INDIAN STATISTICAL INSTITUTE, KOLKATA
End Semester Examination, First Semester 2019-2020
Analysis III, B.Stat 2nd year
Total Marks -50, Time : 3 hours , Date : 29.11.2019

1. Let $\omega = (y + y^2 z)dx + (x - z + 2xyz)dy + (-y + xy^2)dz$ be a 1-form defined on $D := \{x \in \mathbb{R}^3 | |x| < 3\}$.
   (a) Show that $d(\omega) = 0$. [3]
   (b) Find a function $f$ such that $d(f) = \omega$. [7]

2. Let $f(x, y) = x + 4y + 2/xy$.
   (a) Find the critical point(s) of $f(x, y)$. [5]
   (b) Test the nature of the critical point(s) found in part (a). [5]

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that $f$ can not be $1 - 1$.[10]

4. Let $P$ be the parallelogram in $\mathbb{R}^2$ with vertices $(1, 1)$, $(3, 4)$, $(5, 8)$ $(3, 5)$. Find suitable map $T : I^2 \to P$ to convert the integral $\int_H e^{x-y} dxdy$ to an integral over the unit square $I^2$ and evaluate the integral. [10]

5. Let $S$ be the surface given by the quarter of the right-circular cyclinder centered on the $z$-axis, of radius 2 and height 4, which lies in the first octant.
   (a) Let $F(x, y, z) = (x, 0, 0)$ be the vector field. Use the normal which points "outward" from $S$, i.e. on the side away from $z$-axis to compute the flux integral $\int\int_S F.\hat{n} dS$. [5]
   (b) Let $D$ be the 3-dimesnional solid in the first octant given by the interior of the quarter cylinder defined above. Use the divergence theorem to compute the flux of the field $F$ out of the region $D$. [5]
   (c) What is the flux of $F$ across all faces of $D$ ? [5]

6. Use Green's Theorem to evaluate

$$\int_C (sin(y) - x^2)dx - (x - cos(x)y^3)dy$$

where $C$ is the boundary of the rectangle with vertices $(0, 0), (6, 0), (0, 4), (6, 4)$ traversed counterclockwise. [5]

# INDIAN STATISTICAL INSTITUTE

Semestral Examination : First Semester 2019-20

Course : B Stat IInd Year

Subject : Elements of Algebraic Structures

Date : 25/11/2019 Maximum Marks : 50 Duration : 3 Hours

**CLEARLY STATE THE RESULTS WHICH YOU MAY USE. USE SEPARATE PAGE FOR EACH QUESTION. IF YOU WISH YOU MAY ANSWER MORE THAN SIX QUESTIONS BUT ONLY SIX WILL BE NECESSARILY GRADED. AT THE TOP OF YOUR ANSWER SCRIPT YOU CAN MENTION WHICH SIX. IF NOTHING IS MENTIONED FIRST TWO FROM EACH SECTION WILL BE GRADED.**

Section A (Answer any two)

1. Up to isomorphism how many groups are there of order 2019? Prove your assertion. [2+8=10]

2. Let $\pi$ be a permutation of $\{1, \ldots, n\}$. Then show that $sign(\pi) = (-1)^{\#\{(i,j):i<j,\pi(j)<\pi(i)\}}$. [10]

3. Consider the collection of all possible necklaces that one can make with 13 beads where for each bead there are three possible colors Red, Yellow, Green. If one picks a necklace at random what is the probability that it does not contain any red bead? [10]

Section B (Answer any two)

4. Let $R$ be a UFD and $F$ its quotient field. Let $f \in R[X]$ be a monic polynomial and $g \in F[X]$ be a monic factor of $f$. Then show that $g \in R[X]$. Conclude that $X^3 - X - \frac{1}{3}$ is irreducible over $\mathbb{Q}$. [5+5=10]

5. Let $G, H$ be finite abelian groups such that for all natural numbers $n$ both $G, H$ have same number of elements of order $n$. Show that $G$ is isomorphic with $H$. [10]

6. Let $k$ be a field. For a polynomial $P(X,Y) = \sum a_{ij}X^iY^j$, let $\phi(P) : k \times k \to k$ be the map $\phi(P)(x,y) = \sum a_{ij}x^iy^j$. We know that $\Phi : P \mapsto \phi(P)$ is a homomorphism from $k[X,Y]$ to $k^{k\times k}$. Show that $\Phi$ is one to one iff cardinality of $k$ is infinite. [3+7=10]

