INDIAN STATISTICAL INSTITUTE

Mid-semester Examination

First semester

B. Stat - Second year 2018-2019

Analysis III

Date: September 3, 2018 — Maximum Marks: 40

Duration: 2 hours

Answer all questions.

You must state clearly any result you use.

(1) Suppose A is a subset of $\mathbb{R}^n$. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$f(x) = \inf_{y \in A} \|x - y\|.$$

Prove that f is a continuous function. Determine the set $f^{-1}(0)$. 7

(2) Consider the series

$$I + A + A^2 + \cdots + A^n + \dots \quad A \in M_n(\mathbb{R})$$

Prove that the series converges if $\|A\|_{op} < 1$. Determine the sum of the series when it converges. 5

(3) (a) Let $\gamma : I \to O(n, \mathbb{R})$ be a differentiable function defined on an open interval I of $\mathbb{R}$ containing 0 such that $\gamma(0) = I_n$. Prove that, $\gamma'(0)$ is a skew-symmetric matrix.

(b) Prove that given a skew symmetric matrix A, there exists a function $\gamma : I \to O(n, \mathbb{R})$ such that $\gamma(0) = I_n$ and $\gamma'(0) = A$. 9

(4) (a) Let $f : M_n(\mathbb{R}) \to \mathbb{R}$ be defined by $f(A) = \text{trace }(AA^T)$ for $A \in M_n(\mathbb{R})$. Compute the derivative map of f at the identity matrix. 7

(5) Let $Q = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$. Consider the function $f : Q \to \mathbb{R}^2$ defined by

$$f(x, y) = (e^{xy}, y/x)), \quad (x, y) \in Q.$$

Compute the Jacobian of f at an arbitrary point of Q. Prove that the image of f is an open subset of $\mathbb{R}^2$. Is f a diffeomorphism onto its image? 9

(6) Use Lagrange's multiplier method to show that $f(x, y, z) = z^2$ has only one local extrema on the surface $x^2 + y^2 - z = 0$. Show that this is a minimum. Why did we not find a maximum of f on the surface? 8

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination, First Semester 2018-2019
Elements of Algebraic Structures, B.Stat II year
Total Marks 30.

1. Let S be a set and $\mathcal{P}(S)$ be the set of subsets of S. Define

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

   (a) (2-points) Show that $(\mathcal{P}(S), \triangle)$ is an abelian group.
   (b) (2-points) Show that there are groups G of infinite order such that each element has finite order.

2. Determine whether the following pairs of groups are isomorphic.
   (a) (2-points) $(\mathbb{R}^+, .)$ and $(\mathbb{R}, +)$.
   (b) (3-points) $(\mathbb{Z}[x], +)$ and $(\mathbb{Q}^+, .)$.
   (c) (1-point) $(\mathcal{P}(S), \triangle)$ and $\mathbb{Z}/32\mathbb{Z}$ where S is a set of cardinality 5.

3. (4+2+2-points) Determine the possible orders for elements in S_6 and A_6. What is the maximum possible order for an element in S_7 ? What is the maximum possible order for an element in A_7.

4. Write down the class equations for the following groups.
   (a) (3-points) A_4.
   (b) (3-points) D_5.

5. Let G be a group such that $|G| = 15$. Let $X_5 := \{x \in G | x^5 = e\}$ and $X_3 := \{x \in G | x^3 = e\}$
   (a) (3-points) Show that $5 || X_5|$ and $3 || X_3|$.
   (b) (4-points) Let n_5 be the number of distinct subgroups H of G such that $|H| = 5$. Show that $|X_5| = 4n_5 + 1$. Conclude that $n_5 = 1$.
   (c) (3-points) Let n_3 be the number of distinct subgroups H of G such that $|H| = 3$. Show that $|X_3| = 2n_3 + 1$. Conclude that $n_3 = 1$.
   (d) (3-points) Show that G is abelian.

INDIAN STATISTICAL INSTITUTE

Mid Semester Examination: 2018-19

B STAT, SECOND YEAR

Statistical Methods - III

Date: 05/09/2018 Maximum Marks: 30 Duration: 2 hours

1. State and prove Cramer-Rao inequality that provides a lower bound to an unbiased estimator of a parametric function $\gamma(\theta)$ for $\theta \in \Theta$. (6)

2. Suppose $X_1, X_2, \ldots, X_n$ be a random sample from
$$f(x) = \frac{1}{2\theta^3} e^{-\frac{x}{\theta}} x^2;\ x > 0.$$
Find MVUE of θ, deriving all intermediate steps and mention clearly any result and/or theorem that you want to use. (8)

3. It takes me around 65 minutes to come to ISI from my home by public bus. Usually I reach ISI by 8 : 30 am and I return home by 10 pm. Traffic is more in the evening than in the morning. To do time management properly, I plan to study the average time required to reach ISI and that for reaching home after day's work and the corresponding delay incurred.

   (a) If I want to compare the time taken to reach ISI to that reaching home, how do you frame this query as a statistical testing problem? Justify your answer.

   (b) After around two months, I observed that although I usually reach by 8 : 30 am, on few occasions I was late. So I record the time randomly for a few days so as to do a statistical test to come to a conclusion. For this, I adopted two schemes.
   Scheme 1: I randomly selected 10 days and observed the number of times I got delayed to reach ISI.
   Scheme 2: I record the exact time of delay for four randomly selected days.

i. For **Scheme 1**, frame this as a problem of statistical testing of hypothesis. Describe an appropriate test based on the data obtained through **Scheme 1**.

ii. It is known that delay to reach ISI follows an exponential distribution with θ. So for **Scheme 2**, frame this as a problem of statistical testing of hypothesis. Describe an appropriate test based on the data obtained through **Scheme 2**.

iii. Now fix the level of significance α as $\min\{P(\text{type I error under Scheme 1}), 0.05\}$. Do you want to prefer the test based on **Scheme 1** or that based on **Scheme 2**? Justify your answer. $(3 + 6 + 6 + 6 = 21)$

[If I reach ISI at 8:30 a.m., this means that I am late for 10 minutes.]

INDIAN STATISTICAL INSTITUTE

First Semester Mid-term Examination: 2018 – 19

B. Stat 2nd Year
Probability theory 3

Date: 6 September, 2018
Maximum Marks: 60 **Duration: 2 hours 30 minutes**

Answer **any four** questions. Each question carries 15 marks. The solution of each question has to be in one place, that is, answering different parts in separate places is not allowed.

1. Suppose that $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3) \ldots$ are i.i.d. random vectors from bivariate normal with mean zero, variance one and correlation $\rho \in (-1, 1)$, that is,
$$\mathrm{E}(X_1) = \mathrm{E}(Y_1) = 0\,,$$
$$\mathrm{Var}(X_1) = \mathrm{Var}(Y_1) = 1\,,$$
and
$$\mathrm{Corr}(X_1, Y_1) = \rho\,.$$
Show that as $n \to \infty$,
$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \rho\right) \Rightarrow Z\,,$$
where Z follows $N(0, \sigma^2)$ for some σ . Calculate σ^2.

2. Let $X_1, \ldots, X_n$ be i.i.d. from standard normal. Fix $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that
$$\sum_{i=1}^{n} \alpha_i^2 = 1\,.$$
Define
$$Y = \sum_{i=1}^{n} \alpha_i X_i\,,$$
$$Z = \sum_{i=1}^{n} X_i^2 - Y^2\,.$$
Show that Y and Z are independent.

3. Consider an i.i.d. sample $X_1, \ldots, X_9$ from standard uniform, and denote their order statistics by $X_{(1)} \leq \ldots \leq X_{(9)}$. Find the density of the sample median $X_{(5)}$.

4. Let $X_1, X_2, \ldots$ be independent random variables, where X_n follows normal with mean 2^{-n} and standard deviation $2^{-n/2}$ for each $n \geq 1$. Show that as $n \to \infty$,

$$\sum_{i=1}^{n} X_i \Rightarrow Y\,,$$

for some random variable Y. Find the distribution of Y.

5. Suppose that $X_1, X_2, \ldots$ are independent random variables such that

$$X_n \Rightarrow Y\,,$$

for some random variable Y. Show that as $n \to \infty$,

$$X_n - X_{n+1} \Rightarrow Y - Z\,,$$

where Z is an independent copy of Y, that is, Z is independent of Y and

$$Z \stackrel{d}{=} Y\,.$$

Indian Statistical Institute

Mid-semester Examination 2018

Course name: **Microeconomics**

Subject name: **Economics**

Date: **7 September 2018**

Maximum marks: **80**

Duration: **2 hours**

1. This question pertains to a situation in which a particular commodity, like rice, is both available at a subsidised rate from a fair price shop (ration shop) and at a higher price from the open market. Suppose a consumer can buy a certain (fixed) quantity of rice at a lower price from the ration shop (that is, there is a ration quota). In addition, he can buy more of rice (assume a uniform quality of rice) from the open market at a higher price. (You may assume that consumer's preferences are represented by standard downward sloping, smooth, convex indifference curves.)

(i) Graphically depict the consumer's equilibrium and briefly describe it. **(6 points)**

(ii) Suppose rice is a normal good. What will happen to the quantity of rice purchased from the open market (over and above the ration quota) in equilibrium if there is a cut in the ration quota? Briefly explain. How will your conclusion change if rice is an inferior good? **(4 + 3 = 7 points)**

(iii) Suppose rice is a normal good. What will happen to the quantity purchased in the open market (over and above the ration quota) if the subsidised price (price at which the ration quota rice could be bought) is increased (but is still lower than the open market price)? How will your conclusion change if rice is an inferior good? Briefly explain. **(4 + 3 = 7 points)**

2. Suppose two schemes of subsidising the consumption of a commodity costs the same to the Government - one scheme gives a subsidy on the per unit price of the commodity and the other considers giving a lump sum grant to the consumer. Under which scheme is the consumption of the commodity higher? Under which scheme is the utility of the consumer higher? **(10 points)**

3. Answer "true" or "false", and give a brief defence for your answer for any **four** of the following statements: **(5 x 4 = 20 points)**

(i) Negative income effect is necessary but not sufficient for Giffen's paradox.

(ii) As $\frac{p_1}{p_2}$ rises, income-consumption curve (ICC) shifts downwards/rightwards.

(iii) The income-consumption curve (ICC) does not necessarily begin from the origin.

(iv) In a two-good world, if one of the goods is inferior, the other good has to be income elastic (that is, its income elasticity will be greater than 1).

(v) If the price-consumption curve (PCC) is horizontal, then the price elasticity of demand for good 1 is greater than 1.

(vi) If each of the 100 buyers has price elasticity of demand equal to 3, the price elasticity of demand of 100 buyers taken together will be greater than 3.

4. Suppose Susan says, "I like both tea and biscuits, but prefer to avoid eating them together."

(i) Draw an indifference map that illustrates the preferences of Ankit. **(3 points)**

(ii) Propose a utility function that could possibly depict the preferences you have drawn. **(3 points)**

(iii) Consider the utility function you propose in (ii). Suppose the price of a cup of tea is Rs. 5 and that of a biscuit is Rs. 2 and the money Susan has to spend on these two goods is Rs. 30 . What will be Susan's equilibrium choices of cups of tea and number of biscuits? **(4 points)**

5. Consider the utility function $U(x_1, x_2) = \sqrt{x_1 x_2}$.

(i) Draw indifference curves for $U = 2$ and $U = 3$. **(3 points)**

(ii) Define and briefly describe "marginal rate of substitution" (MRS). What happens to MRS as we move downwards along a standard smooth, downward sloping, convex indifference curve? Briefly discuss using the given utility function. **(3 points)**

(iii) Calculate MRS when $x_1 = 4$ and $x_2 = 9$. Suppose x_1 increases from 4 to 6. How does MRS change? **(3 points)**

(iv) Consider $V(x_1, x_2) = [U(x_1, x_2)]^2$. What is the MRS when $x_1 = 4$ and $x_2 = 9$? **(3 points)**

6. Suppose the consumer has a demand function for a particular kind of biscuit of the form:

$$x_1 = \frac{m}{2p_1}.$$

Initially his income is Rs.120 per week and the price of biscuit is Rs.3 per unit. Now suppose the price of biscuit falls to Rs.2 per unit. What is the *total* change in demand? Can you decompose the total change in demand into substitution and income effects (in the Slutsky sense)? **(8 points)**

Indian Statistical Institute
Mid-Semester Examination: 2018-2019
Course Name: B.Stat II, Subject Name: Physics I

Date: September 07, 2018 | Duration: 1 hr 30 minute

Answer any three questions. Use of Calculator is allowed. | Maximum Marks: 30

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1. (a) A particle of mass m moves along a trajectory given by $x = x_0 cos\,\omega_1 t$, $y = y_0 sin\,\omega_2 t$.

(i) Find the x and y components of the force. Under what condition is the force a central force?

(ii) Find the potential energy as a function of x and y.

(iii) Determine the kinetic energy of the particle. Show that the total energy of the particle is conserved.

(b) Use Lagrange's equations to find the equation of motion of a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis (See Fig. 1). Find the period of oscillation of the compound pendulum. [5 + 5]

2. (a) Two particles of masses M_1 and M_2 are located on a frictionless double inclined plan and connected by an inextensible massless string passing over a smooth peg (See Fig. 2).

(i) Using the principle of virtual work, show that for equilibrium $\frac{sin\alpha}{sin\beta} = \frac{M_2}{M_1}$, where α and β are angles of the incline.

(ii) Use D'Alembert principle to describe the motion of the masses.

