph is planar



5. (10 marks) If G is a 2-vertex-connected graph and u, v are two vertices in G then there is a cycle in G passing through both u and v.

6. (10 marks) Let G be a graph with minimum degree 2. Show that there exists a connected graph with same degree sequence.

7. (10 marks) Let G be an arbitrary planar graph with v vertices, e edges, f regions, and m connected components. Prove that $v - e + f = m - 1$.

8. (10 marks) In a Discrete Math class there are 35 Mtech students and 35 JRF students. The instructor has decided that every Mtech student should team up with another JRF student and each team should do a different project. Of course every student should be in exactly one

team. To form the teams each student has been asked to present a set of exactly 4 students with whom he/she would want to team up. Thus every Mtech student must give a list of 4 JRF student of his/her choice and similarly every JRF student should give a list Mtech students of his/her choice. But the students have been asked to ensure that if student A is in the list of student B then student B must also be in the list of student A, so that it is mutually agreeable. Prove that whatever the submitted lists of the students may be the instuctor can surely form 35 teams such that everybody is happy. That is, for every team the Mtech student is in the list submitted by the JRF student and the JRF student is in the list submitted by the Mtech student.

9. (10 marks) Given an undirected graph the distance between any two vertices is the smallest length of the path that connects the two vertices. *Diameter* of an undirected graph is the maximum distance among all the pairs of vertices. So if the diameter of a graph is k that means any vertex can be reached from any other vertex by a path of length at most k.

   Among all the graphs of diameter 2 with n vertices, which one has the largest independent set? How large is it? Give proofs.

10. (10 marks) The *Lucas Sequence* $1, 3, 4, 7, 11, 18, 29, \ldots$ is defined by $a_1 = 1, a_2 = 3, a_n = a_{n-1} + a_{n-2}$. Give a closed form expression for the nth term of the Lucas Sequence.

11. (10 Marks) For a graph G, the chromatic number of G, denoted as $\chi(G)$, is the minimum number of colors required for coloring the vertices of G such that no two adjacent vertices have the same color.

    The independent number of G, denoted as $\alpha(G)$, is the maximum number of vertices of G that form an independent set, that is, no two vertices in the independent set are adjacent.

    Prove that: $\alpha(G)\chi(G) \geq n$.

INDIAN STATISTICAL INSTITUTE

Semestral Examination

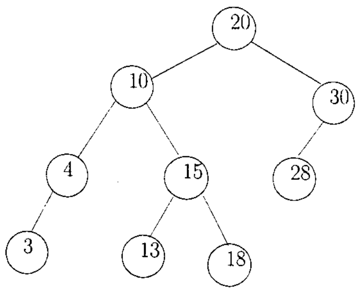
M. Tech (CS) - I Year (Semester - I)

Data and file structures

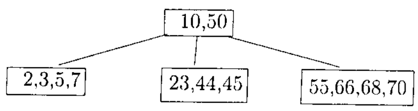
Date : 21.11.19 Maximum Marks : 100 Duration : 3:30 Hours

Note : You may answer any part of any question, but maximum you can score is 100.

1. What is the resultant AVL tree after inserting key 17 in the following AVL tree? [5]



2. Following B-tree of order 5 (at most 5 children) is given. Insert 17, 6, 21, 67 in this order following the algorithm for B-tree insertion. [6]



3. Prove or disprove the following statements. [4+6=10]

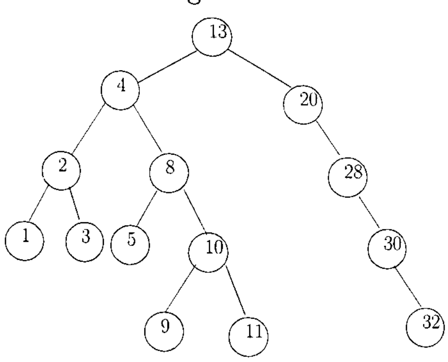
   (a) All Red-Black trees satisfy the AVL tree property.

   (b) It is possible to color nodes of any AVL tree using colors red and black to make the same tree a Red-Black tree.

4. Draw the Cartesian tree for the following list.

   93, 84, 33, 64, 62, 83, 27 [5]

5. Consider the following splay tree. Perform a delete operation for key 3 using the splay tree deletion algorithm. [6]



6. Let A and B be two Red-Black trees. All key values of A are less than any key value of B. The height of A is m and the height of B is n. Write an efficient algorithm to merge these two trees. What is the time complexity of your algorithm? [10+3=13]

7. Consider the following problem of *Range Minimum Query*.

   Given a fixed array A of size n and two indices $i \leq j$, what is the smallest element out of $A[i], A[i+1], \ldots, A[j]$?

