Domain Restrictions in Strategy-proof Social Choice

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To my beloved wife, Akhina Haridas.

"And, when you want something, all the universe conspires in helping you to achieve it."

Paulo Coelho, The Alchemist.

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"Sed quis custodiet ipsos custodes?"

Juvenal, Satires (Satire VI, lines 347–348).

1
Prologue

1.1 Background

THIS FAMOUS PHRASE, which literally translates to "*Who will guard the guards themselves?*", was coined by the Roman poet Juvenal satirically referring to the optimistic view of trusting the guardians of the state as a solution to deal with the problem of marital fidelity. Juvenal suggests that keeping wives under guard may not be a solution to prevent infidelity - because guards themselves might not be trustworthy. Juvenal suggests that keeping wives under guard may not be a solution to prevent infidelity - because guards themselves might not be trustworthy. In present times, this phrase is intimately connected to the corruption and nepotism in the government, law enforcement, and other forms of social institutions. Hence, a central concern of modern societies is to design social institutions that induce *truthful* behaviour from individuals.

Most social institutions resort to collective decision making procedures such as voting when designing policies. The scientific study of collective decision making procedures is known as *social choice theory*. The subject was pioneered by the early works of French mathematicians, Jean-Charles de Borda ([16]) and Marquis de Condorcet ([25]), who initiated a formal analysis of these problems in terms of voting and related procedures. Kenneth Arrow and Leo Hurwicz later developed a mathematical formulation of these issues using a much general framework ([4], [47] and [48]).

A collective decision making procedure is formally called a *social choice function*. Individuals report their preferences over the social alternatives (policies, public facilities, candidates in an election and so on). At every profile of individual preferences, a social choice function chooses an alternative that is, in some sense, *optimal* for the society. Thus, a social choice function can be thought of as embodying the welfare judgements of a social planner. However, the planner would be unaware of the *true* profile of individual preferences, and she must rely on the individuals' reports about their preferences. This requires the planner to design social choice functions so that individuals report their preferences truthfully.

We consider a few examples in order to illustrate these notions more clearly. Suppose that the government proposes a public project, such as a highway, flyover, hospital etc. When the government decides whether the project should be undertaken, it performs a cost-benefit analysis of these projects. The possible benefits of such a project would be lower commuting time, increase in property values, and an overall improvement in social welfare. Since such projects would be usually funded by an increase in taxes, the planner must also have certain fairness considerations in her mind, such as taxing individuals who live near this proposed public facility higher than the ones who live far from it. These aspects of this decision involve information privately held by the individuals such as individuals' valuation of each project. Another class of examples is that of voting problems. A set of voters must elect a candidate for a political position. In this case, the set of social goals can be identified with different political agendas (say, going from the left to the right on a uni-dimensional political spectrum). Again, as in the previous example, the individual preferences over the candidates might be private information.

In order to capture the welfare judgements of the social planner, it is natural to impose certain desirable properties on a social choice function. A social choice function is called *unanimous* if whenever all the agents in a society unanimously agree on their best alternative, that alternative is chosen. A social choice function is called *strategy-proof* if no agent can benefit by misreporting her preferences. Throughout the present thesis, we focus our attention on unanimous and strategy-proof social choice functions.

Designing social choice functions that possess the desirable properties of unanimity and strategy-proofness leads to the famous Gibbard-Satterthwaite impossibility theorem ([43], [75]). It says that if the range of the social choice function contains at least three alternatives, then every unanimous and strategy-proof social choice function is a *dictatorial rule*. A dictatorial rule always selects the most preferred alternative of some particular individual, called a *dictator*, in the society. An assumption that lies at the heart of this impossibility result is that the individual preferences are *unrestricted*. In other words, an individual can misreport any plausible preference in place of his *true* preference.

Researchers in mechanism design have persistently looked for ways to bypass the Gibbard-Satterthwaite impossibility result. The present thesis is concerned with one such approach where we consider restrictions on the domain of admissible preferences. Domain restrictions naturally arise in several practical scenarios. However, this approach has proved to be a double-edged sword in the sense that it leads to both impossibility and possibility results. This is because the unrestricted domain assumption in the Gibbard-Satterthwaite result is by no means a necessary condition for the dictatorial result to hold. This leads us to the notions of dictatorial and non-dictatorial domains. A domain of admissible preferences is called *dictatorial* if every unanimous and strategy-proof social choice function on it is dictatorial. Similarly, a domain is called *non-dictatorial* if it admits non-dictatorial rules as unanimous and strategy-proof social choice functions.

Several restricted domains such as the *free pair at the top domains* ([12], [20]), *linked domains* ([6]), and *circular domains* ([74]) are known to be dictatorial domains. These results imply that it is impossible to sustain a *reasonable* aggregation procedure without strategic manipulation. Hence, these results are usually deemed unsatisfactory as such domain restrictions preclude the possibility of arriving at any *fair* compromise.

On the other hand, consider a situation where the social planner has to locate a *public good* (hospital, shopping mall etc.), i.e., a facility which generates a *positive externality* to individuals. In this case, it is natural that individuals would want to place such a facility closer to their own locations. This means that individual preferences would have a *unique peak* at their own location and it falls as one moves away from its peak. Such preferences are called *single-peaked* and domains containing such preferences are non-dictatorial. Similarly, consider a situation where the social planner has to locate a *public bad* (nuclear power plant, garbage dump etc.), i.e., a facility which generates a *negative externality* to individuals. In this case, it is natural that individuals would want to place such a facility farther away from their own locations. This means that individual preferences would have a *unique dip* at their own location and it rises as one moves away from its dip. Such preferences are called *single-dipped* and domains containing such preferences are non-dictatorial. Next, consider a situation where the government is trying to set the tax level. In such models, it is customary to assume that relatively poorer individuals would prefer a higher tax regime over a relatively lower one as he would benefit from a greater redistribution of income. This means that the domain restriction relevant in such situations allow only for a single reversal of a higher tax regime with a lower one when moving from the preference of a lower income individual to that of a higher income individual. Such a domain of preferences is called *single-crossing* which are also known to be non-dictatorial.

1.2 Motivation

The existing literature on domain restrictions in strategy-proof social choice has been subject to severe criticism due to its limited practical applicability. For instance, most of the existing literature on non-dictatorial domain restrictions like single-peaked domains, single-dipped domains, and single-crossing domains make two customary assumptions: (i) *prior order* assumption, which says that alternatives can be arranged over a uni-dimensional space, and (ii) *maximality* assumption, which says that *all* preferences with the relevant restriction are admissible. In the context of electoral competition, [81] criticizes the prior order assumption on the following accounts:

(i) **Uni-dimensionality**: The uni-dimensionality assumption is a far-fetched one when reconciled with evidence in both two-party and multi-party systems. Voters' ordering over the candidates' positions on different policies and issues might be independent of each other. For example, there would be *no* relation (in a statistical sense) in the voter attitudes towards candidates' views on anti-abortion policies and terrorism in the recently concluded Presidential election in the US. Therefore, each of these attitudes must be placed on different dimensions and a uni-dimensional policy space would be an unrealistic assumption in these situations.

(ii) **Full comparability of alternatives**: The assumption that any two alternatives can be compared with respect to the given prior ordering over the alternatives is very unrealistic. This assumption fails if voters are merely reacting to the association of the candidates to an issue or goal which is positively or negatively valued by them. For instance, the outcome of 2014 General Elections in India was highly influenced by the high profile scandals, such as the Commonwealth Games scam, Coal-gate scandal, 2G scam etc., as the Indian voters associated the then incumbent UPA government to these scandals when bringing the Modi government to power (for more details, see [23] and [82]).

(iii) **Single prior ordering**: The assumption that *all* individuals derive their preferences based a single prior ordering over the alternatives is not applicable to many practical scenarios. This is because different individuals come from different socio-economic backgrounds, and hence they might be concerned about different aspects of an alternative. This leads the individuals to have different prior ordering over the alternatives.

Further, the maximality assumption fails in the domain restriction considered in models of voting ([84], [5],[64], [66], [65], etc.) and taxation and redistribution ([38]).

Among the non-dictatorial domains discussed, single-peaked domains are by far the most popular. A few empirical studies like [58], [41], and [59] find that voters' preferences often violate the very assumption of single-peakedness. For instance, in many practical economic and political situations, voters' preferences are known to be *multi-peaked* (see [28], [32], [78], [31], [34], [80] and so on). However, these empirical studies show that voters' preferences are consistent with mild violations of single-peakedness.

Keeping these practical and empirical considerations in mind, the main motivation of this thesis is to further explore domain restrictions to accommodate these criticisms and thereby, widen the applicability of the standard social choice framework.

1.3 Our contribution

In this section, we provide a brief overview of our contribution to the area of strategy-proof social choice.

1.3.1 Dictatorship on Top-circular Domains

In Chapter 2, we consider domains of admissible preferences with a natural property called *top-circularity*. Several domains with practical applications such as multi-dimensional single-peaked domain in [9], union of a single-peaked and a single-dipped domain, etc. satisfy top-circularity. We show that if such a domain satisfies either the *maximal conflict property* or the *weak conflict property*, then it is dictatorial. We show that this result can be applied to the problem of locating a public facility where the planner does not know whether agents derive positive or negative externality from the facility. The union of a single-peaked and a single-dipped domain captures such situations and such domains are top-circular satisfying the maximal conflict property. It follows from our results that such domains are dictatorial. Further, we obtain the result in [74] as a corollary.

1.3.2 On Single-peaked Domains and Min-max Rules

In Chapter 3, we consider social choice problems where the set of alternatives can be ordered over a real line and the admissible set of preferences of each agent is single-peaked. A preference is called *single-peaked* if the preference falls as one moves away from its top-ranked alternative. First, we show that if all the agents have the same admissible set of single-peaked preferences, then every unanimous and strategy-proof social choice function is tops-only. A social choice function is called *tops-only* if it is insensitive to changes in agents' preferences below the top-ranked alternative. Next, we consider situations where different agents have different admissible sets of single-peaked preferences. We show by means of an example that unanimous and strategy-proof social choice functions need not be tops-only in this situation, and consequently provide a sufficient condition on the admissible sets of preferences of the agents so that unanimity and strategy-proofness guarantee tops-onlyness. Finally, we characterize all domains on which (i) every unanimous and strategy-proof social choice function is a min-max rule ([54]), and (ii) every min-max rule is strategy-proof. As an application of our result, we obtain a characterization of the unanimous and strategy-proof social choice functions on maximal single-peaked domains ([54], [86]), minimally rich single-peaked domains ([61]), maximal regular single-crossing domains ([72], [73]), and distance based single-peaked domains.

1.3.3 Strategy-proof Rules on Partially Single-peaked Domains

In Chapter 4, we consider domains that exhibit single-peakedness only over a subset of alternatives. We call such domains *partially single-peaked domains* and provide a characterization of the unanimous and strategy-proof social choice functions on these domains. As an application of this result, we obtain a characterization of the unanimous and strategy-proof social choice functions on multi-peaked domains ([80], [78], [37]), single-peaked domains with respect to a partial order ([18]), multiple single-peaked domains ([67]) and single-peaked domains on graphs ([76]). As a by-product of our results, it follows that strategy-proofness implies tops-onlyness on these domains. Moreover, we show that strategy-proofness and group strategy-proofness are equivalent on these domains.

1.3.4 On Strategy-proofness and Uncompromisingness

In Chapter 5, we consider a social choice setting where the set of alternatives can be ordered over a real line. In Chapter 3, we have characterized domains where the set of unanimous and strategy-proof rules coincide with the set of min-max rules. Min-max rules satisfy an interesting property called *uncompromisingness* ([17]). A social choice function is *uncompromising* if no agent can influence the outcome by taking extreme positions. It follows from our result in Chapter 3 that a domain is not top-connected single-peaked then unanimous and strategy-proof rules may violate uncompromisingness. In this chapter, we consider arbitrary single-peaked domains (not necessarily top-connected) and provide a general characterization of the unanimous and strategy-proof social choice functions on those domains. We show that every unanimous and strategy-proof social choice function defined on such domains satisfy a property called *weak uncompromisingness*. Weak uncompromisingness implies that whenever an agent's top-ranked alternative moves closer to the outcome, the outcome does not change. Moreover, if an agent moves his top-ranked alternative away from the outcome, the outcome can change only in a restricted way.

As an application of this result, we obtain a characterization of the unanimous and strategy-proof social choice functions on maximal single-peaked domains ([54], [86]), minimally rich single-peaked domains ([61]), maximal regular single-crossing domains ([72], [73]), one-dimensional Euclidean single-peaked domains ([26]), and left (or right) single-peaked domains ([64], [66], [65]).

1.3.5 Social Choice on Domains based on Trees

In Chapter 6, we consider social choice problems where the set of alternatives are arranged over a (fixed) tree. We study unanimous and strategy-proof SCFs when agents' preferences are single-peaked on such a tree. Such preferences naturally arise in situations where a public good (shopping mall, hospital, etc.) has to be located on a road or railroad network. We show that when such domains satisfy a property called top-connectedness, every unanimous and strategy-proof SCF satisfies Pareto property and tops-only property. Further, we show that when such domains satisfy a stronger requirement called strongly connected, every unanimous and strategy-proof SCF is uncompromising. In this setting, uncompromisingness means if the top-ranked alternative of each agent does not "cross" the outcome, i.e., the outcome does not lie on the unique path joining the initial top-ranked alternative to the final one of each agent.

The Chapters are intended to be self-contained. We also try to unify the symbols across the chapters.

2

Dictatorship on Top-circular Domains

2.1 Introduction

2.1.1 Motivation

The coincidence of strategy-proofness and dictatorship has always been an intriguing question since Alan Gibbard and Mark Satterthwaite proposed their impossibility result ([43], [75]) - famously known as the Gibbard-Satterthwaite (GS) Theorem - which states that every unanimous and strategy-proof social choice function (SCF) defined over the unrestricted domain of preferences (provided that there are at least three alternatives) is dictatorial. However, the unrestricted domain assumption in the GS theorem is far from being the necessary condition for dictatorship. A domain of preferences is called *dictatorial* if every unanimous and strategy-proof SCF on it is dictatorial.

Apart from being a generalization of the GS theorem, dictatorial domains have garnered a lot of interest in the literature. At present, there is a sizeable literature on dictatorial domains as seen in the works of [12], [6], [74], and [62]. The main motivation of this chapter is to find dictatorial domains that can be applied to some economic and political environment.

2.1.2 Our contribution

A crucial property of a dictatorial domain is that for every alternative a, there must be at least two preferences $ab\ldots$ and $ac\ldots$ in the domain, where $b \neq c$.[1,2] A domain of practical importance of such type is the one whose *top-graph* comprises of a maximal cycle.[3,4] We call such a domain a *top-circular domain*.

We prove by means of an example that the top-circular domains are not dictatorial. In view of that, we identify two conditions called the *maximal conflict property* and the *weak conflict property* such that if a top-circular domain satisfies either of these two conditions, then it becomes a dictatorial domain. Maximal conflict property requires the existence of two exactly opposite preferences and consequently verifying if some domain satisfies this property is easy. However, weak conflict property is somewhat technical and the corresponding verification is relatively harder. Several domains of practical importance such as the maximal single-peaked domain, the maximal single-dipped domain, and maximal single-crossing domains (with respect to a given ordering over the alternatives) satisfy the maximal conflict property. Also, maximal single-peaked domains satisfy the weak conflict property. Here, maximality refers to the largest possible set of preferences with the corresponding property. We obtain the dictatorial result in [74]

[1]We denote by $ab\ldots$ a preference which places a at the top and b at the second-ranked position.

[2][70] shows that this property is necessary and sufficient for dictatorship on a large class of domains which they call *short-path-connected* domains. However, the domains that we consider are not short-path-connected.

[3]The *top-graph* of a domain is defined as the graph where nodes are alternatives and there is an edge between two alternatives a, b if there are preferences $ab\ldots$ and $ba\ldots$ in the domain.

[4]An undirected graph with nodes $v_1, \ldots, v_k$ is said to contain a maximal cycle if it has the following edges: $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}, \{v_k, v_1\}$.

as a corollary of our result.

We apply this result to the problem of locating a public facility. For certain public facilities such as metro stations, hospitals etc., agents want the facility to be located closer to their own locations, and consequently their preferences can be modeled as single-peaked with respect to a given ordering over the alternatives. On the other hand, for facilities like garbage dumps or nuclear plants, agents want the facility to be located farther from their own locations, and consequently their preferences can be modeled as single-dipped with respect to a given ordering over the alternatives. For both these cases, it is well-known that one can design non-dictatorial rules that satisfy unanimity and strategy-proofness.[5,6]

However, for facilities like shopping malls, factories etc., the social planner may not have clear knowledge on whether the agents want it to be closer or farther away. This is because, some individuals may be concerned about the resulting congestion, pollution etc., whereas some others may want to minimize their commuting distance. In such a situation, the relevant admissible domain is the union of a single-peaked and a single-dipped domain with respect to a given ordering over the alternatives.[7] Our result shows that every unanimous and strategy-proof SCF on such a domain is dictatorial.

2.1.3 Relation to the literature

In this section, we discuss the connection of our result with the vast literature on dictatorial domains. [12] and [77] provide a generalization of Gibbard-Satterthwaite theorem by showing that every *free pair at the top (FPT)* domain is dictatorial.

[5] [54], [9] and [86] characterize the unanimous and strategy-proof SCFs on the single-peaked domains as *min-max rules*.

[6] [60], [8] and [52] characterize the unanimous and strategy-proof SCFs on the single-dipped domains as *voting by extended committees*.

[7] Alternative models that consider similar practical situations exist in the literature. For instance, [83] and [40] partition the set of agents into those who can only have single-peaked preferences and those that can only have single-dipped preferences. On the other hand, [1] considers a situation where the social planner is informed about the location of the agents but agents can have single-peaked preferences with the peak at her location or single-dipped preferences with the dip at her location. Though the domain restriction considered in the aforementioned models are close in spirit with ours, they admit non-dictatorial, unanimous, and strategy-proof SCFs.

A domain satisfies the FPT property if for every two alternatives x and y, it contains a preference which places x at the top and y at the second-ranked position. It is worth noting that such a domain requires at least $m(m-1)$ preferences, where m is the total number of alternatives. Later, in a seminal contribution, [6] further generalizes this result by showing that the same holds if a domain satisfies a much weaker property called the *linked* property.[8] More recently, [62] further generalizes the dictatorial results in [6]. However, all these domain restrictions are different in nature from those we consider in this chapter. This is because, in contrast to these works which put restrictions *only* on the first and second ranked alternatives in a preference, we consider additional restrictions on some other ranked alternatives as well. As a consequence, we obtain dictatorial domains that require fewer preferences.

The structure of the domain restriction that we consider in this chapter is similar to that considered in [50]. They show that any domain containing all clockwise and anti-clockwise preferences with respect to some arrangement of the alternatives on a circle is dictatorial. Such domains are called *circular domains*. Later, [74] generalizes circular domains by placing restrictions only on the first, second and last ranks of a preference and show that such domains are also dictatorial. In a subsequent chapter, [20] independently prove that circular domains (as defined in [50]) are dictatorial.[9] However, our result generalizes all these results in a substantial way.

2.1.4 Remainder

The rest of the chapter is organized as follows. We describe the usual social choice framework in Section 2.2. Section 2.3 presents our main results and Section 2.4 discusses applications of the same. The last section concludes the chapter. All the omitted proofs are collected in the Appendix.

[8] We provide the technical definition of the linked property in Remark 2.2.2.

[9] They also provide two conditions T and T', and show that any domain satisfying those is dictatorial.