(b) Consider a smooth table with a hole as shown in Fig. 3. Let the mass m_1 be resting on the table and is connected to a mass m_2, ($m_2 > m_1$), by an inextensible string of length l. The mass m_2 hangs vertically down and the string passes through the hole on the table as shown in the figure. Find the equations of motion using D'Alembert principle. [5 + 5]

3. (a) A particle of mass M moves on a plane in the field of force given by (in polar coordinates) $F = -\vec{l_r}\, kr\, cos\theta$, where k is constant and $\vec{l_r}$ is the radical unit vector:

(i) Will the angular momentum of the particle about the origin be conserved? Justify your statement.

(ii) Obtain the differential equation of the orbit of the particle.

(b) A planet moves around the sun in an elliptical orbit. Show that the velocity of the planet at its turning point is given by $v = \frac{Gm_s m_p}{l}(\varepsilon \pm 1)$, where m_s and m_p are mass of the sun and the planet, ε = eccentricity. [5 + 5]

4. (a) Consider two particles interacting by way of a central force (potential = $V(r)$, where r is the relative position vector) [see Fig. 4].

(i) Obtain the Lagrangian in the center of mass system and show that the energy and angular momentum are conserved. Prove that the motion is in a plane and satisfies Kepler's second law (that r sweeps out equal areas in equal times).

P.T.O

(ii) Suppose that the potential is $V = \frac{1}{2}kr^2$, where k is a positive constant, and the total energy E is known. Find expression for the minimum and maximum values that r will have in the course of motion.

(b) A particle of mass m is moving in a plane under an inverse square law attractive force. Set up the Lagrangian and obtain the equation of motion. [5 + 5]

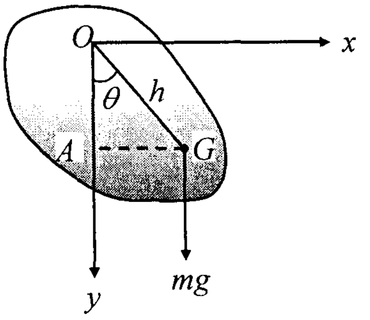

Fig. 1

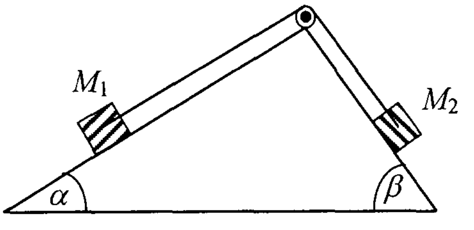

Fig. 2

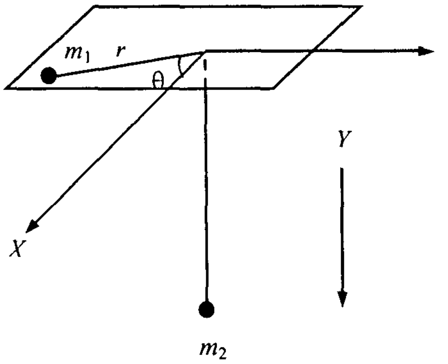

Fig. 3

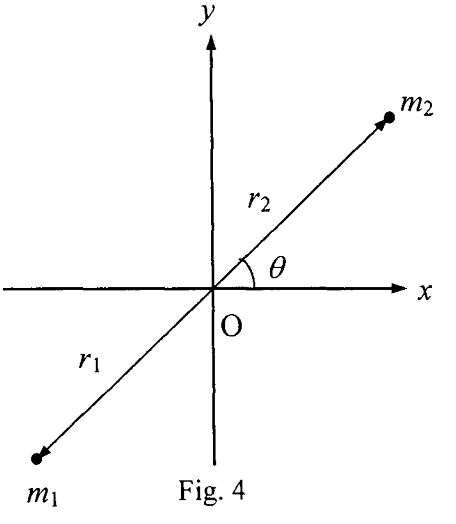

Fig. 4

Indian Statistical Institute

Mid-Semester Examination: 2018-19

Course Name: B. Stat. II, **Subject name:** Molecular Biology

Date: **7th Sept 2018** ***Total Marks:*** **40** ***Duration: 2.0 hrs***

1. How stearic and oleic acids, with the following chemical structures $CH_3(CH_2)_{16}COOH$ and $CH_3(CH_2)_7CH{=}CH(CH_2)_7COOH$ respectively, differ in the metabolism to generate ATP ? [10]

2. Write the metabolic steps in which genetic defects might lead to galactosemia, albinism, alkaptonuria and phenylketonuria phenotypes. [10]

3. Answer any 4 from 6 questions below:

(a) Distinguish between DNA and RNA with respect to their chemical structure and function [5]

(b) In a random copolymer of $(AC)_n$ what will be the frequencies of different triplet codons if $A:C = 1:1$ in the copolymer? [5]

(c). During physical activity we may feel muscle pain: what could be the reason in terms of glucose metabolism? What happens when lemon is added to warm milk and why? [5]

(d) How *Alanine* [CH_3-$CH(NH_2)$-$COOH$] and *Glutamic acid* [$COOH$-CH_2-$CH_2CH(NH_2)$-$COOH$] are metabolized through TCA cycle to generate ATP. [5]

(e) How a mixture of two proteins (mol. wt. 8,000 kD and 16,000 kD) could be separated from each other without denaturation? [5]

(f) How you will determine whether a protein of mol. wt. 42,000 kD consists of single or more polypeptide chains or subunits. [5]

Indian Statistical Institute

Mid-Semester Examination: 2018-19

Course Name: B. Stat. II, **Subject name:** Molecular Biology

Date: **7th Sept 2018** ***Total Marks:*** **40** ***Duration: 2.0 hrs***

1. How stearic and oleic acids, with the following chemical structures $CH_3(CH_2)_{16}COOH$ and $CH_3(CH_2)_7CH{=}CH(CH_2)_7COOH$ respectively, differ in the metabolism to generate ATP ? [10]

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INDIAN STATISTICAL INSTITUTE, KOLKATA
SEMESTER EXAMINATION, FIRST SEMESTER 2018-2019
Elemenst of Algebraic Structures, B. Stat II year
Total Marks -50, Time : 3 hours , Date : dd. mm. yyyy
12. 11. 2018

1. This problem is about the application of Sylow's theorem :

   (a) Show that a group of order 51 is cyclic. [3]

   (b) Show that a group of order 24 is not simple. [4]

   (c) If p, q, r are distinct primes show that no simple group has order pqr. [4]

   (d) Let $x \in GL_2(\mathbb{F}_5)$ be an element of order 5. Show that x is conjugate (as a matrix) to an upper triangular matrix. [3]

   (e) Show that there are 6 Sylow 5-subgroups in $GL_2(\mathbb{F}_5)$. [4]

2. Let $R = M_2(\mathbb{R})$ the ring of 2×2 matrices with entries in $\mathbb{R}$. Show that the ideals in $M_2(\mathbb{R})$ is either the whole ring or the zero ideal. [3]

3. For this problem assume that any constant polynomial $f(t) \in \mathbb{C}[t]$ factors as product of linear polynomials.

   (a) Show that the ideal $\mathfrak{m} := (x - a, y - b) \subset \mathbb{C}[x, y]$ for $a, b \in \mathbb{C}$ is a maximal ideal. [3]

   (b) Show that the maximal ideals of $\mathbb{C}[x, y]/(y - x^2)$ corresponds to maximal ideals $(x - a, y - a^2)$ of $\mathbb{C}[x, y]$ for $a \in \mathbb{C}$. [5]

   (c) Show that the non zero prime ideals of $\mathbb{C}[x, y]/(y - x^2)$ are maximal ideals. [4]

4. Let $\mathbb{Z}[\sqrt{2}] := \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$. Show that $(3) \subset \mathbb{Z}[\sqrt{2}]$ is a maximal ideal and $\mathbb{Z}[\sqrt{2}]/(3)$ is a field with 9 elements. [5]

5. Let $\mathbb{F}_p$ be the field with p elements where p is a prime.

   (a) Show that for every positive integer n there exists an irreducible polynomial $f(x)$ of degree n in $\mathbb{F}_p[x]$ and it divides $x^{p^n} - x$ exactly once. [5+5]

   (b) Show that $x^4 + x^3 + x^2 + x + 1 = (x - 1)^4$ in $\mathbb{F}_5[x]$. [2]

   (c) Show that $x^3 + x + 1$ is irreducible in $\mathbb{F}_2[x]$. [2]

   (d) Show that $x^3 + x + 1$ divides $x^8 - x$ in $\mathbb{F}_2[x]$. [3]

   (e) Show that $x^3 + x + 1$ and $x^3 + x^2 + 1$ are the only two monic irreducible polynomial of degree 3 in $\mathbb{F}_2[x]$. [3]

INDIAN STATISTICAL INSTITUTE

Semestral Examination

First semester

B. Stat - Second year 2018-2019

Analysis III

Date: 16 November, 2018 | Maximum Marks: 60

Duration: 3 hours

Answer all questions.

State clearly any result that you use.

All notations and definitions must be properly explained.

(1) Suppose that $f : [0, a] \to \mathbb{R}$ is a continuous function. Define $g : (0, a) \to \mathbb{R}$ by

$$g(x) = \int_x^a \frac{f(t)}{t}\, dt, \quad x \in (0, a).$$

Prove that

$$\lim_{\varepsilon \to 0} \int_\varepsilon^a g(t)\, dt = \int_0^a f(t)\, dt.$$

Justify each step. 8

(2) (a) Give a sketch of the region $S = \{(x, y) \in \mathbb{R}^2 | |x| + |y| = 1\}$ and evaluate the integral

$$\int\int_S e^{x+y}\, dx\, dy.$$

(b) Use integration to determine the volume of the tetrahedron in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. 8+6

(3) Let $\wedge^n(\mathbb{R}^n)$ denote the vector space of all alternating n-multilinear maps on $\mathbb{R}^n$.

(a) Prove that $\wedge^n(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}$ by giving an explicit isomorphism. Find a basis of $\wedge^n(\mathbb{R}^n)$.

(b) Suppose that $A \in SO(n)$ and let $\ell : \mathbb{R}^n \to \mathbb{R}^n$ be the associated linear map given by $\ell(x) = A.x$ for all $x \in \mathbb{R}^n$. Determine the induced map $\ell^* : \wedge^n(\mathbb{R}^n) \to \wedge^n(\mathbb{R}^n)$. Give a detailed proof. 6+6

(4) Consider the 1-form $\eta = \frac{x\,dy - y\,dx}{x^2+y^2}$ on $\mathbb{R}^2 \setminus (0,0)$.

(a) Prove that η is a closed form.

(b) Evaluate the integral $\int_\gamma \eta$, where γ is the parametrized curve given by

$$\gamma(t) = (\cos 2\pi t, \sin 2\pi t), \quad t \in [0,1].$$

(c) Is η an exact form? Justify your answer. 4+6+5

(5) Consider a singular 2-cube $c : [0,1] \times [0,1] \to \mathbb{R}^3$ defined by:

$$c(\theta, z) = (\cos 2\pi\theta, \sin 2\pi\theta, z)$$

for $(\theta, z) \in [0,1] \times [0,1]$.

(a) Determine the boundary of c as a 1-chain.

(b) Consider the 1-form ω on $\mathbb{R}^3$ given by

$$\omega = y(2z-1)\ dx + x(2z-1)\ dy + xy\ dz.$$

Evaluate the integral

$$\int_c d\omega.$$

6+8

INDIAN STATISTICAL INSTITUTE

First Semester Final Examination: 2018 – 19

B. Stat 2nd Year
Probability theory 3

Date: 19 Nov. 2018 **Maximum Marks: 60** **Duration: 3 hours**

Answer **any five** questions. Each question carries 12 marks. The solution of each question has to be in one place, that is, answering different parts in separate places is not allowed.

1. Suppose that U follows Uniform $(-1,1)$, and given U, the conditional distribution of X is normal with mean U and variance 1. Calculate the covariance between X and U.

2. A coin with probability of head $p \in (0,1)$ is tossed repeatedly. Let X_n denote the number of the toss on which the n-th head is obtained.

   (a) **(4 marks)** Show that as $n \to \infty$,
   $$\frac{1}{n} X_n \to \frac{1}{p} \text{ a.s.}$$

   (b) **(8 marks)** Show that as $n \to \infty$,
   $$\frac{1}{\sqrt{n}}\left(X_n - \frac{n}{p}\right) \Rightarrow Z\,,$$
   where $Z \sim N(0,\sigma^2)$ for some σ. Calculate σ.

3. If $(X_1,\ldots,X_n)$ follows multivariate normal, for some $n \geq 2$, with
   $$\begin{aligned} \mathrm{E}(X_i) &= 0\,,\\ \mathrm{Var}(X_i) &= 1-\frac{1}{n}\,,\\ \mathrm{Cov}(X_i,X_j) &= -\frac{1}{n}\,, \end{aligned}$$
   for all $1 \leq i < j \leq n$, show that
   $$\sum_{i=1}^{n} X_i^2 \sim \chi^2_{n-1}\,.$$

PTO

4. An office receives phone calls according to a Poisson process of rate λ. If exactly 10 calls have come by time t, calculate the conditional distribution of the number of calls received by time $t/2$.

5. Let $(N_t : t \geq 0)$ be a possibly inhomogeneous Poisson process of intensity $(\lambda(t) : t \geq 0)$, where $\lambda(\cdot)$ is a continuous function. If the first arrival time follows Exponential(θ), show that

$$\lambda(t) = \theta \text{ for all } t,$$

and therefore that $(N_t : t \geq 0)$ is a homogeneous Poisson process of rate θ.