   If you are allowed to preprocess the array within limit of $O(n \log n)$ time complexity, propose an algorithm such that query can be solved in $O(1)$ time. Clearly mention the data structure you are using for this algorithm. [10+5=15]

8. You need to maintain an undirected forest G so that edges may be inserted, deleted and connectivity queries may be answered efficiently. Describe a data structure and algorithm so that insert and delete can be done in $O(1)$ time and connectivity queries can be done in $O(n)$ time.

   [15]

9. Propose a heap data structure where both minimum and maximum element can be reported in $O(1)$ time. Moreover, both insertion of a new element and deletion of maximum/minimum element can also be done in $O(\log n)$ time where n is the number of elements currently in the heap. Write a brief algorithm for such insertion and deletion algorithm. [5+5+5=15]

10. Describe the binomial heap data structure. Write an algorithm for uniting two binomial heap to form a new binomial heap in $O(\log n)$ time, where n is the number of elements in the newly formed binomial heap. [6+12=18]

11. Suppose that we insert n keys into a hash table of size m using open addressing and uniform hashing. If the load factor $\alpha = n/m < 1$, prove that the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$. [10]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination

M. Tech (CS) - I Year (Semester - I)

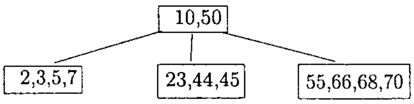
Data and file structures

Date : 15|1|20 Maximum Marks : 100 Duration : 3:30 Hours

Note : You may answer any part of any question, but maximum you can score is 100.

1. For sorting n numbers, only two stack data structure is not sufficient where stack data structure supports only push, pop and Is-Stack-empty. [9]

2. Following B-tree of order 5 (at most 5 children) is given. Insert 17, 6, 21, 67 in this order following the algorithm for B-tree insertion. [6]



3. Prove or disprove the following statements. [4+6=10]

   (a) All Red-Black trees satisfy the AVL tree property.

   (b) It is possible to color nodes of any AVL tree using colors red and black to make the same tree a Red-Black tree.

4. For sufficiently large n (say greater than 1 billion), and for any k (less than $\sqrt{n}$), largest, 2nd largest, upto kth largest elements of an array of n elements can be computed in $O(n)$ time. [9]

6. Let A and B be two Red-Black trees. All key values of A are less than any key value of B. The height of A is m and the height of B is n. Write an efficient algorithm to merge these two trees. What is the time complexity of your algorithm? [10+3=13]

7. Consider the following problem of *Range Minimum Query.*

   Given a fixed array A of size n and two indices $i \leq j$, what is the smallest element out of $A[i]$, $A[i+1], \ldots, A[j]$?

   If you are allowed to preprocess the array within limit of $O(n \log n)$ time complexity, propose an algorithm such that query can be solved in $O(1)$ time. Clearly mention the data structure you are using for this algorithm. [10+5=15]

8. You need to maintain an undirected forest G so that edges may be inserted, deleted and connectivity queries may be answered efficiently. Describe a data structure and algorithm so that insert and delete can be done in $O(1)$ time and connectivity queries can be done in $O(n)$ time.

[15]

9. Describe the binomial heap data structure. Write an algorithm for uniting two binomial heap to form a new binomial heap in $O(\log n)$ time, where n is the number of elements in the newly formed binomial heap. [6+12=18]

10. Suppose that we insert n keys into a hash table of size m using open addressing and uniform hashing. Let $p(n,m)$ be the probability that no collisions occur. Show that $p(n,m) \leq e^{-n(n-1)/2m}$. Argue that when n exceeds $\sqrt{m}$, the probability of avoiding collisions goes rapidly to zero. [Hint: $e^x = 1 + x + x^2/2 + \ldots$.] [12]

Computer Organization
M.Tech.CS-I (2019-20)
Semester Examination
Full Marks: 100

Time: 3 hours

Date: November 29, 2019

Answer Question No. 1, and any _four_ from the following.