PTO

Section C (Answer any two)

7. Let $k \subseteq K$ be a field extension. Let $f, g \in k[X]$. let $r$ be a g.c.d of $f, g$ in $k[X]$ then show that $r$ is also g.c.d of $f, g$ in $K[X]$. If $\alpha \in K$ is a common root of both $f$ and $g$. Then show that degree of $r$ is at least one. [7+3=10]

8. (a) Suppose $K$ is an integral domain and $k$ is a subring that is a field, so that $K$ is a vector space over $k$. Show that if $dim_k K$ is finite then $K$ is a field. [7]

   (b) Let $k \subseteq K$ be a finite extension of the form $K = k(\alpha)$ with $[K : k]$ odd. Prove that $K = k(\alpha^2)$. [3]

9. Let $k \subseteq K$ be a field extension of finite degree and $K = k[\alpha, \beta]$. Show that if $[k(\alpha) : k]$ is relatively prime to $[k(\beta) : k]$, then (i) the minimal polynomial of $\alpha$ over $k$ is irreducible over $k(\beta)$, (ii) $[K : k] = [k(\alpha) : k][k(\beta) : k]$. [5+5=10]

# Probability Theory III
## B. Stat. IInd Year Semester 1
## Indian Statistical Institute

Semestral Examination

Time: 3 Hours Full Marks: 50

Date: November 18, 2019

This is an OPEN NOTE examination. You may use any handwritten notes that you have brought with you, but you are not allowed to share them. Also no notes in printed, photocopied, cyclostyled or any other form (including electronic storage) is allowed. Calculators and any other computing devices are strictly not allowed. State clearly all the results you are using.

1. Let $\{X_n\}$ be a collection of integer valued random variables, which converges in distribution to another random variable $X$. Show that $X$ is also integer valued and for every integer $j$, we have $\mathrm{P}[X_n = j] \to \mathrm{P}[X = j]$. Further show that $\sum_{j\in\mathbb{Z}} |\mathrm{P}[X_n = j] - \mathrm{P}[X = j]| \to 0$. [3+3+3=9]
2. Let $F_n$ be exponential distribution function with mean $1/\lambda_n$. Show that $\{F_n\}$ is tight iff $\{\lambda_n\}$ is bounded away from zero. [6]
3. Let $X$ be a Poisson random variable with mean $\lambda$. Show that, $(X - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard normal random variable as $\lambda \to \infty$. [7]
4. For a continuous real valued function $f$ defined on $[0, 1]$, calculate
$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f\left((x_1 \cdots x_n)^{1/n}\right) dx_1 \cdots dx_n.$$
[6]
5. Let $\{X_k\}$ be an independent sequence of random variables having gamma distribution with mean $3k$ and variance $3k^2$ respectively. Show that $\sum_{k=1}^n \frac{1}{X_k} - \frac{1}{2}\log n$ converges in probability. [7]
6. For an independent sequence of random variables $\{X_n\}$, prove that $\sup X_n < \infty$ almost surely iff $\sum_n \mathrm{P}[X_n > A] < \infty$ for some positive finite number $A$. [6]
7. Suppose that $\{X_n\}$ are independent nonnegative random variables with mean $1/n$ and variance $\sigma_n^2$ respectively, such that $\{n\sigma_n^2\}$ is bounded. If $S_n = X_1 + \cdots X_n$, show that $S_n/\log n \to 1$ in probability. Further show that the convergence happens almost surely as well. [3+6=9]

# INDIAN STATISTICAL INSTITUTE

## Back paper Semester Examination: 2019-20

### B STAT, SECOND YEAR

### Statistical Methods - III

Date: 15.1.2020 Maximum Marks: 100 Duration: 3.5 hours

1. State and prove Cramer-Rao inequality. (8)

2. Suppose $X_1, \ldots, X_n$ be a random sample from

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}; \; x > 0.$$

Find MVUE of $\theta$, deriving all intermediate steps and mention clearly any result and/or theorem that you want to use. (11)