6. A fair die is rolled repeatedly. For $n \geq 1$, let $X_{n1}, \ldots, X_{n6}$ denote the number of times $1, \ldots, 6$ have been obtained, respectively, in the first n rolls. Show that as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \left(\begin{bmatrix} X_{n1} \\ \vdots \\ X_{n6} \end{bmatrix} - \begin{bmatrix} n/6 \\ \vdots \\ n/6 \end{bmatrix} \right) \Rightarrow Z,$$

where Z follows $N_6(0, \Sigma)$ for some 6×6 matrix Σ. Calculate all the entries of Σ.

INDIAN STATISTICAL INSTITUTE
Semester Examination: 2018-19

B STAT, SECOND YEAR

Statistical Methods - III

Date: 22/11/2018 Maximum Marks: 70 Duration: 3.5 hours

1. Suppose $X_1, X_2, \ldots, X_n$ be a random sample from

$$f(x) = \frac{1}{\theta};\ 0 < x < \theta.$$

Consider a statistic $T_1 = X_{(n)}$.

(a) Suggest an unbiased estimator of θ based on T_1, justifying your answer. Find its variance.

(b) Evaluate the lower bound of the variance of an unbiased estimator of θ as given by Cramer-Rao inequality for θ.

(c) Write your comment based on your findings in (a) and (b).

(5+4+3=12)

2. Two ladies claim that they can identify whether sugar is added to tea before or after pouring milk. To test their claim same set of 10 cups of tea are presented to each of them where sugar is added before pouring milk. A lady declares "$before$" or "$after$" if she thinks that sugar is added 'before' or 'after' pouring milk respectively. The results are shown below.

Lady 1 : $after, before, before, before, before, before, before, before, before, after$

Lady 2 : $after, after, after, before, after, after, after, after, before, after$

(a) Suggest a relevant hypothesis to test and write the corresponding null and alternative hypotheses. Note that your hypothesis must involve the opinions of both the ladies.

(b) Describe the testing procedure for the hypothesis suggested by you in (a). (2+5=7)

3. Suppose a random sample of size n is taken from $N(\mu, \sigma^2)$ distribution. Assuming that σ^2 is not known, propose two tests for testing $H_0 : \mu = 10$ against $H_1 : \mu > 10$, based on the following,

   Test 1 : A statistical test using the actual observations.
   Test 2 : Count the number of observations that are greater than 10. Now propose a test based on this count.

   Let α_0 be the probability of type I error based on **Test 2**. Now make the level of significance for **Test 1** as α_0. Which of the two tests would you prefer and why? (5+6+2=13)

4. It takes me around 65 minutes to come to ISI from my home by bus. Usually I reach ISI by 8:30 am. After around two months, I observed that although I usually reach by 8:30 am, on few occasions I was late. Note that, if I reach ISI at 8:40 a.m., this means that I am late by 10 minutes. So I record the time to reach ISI randomly for 10 days so as to do a statistical test to come to a conclusion.

   (a) Frame this as a problem of statistical testing of hypothesis.

   (b) Describe an appropriate test based on the data obtained, assuming that delay in reaching ISI follows an exponential distribution with mean θ.

   (c) Find a 95% confidence interval for θ. (3+5+4=12)

5. Let $\{X_1 \ldots, X_m\}$ be a random sample from $N(\mu_1, \sigma^2)$ and $\{Y_1, \ldots, Y_n\}$ be another random sample, independent of the first one, from $N(\mu_2, \sigma^2)$, all parameters being unknown. Find a $(1-\alpha)100\%$ confidence interval for μ_1/μ_2, where $0 < \alpha < 1$. (7)

6. There are three types of blood groups in human, O, A, B, and AB. Blood samples of 870 randomly selected individuals are taken and their blood groups are determined. It is known that the probabilities that a person has O, A, B, and AB blood groups

blood group	probability	observed frequency
O	r^2	$n_O = 352$
A	$p^2 + 2pr$	$n_A = 364$
B	$q^2 + 2qr$	$n_B = 120$
AB	$2pq$	$n_{AB} = 34$

Table 1: Blood groups of 870 people

are r^2, $p^2 + 2pq$, $q^2 + 2qr$, and $2pq$ respectively. Table 1 gives the frequency of each blood group with respective probability.

Estimate the parameters p, q, and r using EM algorithm explicitly explaining each step. Note that $p + q + r = 1$, $0 < p, q, r < 1$. Show your calculations up to fourth step. (8)

7. Suppose that when a signal having value μ is transmitted from location A the value received at location B is normally distributed with mean μ and variance 4. That is, if μ is sent, then the value received is $\mu + N$ where N, representing noise, is normal with mean 0 and variance 4. To reduce error, suppose the same value is sent 9 times. The successive values received are $5, 8.5, 12, 15, 7, 9, 7.5, 6.5, 10.5$.

   (a) Suggest MVUE for μ, deriving any associated theorem(s) to justify that your suggested estimator is MVUE.

   (b) Is this estimator also MLE? Justify your answer.

   (c) Suggest also an unbiased estimator of μ^2 and evaluate its variance.

   (d) Numerically calculate the estimates obtained in (a) and (c). Also calculate the variances of the corresponding estimators.

   (e) Find a $(1-\alpha)100\%$ confidence interval for $e^{-\mu}$ (both theoretically and numerically). (4+3+5+5+4=21)

Indian Statistical Institute

First Semester Examination: 2018-2019 (Regular)

Course Name: B.Stat II, Subject Name: Physics I

Date: 26.11.2018 Duration: 3 hours

Answer as many questions as you can. Use of Calculator is allowed. Maximum Marks: 50

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1. (a) A particle is constrained to move two dimensionally on a smooth inclined plane making an angle α ($\alpha < 90°$) to the horizontal. Assume that the line of intersection of the inclined plane with the horizontal plane is the x-axis and a line on the inclined plane drawn perpendicular to the x-axis is the y-axis.

(i) Obtain an expression for the Lagrangian of the particle in terms of the co-ordinates x and y.

(ii) Solve the Lagrangian equations of motion and show that the trajectory of the particle on the inclined plane will be a parabola, if the particle was initially at $x = 0$, $y = 0$ with an initial velocity v_0 along the line $x = y$.

b) A particle of mass m falls from a given height z_0 in time $t_0 = \sqrt{\left(\frac{2z_0}{g}\right)}$ and the distance travelled in time t is given by $z = at + bt^2$, where constants a and b are such that the time t_0 has always the same value. Show that the integral $\int_0^{t_0} L dt$ is an extremum for real values of the coefficients only when $a = 0$ and $b = g/2$. [5 + 5]

2. (a) Consider the motion of a particle of mass m under the influence of a force $F = -K\mathbf{r}$, where K is a positive constant and $\mathbf{r}$ is the position vector of the particle.

(i) Prove that the motion of the particle lies in a plane.

(ii) Find the position of the particle as a function of time, assuming that at $t = 0$, $x = a$, $y = 0$, $V_x = 0$, $V_y = V$.

(iii) Show that the orbit is an ellipse.

(iv) Find the period.

(v) Does the motion of the particle obey Kepler's laws of planetary motion?

(b) Derive the Hamiltonian function and the equations of motion for a two-dimensional isotropic harmonic oscillator, using (i) Cartesian co-ordinates, and (ii) Polar co-ordinates. [5 + 5]

3. (a) A gas undergoes a reversible adiabatic compression from pressure 0.5 MPa and volume 0.2 m^3 to volume 0.05 m^3 according to the law, $pv^{1.3}$ = constant. Determine the change in enthalpy, internal energy and entropy, and the heat transfer and work transfer during the process.

(b) Show that for a van der Waals' gas:

(i) $\left(\frac{\partial C_v}{\partial v}\right)_T = 0$, (ii) $(S_2 - S_1)_T = R\ln\frac{v_2 - b}{v_1 - b}$, (iii) $T(v-b)^{R/c_v}$ = constant for an isentropic, (iv) $c_p - c_v = \frac{R}{1 - 2a(v-b)^2 / RTv^3}$, (v) $(h_2 - h_1)_T = (p_2v_2 - p_1v_1) + a\left(\frac{1}{v_1} - \frac{1}{v_2}\right)$.

The symbols carry their usual meanings in thermodynamics. [5 + 5]

P.T.O

4. (a) A block of iron weighing 100 kg and having a temperature of 100°C is immersed in 50 kg of water at a temperature of 20°C. What will be the change of entropy of the combined system of iron and water? Specific heats of iron and water are 0.45 and 4.18 *kJ/kg* K, respectively.

(b) Two Carnot engines *A* and *B* are connected in series between two isothermal reservoirs maintained at 1000 K and 100 K, respectively. Engine *A* receives 1680 *kJ* of heat from high-temperature reservoir and rejects heat to the Carnot engine *B*. Engine *B* takes in heat rejected by engine *A* and rejects heat to the low-temperature reservoir. If engines *A* and *B* have equal thermal efficiencies, determine (i) the heat rejected by engine *B*, (ii) the temperature at which heat is rejected by engine *A*, and (iii) the work done during the process by engines *A* and *B* respectively. If engines *A* and *B* deliver equal work, determine (iv) the amount of heat taken in by engine *B*, and (v) the efficiencies of engines *A* and *B*. [5 + 5]

5. (a) For the Maxwell-Boltzmann distribution of particles of an isolated system in equilibrium obtain the following expressions:

(i) $P_r = \frac{n_r}{N} = \frac{g_r e^{-\beta E_r}}{Z}$, (ii) $\overline{U} = \frac{U}{N} = -\frac{\partial \log Z}{\partial \beta}$, where P_r is the probability that a particle be in any of the quantum states g_r of energy E_r, Z is the partition function of the particle, and all other symbols have their usual meaning.

(b) There are 1000 molecules of an ideal gas in a box of 1 litre volume. Assuming that the particles do not preferentially occupy any region, find the probability of finding the particles in a particular region of volume 100 cc. For what value of *n* (the no. of molecules) is this probability maximum?

[5 + 5]

6. (a) The Fermi energy for sodium at T = 0 K is 3.1 eV. Find its value for aluminium. The free electron density in aluminium is about 8 times that in sodium.

(b) Find the most probable, average and root mean square speed of O_2 molecules at 25° *C* using Maxwell-Boltzmann distribution law.

(c) A gas has only two particles, each one of them can be in one of three quantum states, g_r = 1, 2, and 3. Find the possible number of microstates of the gas according to the three (M-B, B-E, and F-D) statistics.

[3 + 3 + 4]

Indian Statistical Institute

Semester Examination (B. Stat-II, Molecular Biology, Year-2018-19) 26 · 11 · 2018

Answer any five; All questions carry equal marks; Full marks = 50; Time = 2.5 hours

1. (a) If mother and son both are color blind, is it likely that the son inherited the trait from his mother? Show possible genotypes with a pedigree diagram. [5]

(b) How many different DNA sequences are possible with 6 codons: *ATG, TAA, AGG, GGG, CGT* and *TAC* provided there will not be five consecutive "*G*"s within the sequences. (each codon would be used only once). [5]

2. A linear DNA molecule was digested by two restriction enzymes, *EcoRI* and *HindIII,* separately and in combination. Following results were obtained:

Enzymes used	Fragments produced (in kb)
EcoRI	*2.9, 4.5, 7.4, 8.0*
HindIII	3.9, 6.0, 12.9
EcoRI and *HindIII*	1.0, 2.0, 2.9, 3.5, 6.0, 7.4

Draw a restriction site map on the DNA molecule using the above data. [10]

3. (a) In a family of six, both the parents are carriers of sickle cell anemia, a recessive disease caused by a single nucleotide mutation in globin gene. Doctor examined all 4 children and opined that one boy and one girl might suffer from the disease. How will determine the genotypes of all six individuals to suggest that two of the 4 children had sickle cell mutation. (Consider that mutation responsible for sickle cell anemia creates a restriction enzyme site). [8]

(b) Above-mentioned question is an example of recessive disease caused by single gene mutation, so according to Mendel's Law, $1/4^{th}$ of the children may suffer from disease. But in this family, two children might suffer from disease. How will you explain this observation? [2]

4. (a) Why multiple "origin" is required for replication of human genomic DNA?

[5]

(b) Apart from sex chromosomes, mice have 19 autosomes in their genome and consider that each of them has similar size. If two autosomal genes are chosen randomly, what is the chance that they will be on the same chromosome? (consider size of the genes are uniform). [5]

5. A couple has four children and neither of the couple is bald. One of the two sons is bald but neither of the daughters is bald. If one of the daughters marries a non-bald man then what is the chance that their son will become bald at an adult age? Give your answer drawing a pedigree diagram with genotypes. (Assume: baldness is caused by one gene in autosome and one of the two alleles behaves dominantly in male but recessively in female.) [10]

6. Among parents, father is color blind and possesses "O" blood group but mother has normal color vision and "AB" blood group. The woman's father had color blindness. X-linked and autosomal genes determine color blindness and blood group, respectively.