1. Check if the following statements are True or False. Justify your answer (**any four**).

   a) The Single-Cycle datapath design may have a single memory to store both instruction and data.

   b) Pipelined control for datapath improves instruction throughput rather than the individual instruction execution time or latency.

   c) '*Write back*' policy is more efficient than the '*write-through*' policy to handle write-miss in cache memory.

   d) A logical *right-shift instruction* (`srl` in MIPS) always can be used to replace a *signed integer division by a power of 2*.

   e) A 64-bit *Carry-Look-Ahead* (CLA) adder is three times faster than an 8-bit *Ripple-Carry-Adder*. Assume that each gate in the path has equal delay irrespective of the type.

   [5 × 4 = 20]

2. a) Let us consider a benchmark program that takes 100 sec to run on a given machine. Suppose the machine is now enhanced to execute floating point instructions five times faster. What should be the percentage of the floating point instructions in the benchmark to yield an overall speed up of 2 on the enhanced machine?

   b) A 1-bit parity scheme is an error detection code that can detect an error in a single bit of data. Given an input data with n-bit, a 1-bit *odd parity* generator will output '1', when the number of '1's in the n-bit data is even, otherwise it will output '0', so that the total number of '1's in the (n+1) bit data including the parity bit is always odd.
   Show the truth table for generating a single odd-parity bit for a four bit input data. Implement the function using OR, AND and NOT gates only.

   [6 + (4 + 10) = 20]

3. a) Show the flowchart for binary floating point multiplication, mentioning the roles of rounding and normalization.

   b) Using the IEEE 754 (single precision) floating point format, write down the bit pattern for (- ¼). What do you get if you add (- ¼) four times. Check if it is the same with (- ¼) x 4.

   [5+(4+6+5) = 20]

P.T.O

4. a) Compare the pipelined control design with the Multi-Cycle control design.

b) Show the block diagram of a five-stage instruction pipeline, mentioning the operations performed in each stage.

c) Given a sequence of MIPS instructions as follows:

```
lw    $3, 100 ($1)
or    $4, $3, $5
and   $5, $3, $6
sub   $8, $4, $3
```

(i) Show the sequence of instructions in various stages of the pipeline in successive cycles, and hence identify the cases of data hazards.

(ii) What is the minimum number of stalls to handle the data hazards if forwarding is allowed? Justify your answer showing the forwarded data paths.

[5+4+(6+(3+2) = 20]

5. Let us consider a direct-mapped cache design with 32-bit address (for byte addressing), using the address bits to access the cache in the following way:

Tag	Index	Offset	Don't care
31-10	9 - 5	4 - 2	1- 0

(a) Show a schematic diagram for the implementation of the cache.

(b) What is the cache block size, and how many blocks does the cache store?

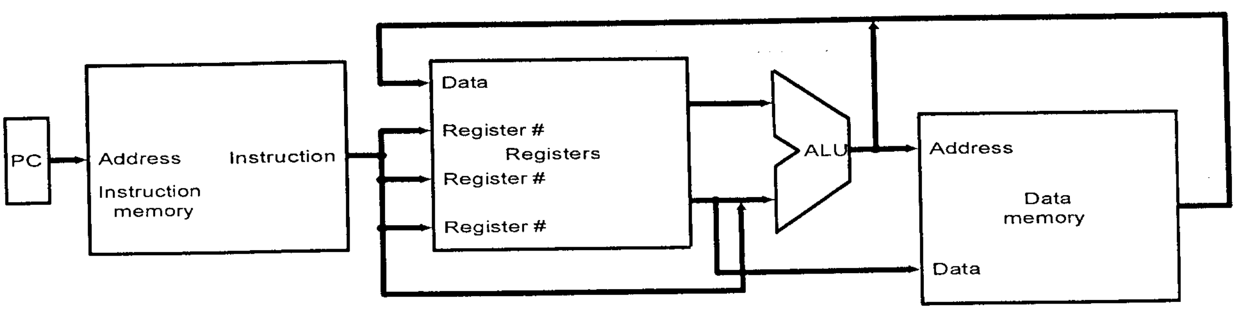
(c) What is the ratio between the data storage bits over the total bits required for such a cache implementation?

(d) Say, initially the cache is empty, and the following cache references are recorded: 0, 4, 16, 132, 232, 160, 1024, 30, 1156.
What is the hit ratio and the number of block replacements? Does a two-way set-associative cache with same capacity can improve the number of block replacements?

[4+2+4+(5+5) = 20]

6. a) Given a subset of MIPS instructions S = { `lw`, `sw`, `Add`, `beq`}, show what do the different groups of bits (0-31) signify for each 32-bit instruction.

b) In the Fig. given below, add the necessary hardware modules (add / shift / multiplexor etc.) with appropriate interconnections, to execute the instructions in S, assuming a single-cycle control design. Show the datapath for each in a separate Fig.