3. A famous mountaineer once told that accidents occur more frequently while descending after summit than while climbing up towards the summit. In 1996, 8 people died while climbing down after reaching the summit whereas 5 people died while ascending. Based on the given data, what is your opinion? Justify it. (10)

4. Suppose that when a signal having value $\mu$ is transmitted from location $A$, the value received at location $B$ is normally distributed with mean $\mu$ and variance 4. That is, if a value $\mu$ is sent, then the value received is $\mu + N$ where $N$, representing noise, is normal with mean 0 and variance 4. To reduce error, suppose the same value is sent 9 times. The successive values received are $5, 8.5, 12, 15, 7, 9, 7.5, 6.5, 10.5$. (6+5+7+7=25)

   (a) Assuming that the values are independent, suggest MVUE for $\mu$ and justify your answer.

   (b) Is this estimator also the MLE? Justify your answer.

   (c) Suggest an unbiased estimator of $\mu^2$ and find its variance.

   (d) Numerically calculate the estimates obtained in (a) – (c).

5. There are three types of blood groups in humans, $O$, $A$, $B$, and $AB$. Blood samples of 870 randomly selected individuals are taken and their blood groups are determined. It is known that the probabilities that a person has $O$, $A$, $B$, and $AB$ blood groups are $r^2$, $p^2 + 2pq$, $q^2 + 2qr$, and $2pq$ respectively. Table 1 gives the frequency of each blood group with respective probability.

| blood group | probability | observed frequency |
|---|---|---|
| $O$ | $r^2$ | $n_O = 352$ |
| $A$ | $p^2 + 2pr$ | $n_A = 364$ |
| $B$ | $q^2 + 2qr$ | $n_B = 120$ |
| $AB$ | $2pq$ | $n_{AB} = 34$ |

Table 1: Blood groups of 870 people

Estimate the parameters $p$, $q$, and $r$ using EM algorithm explicitly explaining each step. Note that $p+q+r = 1,\ 0 < p, q, r < 1$. Show your calculations up to three iterations. (17)

6. Suppose a random sample of size $n$ is taken from $N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ distribution. Assume that all parameters are unknown.

   (a) Describe a test for testing $H_0 : \mu_1 = \mu_2 + \xi_0$ against $H_1 : \mu_1 > \mu_2 + \xi_0$, where $\xi_0$ is a known quantity based on prior information.

   (b) Find a $(1-\alpha)100\%$ confidence interval for $\mu_1/\mu_2$. (8+9=17)

7. Currently in Kolkata East-West metro work is going on. To see the progress, a central team came to Kolkata and observed that the time taken (in years) to complete 70% of work for a complete metro station for 9 ongoing sites are:

$$5.2, 6.1, 4.2, 3.8, 2.6, 3.0, 4.5, 4.4, 3.2$$

Previous experience in Delhi indicates that an average of 3.6 years can be considered to be an indication of good progress. Do you think that progress in building metro station in Kolkata is comparable to that in Delhi? (12)

# INDIAN STATISTICAL INSTITUTE

# Semester Examination: 2019-20

## B STAT, SECOND YEAR

## Statistical Methods - III

Date: 21/11/2019 Maximum Marks: 70 Duration: 3.5 hours

1. Suppose $X_1, X_2, \ldots, X_n$ be a random sample from

$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-x^2/2\sigma^2}; \; -\infty < x < \infty.$$

   (a) Suggest an unbiased estimator of $\sigma^2$ based on $X_1, X_2, \ldots, X_n$, justifying your answer. Find its variance.

   (b) Evaluate the lower bound of the variance of an unbiased estimator of $\sigma^2$ using Cramer-Rao inequality.

   (c) Write your comment based on your findings in (a) and (b).

   (6+6+3=15)

2. Let $X_i,\ i = 1, \ldots, n$ be *i.i.d.* $Poisson(\lambda)$ random variables. Show that $1/\lambda$ is not estimable. (5)

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from $Uniform\,(0, \theta)$ distribution. To test $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$, consider the critical region $\{(x_1, \ldots, x_n) : x_{(n)} > c\}$, where $x_{(n)} = \max\limits_{i=1,\ldots,n} \{x_1, \ldots, x_n\}$. Find $c$ such that the probability of Type I error of this test is $\alpha$. Also derive the power function of the test. $(5 + 5 = 10)$

4. Two ladies claim that they can identify whether sugar is added to tea before or after pouring milk. To test their claim, a set of 10 cups of tea are presented to each of them. None of the ladies knew that actually sugar is added before pouring milk. A lady declares "*before*" or "*after*" if she thinks that sugar is added 'before' or 'after' pouring milk respectively. The results are shown below.