Answer (a) What are the genotypes of the father and mother? (b) What proportion of their children will have color blindness and type "B" blood group? (c) What proportion of their children will have color blind and type "AB" blood group? (3+4+3)

Indian Statistical Institute

Final Examination 2018 for B2

Course name: **Microeconomics**

Date: **26 November 2018**

Maximum marks: **80**

Duration: **3 hours**

Instruction: **Answer all questions.**

1. Suppose a competitive firm has a total cost function $C(q) = 450 + 15q + 2q^2$. If the market price is Rs. 115 per unit, find the level of output produced by the firm and the level of profit. **(5 points)**

2. A monopolist faces the following demand curve: $P = 120 - .02Q$, where Q is weekly production and P is price measured in rupees per unit. The firm's cost function is given by $C = 60Q + 25,000$. What is the level of production, price and total profit per week? **(5 points)**

3. Suppose a profit-maximizing monopolist is producing 800 units of output and charging Rs. 40 per unit. If the absolute value of the elasticity of demand for the product is 2, find the marginal cost of the last unit produced. **(5 points)**

4. Suppose firms in a competitive market have cost function $C(q) = q^2 + 1$. What is the breakeven level of price (that is the price such that entry of new firms occur in the long run if price is above this level)? **(5 points)**

5. Let $MP_L = 100K - L$; $MP_K = 100L - K$, where L is labour and K is capital used in production. Let the prices of labour and capital respectively be $w = 5$ and $r = 5$, and the total cost be $C = 1000$. What are the output-maximizing levels of labour and capital employment? **(5 points)**

6. Consider a constant-elasticity-of-substitution (CES) production function. Find the MRTS for such a function. Interpret the parameter δ as the share of labour relative to capital (or the 'distribution' parameter) in this context. **(10 points)**

PTO

7. "If the average productivity of labour is rising, the corresponding marginal productivity of labour will also be rising". Is this statement true or false. Give reasons for your answer. **(10 points)**

8. Consider a production technology that uses two inputs, labour (L) and capital (K), to produce output (q). If this production function is linearly homogeneous, then the average and marginal productivity of both the inputs will just be functions of the K/L ratio. **(10 points)**

9. Elucidate the statement using a Cobb-Douglas production function: "Diminishing returns to a factor is consistent with increasing returns to scale." **(10 points)**

10. A firm has a production function given by $f(x_1, x_2) = \min\{2x_1 + x_2. x_1 + 2x_2\}$. What is the cost function for this technology? **(10 points)**

11. Suppose trade unions are successful in bargaining and increasing the salaries of engineers working at ONGC (Oil and Natural Gas Corporation) India. Does the share of labour income relative to capital income at ONGC, necessarily rise? Why or why not? **(5 points)**

12. Consider a perfectly competitive industry that mainly employs unskilled labour as its input (like the textile industry, for example) so that the input price is likely to remain constant, even with increased demand for inputs, in the long run. What will the shape of the long run industry supply curve be? Illustrate graphically. **(5 points)**

13. Two factors of production A and B have marginal productivities 3 and 2 respectively, and prices 5 and 4 respectively. Should the employment of any factor increase in equilibrium? Explain. **(5 points)**

14. When can a monopolist firm produce 0 output in equilibrium? When can a perfectly competitive firm produce 0 output in equilibrium? **(5 points)**

INDIAN STATISTICAL INSTITUTE

Back paper Semester Examination: 2018-19

B STAT, SECOND YEAR

Statistical Methods - III

Date: 3/12/2018 | Maximum Marks: 100 | Duration: 3.5 hours

1. State and prove Cramer-Rao inequality. (7)

2. Suppose $X_1, X_2, \ldots, X_n$ be a random sample from

$$f(x) = \frac{1}{2\theta^3} e^{-\frac{x}{\theta}} x^2;\ x > 0.$$

   Find MVUE of θ, deriving all intermediate steps and mention clearly any result and/or theorem that you want to use. (9)

3. In mountaineering expeditions, a general notion is that accidents occur more frequently while descending after summit than while climbing up towards summit. It is known that in 1996, 8 people died while climbing down after summit whereas 5 people died while ascending. Based on the given data, what is your opinion? Justify it. (9)

4. In a region it is observed that there are four species of Iris flowers, viz Sp1, Sp2, Sp3, and Sp4. A person selects a flower at random and assigns its species by looking at some characteristics. The frequency distribution of iris flowers and their respective probabilities are given in the following table:

Species	probability	observed frequency
Sp1	$\frac{1}{2} + \frac{1}{4}\theta$	120
Sp2	$\frac{1}{2} - \frac{1}{4}\theta$	20
Sp3	$\frac{1}{2} - \frac{1}{4}\theta$	18
Sp4	$\frac{1}{4}\theta$	42

   Estimate θ using EM algorithm. (12)

5. Suppose a random sample of size n is taken from $N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ distribution. Assume that all parameters are unknown.

   (a) Describe a test for testing $H_0 : \mu_1 = \mu_2 + \xi_0$ against $H_1 : \mu_1 > \mu_2 + \xi_0$, where ξ_0 is a known quantity based on prior information.

   (b) Find a $(1-\alpha)100\%$ confidence interval for (1) $\mu_1 - \mu_2$ and (2) μ_1/μ_2. (8+7+10=25)

6. Currently in Kolkata East-West metro work is going on. To see the progress, a central team came to Kolkata and observed the time taken (in years) to complete 70% of work for a complete metro station for 9 ongoing sites are:

$$5.2, 6.1, 4.2, 3.8, 2.6, 3.0, 4.5, 4.4, 3.2$$

   Previous experience in Delhi indicates that an average of 3.6 years can be considered to be a good progress. Do you think that progress in building metro station in Kolkata can be comparable to that in Delhi? (10)

7. Suppose that when a signal having value μ is transmitted from location A the value received at location B is normally distributed with mean μ and variance 4. That is, if μ is sent, then the value received is $\mu + N$ where N, representing noise, is normal with mean 0 and variance 4. To reduce error, suppose the same value is sent 9 times. The successive values received are $5, 8.5, 12, 15, 7, 9, 7.5, 6.5, 10.5$.

   (a) Suggest MVUE for μ, deriving any associated theorem(s) to justify that your suggested estimator is MVUE.

   (b) Is this estimator is also MLE? Justify your answer.

   (c) Suggest also an unbiased estimator of μ^2 and evaluate its variance?

   (d) Numerically calculate the estimates obtained in (a) – (c) and the corresponding variances.

   (e) Find a $(1-\alpha)100\%$ confidence interval for $e^{-\mu}$ (both theoretically and numerically). (6+5+7+6+4=28)

INDIAN STATISTICAL INSTITUTE, KOLKATA
BACK PAPER, FIRST SEMESTER 2018-2019
Elemenst of Algebraic Structures, B. Stat II year
Total Marks -100, Time : 3 hours , Date : 01. 01. 2019

1. Let G be a group and k be a positive integer. If G has exactly one subgroup H of cardinality k, prove that H is normal. [5]

2. How many $\mathbb{Z}/5\mathbb{Z}$ actions are there on $X := \{1, 2, 3, 4, 5, 6, 7\}$. [5]

3. Write down the class equation of S_4. [10]

4. Find all ring homomorphisms $\phi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$. [5]

5. Let R_1 and R_2 be two rings and suppose $\phi : R_1 \to R_2$ be a ring homomorphism. Then show the following :

   (a) $Image(\phi)$ is a subring of R_2. [2]

   (b) Given an ideal $I \subset R_2$, the set $\phi^{-1}(I)$ is an ideal in R_1. [3]

   (c) If $u \in R_1$ is a unit then $\phi(u) \in R_2$ is a unit. [3]

   (d) Give example of R_1 and R_2 and an ideal $I \subset R_1$ such that $\phi(I)$ is not an ideal of R_2. [2]

   (e) Give example of R_1 and R_2, ϕ and an element $u \in R_1$ such that $\phi(u)$ is a unit but u is not. [5]

6. Let $a \in \mathbb{Z}$ and $ev_a : \mathbb{Z}[x] \to \mathbb{Z}$ be the map defined by $ev_a(f(x)) = f(a)$. Show that ev_a is a ring homomorphism. Find out the kernel of ev_a. Show that any ring homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ is of the form ev_a for some $a \in \mathbb{Z}$. [10]

7. Let $p \in \mathbb{Z}$ prime such that $x^2 + 1$ is irreducible in $\mathbb{Z}/p\mathbb{Z}[x]$. Show that the ideal generated by p in $\mathbb{Z}[i]$ denoted by (p) is a maximal ideal. Conclude that p is irreducible in $\mathbb{Z}[i]$. [10]

8. Let R be a commutative ring and $I \subset R$ be an ideal. Show that there exists a bijection between prime (resp. maximal) ideals of R/I and prime (resp.) maximal ideals of R containing I. Find out prime ideals of $\mathbb{Z}/m\mathbb{Z}$ where m is an integer. [10]

9. Show that $2x^7 - 16x^6 + 4x^2 - 9$ is irreducible in $\mathbb{Q}[x]$. [10]

10. Show that $L := \mathbb{Q}[x]/(x^4 + x^3 + x^2 + x + 1)$ is a finite dimensional vector space over $\mathbb{Q}$. Find out the dimension of L as a $\mathbb{Q}$ vector space. [10]

11. Let F be a finite field. Show that $|F| = p^n$ for some prime p and some positive integer n. [10]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination

First semester

B. Stat - Second year 2018-2019

Analysis III

Date: 3/01/2018

Maximum Marks: 100

Duration: 3 hours

Answer all questions.

State clearly any result that you use.

All notations and definitions must be properly explained.

(1) Let $O(n)$ be the set of all orthogonal $n \times n$ matrices over $\mathbb{R}$ and let I denote the identity matrix in $O(n)$.

(a) Prove that $O(n)$ is compact.

(b) Prove that I and $-I$ can not be joined by a path in $O(n)$ if n is odd.

(c) Show that the statement in (b) is not true if n is even. 6+4+2

(2) Let A be a subset of $\mathbb{R}^2$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x) = d(x, A)$, where $d(x, A) = \inf\{\|x - a\| : a \in A\}$. Prove that f is continuous and show that $f^{-1}(0)$ equal to the closure of A. 5+5

(3) Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$g(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove that g is continuous but not differentiable. 5+5

(4) Prove that the function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\phi(x,y) = (x + y^2 + 2x^3y + x^6, y + x^3)$$

is a diffeomorphism. 10

(5) Find the critical points of the function $f(x,y) = x^2 - y^2$ on the surface $S = \{(x,y,z) \in \mathbb{R}^3 | x^2 + 2y^2 + 3z^2 = 1\}$. Determine the nature of the critical points (that is, maxima, minima or a saddle point). 10

PTO

(6) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree k, that is, $f(tx) = t^k f(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

(a) Prove that

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf.$$

(b) Suppose further that f is compactly supported. Define a function $\widehat{f}$ on $\mathbb{R}^n$ by

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)\ e^{-2\pi i x.y} dx, \quad y \in \mathbb{R}^n.$$

Show that $\widehat{f}$ is also homogeneous. Find the degree of homogeneity. 6+6

(7) Calculate the area of the ellipse $E = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$. 6

(8) (a) Let $e_1, e_2, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$ and let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be the dual basis. Determine the value of

$$\varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k}(v_1, \ldots, v_k),$$

where $v_1, v_2, \ldots, v_k$ are vectors in $\mathbb{R}^n$. 4

(b) Consider the 2-form

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

on $\mathbb{R}^3$. Prove that $\int_\sigma \omega \neq 0$, where $\sigma : [0, 1] \times [0, 1] \to \mathbb{R}^3$ is defined by $\sigma(u, v) = (\cos 2\pi u \cos \pi v, \sin 2\pi u \cos \pi v, \sin \pi v)$. 8

(9) Consider the 2-surface

$$\sigma(u, v) = ((2 + \cos 2\pi v) \cos 2\pi u, (2 + \cos 2\pi v) \sin 2\pi u, \sin \pi v).$$

(a) Write down the boundary of σ as a singular 1-chain. (b) Prove that for any 1-form ω of class C^2, $\int_\sigma d\omega = 0$. 6+4

(10) (a) Let $\omega = f\, dx + g\, dy$ be a smooth 1-form on $\mathbb{R}^2$ such that $d\omega = 0$. Define a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\varphi(x, y) = \int_0^y g(0, t)\, dt + \int_0^x f(s, y)\, ds.$$

Prove that $d\varphi = \omega$.

(b) Find a function φ such that $d\varphi = (1 + y) \cos x\ dx + (\sin x + y) dy$. 6+2

INDIAN STATISTICAL INSTITUTE

Back-paper[1] Examination
First semester
B. Stat - Second year 2018-2019
Analysis III

Date: January 31, 2019 | Maximum Marks: 100
Duration: 3 hours

Answer all questions.
For full credit, you have to state the theorems clearly if you use them.

(1) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then show that the derivative of f at a point x satisfies the following relation.

$$Df_x(v) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i},$$

where $x = (x_1, \ldots, x_n$ and $v = (v_1, \ldots, v_n)$. 7

(2) Consider the subset S of $\mathbb{R}^3$ defined as follows:

$$S\{(x,y,z) \in \mathbb{R}^3 | x^2 - y^2 - z^2 = 1\}$$

(a) Determine whether the following are true: (a) S is closed. (b) S is compact.
(b) Prove that S does not intersect the plane $x = 0$. Is S path-connected? Justify your answer. 16

(3) Check the continuity of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \quad = \quad \begin{cases} xy \sin \frac{1}{xy}, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

7

(4) Let $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a function defined by $f(A) = A^2 + A^T$, where A is any $n \times n$ matrix.
Find the derivative of f at the identity matrix I. Prove that there exists an $\varepsilon > 0$ such that f is injective on the ε-ball about I. 12

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ be two functions defined as follows:

$$f(u,v) = (u^2, v^2, uv), \quad \text{and} \quad g(x,y,z) = (\cos x \sin y, \cos y \sin z, \cos z \sin x),$$

where $(u,v) \in \mathbb{R}^2$ and $(x,y,z) \in \mathbb{R}^3$.
(a) Prove that $h = g \circ f$ is a differentiable function. Find the Jacobian of h at (a,b).
(b) Using the chain rule, find the expression for $\frac{\partial h}{\partial u}$ at (a,b).