2.2 The model

Let $N = \{1, \ldots, n\}$ be a set of agents, who collectively choose an element from a finite set $X = \{x_1, x_2, \ldots, x_m\}$ of at least three alternatives. A *preference* P over X is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on X. For two alternatives $x, y \in X$ and a preference P, we write xPy to mean that x is preferred to y according to P.[10] We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k - 1$. For ease of presentation, by $ab \ldots c \ldots d \ldots$, we denote a preference P where $r_1(P) = a, r_2(P) = b$ and cPd. Also, by $ab \ldots c$, we denote a preference P where $r_1(P) = a$, $r_2(P) = b$, and $r_m(P) = c$. We denote by $\mathcal{D} \subseteq \mathbb{L}(X)$ a set of admissible preferences over X. A preference profile, denoted by P_N, is defined as an element of $\mathcal{D}^n$.

For simplicity, we do not use braces for singleton sets, for instance, we use the notation i to mean $\{i\}$.

Definition 2.2.1 *A social choice function (SCF) f on a domain $\mathcal{D}$ is defined as a mapping $f : \mathcal{D}^n \to X$.*

Definition 2.2.2 *An SCF $f : \mathcal{D}^n \to X$ is unanimous if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.*

Definition 2.2.3 *An SCF $f : \mathcal{D}^n \to X$ is manipulable if there exists a profile $P_N \in \mathcal{D}^n$, an agent $i \in N$, and a preference $P'_i \in \mathcal{D}$ of agent i such that $f(P'_i, P_{-i}) P_i f(P_N)$. An SCF f is strategy-proof if it is not manipulable.*

Definition 2.2.4 *An SCF $f : \mathcal{D}^n \to X$ is dictatorial if there exists an agent $i \in N$ such that for all profiles $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.*

Definition 2.2.5 *A domain $\mathcal{D}$ is called dictatorial if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.*

[10] More formally, since P is a binary relation on X, xPy means that the pair $(x, y) \in P$.

Definition 2.2.6 *A domain $\mathcal{D}$ is regular if for all $x \in X$, there exists $P \in \mathcal{D}$ such that $r_1(P) = x$.*

Remark 2.2.1 *All the domains we consider in this chapter are regular.*

Now, we introduce a few graph theoretic notions. A *graph* G is defined as a pair $\langle V, E \rangle$, where V is the set of *nodes* and $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$ is the set of *edges*. A *cycle* in a graph $G = \langle V, E \rangle$ is defined as a sequence of nodes $(v_1, \ldots, v_k, v_1)$ such that the nodes $v_1, \ldots, v_k$ are all distinct and $\{v_i, v_{i+1}\} \in E$ for all $i = 1, \ldots, k$, where $v_{k+1} \equiv v_1$.

All the graphs we consider in this chapter are of the kind $G = \langle X, E \rangle$, i.e., whose node set is the set of alternatives.

Definition 2.2.7 *The top-graph of a domain $\mathcal{D}$ is defined as the graph $\langle X, E \rangle$ such that $\{x, y\} \in E$ if and only if there exist two preferences $P, P' \in \mathcal{D}$ with $r_1(P) = r_2(P') = x$ and $r_2(P) = r_1(P') = y$.*

Now, we introduce the notion of a top-circular domain.

Definition 2.2.8 *A domain $\mathcal{C}$ with top-graph $\langle X, E \rangle$ is called top-circular if $\{x_i, x_j\} \in E$ for all i, j with $|i - j| \in \{1, m - 1\}$.*

Below, we present a top-circular domain and its top-graph.

Example 2.2.1 *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. Consider the domain given in Table 2.2.1. Figure 2.2.1 presents the top-graph of this domain. Note that this graph contains a maximal cycle given by $(x_1, x_2, \ldots, x_5, x_1)$. Further, note that such a graph may contain some additional edges like $\{x_1, x_3\}$ and $\{x_2, x_5\}$.*

Remark 2.2.2 *[6] introduce the notion of a linked domain and show that every linked domain is dictatorial. A domain is called linked if the alternatives can be ordered as $y_1, \ldots, y_m$ such that the top-graph $\langle X, E \rangle$ of the domain has the following property:*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_5$	$\mathbf{x}_1$	$\mathbf{x}_1$	$\mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{x}_5$
$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{x}_4$	$\mathbf{x}_3$	$\mathbf{x}_5$	$\mathbf{x}_4$	$\mathbf{x}_1$	$\mathbf{x}_5$	$\mathbf{x}_3$	$\mathbf{x}_1$	$\mathbf{x}_5$	$\mathbf{x}_2$
x_5	x_4	x_5	x_4	x_2	x_1	x_3	x_1	x_2	x_3	x_5	x_4	x_3	x_4
x_4	x_5	x_1	x_1	x_1	x_5	x_1	x_3	x_3	x_4	x_2	x_5	x_4	x_3
x_3	x_3	x_4	x_5	x_5	x_2	x_2	x_2	x_4	x_2	x_4	x_2	x_1	x_1

Table 2.2.1 A top-circular domain

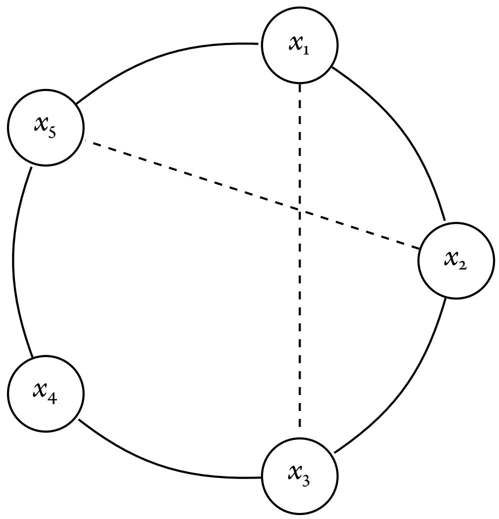


Figure 2.2.1 Top-graph of a top-circular domain

$\{y_1, y_2\} \in E$, and for all $3 \leq l \leq m$, $\{\{y_j, y_l\}, \{y_k, y_l\}\} \subseteq E$ for some $j, k < l$ with $j \neq k$. It can be verified the the top-graph in Figure 2.2.1 satisfies this property with respect to the order $x_5, x_1, \ldots, x_4$. Thus, it follows from [6] that every unanimous and strategy-proof SCF on this domain is dictatorial. However, note that any arbitrary top-circular domain need not be linked.

2.3 Main result

In this section we present the main result of this chapter. We first show by means of an example that an arbitrary top-circular domain need not be dictatorial. The domain provided in this example is a two-dimensional single-peaked domain given in [9].

Example 2.3.1 *Suppose that each alternative has two dimensions, and in each dimension, it can take two values* $\{0, 1\}$. *In other words, the set of alternatives is* $X = \{0, 1\} \times \{0, 1\}$. *Suppose further that the domain is the maximal set of preferences satisfying separability. Separability in this case says that if* (x, y) *appears as a top-ranked alternative at some preference, then* $(1 - x, 1 - y)$ *must appear as the bottom-ranked alternative. It can be verified that the top-graph of this domain has the following cycle* $((1, 1), (1, 0), (0, 0), (0, 1), (1, 1))$. *Thus, this domain is top-circular. However, it is well-known that this domain admits unanimous, strategy-proof, and non-dictatorial SCFs.*

In view of Example 2.3.1, we present below two conditions, and show that if a top-circular domain satisfies either of the two, then it is dictatorial.

Definition 2.3.1 *A domain* $\mathcal{D}$ *satisfies the maximal conflict property if there exist* $P, P' \in \mathcal{D}$ *such that* $r_k(P) = r_{m-k+1}(P') = x_k$ *for all* $k = 1, \ldots, m$.

Thus, the maximal conflict property ensures the existence of two exactly opposite preferences.

Definition 2.3.2 *A domain* $\mathcal{D}$ *satisfies the weak conflict property if*

(i) $\{x_1 x_2 \ldots x_m, x_m x_{m-1} \ldots x_1\} \subseteq \mathcal{D}$, *and*

(ii) for all $k = 2, \ldots, m-1$, *there are two preferences* $P = x_k x_{k-1} \ldots x_1 \ldots x_{k+1} \ldots$ *and* $P' = x_k x_{k+1} \ldots x_m \ldots x_{k-1} \ldots$ *in the domain* $\mathcal{D}$.

Condition (i) in Definition 2.3.2 is a weaker version of the maximal conflict property. It requires the existence of two preferences that are opposite with respect to their first, second, and last-ranked alternatives. For an intuitive explanation of Condition (ii), assume that the alternatives are arranged on a circle in the following (clockwise) order: $x_1, x_2, \ldots, x_m$. Then, for every alternative x_k, where $k \neq 1, m$, it ensures the existence of two preferences with x_k as the top-ranked alternative such that in one of them, preference decreases in the anti-clockwise direction (*only*) over the alternatives x_{k-1}, x_1 and x_{k+1}, and in the other, it decreases in the clockwise direction (*only*) over the alternatives x_{k+1}, x_m and x_{k-1}. It further says that in those two preferences, the second-ranked alternatives must be the *next* alternatives in the corresponding direction (i.e., x_{k-1} in the anticlockwise direction and x_{k+1} in the clockwise direction).

In the following, we illustrate the notion of a top-circular domain with the maximal conflict property by means of an example.

Example 2.3.2 *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$*. Then, the domain* $\mathcal{C} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}\}$ *as given in Table 2.3.1 is a top-circular domain satisfying the maximal conflict property.*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_7$	$\mathbf{x}_1$
$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{x}_4$	$\mathbf{x}_3$	$\mathbf{x}_5$	$\mathbf{x}_4$	$\mathbf{x}_6$	$\mathbf{x}_5$	$\mathbf{x}_7$	$\mathbf{x}_6$	$\mathbf{x}_1$	$\mathbf{x}_7$
$\mathbf{x}_3$	x_6	x_5	x_6	x_2	x_1	x_7	x_1	x_7	x_7	x_3	$\mathbf{x}_5$	x_5	x_4
$\mathbf{x}_4$	x_5	x_1	x_1	x_6	x_7	x_1	x_3	x_3	x_1	x_5	$\mathbf{x}_4$	x_3	x_2
$\mathbf{x}_5$	x_3	x_4	x_4	x_5	x_2	x_2	x_2	x_4	x_4	x_4	$\mathbf{x}_3$	x_4	x_3
$\mathbf{x}_6$	x_7	x_7	x_7	x_1	x_6	x_6	x_7	x_1	x_2	x_1	$\mathbf{x}_2$	x_2	x_6
$\mathbf{x}_7$	x_4	x_6	x_5	x_7	x_5	x_3	x_6	x_2	x_3	x_2	$\mathbf{x}_1$	x_6	x_5

Table 2.3.1 A top-circular domain satisfying the maximal conflict property

Now, we present an example of a top-circular domain with the weak conflict property.

Example 2.3.3 *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$*. Then, the domain* $\mathcal{C} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}\}$ *as given in Table 2.3.2 is a top-circular domain satisfying the weak conflict property.*

Remark 2.3.1 *[50] consider domains containing all clockwise and anti-clockwise preferences with respect to some arrangement of the alternatives on a circle and show that*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_7$	$\mathbf{x}_1$
$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{x}_4$	$\mathbf{x}_3$	$\mathbf{x}_5$	$\mathbf{x}_4$	$\mathbf{x}_6$	$\mathbf{x}_5$	$\mathbf{x}_7$	$\mathbf{x}_6$	$\mathbf{x}_1$	$\mathbf{x}_7$
x_3	x_6	x_5	x_6	$\mathbf{x}_7$	$\mathbf{x}_1$	$\mathbf{x}_7$	$\mathbf{x}_1$	x_2	$\mathbf{x}_1$	x_3	x_4	x_5	x_4
x_6	$\mathbf{x}_3$	$\mathbf{x}_7$	$\mathbf{x}_1$	x_6	x_7	x_1	$\mathbf{x}_6$	x_3	$\mathbf{x}_7$	x_2	x_2	x_3	x_5
x_4	x_4	x_4	x_5	x_5	$\mathbf{x}_5$	x_2	x_2	$\mathbf{x}_7$	x_4	x_4	x_5	x_4	$\mathbf{x}_2$
x_5	x_7	$\mathbf{x}_1$	$\mathbf{x}_4$	x_1	x_6	$\mathbf{x}_3$	x_7	x_1	x_2	x_1	x_3	x_2	x_6
$\mathbf{x}_7$	x_5	x_6	x_7	$\mathbf{x}_2$	x_2	x_6	x_3	$\mathbf{x}_4$	x_3	$\mathbf{x}_5$	$\mathbf{x}_1$	$\mathbf{x}_6$	x_3

Table 2.3.2 A top-circular domain satisfying the weak conflict property

these domains are dictatorial. Later, [20] independently show the same result. It is worth noting that these are top-circular domains satisfying the weak conflict property.

Now, we proceed to present our main results.

Theorem 2.3.1 *Let $\mathcal{C}$ be a top-circular domain satisfying the maximal conflict property. Then, $\mathcal{C}$ is a dictatorial domain.*

Theorem 2.3.2 *Let $\mathcal{C}$ be a top-circular domain satisfying the weak conflict property. Then, $\mathcal{C}$ is a dictatorial domain.*

The proofs of Theorem 2.3.1 and 2.3.2 are relegated to the Appendix.

Remark 2.3.2 *[20] introduce two properties called T and T' and show that every domain satisfying those two properties is dictatorial. It can be verified that a top-circular domain satisfying the maximal conflict or the weak conflict property does not satisfy properties T and T'.*

2.4 Applications

2.4.1 Locating a public facility

In this section, we consider the problem of locating a public facility when the social planner does not have any information whether it generates positive or negative externality for the agents. As argued in Section 2.1, the relevant domain restriction in such problems is the union of the single-peaked and the single-dipped domain. In what follows, we describe such domains formally and show that they are dictatorial.

Definition 2.4.1 *A preference* $P \in \mathbb{L}(X)$ *is called single-peaked if* $r_1(P) = x_i$ *and* $[j < k \leq i$ *or* $i \leq k < j]$ *imply* $x_k P x_j$*. A domain* $\mathcal{D}_p$ *is called single-peaked if it contains all single-peaked preferences.*

Definition 2.4.2 *A preference* $P \in \mathbb{L}(X)$ *is called single-dipped if* $r_m(P) = x_i$ *and* $[j < k \leq i$ *or* $i \leq k < j]$ *imply* $x_j P x_k$*. A domain* $\mathcal{D}_d$ *is called single-dipped if it contains all single-dipped preferences.*

A domain $\mathcal{D}$ is called the union of the single-peaked and the single-dipped domain if $\mathcal{D} = \mathcal{D}_p \cup \mathcal{D}_d$. It is easy to verify that the union of the single-peaked and the single-dipped domain is a top-circular domain satisfying the maximal conflict property. Thus, we have the following corollary of Theorem 2.3.1.

Corollary 2.4.1 *Let* $\mathcal{D}$ *be the union of the single-peaked and the single-dipped domain. Then,* $\mathcal{D}$ *is a dictatorial domain.*

2.4.2 Circular domains

The notion of circular domains is introduced in [74], where he shows that a circular domain is dictatorial. However, we obtain this result as a corollary of our result.

Definition 2.4.3 *A domain* $\mathcal{D}$ *is called circular if it is a top-circular domain satisfying the property that for all* $k = 1, \ldots, m$*, there are two preferences* $x_k x_{k+1} \ldots x_{k-1}$ *and* $x_k x_{k-1} \ldots x_{k+1}$ *in the domain* $\mathcal{D}$*.*

Note that a circular domain is a top-circular domain satisfying the weak conflict property. Thus, we have the following corollary of Theorem 2.3.2.

Corollary 2.4.2 ([74]) *Let* $\mathcal{D}$ *be a circular domain. Then,* $\mathcal{D}$ *is a dictatorial domain.*

2.5 Concluding remarks

In this chapter, we prove that any unanimous and strategy-proof social choice rule on a top-circular domain satisfying either the maximal conflict property or the weak conflict property is dictatorial. Our result is independent from the existing results on dictatorial domains.

Since dictatorial rules are tops-only, Theorem 2.3.1 and 2.3.2 imply that top-circular domains satisfying either the maximal conflict property or the weak conflict property are tops-only. [20] provides sufficient conditions for a domain to be tops-only, however, our domain restrictions do not satisfy their condition. Moreover, since dictatorial rules are also group-strategy-proof, it follows that the notions of strategy-proofness and group-strategy-proofness are equivalent for the domains we consider.

2.6 Appendix

In this section, we prove Theorem 2.3.1 and Theorem 2.3.2. The following proposition in [6] allows us to restrict our attention to the case of two agents.

Proposition 2.6.1 ([6]) *Let $\mathcal{D}$ be a regular domain such that every unanimous and strategy-proof SCF $f : \mathcal{D}^2 \to X$ is dictatorial. Then, every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.*

The following proposition in [71] allows us to restrict our attention to *minimal* top-circular domains satisfying either the maximal conflict or the weak conflict property.[11]

Proposition 2.6.2 ([71]) *A superset of a regular dictatorial domain is also dictatorial.*

Now, we introduce the notion of option sets, which we use in our proofs.

[11] A top-circular domain is *minimal* if none of its subsets is top-circular.

Definition 2.6.1 *Given an SCF $f : \mathcal{D}^2 \to X$, we define the option set of agent $i \in \{1,2\}$ at preference $P_j \in \mathcal{D}$ of agent $j \in \{1,2\} \setminus i$, denoted by $O_i(P_j)$, as $O_i(P_j) = \bigcup_{P_i \in \mathcal{D}} f(P_i, P_j)$.*

Remark 2.6.1 *Note that if an SCF $f : \mathcal{D}^2 \to X$ is unanimous, then $r_1(P_j) \in O_i(P_j)$ for all $P_j \in \mathcal{D}$. Furthermore, if f is strategy-proof, then for all $i, j \in \{1,2\}; i \neq j$ and all $(P_1, P_2) \in \mathcal{D}^2$, $f(P_1, P_2) = \max_{P_i} O_i(P_j)$, where $\max_{P_i} O_i(P_j) = x$ if and only if $x \in O_i(P_j)$ and xP_iy for all $y \in O_i(P_j) \setminus x$.*

Remark 2.6.2 *Note that an SCF $f : \mathcal{D}^2 \to X$ is dictatorial if and only if there is $i \in \{1,2\}$ such that $O_i(P_j) = \{r_1(P_j)\}$ for all $P_j \in \mathcal{D}$.*

For all the subsequent results, let $\mathcal{C}$ be a minimal top-circular domain. Suppose $f : \mathcal{C}^2 \to X$ is a unanimous and strategy-proof SCF and $O_i(P_j)$ is the corresponding option set of agent i at a preference P_j of agent $j \in \{1,2\} \setminus i$. We prove a sequence of lemmas that we use in the proofs of Theorem 2.3.1 and Theorem 2.3.2.

The following lemma establishes a property of a minimal top-circular domain. We assume for this lemma that $0 \equiv m$ and $m+1 \equiv 1$.

Lemma 2.6.1 *Let $P_2, P_2' \in \mathcal{C}$ be such that $r_1(P_2) = r_1(P_2') = x_k$. Then, for all $j \in \{k-1, k+1\}$, $x_j \in O_1(P_2)$ if and only if $x_j \in O_1(P_2')$.*

Proof: Assume for contradiction that there exist $P_2, P_2' \in \mathcal{C}$ with $r_1(P_2) = r_1(P_2') = x_k$ such that $x_j \in O_1(P_2)$ and $x_j \notin O_1(P_2')$ for some $j \in \{k-1, k+1\}$. Consider $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_j$ and $r_2(P_1) = x_k$. Such a preference exists in $\mathcal{C}$ as $|j-k| = 1$. Then, by the strategy-proofness of f, $f(P_1, P_2) = x_j$ and $f(P_1, P_2') = x_k$. This means agent 2 manipulates at (P_1, P_2) via P_2', a contradiction. This completes the proof of the lemma. ∎

The subsequent lemmas establish few crucial properties of a minimal top-circular domain $\mathcal{C}$ such that $\{x_1x_2 \ldots x_m, x_mx_{m-1} \ldots x_1\} \subseteq \mathcal{C}$. Note that if a minimal top-circular domain $\mathcal{C}$ satisfies either the maximal conflict property or the weak conflict property, then such two preferences are there in $\mathcal{C}$.