P.T.O

c) The module *coExam* given below is executed with the module *testbench* using a Verilog simulator. What output will you see?

Note that in the module coExam, **and, nand nor, not** denote concurrent module instantiations, with the first port as the gate output.

Verilog file coExam.v

```
// Module definition
module coExam(A, B, C, D, Y);

// port declaration
input A, B, C, D;
output Y;

// internal wire declaration
wire e, f, g;
// gate instantiations
and i1(e, A, B);
nand i2(f, C, D);
nor i3(g, e, f);
not i4(Y, g);

endmodule
```

Verilog file testbench.v

```
module testbench;
reg TA, TB, TC, TD;
wire TY;
// module instantiation
coExam inst(TA, TB, TC, TD, TY);
initial // drive test vectors
begin
      TA = 1'b0; TB = 1'b0;
      TC = 1'b0; TD = 1'b0;
  #10 TD = 1'b1;
  #10 TC = 1'b1; TD = 1'b0;
  #10 TD = 1'b1;
  #10 TB = 1'b1;
      TC = 1'b0; TD = 1'b0;
end
initial // monitor output
begin
  // Log based data monitor
  $monitor($time, ": ",
    TA, TB, TC, TD, " : ", TY);
  // Control simulation time
  #100 $finish;
end

endmodule
```

[(4+7+4)+5 = 20]

INDIAN STATISTICAL INSTITUTE

End-Semester Examination (First Semester) : 2019-2020

Course Name : M.TECH. (CS) YEAR I

Subject name : ELEMENTS OF ALGEBRAIC STRUCTURES

Date : November 29, 2019 Maximum Marks : 50 Duration : 3 hours

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Answer any 10 questions. All questions carry equal marks. Notations are used as in the class.

1. Let X be a non-empty set. Consider two sets A and B defined as follows:

   $$A = \{Y : Y \subseteq X\},\ B = \{f \mid f : X \longrightarrow \{0,1\}\}.$$

   Does there exist a bijection between the sets A and B? Justify your answer.

2. Prove that the set $Aut\ G$ of automorphisms of a group G forms a group, the law of composition being composition of functions.

3. Show the following:

   (a) Every subgroup of index 2 is normal.

   (b) A subgroup of index 3 may not be normal.

4. Prove that the center of the product of two groups is the product of their centers.

5. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ over the field $\mathbb{R}$. Show that the functions $x^3, sin\ x, cos\ x$ are linearly independent.

6. Let T be a linear operator on a vector space V over a field F, and let $c \in F$. Let W be the set of all eigenvectors of T with eigenvalue c, together with the 0 vector. Prove the following:

   (a) W is a subspace of V.

   (b) $T(W) \subseteq W$.

7. Let I, J be ideals of a ring R.

   (a) Prove that $I \cap J$ is an ideal.

   (b) Show by example that the set of products $\{xy \mid x \in I, y \in J\}$ need not be an ideal.

8. Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{R}$ be the field of real numbers. Let $a, b \in \mathbb{R}$ be algebraic over $\mathbb{Q}$ of degree m and n, respectively. Suppose m and n are relatively prime. Show that $[\mathbb{Q}(a,b) : \mathbb{Q}] = mn$.

9. Find a splitting field S of the polynomial $x^p - 1$ over $\mathbb{Q}$, where p is a prime number. Find $[S : \mathbb{Q}]$.

10. Is there an integral domain containing exactly 10 elements? Justify your answer.

11. Let A be a $k \times m$ real matrix and let B be an $n \times p$ real matrix. Prove that the map $M \mapsto AMB$ defines a linear transformation from the space $F^{m \times n}$ of $m \times n$ real matrices to the space $F^{k \times p}$.

12. Answer the following:

    (a) Prove that 2 has no multiplicative inverse modulo 6.

    (b) Determine all integers n such that 2 has an inverse modulo n.

INDIAN STATISTICAL INSTITUTE

Back Paper Examination (First Semester) : 2019-2020

Course Name : M.TECH. (CS) YEAR I

Subject name : ELEMENTS OF ALGEBRAIC STRUCTURES

Date : January 16, 2020 Maximum Marks : 100 Duration : 3 hours

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Answer any 8 questions. All questions carry equal marks. 4 marks are reserved for neatness. Notations are used as in the class.