[1]Backpaper for supplementary endsemester examination

12

(6) Use Lagrange's multiplier method to prove that $|a.b| \leq \|a\| \, \|b\|$, where . denotes the dot product of two elements in $\mathbb{R}^n$ and $\| \, \|$ denotes the Euclidean norm of a vector. 12

(7) Let S be the region in $\mathbb{R}^2$ bounded by the curves $xy = 1$, $xy = 2$, $y = x, y = 4x$. Draw a sketch of the region. Use the transformation $x = \sqrt{v/u}$ and $y = \sqrt{uv}$ to find the value of the integral $\int_S x^2y^2 dydx$. 8

(8) (a) Let $\omega = f\,dx + g\,dy$ be a smooth 1-form on $\mathbb{R}^2$ such that $d\omega = 0$. Define a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\varphi(x, y) = \int_0^y g(0, t)\,dt + \int_0^x f(s, y)\,ds.$$

Prove that $d\varphi = \omega$. 10

(9) Let $\omega = (1 + xy^2)dx - x^2y\,dy$ be a 1-form on $\mathbb{R}^2$.

(a) Evaluate the line integral $\int_C \omega$, where C consists of the arc of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$. (Use the parametrization $\gamma(t) = (t, t^2)$ for C)

(b) Use Green's theorem to evaluate $\int_S d\omega$, where S is the region bounded by C and the horizontal line segment C' joining $(-1, 1)$ and $(1, 1)$. 16

INDIAN STATISTICAL INSTITUTE

First Semester Backpaper Examination: 2018 – 19

B. Stat 2nd Year
Probability theory 3

Date: 02/01/19 **Maximum Marks: 100** **Duration: 3 hours**

Answer **any five** questions. Each question carries 20 marks. The solution of each question has to be in one place, that is, answering different parts in separate places is not allowed.

1. Let $X_1, X_2, \ldots$ be uncorrelated random variables such that

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty\,,$$

and for each $n \geq 1$,

$$\mathrm{E}(X_n) = 0\,.$$

Show that the sum

$$\sum_{n=1}^{\infty} X_n$$

converges in L^2.

2. The number of accidents in Kolkata on a particular day follows Poisson(λ). Assume that the number of accidents on different days are independent. Counting from a given day, let Z_n denote the proportion of days, in the first n days, on which there were no accidents.

   (a) **(8 marks)** Show that as $n \to \infty$,

$$Z_n \to e^{-\lambda} \text{ a.s.}$$

   (b) **(12 marks)** Show that as $n \to \infty$,

$$\sqrt{n}\left(Z_n - e^{-\lambda}\right) \Rightarrow X\,,$$

   for some random variable X. State the distribution of X.

PTO

3. Suppose that

$$X \sim N_d(0, \Sigma)\,,$$

for a $d \times d$ non-negative definite matrix Σ. If

$$\Sigma^3 = \Sigma\,,$$

and

$$k = \text{Rank}(\Sigma) \geq 1\,,$$

show that

$$\sum_{i=1}^{d} X_i^2 \sim \chi_k^2\,.$$

4. If Λ follows the standard exponential distribution, that is, Exponential(1), and given Λ, the conditional distribution of X is Poisson with parameter Λ, calculate $\text{E}(X)$.

5. Buses arrive at a stop according to a Poisson process of rate one every 10 minutes. Calculate the expected waiting time of a person for a bus.

6. Suppose that (X, Y) follows bivariate normal with

$$\text{E}(X) = \text{E}(Y) = 0\,,$$

$$\text{Var}(X) = \text{Var}(Y) = 1\,,$$

and

$$\text{Corr}(X, Y) = \rho \in (-1, 1)\,.$$

Calculate the covariance between X and $\mathbf{1}(Y > 0)$.

INDIAN STATISTICAL INSTITUTE, KOLKATA
BACK PAPER, FIRST SEMESTER 2018-2019
Elemenst of Algebraic Structures, B. Stat II year
Total Marks -100, Time : 3 hours , Date : 01-02-2019

1. Let p, q be prime numbers such that $q^n > (p-1)!$. Show that a group of order $p.q^n$ can not be simple. [10]

2. Let G be a finite group acting on a finite set S such that the cardinality of S is even and the number of distinct orbits is odd. Show that the cardinality of G is even.[5]

3. Find out the number of distinct group isomorphisms $\phi : \mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$. Find out the number of distinct group homomorphisms $\phi : \mathbb{Z}/5\mathbb{Z} \to S_5$. [5+5]

4. Write down the class equation of A_4. [10]

5. Let p be a prime.
   (a) Show that $x^p - 1 = (x-1)^p$ in $\mathbb{F}_p[x]$. [3]
   (b) Show that $x^p - x$ has all roots in $\mathbb{F}_p$ and the roots are distinct. [2]
   (c) Show that $x^p - 1$ has only one rational root. [5]

6. Let $\mathbb{Z}[\sqrt{2}] := \{a \in \mathbb{R} | a = x + y\sqrt{2}; x, y \in \mathbb{Z}\}$. Show that the ideal generated by 5 in $\mathbb{Z}[\sqrt{2}]$ is a maximal ideal. [10]

7. Let R be a ring and S be an integral domain and $\phi : R \to S$ a ring homomorphism. Let $r \in R$ be such that $\phi(r) \neq 0$ and let $a(r) := \{x \in R | x.r = 0\}$.
   (a) Show that $a(r)$ is an ideal of R. [2]
   (b) Show that there exists a unique homomorphism $\phi' : R/a(r) \to S$ such that $\phi = \phi' \circ \pi$, here $\pi : R \to R/(a(r))$ is the canonical ring homomorphism.[5]
   (c) Show that there exists exactly one ring homomorphism $\phi : \mathbb{Z}/15Z \to \mathbb{Z}/3\mathbb{Z}$ and the kernel ϕ is exactly $a(r)$ for some r.[3]

8. Show that $\mathbb{C}[x, y]/(xy - 1)$ is an integral domain and it is not isomorphic to $\mathbb{C}[t]$. Show that the maximal ideals of $\mathbb{C}[x, y]/(xy-1)$ are in bijection with the maximal ideals of $\mathbb{C}[x, y]$ of the form $(x - a, y - 1/a)$, where $a \in \mathbb{C}^*$. [10 + 10]

9. Let F be a finite field. Show that $F^* := F \setminus \{0\}$ is a finite cyclic group. Conclude that $x^{|F|} - x$ has all the roots inside F. Here $|F|$ denotes the cardinality of F. [10]

10. Let L be a field of cardinality 4. Find out an irreducible polynomial $f(x) \in \mathbb{F}_2$ such that $\mathbb{F}_2[x]/(f(x)) \cong L$. [5]

Indian Statistical Institute
Mid-Semester Examination
B-Stat (Hons.), 2nd Year, 2nd Semester (2018-19)
Subject : Statistical Methods IV
Date and Time : February 18, 2019, 2.30pm to 4.30pm

Answer all questions. Each question carries 15 marks.
If you use any mathematical result without proof, state the result clearly.

(1). Consider the data points (Y_i, X_i, Z_i) satisfying the linear model $Y_i = \alpha + \beta X_i + \gamma Z_i + \epsilon_i$ for $i = 1, \cdots, 15$, where the ϵ's are i.i.d normal with zero mean. It is given that $\sum_i Y_i^2 = \sum_i X_i^2 = \sum_i Z_i^2 = 4.0$, $\sum_i Y_i = \sum_i X_i = \sum_i Z_i = \sum_i X_i Z_i = 0.0$ and $\sum_i Y_i X_i = \sum_i Y_i Z_i = 2.0$. Compute the least squares estimates for α, β and γ and the unbiased estimates for the variances of these estimates.

(2). Consider n i.i.d observations $T_1, \cdots T_n$ on the life time of an electronic device that has exponential distribution with expected life μ years. Find the maximum liklihood estimate of μ based on these observations and derive an approximate 95% confidence interval for μ after making variance stabilizing transformation of the maximum likelihood estimate.

(3). Consider the simple linear model $Y_i = \beta i^\theta + \epsilon_i$ for $i = 1, \cdots, n$, where the ϵ's are i.i.d normal with zero mean. Determine the values of $-\infty < \theta < \infty$, for which the least squares estimate of β is consistent as $n \longrightarrow \infty$. Justify your answer.

(4). Consider 11 variables $Y, X_1, \cdots, X_{10}$ such that the correlation coeficient between any two variables is 0.75. Determine the multiple correlation coefficient between Y and $X_1, \cdots, X_{10}$.

Indian Statistical Institute

Mid-Semester Examination

B-Stat (Hons.), 2nd Year, 2nd Semester (2018-19)

Subject : Statistical Methods IV

Date and Time : February 18, 2019, 2.30pm to 4.30pm

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INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : 2018-19

Introduction to Stochastic Processes

B.Stat 2nd year

19^{th} February, 2019

Maximum marks: 50 Duration: 2hr 45 min

1. For each of the following statements, prove if it is true, or provide a counterexample if it is false :

   (a) If x is a recurrent state and y is a transient state in the same Markov chain, then the transition probability p_{xy} has to be 0.

   (b) Simple symmetric random walk on $\mathbb{Z}^d$ for $d \geq 2$ is an irreducible Markov chain with period d.

   (c) A transition matrix is called doubly stochastic if each of the column sums is 1 (in addition to each row sum being 1). If a finite Markov chain has a doubly stochastic transition matrix, it necessarily has the uniform distribution (over the state space) as a stationary distribution.

   (d) If the state space of a Markov chain has infinitely many disjoint irreducible closed sets, either all of them has to be finite, or all of them has to be infinite.

   (e) If an irreducible Markov chain with transition matrix P is periodic with period $d \geq 2$, it is possible to make it aperiodic if we change exactly two entries of P suitably (while still keeping the matrix stochastic). **5 x 4=20**

2. We define the following random walk on $\mathbb{Z}$, where a particle starts from the origin, and at each step either takes one step to the right with probability 2/3 or takes two steps to the left with probability 1/3.

   (a) Write down the transition matrix for this process.

   (b) Show that the Markov chain induced is irreducible.

   (c) For any $x, y \in Z$, we can find $r = r(x, y) \in \{0, 1, 2\}$ such that $x - y - r$ is a multiple of 3. Show that for any natural number n, $P^n(x, y) > 0$ iff $n - r$ is a multiple of 3 and $n \geq |x - y|$. Write down the expression of $P^n(x, y)$ when it is non-zero.

   (d) Show that the Markov chain induced is recurrent.

   (e) Is the chain null-recurrent or positive recurrent?

   (f) For any $x, y \in Z$, we take the sub-sequence of $P^n(x, y)$ whenever it is non-zero. Does it converge? If so, find the limit. **1+3+(3+2)+5+4+4=22**

3. For a branching process starting from one individual with offspring distribution having mean 2 and variance 1, if X_n indicates number of children of n-th generation, calculate the variance of X_n. $\boxed{8}$

4. A spider and an ant are at the opposite corners of a cube. The spider starts crawling across the boundary edges, and it takes one minute to cross any edge. After crossing each edge, the spider starts crawling across one of the three edges meeting at the corner with probability 1/3 (it does not remember any of its past journey). It stops when it meets the ant (which does not move).

   (a) For any two corners, we define their distance as the minimum number of edges the spider can take while travelling from one to another. Show that, if X_n indicates the distance between the spider and the ant after n minutes, it follows a Markov chain, and write down the transition matrix explicitly.

   (b) If T is the time after which the spider meets the ant, show that $\frac{T-1}{2}$ follows geometric distribution. Hence or otherwise find out $E(T)$. $\boxed{4+(5+4)=13}$

You are allowed to use your own handwritten note on one A4 sheet.

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : 2018-19

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B.Stat 2nd year

19^{th} February, 2019

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You are allowed to use your own handwritten note on <u>one</u> A4 sheet.

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination
B. Stat II year, 2nd Sem, AY 2018–2019
Discrete Mathematics

Date: 20. 02. 2019, Time: 3 Hours (2:30 PM to 5:30 PM)

Total Marks: 72, Buffer Marks: 12, Maximum Marks: 60

Please try to write all the part answers of a question at the same place.

1. (a) Is the *complement* of a set unique? Justify.

   (b) Count the numbers of *equivalence* relations and *partial order* relations on the set $A = \{1, 2, 3\}$.

   (c) Does $\mathbb{R}$ and $\mathbb{C}$ have the same *cardinality*? Prove your claim.

   [2 + (4 + 4) + 4 = 14]

2. (a) Is the *least* element in a POSET necessarily unique? Justify.

   (b) Can there exist a POSET with multiple *minimal* elements, but only one *least* element? Justify.

   (c) Find the fallacy in the following application of *strong induction*. Claim: Given $a \in \mathbb{R}^+$, one has that $a^n = 1$, $\forall n \in \mathbb{N}$ (assume that $\mathbb{N}$ includes 0). In the proof, show the base case for $n = 0$. Assume that $\forall k \leq n$, it holds. And now show that $a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$.

   (d) If a *logical theory* is *inconsistent*, what can we say about its *completeness*?

   [2 + 3 + 3 + 2 = 10]

3. (a) Count the number of *arrangements* of n distinct letters in n distinct envelopes so that exactly one letter goes to the correct envelope.

   (b) Count the number of integral solutions of the equation

   $$x_1 + x_2 + x_3 + x_4 = 18,$$

   that satisfy

   $$1 \leq x_1 \leq 5, \quad -2 \leq x_2 \leq 4, \quad 0 \leq x_3 \leq 5, \quad 3 \leq x_4 \leq 9$$

   using two methods: *inclusion-exclusion* principle and the method of *generating functions.*

   [4 + (8 + 4) = 16]

4. Use *generating functions* to

   (a) count the number of n-bit sequences where both zeros and ones appear even number of times.

   (b) evaluate $2 + 8 + 24 + 64 + 160 + 384 + \ldots$ up to n terms.