**Lady 1** : *after*, *before*, *before*, *before*, *before*, *before*, *before*, *before*, *before*, *after*

**Lady 2** : *after*, *after*, *after*, *before*, *after*, *after*, *after*, *after*, *before*, *after*

(a) Suggest a relevant hypothesis to test and write the corresponding null and alternative hypotheses with justification. Note that your hypothesis must involve the opinions of both the ladies.

(b) Describe a testing procedure for the hypothesis suggested by you in (a). (3+6=9)

5. Currently in Kolkata East-West metro work is going on. To see the progress, a central team came to Kolkata and observed that the time taken (in years) to complete 70% of work for a complete metro station for 9 ongoing sites are:

$$5.2, 6.1, 4.2, 3.8, 2.6, 3.0, 4.5, 4.4, 3.2$$

Previous experience in Delhi indicates that an average of 3.6 years can be considered to be an indication of good progress. Do you think that progress in building metro station in Kolkata is comparable to that in Delhi? (9)

6. In a region it is observed that there are four species of Iris flowers, viz Sp1, Sp2, Sp3, and Sp4. A person selects a flower at random and assigns its species by looking at some characteristics. The frequency distribution of Iris flowers and their respective probabilities are given in the following table:

| Species | probability | observed frequency |
|---|---|---|
| Sp1 | $\frac{1}{2} + \frac{1}{4}\theta$ | 120 |
| Sp2 | $\frac{1}{4}(1-\theta)$ | 20 |
| Sp3 | $\frac{1}{4}(1-\theta)$ | 18 |
| Sp4 | $\frac{1}{4}\theta$ | 42 |

Estimate $\theta$ using EM algorithm. Show numerical calculations up to 3 iterations. (12)

7. Let $\{(X_i, Y_i),\ i = 1, \ldots, n\}$ be a random sample from $N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, all parameters being unknown. Find a $(1-\alpha)100\%$ confidence interval for $\mu_1/\mu_2$, where $0 < \alpha < 1$. (10)

8. Let $\{(x_i, y_i),\ i = 1, \ldots, n\}$ be a random sample where the study variable $y$ takes the values 0 or 1; $x$ may be either categorical or continuous variable. A statistician wants to take logarithm of odds and then consider a linear model based on $x$ variable.

   (a) What is the justification of taking logarithm of odds while studying the dependence of $y$ on $x$ based on a linear equation?

   (b) Explain how will you estimate the parameters of the model in such a situation? (4+6=10)

Indian Statistical Institute
First Semester Examination 2019
Course Name: B Stat II Year
Subject: Economics I (Microeconomics)

Date 27.11.2019 Maximum Marks: 60 Duration: 2 Hours

Answer all the following questions

1a. Suppose an individual firm in a perfectly competitive industry has the short run cost function $C = 0.1q^2 + q + 10 \quad \forall q \geq 0$ . Derive the short run supply function of the firm specifying the relevant set of values of P for which the supply function is valid. Explain your answer.

b. What is the quantity of supply offered and the maximum profit earned by the firm when P = 2? How do you explain your finding about the level of maximum profit at this price?

[8+(7+7)=22]

2a. (i) Suppose a perfectly competitive industry consists of 100 identical firms each of which has the cost function $C = 0.1q^2 + q + 10$. Derive the aggregate supply function of the industry in the short run. (ii) How will the supply function change if a specific tax at the rate t = 0.2 is imposed on the sales of the good.

b. Every individual producer of a good in a perfectly competitive market has the long-run cost function given by $C = q^2 + 1$, where C and $q$ denote total cost and output respectively of an individual firm. The market demand function is given by $Q = 102 - P$ , where $Q$ and $P$ denote market demand and price respectively. Derive the values of $P$, $Q$ and the number of firms in the industry in the long-run industry equilibrium. [(5+5)+12=22]