1. (a) Let R_1 and R_2 be two symmetric relations on a non-empty set S. Show that $R_1 \circ R_2$ is symmetric iff $R_1 \circ R_2 = R_2 \circ R_1$.

   (b) Give an example of an equivalence relation on the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ such that the relation induces exactly four equivalence classes.

2. Let $\mathbb{R}$ denote the set of real numbers.

   (a) A binary operation $*$ is defined on $\mathbb{R}$ as follows:
   $$x * y = |x + y| \text{ for all } x, y \in \mathbb{R}.$$
   Is $*$ associative? Justify your answer.

   (b) Let $G = \{(a, b) : a, b \in \mathbb{R}, a \neq 0\}$. Define a binary operation $*$ on G by,
   $$(a, b) * (c, d) = (ac, b + d),$$
   for all $(a, b), (c, d) \in G$. Show that $(G, *)$ forms a group.

3. (a) Find all the generators of the cyclic group $\mathbb{Z}/10\mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers.

   (b) Let $\mathbb{Q}$ denote the set of rational numbers. Is $(\mathbb{Q}, +)$ cyclic? Justify your answer.

   (c) Let G be a non-cyclic group of order p^2, where p is a prime number. Show that the order of each non-identity element of G is p.

4. (a) Let $f : G \to G'$ be a surjective homomorphism of groups. If H is a normal subgroup of G, then show that $f(H)$ is a normal subgroup of G'.

   (b) Let G be a finite group, and let $f : G \to G$ be an automorphism such that for all $a \in G, f(a) = a$ iff $a = e$. Prove that for any $g \in G$, there exists $a \in G$, such that $g = a^{-1}f(a)$.

5. Let $\mathbb{R}$ denote the set of real numbers. Let $v_1 = (0, 1, 1, 0)$, $v_2 = (1, 0, 1, 0)$ and $v_3 = (-1, -2, 0, 0)$ be three vectors in the vector space $\mathbb{R}^4$ over the field $\mathbb{R}$.

   (a) Show that $\{v_1, v_2, v_3\}$ forms a linearly independent set.

   (b) Extend $\{v_1, v_2, v_3\}$ to a basis of $\mathbb{R}^4$.

6. (a) Let F be a field and V be a vector space over F of dimension n. Let r be a natural number such that $0 \leq r \leq n$. Prove that V contains a subspace of dimension r.

   (b) Let $v_1 = (1, 1)$, $v_2 = (1, -1)$ be two vectors in $\mathbb{R}^2$. Let $V = Span(v_1, v_2)$. Do v_1 and v_2 form an orthonormal basis of V? If not, find an orthonormal basis for V.

7. Find all the subrings of the ring of integers $\mathbb{Z}$ under usual addition and multiplication.

8. Let R be a ring with 1.

   (a) Let $r \in R$ be such that r has an inverse in R. Show that r cannot be a zero divisor.

   (b) Consider the polynomial ring $R[x]$ and the ideal $\langle x \rangle$ of $R[x]$ generated by $x \in R[x]$. Show that $R[x]/\langle x \rangle$ is isomorphic to R.

9. (a) Let $\mathbb{Q}$ be the field of rational numbers, and $\mathbb{R}$ be the field of real numbers. Find an element $r \in \mathbb{R}$ such that $\mathbb{Q}(\sqrt{2}, \sqrt[3]{7}) = \mathbb{Q}(r)$.

   (b) Consider the field extension $\mathbb{Q}(\sqrt{3}, \sqrt{7}) \supseteq \mathbb{Q}(\sqrt{3})$. Find a basis for the vector space $\mathbb{Q}(\sqrt{3}, \sqrt{7})$ over $\mathbb{Q}(\sqrt{3})$.

10. Find a splitting field S of the polynomial $x^4 - 10x^2 + 21$ over $\mathbb{Q}$. Find $[S : \mathbb{Q}]$ and a basis of S over $\mathbb{Q}$.

INDIAN STATISTICAL INSTITUTE

End Semestral Examination

M. Tech (CS) – I Year, 2019-2020 (Semester - I)

Probability and Stochastic Processes

Date: 18.11.2019 | Maximum Marks: 100 | Duration: 3.5 Hours

Note: This is a 2-page question paper.

Answer as much as you can but the maximum you can score in Group-A is 40 and the maximum you can score in Group-B is 60.