[4 + 6 = 10]

5. (a) Solve the following *recurrence* relation:

$$a_n = 6a_{n-1} - 9a_{n-2} + (n^2+1)3^n, \qquad \forall n \geq 2,$$

where $a_0 = 0$, $a_1 = 1$.

(b) A *divide and conquer* algorithm works on an integer array of size n. For $n \geq 2$, it divides the array into two almost equal halves and *recursively* processes each part. After the *recursive calls* return, it takes constant time 1 (i.e., just one elementary operation) to combine the solutions on the parts. Formulate a recurrence for the time complexity function $t(n)$ and use *induction* on n to show that $t(n) \in O(n)$. Note that n is any positive integer ≥ 2 and not necessarily a power of 2.

[8 + 6 = 14]

6. (a) Let $T(r,n)$ be the number of *onto functions* from a set with cardinality r to a set with cardinality n. Prove the following recurrence using *combinatorial argument*:

$$T(r,n) = nT(r-1,n-1) + nT(r-1,n).$$

Algebraic derivation using any explicit formula would lead to zero credit.

(b) Prove that the number of *p-partitions* of a positive integer n is equal to the number of *partitions* of $n + \binom{p}{2}$ into p *distinct* parts.

[4 + 4 = 8]

Indian Statistical Institute
Semester 2 (2018-2019)
B. Stat 2nd Year
Midsemestral Examination
Differential Equations

Thursday 21.2.2019, 2:30-4:30 PM Total Points: 30

Answers must be justified with clear and precise arguments. More than one uncrossed answers to the same question or part of a question will not be entertained and only the first uncrossed answer will be graded. *If you use any theorem or result proved in class state it explicitly.*

1. Solve the equations $4 + 4 = 8$ pts.

   (a) $dy/dx = (x + y + 4)/(x - y - 6)$,

   (b) $xy'' - y' = 3x^2$.

2. Find the general solution to the equation $(y')^2 = 1 + y^2$. 6 pts.

3. Consider the equation $y'' + P(x)y' + Q(x)y = 0$ where the coefficients $P(x), Q(x)$ are continuous on an interval $[a, b]$ so that Picard's theorem on unique solution is applicable. At some point $x_0 \in (a, b)$ consider two solutions $y_1(x), y_2(x)$ starting from initial conditions $(y_i(x_0), y_i'(x_0)) = \boldsymbol{\xi}_i, i = 1, 2$, where $\boldsymbol{\xi}_i, i = 1, 2$, are two linearly independent (constant) vectors in $\mathbb{R}^2$. 4+4 = 8 pts.

   (a) Consider the Wronskian $W(y_1, y_2; x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$. The derivative of W is easily computed using the fact that y_1, y_2 are solutions to the differential equation given. Show that $W \neq 0$ on $[a, b]$.

   (b) Show that $c_1y_1(x) + c_2y_2(x) = 0$, for all $x \in [a, b]$, where c_1, c_2 are some constants, implies $c_1 = c_2 = 0$.

4. Solve the equations: $4 + 4 = 8$ pts.

   (a) $y'' + 3y' - 10y = 6e^{4x}$,

   (b) $y'' + 4y = 4\cos 2x$.

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination 2018 – 2019
B. Stat. II Year
AGRICULTURAL SCIENCE

Date : 22.02.2019 **Maximum Marks : 30** **Duration :Two hours**

(Attempt any three questions)
(Number of copies of the question paper required : 15)

1. What is MAI? Characterize a rice-growing area in the eastern plateau region using MAI concept.

   4+6

2. Name different meteorological variables that are related to crop production. Write the names of the apparatus used to measure these variables with their unit. Write in details about the Stevenson Screen.

   4+4+2

3. Write short notes on the following
   a) Growing degree days b) Nowcasting c) cessation of the monsoon
   d) Phytoclimate e) Wind vane

   2 X 5

4. What is drought? What are the different types of drought? If an area is susceptible to late monsoon drought, what will be your recommendation for growing crops in that area?

   3+4+3

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2018-2019
Course Name: BStat II
Subject Name: Physics II

Date: 22 February 2019 **Maximum Marks: 40** **Duration: 2 hrs**

Note: Answer as many questions as you can.
Credit will be given for any positive attempt.
Calculators are allowed.

1. (a) Show that an observer moving with relativistic speed will measure a longer time interval from his frame as compared to the time interval in the rest frame. For convenience, you can consider motion along x-axis only.

   (b) An atomic clock is a classic example of time dilation. Its working principle is based on the fact that an atom makes transition between two of its internal energy states and the frequency change due to this transition can be measured in spectroscopes. The average kinetic energy of thermal motion of the atom is $\frac{3}{2}kT$.
   Given that the Boltzmann constant $k = 1.38 \times 10^{-23} m^2 Kg s^{-2} K^{-1}$ and a typical atom's mass $M \approx 1.67 \times 10^{-27} Kg$, find out the observed change in frequency due to this transition at room temperature.
   Note: You don't need to find out the exact numerical values. Results upto correct orders will do.

[5+5=10]

2. (a) (i) Consider a particle moving with velocity u' in the rest frame. Derive the relative velocity u of the particle as measured by an observer moving with velocity v with respect to the rest frame in the same direction.
   (ii) Can you explain from here that the speed of light in vacuum is the maximum attainable speed of a signal?

   (b) The universal speed of light c assumes vacuum as its medium of propagation. When light moves in a stationary transparent medium, its velocity v_m slows down by a factor n, called the refractive index of the medium, so that $v_m = \frac{c}{n}$. Suppose the medium moves with speed v with respect to a stationary observer in the lab frame. Show that for small v, the speed of light v' as measured by the stationary observer in the lab frame is approximately

$$v' \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right).$$

[(5+1)+4=10]

3. (a) A and B leave from a common point and travel in opposite directions with a relative speed $\frac{4}{5}c$. When B's clock shows that a time T has elapsed, he sends out a light signal. When A receives the signal, what time does A's clock show? Calculate first (i) in A's frame and then (ii) in B's frame, and show that you arrive at same results.

   (b) In a given reference frame, event 1 happens at $x = 0$, $ct = 0$, and event 2 happens at $x = 2$, $ct = 1$. Show that the speed of another frame from which the two events will appear as simultaneous is $\frac{c}{2}$.
   Note: You can prove it either mathematically or by logical argument.

[(3+3)+4=10]

4. (a) (i) Derive the expression for kinetic energy of a relativistic point particle moving with velocity v and show that it can be expressed as

$$E_K = m_0c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right],$$

   where symbols have their usual meanings.
   (ii) Can you justify from the above kinetic energy that the speed of any signal cannot exceed the speed of light in vacuum?

   (b) Show that the above expression for relativistic kinetic energy reduces to the usual Newtonian expression $E_K = \frac{1}{2}m_0v^2$ for low velocity limit.

[(7+1)+2=10]

5. (a) Prove that the mass-energy relation of a relativistic point particle is

$$E^2 = p^2c^2 + m_0^2c^4,$$

   where the symbols have their usual meanings.

   (b) Hence show that the dynamical definition of velocity can be given by

$$v = \frac{dE}{dp}$$

   (c) A large mass M, moving at relativistic speed v, collides with a small mass m which is initially at rest. The two masses stick together after collision. Considering $M \gg m$, can you find out what is the approximate mass of the resulting object?

[4+2+4=10]

Indian Statistical Institute
Mid-Semester Examination
Course Name: B.Stat Second Year (2018-19)
Subject Name: Economics II (Macroeconomics)

Date: 22.02.2019 Maximum Marks: 40 Time: 2.5 Hours

Answer the following questions:

1.(a) Suppose in a given year the residents of a country received Rs.50 lakh as interest on their government bond holding, Rs.26 lakh as interest from their bank deposits, Rs.10 lakh as interest from their holding of bonds issued by PSEs. How will you treat the above mentioned items while computing national income and personal income of the country? Explain your answer.

(b) This is a problem on national income accounting. We are comparing two different situations (i) and (ii) in two different countries. In situation (ii), the interest payment on public debt by the public administration and defence is found to be larger by 1000 units compared to that in situation (i). Everything else is the same in the two situations. Compare the levels of aggregate national saving (NS), aggregate private saving (S_{pvt}) and aggregate government saving in the two situations. [7+7=14]

2. (a) The following data are for a hypothetical economy for the year 2001 in billions of rupees:

Net rental income of persons	24.1	Purchase of goods and services by public administration and defence from other firms	1148.4
Depreciation	669.1	Import of goods and services	724.3
Wages and salaries of employees	3780.4	Interest payments received by the residents of the country (of which 100 units received from government bond holding)	724.3
The part of the above mentioned wage bill that is earned by the employees of public administration and defence sector	1000	Proprietors' income	399.5
Personal consumption expenditure	4378.2	Corporate profit	485.8
Total tax collection of the government	625.3	Net factor income from abroad	5.7
The part of the aforementioned tax that is collected as corporate profit tax, social security contribution and personal taxes	100	Government expenditure to compensate the banks for farm loan waiver	10

Donations made by firms to Political parties	525.3	Undistributed corporate profit	0
Gross investment	882.0	PSE Profit	0
Export of goods and services	659.1	Net Foreign Transfer	0
Subsidies paid by the government	9.0		

Government makes only those expenditures which are mentioned in the data given above.

Using the above information

(i) Compute GDP using expenditure method as well as income method. Compute the value of statistical discrepancy.

(ii) Compute personal disposable income

(b) Suppose a firm buys machinery worth Rs.10 lakh in the current period. Suppose the machines are supposed to last for 10 years. In another transaction, the firm buys a plot of land in the current period worth Rs.10 lakh for construction of an additional shed. What are the implications of the two aforesaid transactions in the profit calculation of the firms in the current period? Explain your answer. [20+6=26]

INDIAN STATISTICAL INSTITUTE

Semester Examination : 2018-19

Introduction to Stochastic Processes

B.Stat 2nd year

22^{nd} April, 2019

Maximum marks: 100 Duration: 4 hr

1. For each of the following statements, prove if it is true, or provide a counterexample if it is false :

   (a) If $f(x)$ is a finite-degree polynomial with non-negative coefficients and $f(1) = 1$, $f(P)$ is a stochastic matrix whenever P is a stochastic matrix.

   (b) We fix a state $x \in S$ of a recurrent Markov chain with period $d > 1$. There exists $K = K(x)$ such that $P^{kd}(x, x) > 0$ for all integer $k \geq K$.

   (c) A birth and death process on $\{0, 1, 2, \dots\}$ with birth and death rates indicated as $\mu_x = \lambda_{x-1} = x$ for all $x \in \mathbb{N}$ is transient.

   (d) For a Brownian motion $\{B_t\}_{t\geq 0}$, $\mathbb{P}[\min_{i\in\{1,2,\dots k\}} B(i) \geq 0] \geq 1/2^k$.

   (e) A stochastic matrix is called *regular* if $p_{xy} > 0$ for all $x, y \in S$. For an irreducible aperiodic recurrent Markov chain with transition matrix P, there exists a $d \in \mathbb{N}$ such that P^d is *regular*. **[5 x 7=35]**

2. Show that if a Markov chain with a *regular* transition matrix has the property that for some $j \in S$, $\inf_{i\in S} p_{ij} > 0$, then it is positive recurrent. **[10]**

3. For a non-homogeneous Poisson process $\{N_t\}_{t\geq 0}$ with intensity function $\lambda(t) = t^2$, find out $\mathbb{P}[\tau_2 - \tau_1 > t/3 | N_t = 2]$ **[10]**

4. We have 2 classrooms and $2n$ students- n boys and n girls. We distribute n students to each of the classrooms, and at any stage we pick one student uniformly at random from each classroom and swap them.

   (a) Show that the number of girls in the first classroom is a Markov process and write the transition matrix associated with it.

   (b) Show that the process is irreducible and find out the stationary distribution. Is the process time reversible with respect to it? **[3+(3+7+5)=18]**

5. A second order Markov chain is defined as an S-valued stochastic process $\{X_n\}_{n\in\mathbb{N}}$ which satisfies $\mathbb{P}[X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1] = \mathbb{P}[X_n = x_n, X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}] = \mathbb{P}[X_3 = x_n, X_2 = x_{n-1}, X_1 = x_{n-2}]$ for every n, every $x_1, x_2, \ldots x_n \in S$. In other words, X_n depends 'only' on location of X_{n-1} and X_{n-2}. Here is an example of such a process.

Let a plant have 3 genotypes for the color of its flower- red (TT), white (tt) or pink (Tt). While breeding a new plant, the genotype of the progeny is determined by joining one allele (T or t) obtained from each of its parents. A red flowered plant will pass on T allele, a white flowered plant will pass on t allele, and a pink-flowered one will pass on either allele with probability 1/2. Let X_n be the genotype of n-th plant, which has been crossbred between $n-1$-th and $n-2$-th plant. This process is second order Markov chain.

(a) Show that $Y_n = (X_{2n-1}, X_{2n})$ is a Markov chain.

(b) Write down the state space and the transition probability matrix.

(c) What are the absorbent states?

(d) What are the stationary distributions for this chain?

(e) If $X_1 = X_2 = Tt$, what are the probabilities of eventually reaching one of the absorbent states?

(f) Answer the same question above if $X_1 = TT, X_2 = tt$. Would the answer be same if $X_1 = tt, X_2 = TT$? If not, what would it be? **3+5+1+3+5+(6+4)=27**

6. There are $n \geq 2$ political parties in a country, and a leader L considers changing his affiliation only when he encounters a social turmoil. The social turmoils follow a Poisson process with parameter λ, and L chooses a party at random uniformly from all the choices (including the one he is currently affiliated to).

(a) Show that the number of 'switches' made by leader is also a Poisson process, and find out the parameter.

(b) Show that the number of times L is affiliated to a particular political party immediately after a turmoil also follows a Poisson process.