Lemma 2.6.2 *Let* $\{x_1x_2\dots x_m, x_mx_{m-1}\dots x_1\} \subseteq \mathcal{C}$. *Then, for all* $P_2 \in \{x_1x_2\dots x_m, x_mx_{m-1}\dots x_1\}$, $r_m(P_2) \notin O_1(P_2)$ *implies* $O_1(P_2) = \{r_1(P_2)\}$.

Proof: We prove the lemma for the case where $P_2 = x_1x_2\dots x_m \in \mathcal{C}$, the proof of the same for the other case is analogous. Let $P_2 = x_1x_2\dots x_m \in \mathcal{C}$ and let $r_m(P_2) = x_m \notin O_1(P_2)$. We show $O_1(P_2) = \{r_1(P_2)\}$. Assume for contradiction that $x_j \in O_1(P_2)$ for some $j \neq 1, m$. Let $P_2' \in \mathcal{C}$ be such that $r_1(P_2') = x_1$ and $r_2(P_2') = x_m$. Since $x_m \notin O_1(P_2)$, by Lemma 2.6.1, $x_m \notin O_1(P_2')$. Let $P_1 = x_mx_{m-1}\dots x_1$. By unanimity and strategy-proofness, we must have $f(P_1, P_2') \in \{x_1, x_m\}$ as otherwise, agent 2 manipulates at (P_1, P_2') via a preference which places x_m at the top. Also, since $x_m \notin O_1(P_2')$, we have $f(P_1, P_2') = x_1$. However, since $x_j \in O_1(P_2)$ and $x_jP_1x_1$, it must be that $f(P_1, P_2) \neq x_1$. Because $r_1(P_2) = x_1 = r_1(P_2')$, this means agent 2 manipulates at (P_1, P_2) via P_2', a contradiction. This completes the proof of the lemma. ■

Lemma 2.6.3 *Let* $\{x_1x_2\dots x_m, x_mx_{m-1}\dots x_1\} \subseteq \mathcal{C}$ *and let* $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$ *for all* $P_2 \in \{x_1x_2\dots x_m, x_mx_{m-1}\dots x_1\}$. *Suppose* $\hat{P}_2, \bar{P}_2 \in \mathcal{C}$ *is such that* $r_1(\hat{P}_2) = x_1$ *and* $r_1(\bar{P}_2) = x_m$. *Then,* $O_1(\hat{P}_2) = \{x_1\}$ *if and only if* $O_1(\bar{P}_2) = \{x_m\}$.

Proof: Let $\hat{P}_2, \bar{P}_2 \in \mathcal{C}$ be such that $r_1(\hat{P}_2) = x_1$ and $r_1(\bar{P}_2) = x_m$. It is sufficient to show that $O_1(\hat{P}_2) = \{x_1\}$ implies $O_1(\bar{P}_2) = \{x_m\}$. By strategy-proofness, it is enough to show that $O_1(\bar{P}_2) = \{x_m\}$ where $\bar{P}_2 = x_mx_{m-1}\dots x_1$.

Assume for contradiction that $O_1(\hat{P}_2) = \{x_1\}$ and $O_1(\bar{P}_2) \neq \{x_m\}$. By the assumption of the lemma, $O_1(\bar{P}_2) \neq \{x_m\}$ implies $O_1(\bar{P}_2) = X$. Consider $\bar{P}_2' \in \mathcal{C}$ such that $r_1(\bar{P}_2') = x_m$ and $r_2(\bar{P}_2') = x_1$. Since $O_1(\bar{P}_2) \neq \{x_m\}$, it follows from strategy-proofness that $O_1(\bar{P}_2') \neq \{x_m\}$. We show $x_j \notin O_1(\bar{P}_2')$ for all $j \neq 1, m$. Suppose not. Then, $f(P_1, \bar{P}_2') = x_j$ for some $P_1 \in \mathcal{C}$ with x_j at the top. However, because $O_1(\hat{P}_2) = \{x_1\}$, agent 2 manipulates at $(P_1, \bar{P}_2')$ via $\hat{P}_2$. Since $O_1(\bar{P}_2') \neq \{x_m\}$ and $x_j \notin O_1(\bar{P}_2')$ for all $j \neq 1, m$, it must be that $O_1(\bar{P}_2') = \{x_1, x_m\}$. However, since $O_1(\bar{P}_2) = X$, which in turn means $x_{m-1} \in O_1(\bar{P}_2)$, by Lemma 2.6.1, we must have $x_{m-1} \in O_1(\bar{P}_2')$, a contradiction. This completes the proof of the lemma. ■

2.6.1 Proof of Theorem 2.3.1

In this section, we provide a proof of Theorem 2.3.1. First, we establish a few properties of a top-circular domain satisfying the maximal conflict property.

Lemma 2.6.4 *Let $\mathcal{C}$ satisfy the maximal conflict property. Let $P, P' \in \mathcal{C}$ be such that $r_k(P) = r_{m-k+1}(P') = x_k$ for all $k = 1, \ldots, m$. Then, for all $P_2 \in \{P, P'\}$, $r_m(P_2) \in O_1(P_2)$ implies $O_1(P_2) = X$.*

Proof: We prove this lemma for the case where $P_2 = P$, the proof of the same for the other case is analogous. Let $P_2 = P$. Suppose $x_m \in O_1(P_2)$. We show $O_1(P_2) = X$. We prove this by induction. Since $x_m \in O_1(P_2)$, it is sufficient to show that for all $1 < k \leq m$, $x_k \in O_1(P_2)$ implies $x_{k-1} \in O_1(P_2)$. Assume for contradiction that $x_k \in O_1(P_2)$ but $x_{k-1} \notin O_1(P_2)$ for some $1 < k \leq m$. Consider $P_1 = x_{k-1}x_k \ldots \in \mathcal{C}$. Since $x_k \in O_1(P_2)$ and $x_{k-1} \notin O_1(P_2)$, $f(P_1, P_2) = x_k$. However, this means agent 2 manipulates at (P_1, P_2) via a preference which places x_{k-1} at the top, a contradiction. This completes the proof of the lemma. ■

Remark 2.6.3 *Let $\mathcal{C}$ be a minimal top-circular domain satisfying the maximal conflict property, and let $P, P' \in \mathcal{C}$ be such that $r_k(P) = r_{m-k+1}(P') = x_k$ for all $k = 1, \ldots, m$. Then, it follows from Lemma 2.6.2 that for all $P_2 \in \{P, P'\}$, $r_m(P_2) \notin O_1(P_2)$ implies $O_1(P_2) = \{r_1(P_2)\}$. Again, it follows from Lemma 2.6.4 that for all $P_2 \in \{P, P'\}$, $r_m(P_2) \in O_1(P_2)$ implies $O_1(P_2) = X$. Thus, for all $P_2 \in \{P, P'\}$, we have $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$.*

Lemma 2.6.5 *Let $\mathcal{C}$ satisfy the maximal conflict property. Further, let $P, P' \in \mathcal{C}$ be such that $r_k(P) = r_{m-k+1}(P') = x_k$ for all $k = 1, \ldots, m$. Then, for all $P_2 \in \{P, P'\}$, $O_1(P_2) = \{r_1(P_2)\}$ implies $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$.*

Proof: It is enough to prove the lemma for the case where $P_2 = P$, the proof for the other case is analogous. Let $P_2 = P$. Suppose $O_1(P_2) = \{r_1(P_2)\}$. We show $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$. By strategy-proofness, we have $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_1$. Moreover, by Lemma 2.6.3 and Remark

2.6.3, we have $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_m$. Take $j \neq 1, m$ and $\hat{P}_2 \in \mathcal{C}$ with $r_1(\hat{P}_2) = x_j$. We show $O_1(\hat{P}_2) = \{r_1(\hat{P}_2)\}$.

First, we show $O_2(P_1) = O_2(P_1') = X$, where $r_{m-k+1}(P_1) = r_k(P_1') = x_k$ for all $k = 1, \ldots, m$. We show this for P_1, the proof of the same for P_1' is analogous. Since $O_1(P_2) = \{x_1\}$, we have $f(P_1, P_2) = x_1$. Because $r_m(P_1) = x_1$, this means $r_m(P_1) \in O_2(P_1)$. By Lemma 2.6.4, this means $O_2(P_1) = X$.

Now, we complete the proof of the lemma. Assume for contradiction that $x_l \in O_1(\hat{P}_2)$ for some $x_l \neq r_1(\hat{P}_2) = x_j$. Since $r_{m-k+1}(P_1) = r_k(P_1') = x_k$ for all $k = 1, \ldots, m$, we must have either $x_l P_1 x_j$ or $x_l P_1' x_j$. Assume without loss of generality that $x_l P_1 x_j$. Since $O_2(P_1) = X$ and $r_1(\hat{P}_2) = x_j$, $f(P_1, \hat{P}_2) = x_j$. Let $\hat{P}_1 \in \mathcal{C}$ such that $r_1(\hat{P}_1) = x_l$. Since $x_l \in O_1(\hat{P}_2)$ and $r_1(\hat{P}_1) = x_l$, we have $f(\hat{P}_1, \hat{P}_2) = x_l$. This means agent 1 manipulates at $(P_1, \hat{P}_2)$ via $\hat{P}_1$, a contradiction. Therefore, $O_1(\hat{P}_2) = \{r_1(\hat{P}_2)\}$, which completes the proof of the lemma. ■

Now we are ready to prove Theorem 2.3.1.

Proof: [Proof of Theorem 2.3.1] In view of Propositions 2.6.1 and 2.6.2, it sufficient to show that a minimal top-circular domain with the maximal conflict property is dictatorial for two agents. Consider $P_2 \in \mathcal{C}$ such that $r_k(P_2) = x_k$ for all $1 \leq k \leq m$. By Remark 2.6.3, we have $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$. Suppose $O_1(P_2) = \{r_1(P_2)\}$. Then, by Lemma 2.6.5, it follows that $O_1(P_2') = \{r_1(P_2')\}$ for all $P_2' \in \mathcal{C}$, which implies agent 2 is the dictator.

Now, suppose $O_1(P_2) = X$. Consider $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_m$. Since $O_1(P_2) = X$, we have $f(P_1, P_2) = x_m$. We claim $O_2(P_1) = \{r_1(P_1)\}$. Assume for contradiction that $x_j \in O_2(P_1)$ for some $j \neq m$. Since $r_m(P_2) = x_m$, we have $x_j P_2 x_m$. However, since $x_j \in O_2(P_1)$, agent 2 manipulates at (P_1, P_2) via some preference $\bar{P}_2$ with $r_1(\bar{P}_2) = x_j$. Therefore, $O_2(P_1) = \{r_1(P_1)\}$. By Lemma 2.6.5, this means $O_2(P_1) = \{r_1(P_1)\}$ for all $P_1 \in \mathcal{C}$, which implies agent 1 is the dictator. This completes the proof of the theorem. ■

2.6.2 Proof of Theorem 2.3.2

In this section, we provide a proof of Theorem 2.3.2. First, we establish a few properties of a top-circular domain satisfying the weak conflict property.

Lemma 2.6.6 *Let $\mathcal{C}$ satisfy the weak conflict property. Suppose $P_2 \in \{x_1x_2 \dots x_m, x_mx_{m-1} \dots x_1\} \subseteq \mathcal{C}$. Then, $r_m(P_2) \in O_1(P_2)$ implies $O_1(P_2) = X$.*

Proof: It is enough to prove the lemma for $P_2 = x_1x_2 \dots x_m \in \mathcal{C}$, the proof for the other case is analogous. Suppose $x_m \in O_1(P_2)$. We show $O_1(P_2) = X$. We prove this by induction. By unanimity, $x_1 \in O_1(P_2)$. Therefore, it is sufficient to show that for all $1 \leq k < m$, $x_k \in O_1(P_2)$ implies $x_{k+1} \in O_1(P_2)$. Assume for contradiction that $x_k \in O_1(P_2)$ and $x_{k+1} \notin O_1(P_2)$ for some $1 \leq k < m$. Let $\hat{P}_2 = x_kx_{k+1} \dots x_m \dots x_{k-1} \dots \in \mathcal{C}$. Note that since $x_m \in O_1(P_2)$ and $r_m(P_2) = x_m$, by strategy-proofness, it must be that $x_m \in O_1(\hat{P}_2)$. Let $P_1 = x_{k+1}x_{k+2} \dots x_m \dots x_k \dots \in \mathcal{C}$. By unanimity and strategy-proofness, $f(P_1, \hat{P}_2) \in \{x_k, x_{k+1}\}$, as otherwise agent 2 manipulates at $(P_1, \hat{P}_2)$ via some preference with x_{k+1} at the top. Suppose $f(P_1, \hat{P}_2) = x_k$. Since $x_mP_1x_k$ and $x_m \in O_1(\hat{P}_2)$, this means agent 1 manipulates at $(P_1, \hat{P}_2)$ via some preference with x_m at the top. Therefore, we have $f(P_1, \hat{P}_2) = x_{k+1}$. Now, let $P'_1 = x_{k+1}x_k \dots \in \mathcal{C}$. Then, since $f(P_1, \hat{P}_2) = x_{k+1}$ and $r_1(P_1) = r_1(P'_1) = x_{k+1}$, by strategy-proofness, $f(P'_1, \hat{P}_2) = x_{k+1}$. Also, because $x_k \in O_1(P_2)$ and $x_{k+1} \notin O_1(P_2)$, we have $f(P'_1, P_2) = x_k$. Therefore, agent 2 manipulates at $(P'_1, \hat{P}_2)$ via P_2, a contradiction. This completes the proof of the lemma. ■

Remark 2.6.4 *Let $\mathcal{C}$ satisfy the weak conflict property. Then, by using arguments similar to the ones employed in Remark 2.6.3, it follows from Lemma 2.6.2 and Lemma 2.6.6 that for all $P_2 \in \{x_1x_2 \dots x_m, x_mx_{m-1} \dots x_1\}$, $O_1(P_2) \in \{\{r_1(P_2)\}, X\}$.*

Lemma 2.6.7 *Let $\mathcal{C}$ satisfy the weak conflict property. Further, let $P_2 \in \{x_1x_2 \dots x_m, x_mx_{m-1} \dots x_1\}$. Then, $O_1(P_2) = \{r_1(P_2)\}$ implies $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$.*

Proof: We prove this lemma for the case where $P_2 = x_1x_2 \ldots x_m$, the proof for the case where $P_2 = x_m x_{m-1} \ldots x_1$ is analogous. Let $P_2 = x_1x_2 \ldots x_m$. Suppose $O_1(P_2) = \{x_1\}$. We show $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$. By strategy-proofness, we have $O_1(\bar{P}_2) = \{r_1(\bar{P}_2)\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_1$. By Lemma 2.6.3 and Remark 2.6.4, $O_1(P_2) = \{x_1\}$ implies $O_1(\bar{P}_2) = \{x_m\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_m$. We prove the lemma using induction. Take $1 \leq j < m$. Suppose $O_1(\bar{P}_2) = \{x_j\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_j$. We show $O_1(\hat{P}_2) = \{x_{j+1}\}$ for all $\hat{P}_2 \in \mathcal{C}$ with $r_1(\hat{P}_2) = x_{j+1}$. Take $\hat{P}_2 \in \mathcal{C}$ with $r_1(\hat{P}_2) = x_{j+1}$. We show $O_1(\hat{P}_2) = \{x_{j+1}\}$. By strategy-proofness, it is enough to show this for $\hat{P}_2 = x_{j+1}x_j \ldots$.

First, we claim $x_k \notin O_1(\hat{P}_2)$ for all $k \neq j, j+1$. Assume for contradiction that $x_k \in O_1(\hat{P}_2)$ for some $k \neq j, j+1$. Then, $f(P_1, \hat{P}_2) = x_k$ for some $P_1 \in \mathcal{C}$ with $r_1(P_1) = x_k$. However, since $O_1(\bar{P}_2) = \{x_j\}$ for all $\bar{P}_2 \in \mathcal{C}$ with $r_1(\bar{P}_2) = x_j$, agent 2 manipulates at $(P_1, \hat{P}_2)$ via some preference $\bar{P}_2$ with $r_1(\bar{P}_2) = x_j$.

Now, we show $x_j \notin O_1(\hat{P}_2)$. Assume for contradiction that $x_j \in O_1(\hat{P}_2)$. Let $\hat{P}'_2 = x_{j+1}x_{j+2} \ldots x_m \ldots x_j \ldots$. Then, by Lemma 2.6.1, $x_j \in O_1(\hat{P}'_2)$. Take $P_1 \in \mathcal{C}$ such that $r_1(P_1) = x_j$. Then, because $x_j \in O_1(\hat{P}'_2)$, $f(P_1, \hat{P}'_2) = x_j$. Now, take $P_2 \in \mathcal{C}$ with $r_1(P_2) = x_m$. Since $O_1(P_2) = \{x_m\}$, we have $f(P_1, P_2) = x_m$. This means agent 2 manipulates at $(P_1, \hat{P}'_2)$ via P_2. This completes the proof of the lemma. ■

Proof:[Proof of Theorem 2.3.2] The proof of Theorem 2.3.2 follows by using analogous arguments as for the proof of Theorem 2.3.1. ■

3

On Single-peaked Domains and Min-max Rules

3.1 Introduction

3.1.1 Background

The celebrated Gibbard-Satterthwaite ([43], [75]) theorem has drawn severe criticism as it assumes that the admissible domain of each agent is unrestricted. However, it is well established that in many economic and political applications, there are natural restrictions on such domains. For instance, in the models of locating a firm in a unidimensional spatial market ([46]), setting the rate of carbon dioxide emissions ([15]), setting the level of public expenditure ([69]), and so on, preferences admit a natural restriction widely known as *single-peakedness*. Roughly speaking, single-peakedness of a preference implies that there is a prior order over the alternatives such that the preference decreases as one moves away (with respect to the prior order) from her best alternative.

3.1.2 Motivation and Contribution

The study of single-peaked domains dates back to [15], where it is shown that the pairwise majority rule is strategy-proof on such domains. Later, [54] and [86] characterize the unanimous and strategy-proof SCFs on these domains.[1,2] However, their characterization rests upon the assumption that the set of admissible preferences of each agent in the society is the *maximal* single-peaked domain, i.e., it contains *all* single-peaked preferences with respect to a given prior order over the alternatives. Note that demanding the existence of all single-peaked preferences is a strong prerequisite in many practical situations.[3] This motivates us to analyze the structure of the unanimous and strategy-proof SCFs on domains where agents have arbitrary (but same) admissible sets of single-peaked preferences. We show that every unanimous and strategy-proof SCF on such domains satisfies the *Pareto property* and *tops-onlyness*.[4]

While single-peakedness is a natural condition in many practical scenarios, the assumption that all agents have the *same* set of single-peaked preferences is not justifiable in many contexts. In view of this, we consider the situation where different agents have different admissible sets of single-peaked preferences.

First, we show by means of an example that tops-onlyness is not guaranteed for unanimous and strategy-proof SCFs on such domains. Next, we provide two suffi-

[1] [9] and [24] provide equivalent presentations of this class of SCFs.

[2] A rich literature has developed around the single-peaked restriction by considering various generalizations and extensions (see [9], [29], [76], [55], and [56]).