$E[X]$ and $\mathrm{var}[X]$ denote the expectation and variance of the random variable X, respectively.

Group A

(QA1) (i) State and prove Bayes' Rule.

(ii) Let A, B and C be independent events, with $\Pr(C) > 0$. Prove that A and B are conditionally independent given C.

[(1+4)+5=10]

(QA2) Consider the probability mass function (PMF) of a binomial random variable with parameters n and p. Show that asymptotically, as $n \to \infty$, $p \to 0$, while the product np is fixed at a given value λ, the binomial PMF approaches the PMF of a Poisson random variable with parameter λ. [10]

(QA3) (i) Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\Pr(A_i) > 0$ for all $i = 1, \ldots, n$. Then show that

$$E[X] = \sum_{i=1}^{n} \Pr(A_i)\, E[X \mid A_i].$$

(ii) Show that the expected value of a continuous random variable X satisfies

$$E[X] = \int_0^\infty \Pr(X > x)\, dx - \int_0^\infty \Pr(X < -x)\, dx$$

.

[5+5=10]

(QA4) A branching process starts with a single member. Henceforth, each member generates new members, where the number of new members generated is a Poisson random variable with parameter $\lambda > 0$. Assume that this generating process is independent for each new generation of members.

(i) What is the expected number of members generated after n generations?

(ii) Comment on the behaviour of the expected value calculated vis-a-vis the value of λ and as $n \to \infty$.

[8+2=10]

(QA5) (i) State the weak law and the strong law of large numbers.

(ii) Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables and assume that $E[X_1^4]$ is finite. Under the above assumption, prove the strong law of large numbers.

[(1+1)+8=10]

Group B

(QB1) (i) We toss n coins and each turns up heads with probability p independently of other tosses. There are q rounds of tosses. In the first round, all coins are tossed. In the second round, each coin that shows up heads in the first round, is tossed again. In any subsequent round i, $1 < i \leq q$, each coin that shows up heads in round $i - 1$, is tossed again. Deduce the PMF of the number of heads resulting from the q-th round of toss.

(ii) Let $X_1, \ldots, X_n$ be independent random variables and each X_i is Bernoulli with parameter $p_i > 0$, for all $i = 1, \ldots, n$. Let $X = X_1 + \cdots + X_n$ be their sum and $E[X] = \mu$. Show that the variance of X is maximized if the p_i's are chosen to be all equal to μ/n.

[4+8=12]

(QB2) Let $X_1, \ldots, X_N$ be random variables with the same expectation $E[X]$ and variance $\text{var}(X)$. Assume X_i's and N are positive random variables and they are also independent. Let $T = \sum_{i=1}^{N} X_i$. Show that

- $E[T] = E[N]E[X]$.
- $\text{var}(T) = \text{var}(X)E[N] + (E[X])^2 \, \text{var}(N)$.

[5+7=12]

(QB3) (i) Let the moment generating function of a discrete random variable X be of the form $M_X(s) = \frac{pe^s}{1-(1-p)e^s}$, where $0 < p \leq 1$ is a constant. Find the distribution of X and its expectation.

(ii) Let X and Y be two independent exponential random variables with a common expectation λ. Find the PDF of $Z = X + Y$.

[(5+2)+5=12]

(QB4) Let $X_1, \ldots, X_n$ be independent random variables with $\Pr(X_i = 1) = \Pr(X_i = -1) = \frac{1}{2}$, for all $i = 1, \ldots, n$. Let $X = \sum_{i=1}^{n} X_i$. Then, for any $a > 0$, show that $\Pr(|X| \geq a) \leq 2e^{-a^2/2n}$. [12]

(QB5) Prove that, for $x \geq 0$, as $n \to \infty$.

(i) $$\sum_{k:\left|k-\frac{1}{2}n\right| \leq \frac{1}{2}x\sqrt{n}} \binom{n}{k} \sim 2^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

(ii) $$\sum_{k:|k-n| \leq x\sqrt{n}} \frac{n^k}{k!} \sim e^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

Here $A \sim B$ means A can be approximated by B.

[6+6=12]

[Hints: Can you apply the central limit theorem? Does (i) and (ii) have anything to do with a collection of Binomial and Poisson random variables, respectively?]

(QB6) (i) For a finite Markov chain, show that if one state in a communicating class is transient, then all states are transient in that communicating class.

(ii) In a finite Markov chain, show that (a) at least one state is recurrent; and (b) all recurrent states are positive recurrent.

[4+(4+4)=12]