3. A monopolist can sell in the home market protected by law. The home market demand function is given by $P_h = 120 - (q_h / 10)$ , where $P_h$ and $q_h$ denote respectively the price and quantity demanded in the home market. He can also sell in the perfectly competitive world market at a given price $P_W$. The cost function of the seller is $C = 50q + (q^2 / 20) + 10$, where $C$ and $q$ denote his total cost and total output, respectively. Derive the set of values of $P_W$ for which the producer will sell (i) only in the home market, (ii) only in the world market. [22]

Indian Statistical Institute
Back Paper Examination 2019
Course Name: B Stat II Year
Subject: Economics I(Microeconomics)

Date 16/1/20 Full Marks: 100 Duration: 3 Hours

Answer all the following questions

**1.a.** Consider the optimisation problem, max $U = U(x_1, x_2, \ldots\ldots\ldots, x_n)$, $\frac{\partial U}{\partial x_i} > 0, x_i \geq 0$ (i=1,2,...,n) subject to the budget constraint $\sum_{i=1}^{n} p_i x_i = w$, ~~(i=1,2,......,n)~~.
Suppose the utility function is strictly quasiconcave. Derive the optimality condition for the following two cases: (i) When there is interior solution and (ii) When there is corner solution. Explain your answers with the help of diagrams.

**b.** Consider the utility function of a consumer given by $U = X_1X_2$. Suppose $P_1$ and $P_2$ denote the prices of good 1 and good 2, respectively and $W$ denotes the budget level. (i) Compute the equation of the income expansion path. (ii) Derive the Walrasian demand function $X_i^* = X_i^*(P_1, P_2, W)$. (iii) Compute the optimum values of $X_1$, $X_2$ and $U$, when $P_1 = 1$, $P_2 = 2$ and $W = 40$. (iv) Compute the change in the level of $X_1^*$ as $P_1$ rises to 2. Break up this change into substitution effect and income effect.
[10 + 15=25]

**2.** Suppose the production function of an individual firm as given by q = f (x₁, x₂,......, xₙ), $\frac{\partial f}{\partial x_i} > 0$ is twice continuously differentiable and strictly concave. Derive the cost function of the firm C = C(q). Derive the properties of the cost function C = C(q), average variable cost function AVC = AVC(q), marginal cost function MC = MC(q) and average cost function AC = AC(q). Present these cost functions in the same diagram. [25]

**3.** Is it possible for a profit maximizing firm in a perfectly competitive market to produce and sell positive output while suffering a loss in the short-run? Explain. [25]

**4. a.** Explain why supply function does not exist under monopoly.

**b.** Suppose a discriminating monopolist sells its output in two segregated markets. The demand functions faced by the monopolist in the two markets are $p_1 = 40 - 5q_1$ and $p_2 = 18 - 0.25q_2$. The marginal cost function of the monopolist is MC = 10. Compute the quantities sold in the two markets such that the monopolist's profit is maximized. What are the prices charged in the two markets? Also compute relevant levels of price elasticities in the two markets.
[10+15 = 25]

Indian Statistical Institute

First Semester Examination: 2019-2020

Course Name: B.Stat II, Subject Name: Physics I

Date: 27 November, 2019 (Morning) Duration: 3 hours

Answer as many questions as you can. Use of Calculator is allowed. Maximum Marks: 50

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1. (a) Obtain the Hamiltonian and Hamilton's equations for a charged particle in an electromagnetic field.

b) Prove that the shortest distance between two points on the surface of the sphere is the arc of the great circle connecting them. [5 + 5]

2. (a) A pendulum of length $l$ and mass $m$ is mounted on a block of mass $M$. The block can move freely without friction on a horizontal surface as shown in Fig. 1.

(i) Show that the Lagrangian for the system is

$$L=\left(\frac{M+m}{2}\right)(\dot{x})^2+ml\cos\theta\dot{x}\dot{\theta}+\frac{m}{2}l^2\left(\dot{\theta}\right)^2+mgl\cos\theta$$

(ii) Show that the approximate form for this Lagrangian, which is applicable for a pendulum swinging with a small magnitude, is $L=\left(\frac{M+m}{2}\right)(\dot{x})^2+ml\dot{x}\dot{\theta}+\frac{m}{2}l^2\left(\dot{\theta}\right)^2+mgl\left(1-\frac{\theta^2}{2}\right)$

(iii) Find the equations of motion that follow from the simplified Lagrangian obtained in (ii).