(c) At the beginning of the career, if the time till death of L follows exponential process with mean μ, show that the number of switches in his lifetime follows a Geometric process. What is the expected number of switches? **4+4+12=20**

You are allowed to use your own handwritten notes.

INDIAN STATISTICAL INSTITUTE
Second Semester Examination: 2018-2019
Course Name: BStat II
Subject Name: Physics II

Date: 24 April 2019 Maximum Marks: 75 Duration: 3 hrs

Note: Answer as many questions as you can.
Credit will be given for any positive attempt.

1. (a) Show that a moving rod appears shorter to a stationary observer. For convenience, you can consider motion along x-axis only.

   (b) The universal speed of light c assumes vacuum as its medium of propagation. When light moves in a stationary transparent medium, its velocity v_m slows down by a factor n, called the refractive index of the medium, so that $v_m = \frac{c}{n}$. Suppose the medium moves with speed v with respect to a stationary observer in the lab frame in the direction light moves. Show that for small v, the speed of light v' as measured by the stationary observer in the lab frame is approximately

$$v' \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right).$$

[7+8=15]

2. (a) The kinetic energy of a relativistic point particle moving with velocity v is given by

$$E_K = m_0 c^2 \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right],$$

   where symbols have their usual meanings. Show that the above expression reduces to the usual Newtonian expression $E_K = \frac{1}{2} m_0 v^2$ for low velocity limit.

   (b) A particle of mass m oscillates relativistically along x-axis under a force $F = -m\omega^2 x$ with an amplitude of oscillation b. Show that the period of oscillation is given by

$$T = \frac{4}{c}\int_0^b \frac{\gamma}{\sqrt{\gamma^2 - 1}}dx,$$

   where $\gamma = 1 + \omega^2(b^2 - x^2)/2c^2$.

[5+10=15]

3. (a) i. Using 4-vector notation, prove that the 4-momentum is a Lorentz-invariant quantity.

   ii. Can you give a relativistic interpretation of total energy from here? Justify your answer mathematically.

(b) Two objects A and B travel with velocities $\frac{4}{5}c$ and $\frac{3}{5}c$ respectively (with respect to a stationary observer sitting on the earth) along the same straight line in the same direction. How fast (with respect to the stationary observer) should another object C travel between them, so that it feels that both A and B are approaching C at the same speed?

[(5+3)+7=15]

4. (a) Prove that the electric potential $V \propto \frac{1}{r}$ is a solution of the Laplace's equation $\nabla^2 V = 0$.

   (b) Show that the work done due to a continuous charge distribution can be expressed as $W = \frac{\epsilon_0}{2}\int E^2 d\tau$, where the integration is taken over the entire space.

[7+8=15]

5. (a) Show that for any two vectors $\vec{A}_1$ and $\vec{A}_2$, the following relations hold good:

   i. $\vec{\nabla}.(\vec{A}_1 \times \vec{A}_2) = \vec{A}_2.(\vec{\nabla} \times \vec{A}_1) - \vec{A}_1.(\vec{\nabla} \times \vec{A}_2)$

   ii. $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}_1) = \vec{\nabla}(\vec{\nabla}.\vec{A}_1) - \nabla^2 \vec{A}_1$.

   (b) Derive Poisson's equation for magnetic field $\nabla^2 \vec{A} = -\mu_0 \vec{J}$, where $\vec{A}$ and $\vec{J}$ are magnetic potential and volume current density respectively.
   Hint: In answering (b) you can take help of (a), if required.

[(5+5)+5=15]

6. (a) Is the vanishing of the curl of electric field still valid in presence of a variable magnetic field? Justify your answer briefly.

   (b) Show that the electric field in electrodynamics is corrected to $\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$, where V and $\vec{A}$ are scalar and vector potentials respectively.

   (c) The electromagnetic field tensors are given by the matrix elements of two traceless antisymmetric matrices

$$F^{\mu\nu} \equiv \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

   and $G^{\mu\nu} \equiv \vec{E}/c \rightarrow \vec{B}, \vec{B} \rightarrow -\vec{E}/c$ is its dual.
   Derive the good old expressions for (i) Ampere's law and (ii) Faraday's law using the relativistic electrodynamics formulas

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \qquad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0,$$

   where J^μ is the 4-conserved current.

[2+3+(5+5)=15]

Indian Statistical Institute
End Semester Examination 2019
Course: BStat II Year
Subject: Economics II (Macroeconomics)

Date : 24.04.19

Maximum Marks: 60

Time: 2.5 Hours

Answer all questions

1. In a Simple Keynesian Model, the income earners are classified into two groups, namely, Group 1 and Group 2. The former earns a fixed income of 2000 units, while the latter earns the remaining part of the GDP, $(Y - 2000)$, where Y denotes GDP of the economy. Group 1's apc is 0.75, while that of Group 2 is 0.5. Average import propensity of Group 2 is 0.25, but Group 1 does not spend on imported goods. Regarding other components of aggregate final demand, the following assumptions are made. $X = \bar{X} > 0$ and $I = \bar{I} > 0$ half of which is spent on imported goods and $\bar{G} = 0$. Now, suppose $\bar{G}$ rises from 0 to 400 units, which is spent on domestic goods only and this additional government expenditure is financed by means of a lump sum tax of 400 units imposed on Group 1. Compute the resulting change in the equilibrium level of Y. Describe the multiplier process. [20]

2. Consider a Simple Keynesian Model for an open economy, where the marginal propensity to consume and the marginal propensity to invest with respect to GDP (denoted Y) are 0.8 and 0.3 respectively. It is given that 20 percent of aggregate planned consumption expenditure is spent on imported goods. A fixed fraction of investment expenditure is also spent on imported goods. An exogenous increase in export demand, $\bar{X}$, by 1 unit is found to raise Y by (25/6) units. Derive the value of the fixed fraction of investment expenditure that is spent on imported goods. (For simplicity, do not incorporate government's activities). [20]

3. Consider a Simple Keynesian Model given by C = 160 + 0.8Y, G = 30, X = 10 and M = 50 + 0.2Y, where Y denotes GDP and all other symbols have their usual meanings. Aggregate planned investment denoted I is constrained by the supply of credit. Now, to hold the exchange rate e (price of foreign currency in terms of domestic currency) at the given target level, the central bank buys 100 units of foreign exchange in the given period. CRR and currency-deposit ratio are 0.5 and 0.5 respectively. Find out the resulting impact on the supply of high-powered money, money supply and Y. [20]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: (2018 – 2019)

B. Stat II Year

Agricultural Science

Date 24.04.19 Maximum Marks 50 Duration 3:00 hours.

(Attempt any five questions)

(Number of copies of the question paper required 20)

1. a) Define the term 'parent material'.
   b) What is the basic difference between granular and crumb structure? What determines soil texture? "Soil texture is basic physical property whereas structure is not"-justify.
   c) What is air filled porosity?
   d) Give the definition of nutrients. What are the cationic and anionic micronutrients found in soil? Why they are so called 'micronutrient'?
   e) What is complex fertilizer? 1+4+1+3+1

2. a) How bulk density value varies with soil porosity?
b) A 1200 g moist soil sample having a volume of 800 cm^3 is oven dried to a dry mass of 1000 g (dry wt of soil). If the particle density is 2.65 Mg/m^3, calculate bulk density and porosity (% of pore space).
c) The maximum water holding capacity, field capacity of a soil for 0-30 cm soil depth are 48% and 25% respectively. If the bulk density of the soil is 1.5 Mg/m^3, calculate i) drainable water and ii) available water for 30 cm soil depth.
d) What is available water? What about the availability of 'capillary water' found in soil?
1+3+3+3

3. Write the different types of rice with their suitable varieties. Briefly describe the cultural practices associated with the transplanted rainfed rice.
3+7

4. What are the elements essential for plants? Briefly mention the criteria for the essentiality of plant nutrients. Calculate the quantity of VC, Urea, SSP and KCL required for 1.5 hectare potato crop to supply the nutrient requirement of 200 kg N, 120 kg P_2O_5 and 120 kg K_2O per hectare. Note that 50% of required N should be given through VC.
2+2+6

5. What are the growth related and yield attributing chatercters of rice. Estimate the yield per hectare of mustard crop from the following data. Mustard was sown at 30x20 cm apart.
(i) Average no. of branches/plant – 25, (ii) Average no. of silique/plant – 132, (iii) Average no. of seeds/siliqua – 8, (iv) Average siliqua length - 3 cm, (v) Test weight - 28 g.
4+6

6. Wheat-Mustard intercropping experiment was done in 2:1 and 2:2 row replacement series system and a data set are given in the following table. In your opinion, which combination is the best intercropping system ? 10

Cropping System	Wheat yield in kg/ha	Mustard yield in kg/ha
Wheat Sole	5624	-
Mustard Sole	-	4533
Wheat+Mustard (2:2)	2463	2123
Wheat+Mustard (2:1)	2122	1572

7. Write the differences between: 2 x 5

a) Physical weathering and chemical weathering
b) Inter cropping and mixed cropping
c) Mass flow and diffusion
d) Macro and micro nutrients
e) Field capacity (FC) and permanent wilting point (PWP)

Indian Statistical Institute
Semester 2 (2018-2019)
B. Stat 2nd Year
Semestral Examination
Differential Equations

Date: 26.4.2019, 2:30-5:30 PM **Total Points: 70**

Answers must be justified with clear and precise arguments. More than one uncrossed answers to the same question or part of a question will not be entertained and only the first uncrossed answer will be graded. If you use any theorem or result proved in class state it explicitly.

1. (a) Consider the differential equation $x^2y'' - 3xy' - 5y = 0$. Try solutions of the form x^m to find the general solution on any interval not containing zero. 7 pts.

   (b) Find the general solution of the (nonlinear) differential equation $y'' + (y')^2 = 0$. 7 pts.

2. Find a particular solution of $y'' + y = 1/\sin x$. The method of variation of parameters may be used which finds linearly independent solutions y_1, y_2 to the homogeneous equation and then assumes a particular solution of the form $y_1v_1(x) + y_2v_2(x)$ where v_1, v_2 are assumed to satisfy $v_1'y_1 + v_2'y_2 = 0$. 14 pts.

3. Consider the equation $x^2y'' + xy' + (x^2 - 1)y = 0$ near the regular singular point zero, actually to the right of zero.

   (a) Find the Frobenius series solution corresponding to the larger root of the indicial equation (coefficients upto x^4 must be computed exactly). 7 pts.

   (b) Show that there is no other Frobenius series solution, i.e. the other solution will have a $\log x$ term present. 7 pts.

4. Find the general solution to the linear first order system $x_1' = x_1 - x_2, x_2' = x_1 + 3x_2$. The final answer should be expressed using two parameters which can be varied independently of each other. 14 pts.

5. A uniform flexible chain of given length hangs between two points. Find its shape if it hangs in such a way as to minimize its potential energy, that is find the curve that minimizes $\int_{x_1}^{x_2} y ds$ subject to $\int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx = c$. The answer must be clearly expressed in terms of the $\cosh x$ function where $\cosh x = (e^x + e^{-x})/2$. You may use the following $\int dy/\sqrt{y^2 - a^2} = \cosh^{-1}(y/a)$. 14 pts.

INDIAN STATISTICAL INSTITUTE
Semester Examination
B. Stat II year, 2nd Sem, AY 2018–2019
Discrete Mathematics

Date: 29. 04. 2019, Time: 3 Hours (2:30 PM to 5:30 PM)

Total Marks: 90, Buffer Marks: 10, Maximum Marks: 80

Please try to write all the part answers of a question at the same place.

1. (a) Sometimes an infinite set is informally defined as "a set that is not finite." This is fallacious, since this is a circular definition. So how to formally define an infinite set?

   (b) Describe *Russel's paradox.*

   (c) Does *Zermelo's well-ordering theorem* imply the *axiom of choice*? What about the converse?

   (d) Are *propositional logic* and *first order predicate logic* decidable? Formally prove the correctness of "proof by contradiction" method.

   [3 + 3 + (2 + 2) + (2 + 4) = 16]

2. (a) Derive the generating function of the number of distributions of r indistinguishable balls into n indistinguishable cells with no cells empty. Can you derive a closed-form expression, if the cells become distinguishable?

   (b) What is the number of combinations of at least one out of n_i indistinguishable objects of type i, $i = 1, 2, \ldots, k$?

   (c) Prove that the number of integer-partitions of a positive integer n into distinct odd parts is equal to the number of self-conjugate partitions.

   [(4 + 3) + 3 + 6 = 16]

3. (a) Every point in a plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all of its vertices have the same color.

   (b) Prove that 9 is the minimum integer n such that every coloring of the edges of K_9 with red and blue will yield either a red triangle or a blue quadrilateral.

   (c) Suppose m is the largest size of an antichain A in a POSET X such that A is neither the set of all maximal elements, nor the set of all minimal elements, of X. Prove that X can be partitioned into at least m many chains.

   [6 + 6 + 6 = 18]

4. (a) What is the expression of the number of distinct labeled unrooted trees with $n \geq 2$ vertices having vertex degrees $d_1, d_2, \ldots, d_n$? (No proof required).

(b) Recall that we had cleverly used the construction in the proof of *Caley's theorem* (establishing bijection between trees and sequences of integers from $\{1, 2, \ldots, n\}$) to derive the above expression. Now, can you prove Caley's theorem, just assuming the above expression?