[3] See, for instance, the domain restriction considered in models of voting ([84], [5]), taxation and redistribution ([38]), determining the levels of income redistribution ([44], [79]), and measuring tax reforms in the presence of horizontal inequity ([45]). Recently, [63] shows that under mild conditions these domains form subsets of the maximal single-peaked domain.

[4] [20] provide a sufficient condition on a domain so that every unanimous and strategy-proof SCF on it is tops-only. However, an arbitrary single-peaked domain does not satisfy their condition.

cient conditions, called *left-connected* and *right-connected*, on the admissible sets of single-peaked preferences of the agents so that unanimity and strategy-proofness imply tops-onlyness. A set of single-peaked preferences is called left-connected (right-connected) if for every two consecutive alternatives with respect to the prior order over the alternatives, say x and $x+1$, we have a preference that places $x+1$ at the top and x at the second-ranked position (x at the top and $x+1$ at the second-ranked position). Finally, we show by means of examples that the exact structure of unanimous and strategy-proof SCFs depends heavily on the domain. In order to obtain a tractable structure of such SCFs, we restrict our attention to *top-connected single-peaked domains* and provide a characterization of the unanimous and strategy-proof SCFs on those. A domain is top-connected if the admissible sets of preferences of each agent is both left-connected and right-connected.[5]

The unanimous and strategy-proof SCFs on the maximal single-peaked domain are known as *min-max rules* ([54], [86]). Min-max rules are quite popular for their desirable properties like *tops-onlyness*, *Pareto property*, and *anonymity* (for a sub-class of min-max rules called *median rules*). Owing to the desirable properties of min-max rules, [11] characterize maximal domains on which a *given* min-max rule is strategy-proof. Recently, [3] provide necessary and sufficient conditions for the comparability of two min-max rules in terms of their vulnerability to manipulation. Motivated by the importance of the min-max rules, we characterize all domains on which (i) every unanimous and strategy-proof social choice function is a min-max rule, and (ii) every min-max rule is strategy-proof. We call such a domain a *min-max domain*.

Note that min-max domains do *not* require that the admissible preferences of all the agents are the same. Furthermore, it is worth noting that in a social choice problem with m alternatives, the number of preferences of each agent in a min-max domain can range from $2m-2$ to 2^{m-1}, whereas that in the maximal single-peaked domain is exactly 2^{m-1}. Thus, on one hand, our result characterizes the unanimous and strategy-proof SCFs on a large class of single-peaked domains, and on the other

[5]The top-connectedness property is well studied in the literature (see [12], [6], [20], [21], [22], [63] and [70]).

hand, it establishes the full applicability of min-max rules as strategy-proof SCFs.

3.1.3 Applications

An outstanding example of a top-connected single-peaked domain is a *top-connected regular single-crossing domain*.[6,7] [73] shows that an SCF is unanimous and strategy-proof on a *maximal* single-crossing domain if and only if it is a min-max rule.[8] In contrast, our result shows that an SCF is unanimous and strategy-proof on a *top-connected regular* single-crossing domain if and only if it is a min-max rule. Thus, we extend [73]'s result in two ways: (i) by relaxing the maximality assumption on a single-crossing domain, and (ii) by relaxing the assumption that every agent has the same set of preferences. However, we assume the domains to be regular. Note that in a social choice problem with m alternatives, the number of admissible preferences of each agent in a top-connected regular single-crossing domain can range from $2m-2$ to $m(m-1)/2$, whereas that in the maximal single-crossing domain is exactly $m(m-1)/2$.

Other important examples of top-connected single-peaked domains include *minimally rich single-peaked domains* ([61]) and *distance based single-peaked domains*. A single-peaked domain is minimally rich if it contains all *left single-peaked* and all *right single-peaked* preferences.[9,10] Further, a single-peaked domain is called distance based if the preferences in it are derived by using some type of distances between the alternatives. It follows from our result that an SCF is unanimous and strategy-proof on these domains if and only if it is a min-max rule.

[6] A domain is *regular* if every alternative appears as the top-ranked alternative of some preference in the domain.

[7] Single-crossing domains appear in models of taxation and redistribution ([68], [53]), local public goods and stratification ([85], [35], [39]), coalition formation ([30], [51]), selecting constitutional and voting rules ([10]), and designing policies in the market for higher education ([36]).

[8] [73] provides a different but equivalent functional form of these SCFs which he calls *augmented representative voter schemes*.

[9] A single-peaked preference is called left (or right) single-peaked if every alternative to the left (or right) of the peak is preferred to every alternative to its right (or left).

[10] Such preferences appear in directional theories of issue voting ([81], [64], [66], [65]).

3.1.4 Remainder

The rest of the chapter is organized as follows. We describe the usual social choice framework in Section 3.2. In Section 3.3, we study the structure of the unanimous and strategy-proof SCFs on single-peaked domains, and in Section 3.4, we characterize such SCFs on top-connected single-peaked domains. Section 3.5 characterizes min-max domains. In Section 3.6, we discuss some applications of our results, and we conclude the chapter in the last section. All the omitted proofs are collected in the Appendix.

3.2 Preliminaries

Let $N = \{1, ..., n\}$ be a set of at least two agents, who collectively choose an element from a finite set $X = \{a, a+1, \ldots, b-1, b\}$ of at least three alternatives, where a is an integer. For $x, y \in X$ such that $x \leq y$, we define the intervals $[x, y] = \{z \in X \mid x \leq z \leq y\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *preference* P over X is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on X. We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k - 1$.

Definition 3.2.1 *A preference $P \in \mathbb{L}(X)$ is called single-peaked if for all $x, y \in X$, $[x < y \leq r_1(P)$ or $r_1(P) \leq y < x]$ implies yPx.*

For an agent i, we denote by $\mathcal{S}_i$ a set of admissible single-peaked preferences. A set $\mathcal{S}_N = \prod_{i \in N} \mathcal{S}_i$ is called a *single-peaked domain*. Note that we do not assume that the set of admissible preferences are the same across all agents. An element $P_N = (P_1, \ldots, P_n) \in \mathcal{S}_N$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \bigcup_{i \in N} r_1(P_i)$. A set $\mathcal{S}_i$ of admissible preferences of agent i is *regular* if for all $x \in X$, there exists a preference $P \in \mathcal{S}_i$ such that $r_1(P) = x$.

Definition 3.2.2 *A social choice function (SCF) f on $\mathcal{S}_N$ is a mapping $f : \mathcal{S}_N \to X$.*

Definition 3.2.3 *An SCF $f : \mathcal{S}_N \to X$ is unanimous if for all $P_N \in \mathcal{S}_N$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.*

Definition 3.2.4 *An SCF $f : \mathcal{S}_N \to X$ satisfies the Pareto property if for all $P_N \in \mathcal{S}_N$ and all $x, y \in X$, xP_iy for all $i \in N$ implies $f(P_N) \neq y$.*

Remark 3.2.1 *Note that since $\mathcal{S}_i$ is single-peaked for all $i \in N$, an SCF $f : \mathcal{S}_N \to X$ satisfies Pareto property if $f(P_N) \in [\min(\tau(P_N)), \max(\tau(P_N))]$ for all $P_N \in \mathcal{S}_N$.*

Definition 3.2.5 *An SCF $f : \mathcal{S}_N \to X$ is manipulable if there is $P_N \in \mathcal{S}_N$, $i \in N$, and $P'_i \in \mathcal{S}_i$ such that $f(P'_i, P_{N\setminus i})P_if(P_N)$. An SCF f is strategy-proof if it is not manipulable.*

Definition 3.2.6 *An SCF $f : \mathcal{S}_N \to X$ is called group manipulable if there is $P_N \in \mathcal{S}_N$, a non-empty coalition $C \subseteq N$, and a preference profile $P'_C \in \mathcal{S}_C$ of the agents in C such that $f(P'_C, P_{N\setminus C})P_if(P_N)$ for all $i \in C$. An SCF $f : \mathcal{S}_N \to X$ is called group strategy-proof if it is not group manipulable.*

Definition 3.2.7 *An SCF $f : \mathcal{S}_N \to X$ is called tops-only if for all $P_N, P'_N \in \mathcal{S}_N$ such that $r_1(P_i) = r_1(P'_i)$ for all $i \in N$, we have $f(P_N) = f(P'_N)$.*

Definition 3.2.8 *An SCF $f : \mathcal{S}_N \to X$ is called uncompromising if for all $P_N \in \mathcal{S}_N$, all $i \in N$, and all $P'_i \in \mathcal{S}_i$:*

(i) if $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$, then $f(P_N) = f(P'_i, P_{N\setminus i})$, and

(ii) if $f(P_N) < r_1(P_i)$ and $f(P_N) \leq r_1(P'_i)$, then $f(P_N) = f(P'_i, P_{N\setminus i})$.

Remark 3.2.2 *If an SCF satisfies uncompromisingness, then by definition, it is tops-only.*

Definition 3.2.9 *Let $\beta = (\beta_S)_{S \subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{S}_N \to X$ is called a min-max rule with respect to β if*

$$f^\beta(P_N) = \min_{S \subseteq N} \{ \max_{i \in S} \{ r_1(P_i), \beta_S \} \}.$$

Remark 3.2.3 *Every min-max rule is uncompromising.*[11]

3.3 SCFs on Single-peaked Domains

In this section, we establish that every unanimous and strategy-proof SCF on a class of single-peaked domains is tops-only.

First, we state an important result that follows from [7].

Theorem 3.3.1 ([7]) *Every strategy-proof SCF on a single-peaked domain is group strategy-proof.*

For a set of preferences $\mathcal{D}$, we denote by $\tau(\mathcal{D})$ the set of alternatives that appear as a top-ranked alternative in some preference in $\mathcal{D}$, that is, $\tau(\mathcal{D}) = \cup_{P \in \mathcal{D}} \{r_1(P)\}$.

Our next corollary follows from Theorem 3.3.1.

Corollary 3.3.1 *Let $\mathcal{S}_N$ be a single-peaked domain such that $\tau(\mathcal{S}_i) = \tau(\mathcal{S}_j)$ for all $i, j \in N$. Then, every unanimous and strategy-proof SCF $f : \mathcal{S}_N \to X$ satisfies Pareto property.*

The proof of Corollary 3.3.1 is rather straight-forward, but for the sake of completeness, we provide a proof.

Proof: By Theorem 3.3.1, f must be group strategy-proof. Suppose that the corollary does not hold. Then, without loss of generality we can assume that there is a profile $P_N \in \mathcal{S}_N$ where $r_1(P_1) \leq r_1(P_j)$ for all $j \in N$ such that $f(P_N) < r_1(P_1)$. Note that this means $r_1(P_1) P_j f(P_N)$ for all $j \in N$. Let $\bar{P}_N$ be such that $r_1(\bar{P}_j) = r_1(P_1)$ for

[11] For details, see [86].

all $j \in N$. Note that such a profile $\bar{P}_N$ is in $\mathcal{S}_N$ since $\tau(\mathcal{S}_i) = \tau(\mathcal{S}_j)$ for all $i, j \in N$. By unanimity, $f(\bar{P}_N) = r_1(P_1)$. This means all the agents together manipulate f at P_N via $\bar{P}_N$, a contradiction. This completes the proof of Corollary 3.3.1. ■

Our next theorem shows that if every agent in a society has the same set of single-peaked preferences, then each unanimous and strategy-proof SCF on that domain is tops-only.

Theorem 3.3.2 *Let $\mathcal{S}$ be a(ny) set of single-peaked preferences. Then, every unanimous and strategy-proof SCF $f : \mathcal{S}^n \to X$ satisfies tops-onlyness.*

Note that the set of single-peaked preferences $\mathcal{S}$ in Theorem 3.3.2 need not be regular. The proof of Theorem 3.3.2 is relegated to the Appendix.

Now, we consider situations where different agents can have different sets of single-peaked preferences. First, we show by means of an example that unanimity and strategy-proofness do not guarantee tops-onlyness on such domains.

Example 3.3.1 *Let $N = \{1, 2, 3\}$ and let $X = \{x_1, x_2, x_3\}$, where $x_1 < x_2 < x_3$. Suppose that $\mathcal{S}_1 = \{x_1x_2x_3, x_2x_1x_3, x_2x_3x_1, x_3x_2x_1\}$, $\mathcal{S}_2 = \{x_1x_2x_3, x_2x_1x_3, x_3x_2x_1\}$, and $\mathcal{S}_3 = \{x_1x_2x_3, x_2x_3x_1, x_3x_2x_1\}$ where by $x_1x_2x_3$ we mean a preference P such that $x_1Px_2Px_3$.*

Consider the SCF on this single-peaked domain as given in Table 3.4.1. The table is self-explanatory.

It is easy to verify that the SCF given in Table 3.3.1 is unanimous and strategy-proof. However, since $f(x_2x_1x_3, x_3x_2x_1, x_1x_2x_3) = x_1$ and $f(x_2x_3x_1, x_3x_2x_1, x_1x_2x_3) = x_3$, this SCF is not tops-only.

In view of Example 3.3.1, we look for additional conditions on the set of (admissible) single-peaked preferences of each agent so as to ensure tops-onlyness for every unanimous and strategy-proof SCF on the respective domains. In what follows, we introduce the notion of left-connected and right-connected single-peaked domains.

$P_3 = x_1x_2x_3$			
P_1 \ P_2	$x_1x_2x_3$	$x_2x_1x_3$	$x_3x_2x_1$
$x_1x_2x_3$	x_1	x_1	x_1
$x_2x_1x_3$	x_1	x_1	x_1
$x_2x_3x_1$	x_1	x_1	x_3
$x_3x_2x_1$	x_1	x_1	x_3

$P_3 = x_2x_3x_1$			
P_1 \ P_2	$x_1x_2x_3$	$x_2x_1x_3$	$x_3x_2x_1$
$x_1x_2x_3$	x_2	x_2	x_3
$x_2x_1x_3$	x_2	x_2	x_3
$x_2x_3x_1$	x_2	x_2	x_3
$x_3x_2x_1$	x_3	x_3	x_3

$P_3 = x_3x_2x_1$			
P_1 \ P_2	$x_1x_2x_3$	$x_2x_1x_3$	$x_3x_2x_1$
$x_1x_2x_3$	x_2	x_2	x_3
$x_2x_1x_3$	x_2	x_2	x_3
$x_2x_3x_1$	x_2	x_2	x_3
$x_3x_2x_1$	x_3	x_3	x_3

Table 3.3.1 A non-tops-only SCF

Definition 3.3.1 *A regular set of single-peaked preferences $\mathcal{S}$ is called left-connected if for all $x \in [a+1, b]$, there exists $P \in \mathcal{S}$ such that $r_1(P) = x$ and $r_2(P) = x - 1$. Similarly, a regular set of single-peaked preferences $\mathcal{S}$ is called right-connected if for all $x \in [a, b-1]$, there exists $P \in \mathcal{S}$ such that $r_1(P) = x$ and $r_2(P) = x + 1$.*

We call a single-peaked domain $\mathcal{S}_N$ *left-connected* (*right-connected*) if for each $i \in N$, $\mathcal{S}_i$ is a *left-connected* (*right-connected*) set of single-peaked preferences.

Note that the domain in Example 3.3.1 is neither left-connected nor right-connected. This is because, even though the set of preferences $\mathcal{S}_1$ is both left-connected and right-connected, the set $\mathcal{S}_2$ is left-connected but not right-connected whereas the set $\mathcal{S}_3$ is right-connected but not left-connected.

Theorem 3.3.3 *Let $\mathcal{S}_N$ be a left-connected or right-connected single-peaked domain. Then, every unanimous and strategy-proof SCF $f : \mathcal{S}_N \to X$ satisfies tops-onlyness.*

The proof of Theorem 3.3.3 is relegated to Appendix.

3.4 SCFs on Top-connected Single-peaked Domains

In this section, we characterize the unanimous and strategy-proof SCFs on a class of single-peaked domains.

First, we present an example to show that the structure of the unanimous and strategy-proof SCFs on arbitrary single-peaked domains is quite intractable.

Example 3.4.1 *Fix $x, y \in X$ with $y - x \geq 2$. For all $i \in N$, let $\mathcal{S}_i^{xy}$ be the set of all single-peaked preferences such that for all $P \in \mathcal{S}_i^{xy}$, $r_1(P) \in (x, y)$ implies $r_{y-x}(P) = x$ and $r_{y-x+1}(P) = y$. In other words, the set of preferences $\mathcal{S}_i^{xy}$ is such that if the top-alternative of a preference is in the interval (x, y), then all the alternatives in that interval are ranked in the top $y - x - 1$ positions, and x and y are ranked consecutively after those alternatives. Similarly, let $\mathcal{S}_i^{yx}$ be the set of all single-peaked preferences such that for all $P \in \mathcal{S}_i^{yx}$, $r_1(P) \in (x, y)$ implies $r_{y-x}(P) = y$ and $r_{y-x+1}(P) = x$.*

Let $x = \beta_{n-1} \leq \beta_{n-2} \leq \ldots \leq \beta_1 \leq \beta_0 = y$. Consider the SCF $f^{xy} : \mathcal{S}_N^{xy} \to X$ as given below:

$$f^{xy}(P_N) = \begin{cases} \text{median}\{r_1(P_2), \ldots, r_1(P_n), \beta_0, \ldots, \beta_{n-1}\}, \ \textit{if}\ r_1(P_1) \in [x, y) \\ r_1(P_1), \ \textit{otherwise} \end{cases}$$

The above SCF can be viewed as a partially median rule in the following sense. Consider the median rule[12] *defined over the interval $[x, y]$ for the agents in $N \setminus 1$ given by the parameters $(\hat{\beta}_k)_{k=0,\ldots,n-1}$, where $\hat{\beta}_k = \beta_k$ for all $k = 0, \ldots, n-1$. Then, the rule f^{xy} works as follows: if agent 1's top-ranked alternative lies outside the interval $[x, y)$, then she is the dictator (i.e., the outcome is her top-ranked alternative), otherwise the outcome is determined by $f^{\tilde{\beta}}$. In other words, f^{xy} partitions the set of preference profiles into two subsets, and behaves like a dictatorial rule in one subset and like a (generalized) median rule in the other.*

Similarly, define $f^{yx} : \mathcal{S}_N^{yx} \to X$ as follows:

$$f^{yx}(P_N) = \begin{cases} \text{median}\{r_1(P_2), \ldots, r_1(P_n), \beta_0, \ldots, \beta_{n-1}\}, \ \textit{if}\ r_1(P_1) \in (x, y] \\ r_1(P_1), \ \textit{otherwise} \end{cases}$$

Note that both f^{xy} and f^{yx} are unanimous by definition. We show that f^{xy} is strategy-proof on $\mathcal{S}_N^{xy}$, but manipulable on $\mathcal{S}_N^{yx}$. It follows from similar arguments that f^{yx} is strategy-proof on $\mathcal{S}_N^{yx}$, but manipulable on $\mathcal{S}_N^{xy}$.

Clearly, no agent can manipulate f^{xy} at a profile $P_N \in \mathcal{S}_N^{xy}$ where $r_1(P_1) \notin [x, y)$. Consider a profile $P_N \in \mathcal{S}_N^{xy}$ where $r_1(P_1) \in [x, y)$. Since $f^{xy}(P_N) = \text{median}\{r_1(P_2), \ldots, r_1(P_n), \beta_0, \ldots, \beta_{n-1}\}$ and $\mathcal{S}_i^{xy}$ is single-peaked, by the property of a median rule, an agent $i \neq 1$ cannot manipulate at P_N. Now, consider a preference $P'_1 \in \mathcal{S}_1^{xy}$. If $r_1(P'_1) \in [x, y)$, then $f^{xy}(P'_1, P_{N\setminus 1}) = f^{xy}(P_N)$ and hence agent 1 cannot manipulate. On the other hand, if $r_1(P'_1) \notin [x, y)$, then $f^{xy}(P'_1, P_{N\setminus 1}) = r_1(P'_1) \notin [x, y)$. However, by the definition of $\mathcal{S}_1^{xy}$, uP_1v for all $u \in [x, y)$ and all $v \notin [x, y)$, this means

[12] A *median rule* is a min-max rule that is anonymous. More formally, a min-max rule with respect to parameters $(\beta_S)_{S \subseteq N}$ is a median rule if $\beta_S = \beta_{\bar{S}}$ for all $S, \bar{S} \subseteq N$ with $|S| = |\bar{S}|$. For details see [54].