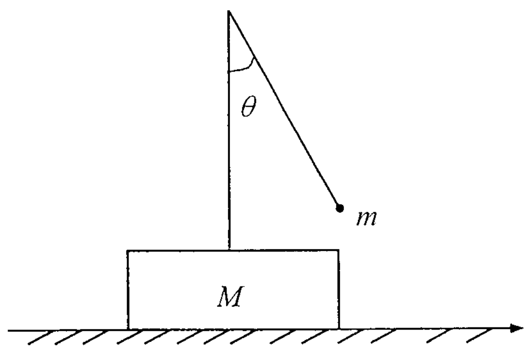


Fig. 1

(b) Calculate the ratio of the mean densities of the earth and the sun from the following approximate data:

$\theta$ = angular diameter of the sun seen from the earth = 0.5°

$l$ = length of 1° of latitude on the earth's surface = 100 $km$.

$t$ = one year = $3 \times 10^7$ $s$, $g$ = 10 $ms^{-2}$. [5 + 5]

3. (a) A uniform rod of mass $M$ and length $2a$ is attached at one end by a cord of length $l$ to a fixed point. Calculate the inclination of the string and the rod when the string plus rod system revolves about the vertical through the pivot with constant angular velocity $\omega$.

(b) Write the Lagrangian function and the equations of motion for a three-dimensional isotropic harmonic oscillator in polar coordinates. [5 + 5]

4. (a) Consider an engine in outer space which operates on the basis of Carnot cycle. The only way in which heat can be transferred from the engine is by radiation. The rate at which heat is radiated is

proportional to the fourth power of the absolute temperature $T_2$ and to the area of the radiating surface. Show that for a given power output and a given $T_1$, the area of the radiator will be a minimum when $T_2 = 0.75\ T_1$. Determine the minimum area of the radiating panel for an output of 2 *kW* if the constant of proportionality is $5.67 \times 10^{-8}\ W/m^2K^4$ and $T_1 = 2000\ K$.

(b) A reversible heat engine operates between two reservoirs at temperatures of 600°*C* and 40°*C*. The engine drives a reversible refrigerator which operates between reservoirs at temperatures of 40°*C* and −20°*C*. The heat transfer to the heat engine is 2000 *kJ* and the net work output of the combined engine refrigerator plant is 360 *kJ*.

(i) Evaluate the heat transfer to the refrigerant and the net heat transfer to the reservoir at 40°*C*.

(ii) Reconsider (i) given that the efficiency of the heat engine and the COP of the refrigerator are each 40% of their maximum possible values. [5 + 5]

5. (a) Prove that the Fermi-Dirac velocity distribution law of *x*-component of velocity among the molecules of an ideal gas is represented by:

$$n(v_x)dv_x = V\left(\frac{4\pi m^2 kT}{h^3}\right)^{1/2} e^{E_F/kT} \times e^{-mv^2/2kT} dv_x$$

(b) One *kg* of ice at −5°*C* is exposed to the atmosphere which is at 20°*C*. The ice melts and comes into thermal equilibrium with the atmosphere. (i) Determine the entropy increase of the universe. (ii) What is the minimum amount of work necessary to convert the water back into ice at −5°*C*? Given that $c_p$ of ice is 2.093 *KJ/kgK* and the latent heat of fusion of ice is 333.3 *kJ/kg*. [5 + 5]

6. (a) Find the Fermi energy at absolute zero for copper. Given that: density of copper = $8.96 \times 10^3\ kg/m^3$, *At. wt.* = $63.5 \times 10^{-3}\ kg/mol$, $N = 6.023 \times 10^{23}/mol$, $h = 6.63 \times 10^{-34}\ Js$, $m = 9.11 \times 10^{-31}\ kg$.

(b) There are about $2.5 \times 10^{28}$ free electrons per $m^3$ in sodium. Calculate its Fermi energy, Fermi velocity and Fermi temperature.

(c) There are 1000 molecules of an ideal gas in a box of 1 *litre* volume. Assuming that the particles do not preferentially occupy any region, find the probability of finding the particles in a particular region of volume 100 *cc*. Find the no. of molecules for which this probability reaches its maximum? [3 + 3 + 4]