[4 + 6 = 10]

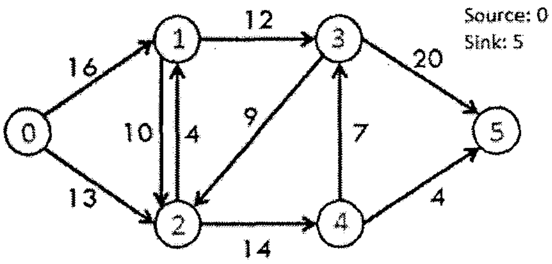
5. (a) Can the finite and the infinite regions/faces created by a planar graph be treated in the same manner in the context of planarity related problems?

   (b) Why do the results on vertex coloring of a planar graph carry to the face coloring as well?

   (c) Is the *hypercube graph* (formed by the vertices and edges of a hypercube) always Hamiltonian? Is it always Eulerian?

[4 + 4 + (4 + 2) = 14]

6. (a) Apply Ford-Fulkerson algorithm to find the maximum flow in the given network (numbers along edges show the capacities).



   Show the flow along each edge for every iteration.

   (b) Instead of standard coloring of the Rubik's cube, suppose we create variants by using a palette of 8 colors and also take out the restriction that each face must have different colors. How many distinct coloring is possible, considering the rotation symmetry?

[8 + 8 = 16]

Indian Statistical Institute
Semestral Examination
B-Stat (Hons.), 2nd Year, 2nd Semester (2018-19)
Subject : Statistical Methods IV
Date and Time : May 3, 2019, 10.30am to 1.30pm

Answer all questions. Each question carries 15 marks.
The assignments will carry 10 marks.
If you use any mathematical result without proof, state the result clearly.

(1) For $1 \leq i \leq 5$, consider the data points (i, Y_i) satisfying the linear model $Y_i = \beta i + \epsilon_i$, where $Y_1 = 2.5$, $Y_2 = 4.4$, $Y_3 = 5.6$, $Y_4 = 8.7$, $Y_5 = 9.6$, and the ϵ_i's are i.i.d normal with zero mean. Compute the maximum likelihood and the least absolute deviations estimates for β.

(2) In question (1) above, describe how one can get an estimate of the variance of the least absolute deviation estimate of β.

(3). Consider the p.d.f $f(x) = c\exp\{-(x-\mu)^4\}$, where $-\infty < x, \mu < \infty$ and c is an appropriate constant. Determine c. Find the asymptotic relative efficiency of sample mean relative to sample median as estimates of μ based on i.i.d. observations from this p.d.f.

(4). Consider two continuous strictly increasing distribution functions F and G. Suppose that we have two i.i.d. observations from each of them, and we compute the Kolmogorov-Smirnov statistic based on their empirical distributions. What are the possible values of Kolmogorov-Smirnov statistic? Compute the probabilities for those values under the null hypothesis $F = G$.

(5). Consider i.i.d. observations $X_1, X_2, \cdots, X_n$ uniformly distributed in the interval $[0, \theta]$. Describe the most powerful test for testing $H_0 : \theta = 1$ against the alternative $H_A : \theta = 2$ with size 5%. Is the test consistent? Justify your answer.

(6). In a class of 400 students, 60 scored above 80% in both mathematics and physics, while 200 scored below 80% in both mathematics and physics. 40 students scored above 80% in physics but failed to score above 80% in mathematics. Carry out an appropriate test to determine if there is significant evidence in the data indicating dependence in the performance in physics and mathematics.

Indian Statistical Institute
Back Paper Examination 2019
Course: B-Stat II Year
Subject: Economics II (Macroeconomics)

Date: 12.07.19

Maximum Marks 100 — Duration 3 Hours

Answer all questions

1. (i) Consider a Simple Keynesian Model for an open economy without government activities. Suppose 20 per cent and 60 per cent of aggregate planned consumption expenditure and aggregate planned investment expenditure respectively are spent on imported goods. Suppose the marginal propensity to consume and marginal propensity to invest (net) with respect to NDP (Y) are 0.8 and 0.3 respectively. Given the above information, answer the following questions under the assumption that all the relations in the model are linear:

a. Compute the marginal propensity to import with respect to Y and check if the equilibrium is stable.

b. Compute the autonomous expenditure multiplier in this model.

(ii) Suppose in a simple Keynesian model for a closed economy with government C = 0.75Y, G = 0 and I = 2000 – 100i, i_c= 8, m = 1 and m_d = 1 (where i_c denotes the repo rate and m_d and m are the mark-ups applied to the repo rate and the deposit rate respectively by the commercial banks to determine their deposit rate and the lending rate; all other symbols have their usual meanings) . Write down the equation of the aggregate demand function. Derive the equilibrium value of Y. To attain full employment, Y has to rise by 2000 units. By how much the central bank has to lower the repo rate to achieve full employment? **[8+4+13=25]**

2.a) Consider a closed economy without government. Suppose in a given period the households of the economy purchased goods and services worth Rs.20000 crore. Firms purchased goods worth Rs. 10000 crore and added them to their stock of fixed capital and inventory. However, all the firms together produced goods and services worth Rs.35000 crore. They, therefore, could not sell Rs.5000 crore worth of goods and had to hold them involuntarily in their inventory. What is the aggregate level of investment made by firms in this case? Does the identity $\text{GDP} \equiv C + I$ hold in this case?

b) Consider the police force of a country. Its job is to maintain law and order of the country. Suppose that in a given year it spent Rs.10 crore as interest on its outstanding loans to households, Rs. 20 crore as wages and salaries, Rs.10 core on stationeries, fuel and power and Rs.30 crore to acquire buildings, equipments and cars. What is the GVA of this sector? Did it contribute anything to G? Did it contribute anything to transfer payments?

c) Consider a closed economy with government. Suppose that in a certain year the private and public sector enterprises produced goods and services worth Rs.10 crore. In the same year they sold goods and services worth Rs.8 crore to households and Rs.5 crore to government. Besides these, the government paid Rs.10 crore as wages and salaries and Rs. 1 crore as interest to households. What is the aggregate firm investment in the given year? Explain. Calculate the GDP of the economy using the spending approach. **[10+10+10=30]**

3. Consider a simple Keynesian model for a closed economy without government. At GDP (Y) = 1000, producers have to sell 20 units from their stock to meet the customers' demand fully. It is given that the equilibrium level of GDP is 1040. Find out the impact of an increase in autonomous expenditure on the equilibrium level of Y, assuming the expenditure function to be linear. **[20]**

4. Suppose the CRR and the currency-deposit ratio in an economy are 0.25 and 0.5 respectively. Start with an initial equilibrium situation, where the banks are fully loaned up. The economy has a fixed exchange rate regime. Now suppose the central bank buys 100 units of foreign exchange from the market. In the light of the information given above, answer the following questions:

(a) Show the resulting changes in the stocks of high-powered money and money supply, when the amount of excess demand for bank credit existing in the initial equilibrium situation is adequate for the money multiplier process to work out fully.

(b) How will your answer to (a) change, if the excess demand for credit in the initial equilibrium situation were 40 units?

(c) How will your answer to (b) change, if the change in money supply were due to a commercial bank having taken a loan of 60 units from the central bank? **[8+8+9=25]**

INDIAN STATISTICAL INSTITUTE

Backpaper Examination : 2018-19

Introduction to Stochastic Processes

B.Stat 2nd year

Maximum marks: 100 Duration: 3 hr 30 min

1. Two irreducible Markov chains on same state space with transition matrices P and P' had periods d and d'. Show that the Markov chain with transition matrix $1/2(P+P')$ has period $gcd(d,d')$. **10**

2. $N = 2^k$ Gamblers take part in a knock-out gambling competition with capital $x_1, x_2, \ldots, x_N$ $(k, x_i \in \mathbb{N})$. In the first round, the gamblers are paired randomly, and every pair plays a match between them- each keep on betting unit capital against each other on the outcome of an unbiased coin till one loses all of his/her capital. The winner moves on the next round with the added capital and after k rounds, one champion wins it all. Given that a player P has won j th round, show that the probability of P winning is proportional to the capital of P at that stage. Using this or otherwise, calculate the probability of i-th gambler becoming the champion and show that it does not depend on the pairing. **7+5=12**

3. We define a simple random walk on $\mathbb{N} \cup \{0\}$ reflected at 0, i.e. a particle begins at 0, and at any stage if it is at $x \in \mathbb{N}$, it jumps to $x+1$ with probability $p \in (0,1)$ and to $x-1$ with probability $1-p$. If the particle is at 0 at any stage, it simply jumps back to 1.

   (a) Write down the transition matrix and show that the process is irreducible.

   (b) For what values of p is it transient? For those values find out $\rho_{00} = \mathbb{P}_0[\tau_0 < \infty]$ as a function of p.

   (c) For those values of p is it positive recurrent? Calculate the stationary distribution for those cases. Is it time-reversible with respect to this distribution?

   4+ (5+3) +(4+7+5)=28

4. The number of offsprings in a branching process follow a Geometric distribution with parameter $p \in (0,1)$ (i.e. the probability of having k offsprings is same as getting k tails before the first head in a series of coin tosses where the probability of landing head is p). Calculate the extinction probability for the same. **12**

5. For two independent Poisson processes $X_1(t)$ and $X_2(t)$ on $\mathbb{R}^+ \cup \{0\}$ with parameter λ, we define $X(t) = X_1(t)\mathbb{1}_{t\geq 0} - X_2(t)\mathbb{1}_{t<0}$. Show that

   (a) $\{X(t)\}_{t\in\mathbb{R}}$ has stationary independent increments, and

   (b) $X(t_2) - X(t_1)$ follows Poisson distribution with parameter $\lambda(t_2 - t_1)$ for $t_2 > 0 > t_1$. **6+4=10**

6. On an 5x5 chessboard, a lone knight start from any square and it jumps uniformly randomly to a square it is legally allowed to. Find out the expected number of steps for it to return to the square from which it started its journey, if that is a corner square. Would it change if it started from the central square instead? If so, how? **11+4=15**

7. We have a single server infinite capacity queue, where the service time is exponentially distributed with parameter μ and the customers arrival time exponentially distributed with parameter λ. However, immediately after arrival, the customers look at the size of the queue $N \geq 0$, and if $N \geq 2$, he/she decides to stay with probability $1/N$, or leaves immediately. Calculate the stationary distribution of this process. **15**

You are allowed to use your own handwritten notes.

Indian Statistical Institute
Back Paper Examination
B-Stat (Hons.), 2nd Year, 2nd Semester (2018-19)
Subject : Statistical Methods IV
Date and Time : July 17, 2019, 02.30pm to 05.30pm

Answer all questions. Each question carries 20 marks.
If you use any mathematical result without proof, state the result clearly.

(1) Consider the data points (Y_i, X_i, Z_i) satisfying the linear model $Y_i = \alpha + \beta X_i + \gamma Z_i + \epsilon_i$ for $i = 1, \cdots, 15$, where the ϵ's are i.i.d normal with zero mean. It is given that $\sum_i Y_i = \sum_i X_i = \sum_i Z_i = \sum_i X_i Z_i = 0.0$, $\sum_i Y_i^2 = \sum_i X_i^2 = \sum_i Z_i^2 = 4.0$ and $\sum_i Y_i X_i = \sum_i Y_i Z_i = 2.0$. Compute the least squares estimates for α, β and γ and the unbiased estimates for the variances of these estimates.

(2). In a class of 400 students, 60 scored above 80% in both mathematics and statistics, while 200 scored below 80% in both mathematics and statistics. 100 students scored above 80% in ~~physics~~ statistics but failed to score above 80% in mathematics. Carry out an appropriate test to determine if there is significant evidence in the data indicating dependence in the performance in physics and mathematics.

(3) For $1 \leq i \leq 5$, consider the data points (i, Y_i) satisfying the linear model $Y_i = \beta i + \epsilon_i$, where $Y_1 = 2.5,\ Y_2 = 4.4,\ Y_3 = 5.6,\ Y_4 = 8.7,\ Y_5 = 9.6$, and the ϵ_i's are i.i.d having common p.d.f $(1/2\sigma)\exp\{-|\epsilon|/\sigma\}$, where $-\infty < \epsilon < \infty$ and $\sigma > 0$ is unknown. Compute the maximum likelihood estimates of β and σ. Describe how one can get an estimate of the variance of the maximum likelihood estimate of β.

(4). Consider $d+1$ variables $Y, X_1, \cdots, X_d$ such that the correlation coefficient between any two variables is $0 < \rho < 1$. Determine the multiple correlation coefficient between Y and $X_1, \cdots, X_d$.

(5). Consider i.i.d. observations $X_1, X_2, \cdots, X_n$ uniformly distributed in the interval $[\theta,\ \theta + 1]$. Describe the most powerful test for testing $H_0 : \theta = 0$ against the alternative $H_A : \theta = 0.5$ with size 5%. Is the test consistent? Justify your answer.

Indian Statistical Institute
Semester 2 (2018-2019)
B. Stat 2nd Year
Backpaper
Differential Equations

Date: 11.07.19 **Total Points:** $5 \times 20 = 100$.

Answers must be justified with clear and precise arguments. More than one uncrossed answers to the same question or part of a question will not be entertained and only the first uncrossed answer will be graded. If you use any theorem or result proved in class state it explicitly.

1. Find the orthogonal trajectories to the family of circles $x^2 + y^2 = 2cx$.
2. For the equation $(x+2)y'' - (2x+5)y' + 2y = (x+1)e^x$ it is seen that $y_1 = e^{2x}$ is a solution of the homogeneous equation. Find a particular solution of the nonhomogeneous equation of the form vy_1 by finding an equation for v.
3. Use variation of parameters to solve $y'' + 2y' + y = e^{-x}\log x$.
4. Find two linearly independent Frobenius series solutions to $xy''+2y'+xy = 0$, to the right of 0.
5. A curve in the first quadrant joins $(0,0)$ and $(1,0)$ and has a given area beneath it. Show that the shortest such curve is an arc of a circle.