$f^{xy}(P_N)P_1f^{xy}(P'_1, P_{N\setminus 1})$, and hence agent 1 cannot manipulate at P_N.

Now, we show that f^{xy} is manipulable on $\mathcal{S}_N^{yx}$. Consider a profile $P_N \in \mathcal{S}_N^{yx}$ where $r_1(P_1) \in (x, y)$ and $r_1(P_j) = x$ for all $j \neq i$. Then, by the definition of f^{xy}, $f^{xy}(P_N) = x$. Let $P'_1 \in \mathcal{S}_1^{yx}$ be such that $r_1(P'_1) = y$. Then, $f^{xy}(P'_1, P_{N\setminus 1}) = y$. However, since $P_1 \in \mathcal{S}_1^{yx}$, by the definition of $\mathcal{S}_1^{yx}$, yP_1x. This means agent 1 manipulates at P_N via P'_1.

It can be verified that the structure of the unanimous and strategy-proof SCFs will get more complicated as we take x and y farther apart (i.e., increase the length of the interval $[x, y]$). This makes the characterization of such SCFs on these domains quite intractable. Therefore, we impose a mild restriction called top-connectedness on single-peaked domains, and characterize the unanimous and strategy-proof SCFs on such domains. We begin with the formal definition of top-connectedness.

Definition 3.4.1 *A set of single-peaked preferences $\mathcal{S}$ is called top-connected if it is both left-connected and right-connected.*

Note that the minimum cardinality of a top-connected set of single-peaked preferences with m alternatives is $2m - 2$. Also, since the maximal set of single-peaked preferences is top-connected, the maximum cardinality of such a set is 2^{m-1}. Thus, the class of top-connected single-peaked preferences is quite large. In what follows, we provide an example of a top-connected set of single-peaked preferences with five alternatives.

Example 3.4.2 *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the set of single-peaked preferences in Table 3.4.1 is top-connected.*

Note that the domains $\mathcal{S}_i^{xy}$ and $\mathcal{S}_i^{yx}$ in Example 3.4.1 are not top-connected. This is because, for instance, there is no preference in $\mathcal{S}_i^{xy}$ with $y - 1$ as the top-ranked alternative and y as the second ranked alternative. A similar argument holds for $\mathcal{S}_i^{yx}$.

Now, we provide a characterization of the unanimous and strategy-proof SCFs on top-connected single-peaked domains.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_5$
$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_3$	$\mathbf{x}_5$	$\mathbf{x}_4$
x_3	x_3	x_4	x_1	x_4	x_4	x_2	x_5	x_2	x_5	x_3	x_3
x_4	x_4	x_1	x_4	x_5	x_5	x_5	x_2	x_1	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_5	x_1	x_1	x_1

Table 3.4.1 A top-connected set of single-peaked preferences

Theorem 3.4.1 *Let $\mathcal{S}_i$ be a top-connected set of single-peaked preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

The proof of Theorem 3.4.1 is relegated to Appendix.

The following corollary is immediate from Theorem 3.4.1.

Corollary 3.4.1 (**[54]**, **[86]**) *Let $\mathcal{S}_i$ be the maximal set of single-peaked preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.5 Min-max Domains

In this section, we introduce the notion of min-max domains and provide a characterization of these domains.

Definition 3.5.1 *Let $\mathcal{D}_i \subseteq \mathbb{L}(X)$ for all $i \in N$ be a regular set of preferences and let $\mathcal{D}_N = \prod_{i \in N} \mathcal{D}_i$. Then, $\mathcal{D}_N$ is called a min-max domain if*

(i) every unanimous and strategy-proof SCF on $\mathcal{D}_N$ is a min-max rule, and

(ii) every min-max rule on $\mathcal{D}_N$ is strategy-proof.

Our next theorem provides a characterization of the min-max domains.

Theorem 3.5.1 *A domain $\mathcal{D}_N$ is a min-max domain if and only if $\mathcal{D}_i$ is a top-connected set of single-peaked preferences for all $i \in N$.*

The proof of Theorem 3.5.1 is relegated to Appendix.

3.6 Applications

3.6.1 Regular Single-crossing Domains

In this subsection, we introduce the notion of regular single-crossing domains and provide a characterization of the unanimous and strategy-proof SCFs on these domains.

Definition 3.6.1 *A set of preferences $\mathcal{S}$ is called single-crossing if there is a linear order $\lhd$ on $\mathcal{S}$ such that for all $x, y \in X$ and all $P, \hat{P} \in \mathcal{S}$,*

$$[x < y, P \lhd \hat{P}, \text{ and } x\hat{P}y] \Rightarrow xPy.$$

Definition 3.6.2 *A single-crossing set of preferences $\mathcal{S}$ is called maximal if there is no single-crossing set of preferences $\mathcal{S}'$ such that $\mathcal{S} \subsetneq \mathcal{S}'$.*

In what follows, we provide an example of a maximal regular single-crossing set of preferences with five alternatives.

Example 3.6.1 *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Then, the set of preferences in Table 3.6.1 is maximal regular single-crossing with respect to the linear order given by $P_1 \lhd P_2 \lhd P_3 \lhd P_4 \lhd P_5 \lhd P_6 \lhd P_7 \lhd P_8 \lhd P_9 \lhd P_{10} \lhd P_{11}$. To see this, consider two alternatives, say x_2 and x_4. Then, x_2Px_4 for all $P \in \{P_1, P_2, P_3, P_4, P_5, P_6\}$ and x_4Px_2 for all $P \in \{P_7, P_8, P_9, P_{10}, P_{11}\}$. Therefore, $x_2\hat{P}x_4$ for some $\hat{P} \in \mathcal{D}$ and $P \lhd \hat{P}$ imply x_2Px_4.*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_1	x_4	x_4	x_4	x_2	x_5	x_5	x_3	x_3
x_4	x_4	x_4	x_1	x_5	x_5	x_5	x_2	x_2	x_2	x_2
x_5	x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_1	x_1	x_1

Table 3.6.1 A maximal regular single-crossing set of preferences

Remark 3.6.1 *Note that a maximal regular single-crossing set of preferences is not unique.*

The following lemmas establish two crucial properties of a (maximal) regular single-crossing set of preferences.

Lemma 3.6.1 ([33], [63]) *Every regular single-crossing set of preferences is single-peaked.*

Lemma 3.6.2 *Every maximal regular single-crossing set of preferences is top-connected.*

The proof of this lemma is left to the reader.

The following corollary follows from Theorem 3.4.1 and Lemma 3.6.2. It characterizes the unanimous and strategy-proof SCFs on maximal regular single-crossing domains.

Corollary 3.6.1 ([73]) *Let $\mathcal{S}_i$ be a maximal regular single-crossing set of preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

The following corollary is obtained from Theorem 3.4.1 and Lemma 3.6.1. It characterizes the unanimous and strategy-proof SCFs on top-connected regular single-crossing domains. Note that in a social choice problem with m alternatives, the cardinality of a top-connected regular single-crossing set of preferences can range from $2m-2$ to $m(m-1)/2$, whereas that of a maximal regular single-crossing set of preferences is exactly $m(m-1)/2$.

Corollary 3.6.2 *Let $\mathcal{S}_i$ be a top-connected regular single-crossing set of preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.6.2 Minimally Rich Single-peaked Domains

In this subsection, we present a characterization of the unanimous and strategy-proof SCFs on minimally rich single-peaked domains. The notion of minimally rich single-peaked domains is introduced in [61]. For the sake of completeness, we present below a formal definition of such domains.

Definition 3.6.3 *A single-peaked preference P is called left single-peaked (right single-peaked) if for all $u < r_1(P) < v$, we have uPv (vPu). Moreover, a set of single-peaked preferences $\mathcal{S}$ is called minimally rich if it contains all left and all right single-peaked preferences.*

Clearly, a minimally rich set of single-peaked preferences is top-connected. So, we have the following corollary from Theorem 3.4.1.

Corollary 3.6.3 *Let $\mathcal{S}_i$ be a minimally rich set of single-peaked preferences for all $i \in N$. Then, an SCF $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.6.3 Distance based Single-peaked Domains

In this subsection, we introduce the notion of single-peaked domains that are based on distances. Consider the situation where a public facility has to be developed at one of the locations $x_1, \ldots, x_m$. Suppose that there is a street connecting these locations, and for every two locations x_i and x_{i+1}, there are two types of distances, a forward distance from x_i to x_{i+1} and a backward distance from x_{i+1} to x_i. An agent bases her preferences on such distances, i.e., whenever a location is strictly closer than another to her most preferred location, she prefers the former to the latter. We show that under some condition on the distances, such a set of preferences is top-connected single-peaked. Below, we present this notion formally.

A *directed graph* G over X is defined as a pair $\langle X, E \rangle$, where X denotes the set of nodes and $E \subseteq X \times X$ denotes the set of edges. The direction of an edge (x, y) is from x to y (this is well-defined since (x, y) is an ordered pair). A graph $G = \langle X, E \rangle$

is called a *directed line graph* if $(x, y) \in E$ if and only if $|x - y| = 1$. Consider the directed line graph $G = \langle X, E \rangle$ on X. A function $d : E \to (0, \infty)$ is called a *distance function* on G. Given a distance function d, define the *distance between two alternatives* x, y as the distance of the path between x and y, i.e., $d(x, y) = d(x, x + 1) + \ldots + d(y - 1, y)$ if $x < y$ and as $d(x, y) = d(x, x - 1) + \ldots + d(y + 1, y)$ if $x > y$. A preference P *respects a distance function* d if for all $x, y \in X$, $d(r_1(P), x) < d(r_1(P), y)$ implies xPy. A set of preferences $\mathcal{S}$ is called *single-peaked with respect to a distance function d* if $\mathcal{S} = \{P \in \mathbb{L}(X) \mid P \text{ respects } d\}$.

A distance function satisfies *adjacent symmetry* if $d(x, x+1) = d(x, x-1)$ for all $x \in X \setminus \{a, b\}$. Below, we provide an example of a set of single-peaked preferences with respect to an adjacent symmetric distance function.

Example 3.6.2 *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. The directed line graph $G = \langle X, E \rangle$ on X and the adjacent symmetric distance function d on E are as given below. Then, the set of preferences in Table 3.6.2 is single-peaked with*

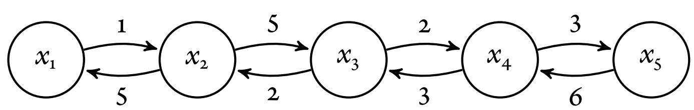


Figure 3.6.1 The directed line graph G on X and an adjacent symmetric distance function d on G

respect to the distance function d.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_5$
$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_1$	$\mathbf{x}_4$	$\mathbf{x}_2$	$\mathbf{x}_5$	$\mathbf{x}_3$	$\mathbf{x}_4$
x_3	x_1	x_3	x_2	x_4	x_3	x_5	x_3
x_4	x_4	x_4	x_5	x_5	x_2	x_2	x_2
x_5	x_5	x_5	x_1	x_1	x_1	x_1	x_1

Table 3.6.2 A set of single-peaked preferences with respected to the distance function d

Let $G = \langle X, E \rangle$ be the directed line graph on X and let $d : E \to (0, \infty)$ be an adjacent symmetric distance function. Then, it is easy to verify that a set of single-peaked preferences with respect to the distance function d is top-connected. Therefore, we have the following corollary from Theorem 3.4.1.

Corollary 3.6.4 *Let $G = \langle X, E \rangle$ be the directed line graph on X and let $d_i : E \to (0, \infty)$ be an adjacent symmetric distance function for all $i \in N$. Suppose that for all $i \in N$, $\mathcal{S}_i$ is the set of single-peaked preferences with respect to the distance function d_i. Then, $f : \mathcal{S}_N \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

3.7 Concluding Remarks

In this chapter, we have studied social choice problems where the admissible sets of preferences of all agents are single-peaked. First, we have shown that if the agents have arbitrary (but same) admissible sets of single-peaked preferences, then every unanimous and strategy-proof SCF on corresponding domains satisfies Pareto property and tops-onlyness. We have further shown that if the admissible sets of preferences of each agent satisfies a mild condition called left-connectedness (or right-connectedness), then the same result holds even when different agents have different admissible sets of single-peaked preferences. Next, we have shown by means of an example that the exact structure of the unanimous and strategy-proof SCFs on such domains is quite intractable, and consequently have provided a full characterization of the unanimous and strategy-proof SCFs under an additional condition that the admissible set of preferences of each agent is top-connected. Outstanding examples of top-connected single-peaked domains are maximal single-peaked domains, minimally rich single-peaked domains, distance based single-peaked domains, and top-connected regular single-crossing domains. Finally, we have introduced the notion of min-max domains, the domains for which the set of unanimous and strategy-proof SCFs coincides with that of min-max rules. We have shown that a domain is a min-max domain if and only if it is a top-connected single-peaked domain.

3.8 Appendix

3.8.1 Proof of Theorem 3.3.2

Proof: Let $P_N \in \mathcal{S}_N$, $i \in N$, and $P'_i \in \mathcal{S}$ be such that $r_1(P_i) = r_1(P'_i)$. It is enough to show that $f(P_N) = f(P'_i, P_{N\setminus i})$. Suppose not. Let $r_1(P_i) = r_1(P'_i) = x$, $f(P_N) = z$, and $f(P'_i, P_{N\setminus i}) = y$. By strategy-proofness, zP_iy and yP'_iz. Since $\mathcal{S}_i$ is single-peaked and $r_1(P_i) = r_1(P'_i)$, this means either $y < x < z$ or $z < x < y$. Assume without loss of generality that $y < x < z$. Let $\bar{N} = \{j \in N \mid r_1(P_j) \geq x\}$ and let $\bar{P}_N \in \mathcal{S}_N$ be such that $\bar{P}_j = P_i$ for all $j \in \bar{N}$, and $\bar{P}_j = P_j$ for all $j \notin \bar{N}$.

Claim 1. $f(P'_i, P_{N\setminus i}) = y$ implies $f(\bar{P}_N) = y$.

By Pareto optimality, $f(\bar{P}_N) \leq x$. If $f(\bar{P}_N) \in (y, x]$ then agents in $\bar{N}$ manipulates f at $(P'_i, P_{N\setminus i})$ via $\bar{P}_N$. On the other hand, if $f(\bar{P}_N) < y$, then agents in $\bar{N}$ manipulate f at $\bar{P}_N$ via $(P'_i, P_{N\setminus i})$. This completes the proof of Claim 1.

Claim 2. $f(P_N) = z$ implies $f(\bar{P}_N) \neq y$.

Because $z\bar{P}_jy$ for all $j \in N$, if $f(\bar{P}_N) = y$, then agents in $\bar{N}$ manipulates f at $\bar{P}_N$ via P_N. This completes the proof of Claim 2.

However, Claim 2 contradicts Claim 1. This completes the proof of the theorem. ■

3.8.2 Proof of Theorem 3.3.3

Proof: We prove the theorem for the case where $\mathcal{S}_N$ is right-connected, the proof of the same for the case where $\mathcal{S}_N$ is left-connected follows from similar arguments.

It is sufficient to show that $f(P_N) = f(P'_i, P_{N\setminus i})$ for all $P_i, P'_i \in \mathcal{S}_i$ and all $P_{N\setminus i} \in \mathcal{S}_{N\setminus i}$ with $r_1(P_i) = r_1(P'_i)$. Assume for contradiction, $f(P_N) \neq f(P'_i, P_{N\setminus i})$. By strategy-proofness, we have $f(P_N)P_if(P'_i, P_{N\setminus i})$ and $f(P'_i, P_{N\setminus i})P'_if(P_N)$. Assume without loss of generality, $f(P'_i, P_{N\setminus i}) < r_1(P_i) < f(P_N)$. Suppose $f(P'_i, P_{N\setminus i}) = y$, $r_1(P_i) = x$, and $f(P_N) = z$. Let $N_1 = \{j \in N \mid r_1(P_j) \geq z\}$. By Pareto optimality, $N_1 \neq \emptyset$. Consider $P^1_N \in \mathcal{S}_N$ such that $r_1(P^1_j) = z - 1$ and $r_2(P^1_j) = z$ for all $j \in N_1$, and $P^1_k = P_k$ for all $k \notin N_1$. By group-strategy-proofness, $f(P^1_N) \in \{z, z-1\}$.

By Pareto optimality, $f(P^1_N) \leq z - 1$. Combining, $f(P^1_N) = z - 1$. Now, let $N_2 = \{j \in N \mid r_1(P_j) = z - 1\}$. Consider $P^2_N \in \mathcal{S}_N$ such that $r_1(P^2_j) = z - 2$ and $r_2(P^2_j) = z - 1$ for all $j \in N_2$, and $P^2_k = P_k$ for all $k \notin N_2$. Using similar logic as before, $f(P^2_N) = z - 2$. Continuing in this manner, we construct a profile $\hat{P}_N \in \mathcal{S}_N$ such that $r_1(\hat{P}_j) = x$ for all $j \in N$ with $r_1(P_j) \geq x$, $\hat{P}_k = P_k$ for all $k \in N$ such that $r_1(\hat{P}_k) < x$, and $f(\hat{P}_N) = x$.

Now, consider the profile $\bar{P}_N \in \mathcal{S}_N$ such that $\bar{P}_i = P'_i$, $\bar{P}_k = P_k$ if $k \in N$ such that $r_1(P_k) < x$, and $r_1(\bar{P}_k) = x$ if $k \in N \setminus i$ such that $r_1(P_k) \geq x$. By Pareto optimality, $f(\bar{P}_N) \leq x$. This, together with group-strategy-proofness from $(P'_i, P_{N\setminus i})$ to $\bar{P}_N$, implies $f(\bar{P}_N) \geq y$. However, by group-strategy-proofness from $\bar{P}_N$ to $(P'_i, P_{N\setminus i})$, we $f(\bar{P}_N) \leq y$. Combining, $f(\bar{P}_N) = y$.

Note that $\hat{P}_j = \bar{P}_j$ for all $j \in N$ with $r_1(P_j) < x$ and $r_1(\hat{P}_j) = r_1(\bar{P}_j)$ for all $j \in N$ with $r_1(P_j) \geq x$. This means the group of agents $\{j \in N \mid r_1(\bar{P}_j) = x\}$ manipulates at $\bar{P}_N$ via $\hat{P}_N$, a contradiction. This completes the proof of the theorem. ■

3.8.3 Proof of Theorem 3.4.1

Proof: (If part) Note that a min-max rule is unanimous by definition (on any domain). We show that such a rule is strategy-proof on $\mathcal{S}_N$. For all $i \in N$, let $\bar{\mathcal{S}}_i$ be the maximal set of single-peaked preferences. By [86], a min-max rule is strategy-proof on $\bar{\mathcal{S}}_N$. Since $\mathcal{S}_i \subseteq \bar{\mathcal{S}}_i$ for all $i \in N$, a min-max rule must be strategy-proof on $\mathcal{S}_N$. This completes the proof of the if part.

(Only-if part) Let $\mathcal{S}_i$ be a top-connected set of single-peaked preferences for all $i \in N$ and let $f : \mathcal{S}_N \to X$ be a unanimous and strategy-proof SCF. We show that f is a min-max rule. First, we establish a few properties of f in the following sequence of lemmas.

By Corollary 3.3.1 and Theorem 3.3.3, f must satisfy the Pareto property and tops-onlyness. Also, by Theorem 3.3.1, f is group strategy-proof. Our next lemma shows that f is uncompromising.

Lemma 3.8.1 *The SCF f is uncompromising.*

Proof: Let $P_N \in \mathcal{S}_N$, $i \in N$, and $P'_i \in \mathcal{S}_i$ be such that $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$. It is sufficient to show $f(P'_i, P_{N\setminus i}) = f(P_N)$. Suppose $r_1(P_i) = x$, $f(P_N) = y$, and $f(P'_i, P_{N\setminus i}) = y'$.

By strategy-proofness, we must have $y' < x$. This is because, if $y' \in [x, y)$, then agent i manipulates at P_N via P'_i. On the other hand, if $y' > y$, then by means of the fact that $r_1(P'_i) \leq y$, agent i manipulates at $(P'_i, P_{N\setminus i})$ via P_i.

Because $y' < x$, we assume without loss of generality that $r_1(P'_i) = y'$ and $\min(\tau(P'_i, P_{N\setminus i})) = y'$.[13] Assume for contradiction that $y \neq y'$.

Let $T = \{j \in N \mid r_1(P_j) < x\}$. For $j \in T$, let $P'_j \in \mathcal{S}_j$ be such that $r_1(P'_j) = x$.

Claim 1. $f(P_N) = y$ implies $f(P'_T, P_{N\setminus T}) = y$.

If T is empty, then there is nothing to show. Suppose T is non-empty. By Pareto property, $f(P'_T, P_{N\setminus T}) \geq x$. If $f(P'_T, P_{N\setminus T}) \in [x, y)$, then the agents in T manipulate f at P_N via $(P'_T, P_{N\setminus T})$. On the other hand, if $f(P'_T, P_{N\setminus T}) > y$, then the agents in T manipulate f at $(P'_T, P_{N\setminus T})$ via P_N. This completes the proof of Claim 1.

Let $T' = T \cup i$. For all $j \in T'$, let $\tilde{P}_j \in \mathcal{S}_j$ be such that $r_1(\tilde{P}_j) = x$.

Claim 2. $f(P'_i, P_{N\setminus i}) = y'$ implies $f(\tilde{P}_{T'}, P_{N\setminus T'}) = x$.

Let T'' be the set of agents whose top-ranked alternative is y' at the profile $(P'_i, P_{N\setminus i})$. More formally, $T'' = i \cup \{j \in N \mid r_1(P_j) = y'\}$. Consider the profile $\bar{P}_N \in \mathcal{S}_N$ such that $r_1(\bar{P}_j) = y' + 1$ for all $j \in T''$ and $\bar{P}_j = P_j$ for all other agents. By tops-onlyness of f, we can assume $r_2(\bar{P}_j) = y'$ for all $j \in T''$. However, since $f(P'_i, P_{N\setminus i}) = y'$, by group strategy-proofness, $f(\bar{P}_N) \in \{y', y' + 1\}$ as otherwise agents in T'' manipulate f at $\bar{P}_N$ via $(P'_i, P_{N\setminus i})$. Since $\min(\tau(\bar{P}_N)) = y' + 1$, by Pareto property,

$$f(\bar{P}_N) = y' + 1.$$

Using similar logic, we can construct a profile $\hat{P}_N \in \mathcal{S}_N$ where $r_1(\hat{P}_j) = y' + 2$ for all agents j with $r_1(\bar{P}_j) = y' + 1$ and $\hat{P}_j = \bar{P}_j$ for all other agents, and conclude

[13] Since $f(P'_i, P_{N\setminus i}) = y'$, if $r_1(P'_i) \neq y'$, then by strategy-proofness, $f(P''_i, P_{N\setminus i}) = y'$ for some $P''_i \in \mathcal{S}_i$ with $r_1(P''_i) = y'$. Similarly, if $r_1(P_j) < y'$ for some $j \in N \setminus i$, then by strategy-proofness, $f(P'_i, P'_j, P_{N\setminus \{i,j\}}) = y'$ for some $P'_j \in \mathcal{S}_j$ with $r_1(P'_j) = y'$.

that

$$f(\hat{P}_N) = y' + 2.$$

Continuing in this manner, we move all the agents j in T' to a preference $\tilde{P}_j \in \mathcal{S}_j$ with $r_1(\tilde{P}_j) = x$ while keeping the preferences of all other agents unchanged and conclude that

$$f(\tilde{P}_{T'}, P_{N \setminus T'}) = x.$$

This completes the proof of Claim 2.

Now, we complete the proof of the lemma. Consider the profiles $(P'_T, P_{N\setminus T})$ and $(\tilde{P}_{T'}, P_{N\setminus T'})$. Note that for an agent j, if $r_1(P_j) > x$, then her preference is the same in both the profiles $(P'_T, P_{N\setminus T})$ and $(\tilde{P}_{T'}, P_{N\setminus T'})$. Moreover, for an agent j, if $r_1(P_j) \leq x$, then her top-ranked alternative is x in both the profiles. Therefore, the top-alternatives of each agent in these two profiles are the same. However, since $f(P'_T, P_{N\setminus T}) \neq f(\tilde{P}_{T'}, P_{N\setminus T'})$, Claim 1 and 2 contradict tops-onlyness of f. This completes the proof of the lemma. ■

The following lemma establishes that f is a min-max rule.

Lemma 3.8.2 *The SCF f is a min-max rule.*

Proof: For all $S \subseteq N$, let $(P^a_S, P^b_{N\setminus S}) \in \mathcal{S}_N$ be such that $r_1(P^a_i) = a$ for all $i \in S$ and $r_1(P^b_i) = b$ for all $i \in N \setminus S$. Define $\beta_S = f(P^a_S, P^b_{N\setminus S})$ for all $S \subseteq N$. Clearly, $\beta_S \in X$ for all $S \subseteq N$. By unanimity, $\beta_\varnothing = b$ and $\beta_N = a$. Also, by strategy-proofness, $\beta_S \leq \beta_T$ for all $T \subseteq S$.

Take $P_N \in \mathcal{S}_N$. We show $f(P_N) = \min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\}$. Suppose $S_1 = \{i \in N \mid r_1(P_i) < f(P_N)\}$, $S_2 = \{i \in N \mid f(P_N) < r_1(P_i)\}$, and $S_3 = \{i \in N \mid r_1(P_i) = f(P_N)\}$. By strategy-proofness and uncompromisingness, $\beta_{S_1\cup S_3} \leq f(P_N) \leq \beta_{S_1}$. Consider the expression $\min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{r_1(P_i), \beta_S\}\}$. Take $S \subseteq S_1$. Then, by Condition (iii) in Definition 3.2.9, $\beta_{S_1} \leq \beta_S$. Since $r_1(P_i) < f(P_N)$ for all $i \in S$ and $f(P_N) \leq \beta_{S_1} \leq \beta_S$, we have $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} = \beta_S$. Clearly, for all $S \subseteq N$ such that $S \cap S_2 \neq \varnothing$, we have $\max\limits_{i\in S}\{r_1(P_i), \beta_S\} > f(P_N)$. Consider $S \subseteq N$ such that $S \cap S_2 = \varnothing$ and $S \cap S_3 \neq \varnothing$. Then, $S \subseteq S_1 \cup S_3$, and hence $\beta_{S_1\cup S_3} \leq$

β_S. Therefore, $\max_{i\in S}\{r_1(P_i), \beta_S\} = \max\{f(P_N), \beta_S\} \geq \max\{f(P_N), \beta_{S_1\cup S_3}\}$. Since $\beta_{S_1\cup S_3} \leq f(P_N)$, we have $\max\{f(P_N), \beta_{S_1\cup S_3}\} = f(P_N)$. Combining all these, we have $\min_{S\subseteq N}\{\max_{i\in S}\{r_1(P_i), \beta_S\}\} = \min\{f(P_N), \beta_{S_1}\}$. Because $f(P_N) \leq \beta_{S_1}$, we have $\min\{f(P_N), \beta_{S_1}\} = f(P_N)$. This completes the proof of the lemma. ■ The proof of the only-if part of Theorem 3.4.1 follows from Lemmas 3.8.1 - 3.8.2. ■

3.8.4 Proof of Theorem 3.5.1

Proof: The proof of the if part follows from Theorem 3.4.1. We proceed to prove the only-if part. Let $\mathcal{D}_N$ be a min-max domain. We show that $\mathcal{D}_i$ is top-connected single-peaked for all $i \in N$. We show this in two steps: in Step 1 we show that $\mathcal{D}_i$ is single-peaked for all $i \in N$, and in Step 2, we show that $\mathcal{D}_i$ is top-connected for all $i \in N$.

Step 1. Suppose that $\mathcal{D}_i$ is not single-peaked for some $i \in N$. Then, there is $Q \in \mathcal{D}_i$ and $x, y \in X$ such that $x < y < r_1(Q)$ and xQy. Consider the min-max rule f^β with respect to $(\beta_S)_{S\subseteq N}$ such that $\beta_S = x$ for all $\emptyset \subsetneq S \subsetneq N$. Take $P_N \in \mathcal{D}_N$ such that $P_i = Q$ and $r_1(P_j) = y$ for all $j \in N \setminus i$. By the definition of f^β, $f^\beta(P_N) = y$. Now, take $P'_i \in \mathcal{D}_i$ with $r_1(P'_i) = x$. Again, by the definition of f^β, $f^\beta(P'_i, P_{N\setminus i}) = x$. This means agent i manipulates at P_N via P'_i, which is a contradiction to the assumption that $\mathcal{D}_N$ is a min-max domain. This completes Step 1.

Step 2. In this step, we show that $\mathcal{D}_i$ satisfies top-connectedness for all $i \in N$. Assume for contradiction that $\mathcal{D}_i$ is not top-connected for some $i \in N$. By definition, $\mathcal{D}_i$ is regular. Since $\mathcal{D}_i$ is single-peaked, for all $P \in \mathcal{D}_i$, $r_1(P) = a$ (or b) implies $r_2(P) = a + 1$ (or $b - 1$). Again, because $\mathcal{D}_i$ is single-peaked, for all $P \in \mathcal{D}_i$ and all $x \in X \setminus \{a, b\}$, $r_1(P) = x$ implies $r_2(P) \in \{x - 1, x + 1\}$. Since $\mathcal{D}_i$ violates top-connectedness, assume without loss of generality that there exists $x \in X \setminus \{a, b\}$ such that for all $P \in \mathcal{D}_i$, $r_1(P) = x$ implies $r_2(P) = x - 1$. Consider the following

SCF:[14]

$$f(P_N) = \begin{cases} x \text{ if } r_1(P_i) = x \text{ and } xP_j(x-1) \text{ for all } j \in N \setminus i, \\ x - 1 \text{ if } r_1(P_i) = x \text{ and } (x-1)P_jx \text{ for some } j \in N \setminus i, \\ r_1(P_i) \text{ otherwise.} \end{cases}$$

It is left to the reader to verify that f is unanimous and strategy-proof. We show that f is not uncompromising, which in turn means that f is not a min-max rule. Let $P_N \in \mathcal{D}_N$ be such that $r_1(P_i) = x$ and $r_1(P_j) = x - 1$ for some $j \neq i$, and let $P'_i \in \mathcal{D}_i$ be such that $r_1(P'_i) = x + 1$. Then, by the definition of f, $f(P_N) = x - 1$ and $f(P'_i, P_{N \setminus i}) = x + 1$. Therefore, because $f(P_N) = x - 1$ and $x - 1 \leq r_1(P_i) \leq r_1(P'_i)$, the fact that $f(P'_i, P_{N \setminus i}) = x + 1$ is a violation of uncompromisingness. This completes Step 2 and the proof of the only-if part. ■

[14] Here $\mathcal{D}_i$ satisfies the *unique seconds* property defined in [6] and the SCF f considered here is similar to the one used in the proof of Theorem 5.1 in [6].

4

Strategy-proof Rules on Partially Single-peaked Domains

4.1 Introduction

4.1.1 Background of the Problem

Most of the subject matter of social choice theory concerns the study of the unanimous and strategy-proof SCFs for different admissible domains of preferences. In the seminal works by [43] and [75], it is shown that if a society has at least three alternatives and there is no particular restriction on the preferences of the individuals, then every unanimous and strategy-proof SCF is *dictatorial*, that is, a particular individual in the society determines the outcome regardless of the preferences of the others. The celebrated Gibbard-Satterthwaite theorem hinges crucially on the assumption that the admissible domain of each individual is unrestricted. However, natural domain restrictions arise in many economic and political applications. For instance, in the models of locating a firm in a unidimensional spatial market ([46]), setting the rate of carbon dioxide emissions ([15]), setting the level of public expenditure ([69]), and so on, preferences admit a natural restriction widely known as *single-peakedness*. Informally, a single-peaked preference with respect to some arrangement of the alternatives over a uni-dimensional space, called a *prior order*, requires that the preference decreases as one moves away (with respect to the prior order) from her best alternative.

The study of single-peaked domains dates back to [15], where it is shown that the pairwise majority rule is strategy-proof on such domains. [54] and [86] have characterized the unanimous and strategy-proof SCFs on such domains as *min-max rules*.[1,2] In Chapter 3, we characterize the domains where the set of unanimous and strategy-proof SCFs coincide with that of min-max rules.

4.1.2 Our Motivation

It is both experimentally and empirically established that in many political and economic scenarios ([58], [41], and [59]), where the preferences of individuals are normally assumed to be single-peaked, they are actually not. Nevertheless, such preferences have close resemblance with single-peakedness. In this chapter, we model such preferences as *partially single-peaked*. Roughly speaking, partial single-peakedness requires the individual preferences to be single-peaked *only* over a subset of alternatives. It is worth noting that the structure of the unanimous and (group) strategy-proof rules on such domains are not explored in the literature. In view of this, our main motivation in this chapter is to develop a general model for partially single-peaked domains and to provide a characterization of the unanimous and (group) strategy-proof rules on those. Below, we present some evi-

[1] [9] and [24] provide equivalent presentations of this class of SCFs.

[2] A rich literature has developed around the single-peaked restriction by considering various generalizations and extensions (see [9], [29], [76], [55], and [56]).

dences of partially single-peaked domains in the literature. In Section 4.4, we will formally define these notions and show that they are special cases of partially single-peaked domains.

Multi-peaked Domains

In many practical scenarios in economics and politics, the preferences of the individuals often exhibit *multi-peakedness* as opposed to single-peakedness. As the name suggests, multi-peaked preferences admit multiple ideal points in a unidimensional policy space. We discuss a few settings where it is plausible to assume that individuals have multi-peaked preferences.

- *Preference for 'Do Something' in Politics*: [28] and [32] consider public (decision) problems such as choosing alternate tax regimes, lowering health care costs, responding to foreign competition, reducing the national debt, etc. They show that a public problem is perceived to be poorly addressed by the status-quo policy, and consequently some individuals prefer both liberal and conservative policies to the moderate status quo. Clearly, such a preference will have two peaks, one on the left of the status quo and another one on the right.

- *Multi-stage Voting System*: [78], [31], [34], etc. deal with multi-stage voting system where individuals vote on a set of issues where each issue can be thought of as a unidimensional spectrum and voting is distributed over several stages considering one issue at a time. In such a model, preference of an individual over the present issue can be affected by her prediction of the outcome of the future issues. In other words, such a preference is not separable across issues. They show that the preferences of the individuals in such scenarios exhibit multi-peaked property.

- *Provision of Public Goods with Outside Options*: [13], [80], and [14] consider the problem of setting the level of tax rates to provide public funding in the education sector, and [49] and [37] consider the same problem in the health insurance market. They show that the preferences of individuals exhibit multi-peaked property due to the presence of outside options (i.e., the public good is also available in a competitive market as a private good). For instance, in the problem of determining educational subsidy, an individual with lower income may not prefer a moderate level of subsidy since she cannot afford to bear the remaining cost for higher education. Thus, her preference in such a scenario will have two peaks - one at a lower level of subsidy so that she can achieve primary education, and another one at a very high level of subsidy so that she can afford the remaining cost for higher education.

- *Provision of Excludable Public Goods*: [42] and [2] consider public good provision models such as health insurance, educational subsidies, pensions, etc. where the government provides the public good to a particular section of individuals, and show that individuals' preferences in such scenarios are multi-peaked.

Single-peaked Domains with respect to Partial Orders

In the literature, single-peaked domains are generally considered with respect to some (prior) linear order. Such a preference restriction requires an individual to order (*a priori*) the whole set of alternatives in a linear fashion. However, it is well-documented in psychology that in many situations individuals are unable to derive a complete ordering over the alternatives. For instance, in the political science literature, it may not be possible for the individuals to unambiguously order the parties who are moderate in their policies (center parties) over the policy spectrum. Similarly, in a public good provision problem where locations are distributed over different geographical regions, even though individuals can derive some prior ordering (based on traffic distance or so) over the locations that are in same region, but they may not be able to do the same for locations in different regions. Such a situation can only be modeled by considering single-peaked domains with respect to prior orderings that are incomplete (or partial). In this respect, our work is closely related to [18] who consider *semi-lattice single-peakedness* - preferences that are single-peaked with respect to a semi-lattice (which is a partial order).

Multiple Single-peaked Domains

[67] introduces the notion of *multiple single-peaked domains*. Such a domain is defined as a union of some domains each of which is single-peaked with respect to some prior orderings over the alternatives. A plausible justification for such a domain restriction is provided by [57] who argues that the alternatives can be ordered differently using different criteria (which he calls an *impartial culture*) and it is not publicly known which individual uses what criterion. On one extreme, such a domain becomes an unrestricted domain if there is no consensus among the individuals on the prior order, and on the other extreme, it becomes a maximal single-peaked domain if all the individuals agree on a single prior order. It is worth noting that such domains can be seen as a special case of partially single-peaked domains.

Single-peaked Domains on Graphs

[76] considers domains that are based on some graph structure over the alternatives (e.g., locating a new station in a rail-road network). They assume that the individuals derive their preferences by using single-peakedness over some spanning tree of the underlying graph. In this chapter, we show that when the underlying graph has some specific structure (involves a cycle or so), then the induced domains become partially single-peaked.

4.1.3 Our Contribution

In this paper, we develop a general model for partially single-peaked domains which capture the non-single-peaked domains that commonly arise in practical scenarios. Formally speaking, we assume that the whole interval of alternatives is divided into subintervals such that every preference in the domain is required to satisfy single-peakedness over each of those subintervals, and is allowed to violate the property outside those. We characterize the unanimous and strategy-proof SCFs on such domains as *partly dictatorial min-max rule (PDMMR)*. Loosely put, a PDMMR acts like a min-max rule over the subintervals where the domain respects single-peakedness and like a dictatorial rule everywhere else.

The class of partially single-peaked domains that we consider in this paper is quite large. It includes single-peaked domains on one extreme and unrestricted domains on the other. To corroborate this fact, we prove that partially single-peaked domains contain *almost* all domains on which (i) every unanimous and strategy-proof SCF is a PDMMR, and (ii) every PDMMR is strategy-proof.

4.1.4 Relation with [67]

In this section, we compare our results with those of [67]. [67] provides a characterization of the unanimous and strategy-proof SCFs on multiple single-peaked domains. We think our results significantly improve that in [67] from both practical and theoretical point of views.

Practical point of view

- Multiple single-peaked domains assume that every preference is single-peaked with respect to some prior ordering. However, this is a strong requirement for practical purposes. For instance, consider the situation where the locations $x_1, \ldots, x_{10}$ are arranged on a street. Suppose further that there is a direct route from x_4 to x_8. This means that a preference with x_4 at the top may have x_8 as its second ranked alternative, and that with x_8 at the top may have x_4 as second ranked alternative and vice-versa. However, it is not possible for the designer to assume any ordering with respect to which such a preference will be single-peaked (particularly, over the alternatives x_5, x_6, and x_7). Thus, such domains violate the basic principle of multiple single-peaked domains which assumes that every agent derives his/her preference with respect to some prior ordering over the alternatives.

- Multiple single-peaked domains require each single-peaked domain to be *maximal*. Such a single-peaked domain requires 2^{m-1} preferences, where m

is the number of alternatives. This is a strong requirement since many domains of practical importance such as Euclidean etc. do not satisfy this condition. In contrast, our result applies to multiple single-peaked domains that require each single-peaked domain to be only top-connected. It is worth noting that the number of preferences in such single-peaked domain can range from $2m - 2$ to 2^{m-1}. This significantly improves the applicability of multiple single-peaked domains.

Theoretical point of view

- In general, a major step in characterizing the unanimous and strategy-proof SCFs on a domain is to show that the domain is tops-only. In case of multiple single-peaked domains, tops-onlyness follows from [20]. However, the same does not follow for partial single-peaked domains.
- It follows from [7] that every unanimous and strategy-proof SCF on multiple single-peaked domain is group strategy-proof. However, the same does not hold for partially single-peaked domains. We establish this independently in this paper.

4.1.5 Other related papers

[19] study a related restricted domain known as a *semi-single-peaked domain*. Such a domain violates single-peakedness around the *tails* of the prior order. They show that if a domain admits an anonymous (and hence non-dictatorial), tops-only, unanimous, and strategy-proof SCF, then it is a semi-single-peaked domain. However, we show that if single-peakedness is violated around the *middle* of the prior order, then there is *no* unanimous, strategy-proof, and anonymous SCF. Thus, our characterization result on partially single-peaked domains complements that in [19]. Recently, [3] provide necessary and sufficient conditions for the comparability of two min-max rules in terms of their vulnerability to manipulation. However, our results identify the min-max rules that are manipulable if single-peakedness is violated over a subset of alternatives.

4.1.6 Remainder

The rest of the chapter is organized as follows. We describe the usual social choice framework in Section 4.2. In Section 4.3, we presents our main results. Section 4.4 provides a few applications of our results, and the last section concludes the chapter. All the omitted proofs are collected in Appendix 4.6.

4.2 Preliminaries

Let $N = \{1, \dots, n\}$ be a set of at least two agents, who collectively choose an element from a finite set $X = \{a, a+1, \dots, b-1, b\}$ of at least three alternatives, where a is an integer. For $x, y \in X$ such that $x \leq y$, we define the intervals $[x, y] = \{z \in X \mid x \leq z \leq y\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. Throughout this chapter, we denote by $\underline{x}$ and $\overline{x}$ two arbitrary but fixed alternatives such that $\underline{x} < \overline{x} - 1$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *preference* P over X is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on X. We denote by $\mathbb{L}(X)$ the set of all preferences over X. An alternative $x \in X$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(X)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k - 1$. For notational convenience, sometimes we denote by $P = xy\dots$ a preference P with $r_1(P) = x$ and $r_2(P) = y$. A domain of admissible preferences, denoted by $\mathcal{D}$, is a subset of $\mathbb{L}(X)$. An element $P_N = (P_1, \dots, P_n) \in \mathcal{D}^n$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \{x \in X \mid r_1(P_i) = x \text{ for some } i \in N\}$.

4.2.1 Domains and Their Properties

In this subsection, we introduce a few properties of a domain and a class of domains.

Definition 4.2.1 *A domain $\mathcal{D}$ of preferences is regular if for all $x \in X$, there exists a preference $P \in \mathcal{D}$ such that $r_1(P) = x$.*

All the domains we consider in this chapter are assumed to be regular.

Definition 4.2.2 *A domain $\mathcal{D}$ satisfies the top-connectedness property if for all $x, y \in X$ with $|x - y| = 1$, there is $P \in \mathcal{D}$ such that $P = xy \ldots$.*

Graph of a Domain

In this subsection, we introduce the notion of the graph of a domain. First, we introduce a few graph theoretic notions. A *directed graph* G is defined as a pair $\langle V, E\rangle$, where V is the set of *nodes* and $E \subseteq V \times V$ is the set of *directed edges*, and an *undirected graph* G is defined as a pair $\langle V, E\rangle$, where V is the set of nodes and $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$ is the set of *undirected edges*. For a graph (directed or undirected) $G = \langle V, E\rangle$, a *subgraph* G' of G is defined as a graph $G' = \langle V, E'\rangle$, where $E' \subseteq E$. For two graphs $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$, the graph $G_1 \cup G_2$ is defined as $G_1 \cup G_2 = \langle V_1 \cup V_2, E_1 \cup E_2\rangle$.

All the graphs we consider in this chapter are of the kind $G = \langle X, E\rangle$, i.e., whose node set is the set of alternatives.

Definition 4.2.3 *A directed (undirected) graph $G = \langle X, E\rangle$ is called the directed (undirected) line graph on X if $(x, y) \in E$ ($\{x, y\} \in E$) if and only if $|x - y| = 1$.*

Definition 4.2.4 *A graph G is called a directed (undirected) partial line graph if G can be expressed as $G_1 \cup G_2$, where $G_1 = \langle X, E_1\rangle$ is the directed (undirected) line graph on X and $G_2 = \langle [\underline{x}, \overline{x}], E_2\rangle$ is a directed (undirected) graph such that $(\underline{x}, y), (\overline{x}, z) \in E_2$ ($\{\underline{x}, y\}, \{\underline{x}, z\} \in E_2$) for some $y \in (\underline{x} + 1, \overline{x}]$ and $z \in [\underline{x}, \overline{x} - 1)$.*

In Figure 4.2.1, we present a directed partial line graph on $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ where $\underline{x} = x_3$ and $\overline{x} = x_6$.

Definition 4.2.5 *The top-graph of a domain $\mathcal{D}$ is defined as the directed graph $\langle X, E\rangle$ such that $(x, y) \in E$ if and only if there exists a preference $P = xy \ldots \in \mathcal{D}$.*

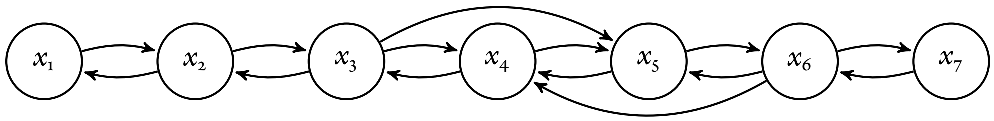


Figure 4.2.1 A directed partial line graph

Note that a domain satisfies the top-connectedness property if and only if its top-graph is the directed line graph on X.

4.2.2 Single-peaked Domains

Definition 4.2.6 *A preference $P \in \mathbb{L}(X)$ is called single-peaked if for all $x, y \in X$, $[x < y \leq r_1(P)$ or $r_1(P) \leq y < x]$ implies yPx. A domain is called single-peaked if each preference in it is single-peaked, and a domain is called maximal single-peaked if it contains all single-peaked preferences.*

Definition 4.2.7 *A domain is called top-connected single-peaked if it is both top-connected and single-peaked.*

4.2.3 Partially Single-peaked Domains

In this section, we consider a class of domains that violates single-peaked property over the interval $[\underline{x}, \overline{x}]$ and exhibits the property everywhere else. We call such domains partially single-peaked domains which are formally defined below.

Definition 4.2.8 *A domain $\tilde{\mathcal{S}}$ is said to satisfy single-peakedness outside $[\underline{x}, \overline{x}]$ if for all $P \in \tilde{\mathcal{S}}$, all $u \notin (\underline{x}, \overline{x})$, and all $v \in X$,*

$$\left[v < u \leq r_1(P) \text{ or } r_1(P) \leq u < v\right] \text{ implies } uPv.$$

To gain more insight about Definition 4.2.8, first consider a preference with top-ranked alternative in $[\underline{x}, \overline{x}]$. Then, Definition 4.2.8 says that such a preference satisfies single-peakedness over the intervals $[a, \underline{x}]$ and $[\overline{x}, b]$. That is, the relative ordering of two alternatives u, v is derived by using single-peaked property whenever both of them are either in the interval $[a, \underline{x}]$ or in the interval $[\overline{x}, b]$. Note that Definition 4.2.8 does not impose any restriction on the relative ordering of an alternative in $[\underline{x}, \overline{x}]$ and any other alternative. Next, consider a preference P such that $r_1(P) \notin [\underline{x}, \overline{x}]$. Suppose, for instance, $r_1(P) \in [a, \underline{x})$. Then, Definition 4.2.8 says that (i) P satisfies single-peakedness over the interval $[a, r_1(P)]$, and (ii) if an alternative u lies in the interval $(r_1(P), \underline{x}]$ or in the interval $[\overline{x}, b]$, then, as required by single-peakedness, it is preferred to any alternative v in the interval $(u, b]$. Thus, Definition 4.2.8 does not impose on P any restriction on the relative ordering of an alternative in $(\underline{x}, \overline{x})$ and an alternative in $[\overline{x}, b]$. Therefore, in particular, Definition 4.2.8 does not impose any restriction on any preference on the relative ordering of two alternatives in the interval $(\underline{x}, \overline{x})$.

Definition 4.2.9 *A domain $\tilde{\mathcal{S}}$ is said to violate single-peakedness over $[\underline{x}, \overline{x}]$ if there exist $Q = \underline{x}y\ldots, Q' = \overline{x}z\ldots \in \tilde{\mathcal{S}}$ such that either $\big[y \in (\underline{x}+1, \overline{x})$ and $z \in (\underline{x}, \overline{x}-1)\big]$ or $\big[y = \overline{x}$ and $z = \underline{x}\big]$.*

Note that since $r_2(Q) > r_1(Q)+1$ and $r_2(Q') < r_1(Q')-1$, both the preferences Q and Q' violate single-peakedness. This, together with the facts that $r_1(Q) = \underline{x}$, $r_1(Q') = \overline{x}$, and $r_2(Q), r_2(Q') \in (\underline{x}, \overline{x})$, implies that a domain with those two preferences violates single-peakedness over $[\underline{x}, \overline{x}]$. In Section 4.3.2, we show that the particular restrictions on the second-ranked alternatives of Q and Q' given in Definition 4.2.9 are necessary for the results we derive in this chapter.

Remark 4.2.1 *Definition 4.2.9 considers violation of single-peakedness only over intervals. It may seem that the possibility of violating this over several intervals is excluded in this definition. However, as we argue in the following, that is not the case. Note that by Definition 4.2.9, if a domain violates single-peakedness over several intervals, then it also violates the same over the minimal interval that contains all those. Thus, for the notion of violation of single-peakedness that we consider in this chapter, it is enough to consider it over an interval.*

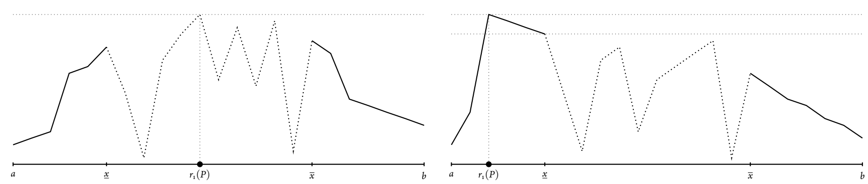
Definition 4.2.10 *A domain $\tilde{\mathcal{S}}$ is called partially single-peaked if* *(i)* *it satisfies single-peakedness outside* $[\underline{x}, \overline{x}]$ *and violates it over* $[\underline{x}, \overline{x}]$*, and*

(ii) *it contains a top-connected single-peaked domain.*

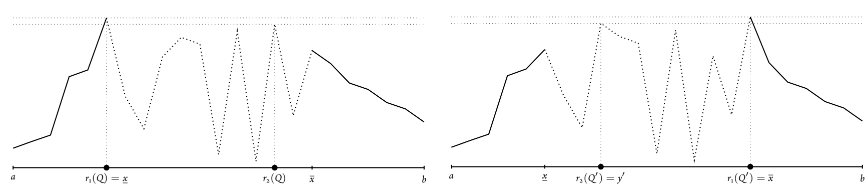
Remark 4.2.2 *Condition (ii) in Definition 4.2.10 may not seem to be essential in modeling non-single-peaked preferences that arise in political and economic scenarios. However, we feel this is not the case. In most political and economic scenarios where a prior ordering over the alternatives exists (naturally), non-single-peaked preferences arise because some individuals may not use that ordering completely in deriving their preferences. However, there is no logical ground to rule out the possibility that some individuals may still use that ordering in deriving their preferences. Thus, one must allow for the single-peaked preferences in such domains.*

We illustrate the notion of partially single-peaked domains in Figure 4.2.2. Figure 4.2.2(a) and Figure 4.2.2(b) present partially single-peaked preferences P with $r_1(P) \in [\underline{x}, \overline{x}]$ and $r_1(P) \in [a, \underline{x})$, respectively. Figure 4.2.2(c) presents partially single-peaked preferences $Q = \underline{x}y\ldots$ and $Q' = \overline{x}z\ldots$ when $y \in (\underline{x}+1, \overline{x})$ and $z \in (\underline{x}, \overline{x}-1)$, and Figure 4.2.2(d) presents those when $y = \overline{x}$ and $z = \underline{x}$. Note that, as explained before, all these preferences are single-peaked over the intervals $[a, \underline{x}]$ and $[\overline{x}, b]$. Furthermore, for the preference depicted in Figure 4.2.2(a), there is no restriction on the ranking of the alternatives in the interval $(\underline{x}, \overline{x})$, and for the one shown in Figure 4.2.2(b), there is no restriction on the ranking of the alternatives in the interval $(\underline{x}, \overline{x})$ except that $\underline{x}$ is preferred to all the alternatives in $(\underline{x}, b]$. Also, for the preferences in Figures 4.2.2(c) and 4.2.2(d), there is no restriction on the ranking of the alternatives in $(\underline{x}, \overline{x})$ other than that on the second-ranked alternatives.

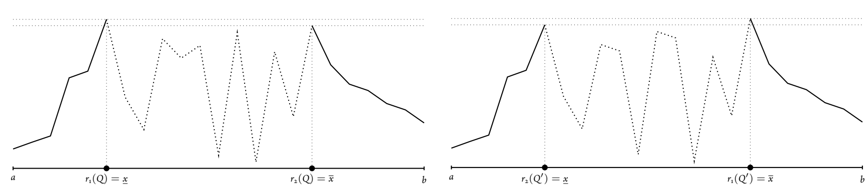
Now, we interpret Definition 4.2.10 in terms of its top-graph. Let G be the top-graph of a partially single-peaked domain. Then, G can be written as $G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the directed line graph on X and $G_2 = \langle [\underline{x}, \overline{x}], E_2 \rangle$ is a directed graph such that $(\underline{x}, r_2(Q)), (\overline{x}, r_2(Q')) \in E_2$ where $r_2(Q) \in (\underline{x}+1, \overline{x}]$ and $r_2(Q') \in [\underline{x}, \overline{x}-1)$. Therefore, G is a directed partial line graph. In Example 4.2.1, we present



(a) Partially single-peaked preference P with $r_1(P) \in [\underline{x}, \overline{x}]$ **(b)** Partially single-peaked preference P with $r_1(P) \in [a, \underline{x})$



(c) Partially single-peaked preferences Q, Q' with $\underline{x} + 1 < r_2(Q) < \overline{x}$ and $\underline{x} < r_2(Q') < \overline{x} - 1$



(d) Partially single-peaked preferences Q, Q' with $r_2(Q) = \overline{x}$ and $r_2(Q') = \underline{x}$

Figure 4.2.2 Partially single-peaked preferences

a partially single-peaked domain with seven alternatives, and in Figure 4.2.3, we present the top-graph of that domain.

Example 4.2.1 *Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7$, and let $\underline{x} = x_3$ and $\overline{x} = x_6$. Then, the domain in Table 4.2.1 is a partially single-peaked domain. To see this, first consider a preference with top-ranked alternative in the interval $[x_3, x_6]$, say P_7. Note that $x_3P_7x_2P_7x_1$ and $x_6P_7x_7$, which means P_7 is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_7]$. Moreover, the position of x_5 is completely unrestricted (here at the bottom) in P_7. Next, consider a preference with top-ranked alternative in the interval $[x_1, x_3]$, say P_2. Once again, note that P_2 is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_7]$. Further, x_3 is preferred to the alternatives*

x_4, x_5, x_6, x_7, and there is no restriction on the relative ordering of the alternatives x_4 and x_5 (here $x_5 P_2 x_4$). Thus, the domain in Table 4.2.1 satisfies single-peakedness outside the interval $[x_3, x_6]$. Now, consider the preferences Q and Q'. Since $r_1(Q) = x_3$, $r_2(Q) = x_5$, $r_1(Q') = x_6$, and $r_2(Q') = x_4$, this domain violates single-peakedness over $[x_3, x_6]$. Finally, note that the domain contains a top-connected single-peaked domain given by $P_1, P_3, P_4, P_5, P_6, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}$, and P_{14}.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	Q	Q'
$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_6}$	$\mathbf{x_7}$	$\mathbf{x_3}$	$\mathbf{x_6}$
$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_3}$	$\mathbf{x_5}$	$\mathbf{x_4}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_7}$	$\mathbf{x_6}$	$\mathbf{x_5}$	$\mathbf{x_4}$
x_3	x_3	x_3	x_1	x_4	x_2	x_3	x_5	x_3	x_3	x_4	x_4	x_5	x_5	x_2	x_3
x_4	x_6	x_4	x_4	x_5	x_5	x_2	x_2	x_2	x_6	x_3	x_3	x_4	x_4	x_6	x_7
x_5	x_5	x_5	x_5	x_6	x_6	x_1	x_6	x_1	x_7	x_2	x_2	x_3	x_3	x_1	x_2
x_6	x_7	x_6	x_6	x_7	x_1	x_7	x_1	x_6	x_2	x_7	x_7	x_2	x_2	x_7	x_1
x_7	x_4	x_7	x_7	x_1	x_7	x_5	x_7	x_7	x_1	x_1	x_1	x_1	x_1	x_4	x_5

Table 4.2.1 A partially single-peaked domain

The top-graph G of the domain in Example 4.2.1 is given in Figure 4.2.3. Note that G is a partial line graph since it can be written as $G_1 \cup G_2$, where G_1 is the directed line graph on $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and G_2 is a directed graph on $\{x_3, x_4, x_5, x_6\}$ having edges (x_3, x_5), (x_4, x_6) and (x_6, x_4).

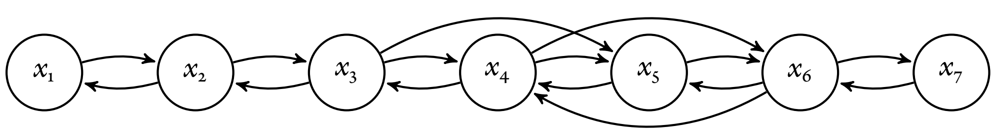


Figure 4.2.3 Top-graph of the domain in Example 4.2.1

4.2.4 Social Choice Functions and Their Properties

In this section, we introduce the notion of social choice functions and discuss their properties.

Definition 4.2.11 *A social choice function (SCF) f on $\mathcal{D}^n$ is a mapping $f : \mathcal{D}^n \to X$.*

Definition 4.2.12 *An SCF $f : \mathcal{D}^n \to X$ is unanimous if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.*

Definition 4.2.13 *An SCF $f : \mathcal{D}^n \to X$ is manipulable if there exists $i \in N$, $P_N \in \mathcal{D}^n$, and $P_i^{'} \in \mathcal{D}$ such that $f(P_i^{'}, P_{N \setminus i}) P_i f(P_N)$. An SCF f is strategy-proof if it is not manipulable.*

Definition 4.2.14 *An SCF $f : \mathcal{D}^n \to X$ is called dictatorial if there exists $i \in N$ such that for all $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.*

Definition 4.2.15 *A domain $\mathcal{D}$ is called dictatorial if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.*

Definition 4.2.16 *Two preference profiles P_N, P_N' are called tops-equivalent if $r_1(P_i) = r_1(P_i')$ for all agents $i \in N$.*

Definition 4.2.17 *An SCF $f : \mathcal{D}^n \to X$ is called tops-only if for any two tops-equivalent $P_N, P_N' \in \mathcal{D}^n$, $f(P_N) = f(P_N')$.*

Definition 4.2.18 *A domain $\mathcal{D}$ is called tops-only if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is tops-only.*

Definition 4.2.19 *An SCF $f : \mathcal{D}^n \to X$ is called uncompromising if for all $P_N \in \mathcal{D}^n$, all $i \in N$, and all $P_i' \in \mathcal{D}$:*

(i) if $r_1(P_i) < f(P_N)$ and $r_1(P_i') \leq f(P_N)$, then $f(P_N) = f(P_i', P_{-i})$, and

(ii) if $f(P_N) < r_1(P_i)$ and $f(P_N) \leq r_1(P_i')$, then $f(P_N) = f(P_i', P_{-i})$.

Remark 4.2.3 *If an SCF satisfies uncompromisingness, then by definition, it is tops-only.*

Definition 4.2.20 *Let $\beta = (\beta_S)_{S\subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{D}^n \to X$ is called a min-max rule with respect to β if*

$$f^\beta(P_N) = \min_{S\subseteq N}\{\max_{i\in S}\{r_1(P_i), \beta_S\}\}.$$

Remark 4.2.4 *Every min-max rule is uncompromising.*[3]

Definition 4.2.21 *A min-max rule $f^\beta : \mathcal{D}^n \to X$ with parameters $\beta = (\beta_S)_{S\subseteq N}$ is a partly dictatorial min-max rule (PDMMR) if there exists an agent $d \in N$, called the partial dictator of f^β, such that $\beta_d \in [a, \underline{x}]$ and $\beta_{N\setminus d} \in [\overline{x}, b]$.*

In Lemma 4.3.1, we explain why the particular agent d is called the partial dictator of f^β.

Remark 4.2.5 *[67] defines partly dictatorial generalized median voter scheme (PDGMVS) on multiple single-peaked domains. It can be shown that PDMMR coincides with PDGMVS on those domains.*[4]

4.3 Results

4.3.1 Unanimous and Strategy-proof SCFs On Partially Single-peaked Domains

In this subsection, we characterize the unanimous and strategy-proof SCFs on partially single-peaked domains as partly dictatorial generalized median voter schemes.

First, we present a lemma that justifies why the agent d in Definition 4.2.21 is called the partial dictator. It shows that a PDMMR chooses the top-ranked alternative of the partial dictator whenever that lies in the interval $[\underline{x}, \overline{x}]$. It further shows that it chooses an alternative in the interval $[a, \underline{x}]$ or $[\overline{x}, b]$ depending on whenever the top-ranked alternative of the partial dictator lies in that interval.

[3] For details, see [86].
[4] For details see the proof of Theorem 3.1 in [67].

Lemma 4.3.1 *Let $f^\beta : \mathcal{D}^n \to X$ be a PDMMR. Suppose agent d is the partial dictator of f^β. Then,*

(i) $f^\beta(P_N) \in [a, \underline{x}]$ *if* $r_1(P_d) \in [a, \underline{x})$,

(ii) $f^\beta(P_N) \in [\overline{x}, b]$ *if* $r_1(P_d) \in (\overline{x}, b]$, *and*

(iii) $f^\beta(P_N) = r_1(P_d)$ *if* $r_1(P_d) \in [\underline{x}, \overline{x}]$.

Proof: First, we prove (i). The proof of (ii) can be established using symmetric arguments. Assume for contradiction that $r_1(P_d) \in [a, \underline{x})$ and $f^\beta(P_N) > \underline{x}$. Since f^β is a min-max rule, f^β is uncompromising. Therefore, $f^\beta(P'_d, P_{N\setminus d}) = f^\beta(P_N)$, where $r_1(P'_d) = a$. Again by uncompromisingness, we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_i) = b$ for all $i \neq d$. Because $f^\beta(P_N) > \underline{x}$, this means $f^\beta(P'_N) > \underline{x}$. However, by the definition of f^β, $f^\beta(P'_N) = \beta_d$. Since $\beta_d \in [a, \underline{x}]$, this is a contradiction. This completes the proof of (i).

Now, we prove (iii). Without loss of generality, assume for contradiction that $r_1(P_d) \in [\underline{x}, \overline{x}]$ and $f^\beta(P_N) > r_1(P_d)$. Using a similar argument as for the proof of (i), we have $f^\beta(P'_N) \geq f^\beta(P_N)$, where $r_1(P'_d) = a$ and $r_1(P'_i) = b$ for all $i \neq d$. This, in particular, means $f^\beta(P'_N) > \underline{x}$. Since by the definition of f^β, $f^\beta(P'_N) = \beta_d$ and $\beta_d \in [a, \underline{x}]$, this is a contradiction. This completes the proof of (iii). ■

Now, we present a characterization of the the unanimous and strategy-proof SCFs on partially single-peaked domains.

Theorem 4.3.1 *Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a PDMMR.*

The proof of the Theorem 4.3.1 is relegated to Appendix 4.6.

Our next corollary is a consequence of Lemma 4.3.1 and Theorem 4.3.1. It characterizes a class of dictatorial domains, and thereby it generalizes the celebrated Gibbard-Satterthwaite ([43], [75]) results. Note that our dictatorial result is independent of those in [6], [74], [62], and so on.

Corollary 4.3.1 *Let $\underline{x} = a$ and $\overline{x} = b$. Then, every partially single-peaked domain is dictatorial.*

4.3.2 A Result on Partial Necessity

In Subsection 4.3.1, we have focused on partially single-peaked domains and have shown that every unanimous and strategy-proof SCF on those is a PDMMR. In this subsection, we look at the converse of this problem, that is, we focus on PDMMR and investigate the class of domains where these rules are unanimous and strategy-proof. We show that the partially single-peaked domains are almost all domains with this property. This indicates that our notion of partial single-peaked domains is quite general. A formal definition is as follows.

Definition 4.3.1 *A domain $\mathcal{D}$ is called a PDMMR domain if*

(i) every unanimous and strategy-proof SCF on $\mathcal{D}^n$ is a PDMMR, and

(ii) every PDMMR on $\mathcal{D}^n$ is strategy-proof.

By Theorem 4.3.1, every partially single-peaked domain is a PDMMR domain. In what follows, we show that a domain has to be very close to a partially single-peaked domain in order to be a PDMMR domain.

Conditions (i) and (ii) in Definition 4.2.10 are obviously strong conditions. In what follows, we show by means of Lemma 4.3.2, Example 4.3.1 and Example 4.3.2 that Condition (i) in the said definition is necessary for a domain to be a PDMMR domain. Regarding Condition (ii), in Example 4.3.3, we provide a domain that fails to satisfy Condition (ii) (but satisfies Condition (i)) and is not a PDMMR domain. Note that this does not prove that Condition (ii) is necessary (even in the presence of Condition (i)) since there are many ways this condition can be violated, and here we consider a particular type of violation of it. Thus, Conditions (i) and (ii) are close to being necessary in an appropriate sense for a PDMMR domain.

Lemma 4.3.2 *Let $\mathcal{D}$ be a PDMMR domain. Then, $\mathcal{D}$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$.*

Proof: First, we show that a preference with top-ranked alternative in $[\underline{x}, \overline{x}]$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Without loss of generality, assume for contradiction that there exists $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [\underline{x}, \overline{x}]$ such that $u\tilde{P}v$ for some $u < v \leq \underline{x}$. Consider the PDMMR $f^{\beta} : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} v \text{ if } S = \{1\}, \\ a \text{ if } \{1\} \subsetneq S, \\ b \text{ if } 1 \notin S. \end{cases}$$

We show that f^{β} is not strategy-proof. Note that agent 1 is the partial dictator of f^{β}. Consider the preference profile $P_N \in \mathcal{D}^n$ such that $r_1(P_1) = a$, $P_2 = \tilde{P}$, and $r_1(P_j) = b$ for all $j \neq 1, 2$. Then, by the definition of f^{β}, $f^{\beta}(P_N) = v$. Let $P_2' \in \mathcal{D}$ be such that $r_1(P_2') = u$. Again, by the definition of f^{β}, $f^{\beta}(P_2', P_{N \setminus 2}) = u$. Since $u\tilde{P}v$, this means agent 2 manipulates at P_N via P_2'.

Now, we show that a preference with top-ranked alternative outside $[\underline{x}, \overline{x}]$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Without loss of generality, assume for contradiction that there exist $\tilde{P} \in \mathcal{D}$ with $r_1(\tilde{P}) \in [a, \underline{x})$ and $u, v \in X$ with $u \notin (\underline{x}, \overline{x})$ such that $\left[v < u \leq r_1(P) \text{ or } r_1(P) \leq u < v\right]$ and $v\tilde{P}u$. If $\left[v < u \leq r_1(\tilde{P})\right]$ and $v\tilde{P}u$, then using a similar argument as for the proof of the necessity of Condition (i), it follows that there is a PDMMR on $\mathcal{D}^n$ that is manipulable. Hence, assume $r_1(\tilde{P}) \leq u < v$ and $v\tilde{P}u$. We distinguish two cases.

Case 1. Suppose $u \leq \underline{x}$.

Consider the PDMMR $f^{\beta} : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} u \text{ if } 1 \in S \text{ and } S \neq N, \\ b \text{ if } 1 \notin S. \end{cases}$$

We show that f^{β} is not strategy-proof. Let $P_N \in \mathcal{D}^n$ be such that $P_1 = \tilde{P}$ and $r_1(P_j) = b$ for all $j \neq 1$. Then, by the definition of f^{β}, $f^{\beta}(P_N) = u$. Let $P_1' \in \mathcal{D}$ be such that $r_1(P_1') = v$. Again, by the definition of f^{β}, $f^{\beta}(P_1', P_{N \setminus 1}) = v$. Since $v\tilde{P}u$, agent 1 manipulates at P_N via P_1'.

Case 2. Suppose $\underline{x} < u$.

Since $u \notin (\underline{x}, \overline{x})$, this means $\overline{x} \leq u$. Consider the PDMMR $f^{\beta} : \mathcal{D}^n \to X$, where

$$\beta_S = \begin{cases} a \text{ if } 1 \in S, \\ u \text{ if } 1 \notin S \text{ and } S \neq \emptyset. \end{cases}$$

We show that f^{β} is not strategy-proof. Let $P_N \in \mathcal{D}^n$ be such that $P_2 = \tilde{P}$ and $r_1(P_j) = b$ for all $j \neq 2$. Then, by the definition of f^{β}, $f^{\beta}(P_N) = u$. Let $P'_2 \in \mathcal{D}$ be such that $r_1(P'_2) = v$. Again, by the definition of f^{β}, $f^{\beta}(P'_2, P_{N\setminus 2}) = v$. Since $v\tilde{P}u$, agent 2 manipulates at P_N via P'_2. ■

Now, we discuss the necessity of the existence of two particular preferences Q, Q' as mentioned in Definition 4.2.9. Recall that Definition 4.2.9 requires two non-single-peaked preferences $Q = \underline{x}y\ldots$ and $Q' = \overline{x}z\ldots$ in $\mathcal{D}$ such that either $\big[y \in (\underline{x}+1, \overline{x})$ and $z \in (\underline{x}, \overline{x}-1)\big]$ or $\big[y = \overline{x}$ and $z = \underline{x}\big]$. Suppose a domain $\mathcal{D}$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Suppose further that it contains a non-single-peaked preference of the form Q, but no preference of the form Q'. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain that is not a PDMMR.

Example 4.3.1 *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. By $P = x_1x_2x_3x_4x_5$, we mean a preference P such that $x_1Px_2Px_3Px_4Px_5$. Consider the domain as follows:*

$$\mathcal{D} = \{x_1x_2x_3x_4x_5, x_1x_3x_4x_5x_2, x_2x_1x_3x_4x_5, x_2x_3x_4x_5x_1, x_3x_2x_1x_4x_5, x_3x_4x_5x_2x_1, x_4x_3x_2x_1x_5, x_4x_5x_3x_2x_1, x_5x_4x_3x_2x_1\}.$$

Note that $\mathcal{D} \setminus \{x_1x_3x_4x_5x_2\}$ is a top-connected single-peaked domain and the preference $x_1x_3x_4x_5x_2$ is of the form Q where $\underline{x} = x_1$ and $\overline{x} \geq x_3$. However, there is no preference in $\mathcal{D}$ of the form Q', that is, no preference Q' with $r_1(Q') \geq x_3$ and $r_2(Q') \in [x_1, r_1(Q') - 1)$. In Table 4.3.1, we present a two-agent SCF that is unanimous and strategy-proof but not a PDMMR.

$P_1 \backslash P_2$	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5

Table 4.3.1 A unanimous and strategy-proof SCF which is not a PDMMR

It is left to the reader to verify that the SCF presented in Table 4.3.1 is unanimous and strategy-proof. Note that it violates tops-onlyness at the preference profiles $(x_3x_4x_5x_2x_1, x_1x_2x_3x_4x_5)$ *and* $(x_3x_4x_5x_2x_1, x_1x_3x_4x_5x_2)$*, and hence it is not a PDMMR.*

Now, suppose that $\mathcal{D}$ contains two non-single-peaked preferences Q and Q', however, they do *not* satisfy Definition 4.2.9 for their second-ranked alternatives. In the following example, we construct a two-agent unanimous and strategy-proof SCF on such a domain $\mathcal{D}$ that is not a PDMMR.

Example 4.3.2 *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$*, where* $x_1 < x_2 < x_3 < x_4 < x_5$*. Let* $\mathcal{D}$ *be the domain given in Example 4.3.1. Consider the domain* $\mathcal{D} \cup \{x_5x_1x_4x_3x_2\}$*. As pointed out in Example 4.3.1,* $\mathcal{D} \setminus \{x_1x_3x_4x_5x_2\}$ *is a top-connected single-peaked domain. Consider the non-single-peaked preferences* $x_1x_3x_4x_5x_2$ *and* $x_5x_1x_4x_3x_2$*. They can be considered as* Q *and* Q' *only if* $\underline{x} = x_1$ *and* $\overline{x} = x_5$*. However, since their second-ranked alternatives are* x_3 *and* x_1*, respectively, they do not satisfy Definition 4.2.9. In Table 4.3.2, we present a two-agent SCF that is unanimous and strategy-proof but not a PDMMR.*

Note that the restriction of the SCF presented in Table 4.3.2 to $\mathcal{D}^2$ *is same as the SCF presented in Table 4.3.1. It is left to the reader to verify that this SCF is unanimous and strategy-proof. However, as pointed out in Example 4.3.1, it violates tops-onlyness, and hence it is not a PDMMR.*

Our next example shows that the requirement of top-connectedness in addition with single-peakedness as given in Condition (ii) is necessary. In fact, we show that

$P_1 \backslash P_2$	$x_1x_2x_3x_4x_5$	$x_1x_3x_4x_5x_2$	$x_2x_1x_3x_4x_5$	$x_2x_3x_4x_5x_1$	$x_3x_2x_1x_4x_5$	$x_3x_4x_5x_2x_1$	$x_4x_3x_2x_1x_5$	$x_4x_5x_3x_2x_1$	$x_5x_4x_3x_2x_1$	$x_5x_1x_4x_3x_2$
$x_1x_2x_3x_4x_5$	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_1
$x_1x_3x_4x_5x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_1
$x_2x_1x_3x_4x_5$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_2x_3x_4x_5x_1$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$x_3x_2x_1x_4x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_3x_4x_5x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3
$x_4x_3x_2x_1x_5$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_4x_5x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_4	x_4
$x_5x_4x_3x_2x_1$	x_2	x_3	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5
$x_5x_1x_4x_3x_2$	x_1	x_1	x_2	x_2	x_3	x_3	x_4	x_4	x_5	x_5

Table 4.3.2 A unanimous and strategy-proof SCF which is not a PDMMR

if a single-peaked domain is not top-connected, then it admits SCFs that are not PDMMRs.

Example 4.3.3 *Consider a domain $\mathcal{D}$ that violates Condition (ii) in the following manner: there exists $u \in [\underline{x}, \overline{x}]$ such that for all $P \in \mathcal{D}$, $r_1(P) = u$ implies $r_2(P) = u - 1$.*

$$f(P_N) = \begin{cases} u \text{ if } r_1(P_i) = u \text{ and } uP_j(u-1) \text{ for all } j \in N \setminus i, \\ u - 1 \text{ if } r_1(P_i) = u \text{ and } (u-1)P_j u \text{ for some } j \in N \setminus i, \\ r_1(P_i) \text{ otherwise.} \end{cases}$$

It is straightforward to show that the SCF f is not a PDMMR as no agent is a partial dictator.

Remark 4.3.1 *It is worth noting that the reason why PDMMR domains are not necessarily partially single-peaked is that the top-connectedness requirement of the single-peaked preferences as given in Condition (ii) is not necessary for the alternatives in the interval $(\underline{x}, \overline{x})$.*

4.3.3 Group Strategy-proofness

In this section, we consider group strategy-proofness and obtain a characterization of the unanimous and group strategy-proof SCFs on partially single-peaked domains. We begin with the definition of group strategy-proofness.

Definition 4.3.2 *An SCF $f : \mathcal{D}^n \to X$ is called group manipulable if there is a preference profile P_N, a non-empty coalition $C \subseteq N$, and a preference profile $P'_C \in \mathcal{D}^{|C|}$ of the agents in C such that $f(P'_C, P_{N \setminus C}) P_i f(P_N)$ for all $i \in C$. An SCF $f : \mathcal{D}^n \to X$ is called group strategy-proof if it is not group manipulable.*

In the following theorem, we present a characterization of the unanimous and group strategy-proof SCFs on partially single-peaked domains. It is worth mentioning that these domains do not satisfy the sufficient condition for the equivalence of strategy-proofness and group strategy-proofness provided in [7].

Theorem 4.3.2 *Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and group strategy-proof if and only if it is a PDMMR.*

Proof: Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a PDMMR where agent d is the partial dictator. It is enough to show that f is group strategy-proof. Clearly, no group can manipulate f at a preference profile $P_N \in \tilde{\mathcal{S}}^n$ where $r_1(P_d) \in [\underline{x}, \overline{x}]$. Consider a preference profile $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, \underline{x})$. We show that f is group strategy-proof at P_N. Since $r_1(P_d) \in [a, \underline{x})$, by the definition of PDMMR, $f(P_N) \in [a, \underline{x}]$. Let $C' = \{i \in N \mid r_1(P_i) \leq f(P_N)\}$ and let $C'' = \{i \in N \mid r_1(P_i) > f(P_N)\}$. Suppose a coalition C manipulates f at P_N. Then, there is $P'_C \in \tilde{\mathcal{S}}^{|C|}$ such that $f(P'_C, P_{N \setminus C}) P_i f(P_N)$ for all $i \in C$. If $f(P'_C, P_{N \setminus C}) < f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C'' = \emptyset$. However, by the definition of PDMMR, $f(P'_C, P_{N \setminus C}) \geq f(P_N)$ for all $C \subseteq C'$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. Again, if $f(P'_C, P_{N \setminus C}) > f(P_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C' = \emptyset$. However, by the definition of PDMMR, $f(P'_C, P_{N \setminus C}) \leq f(P_N)$ for all $C \subseteq C''$ and all $P'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. The proof of the same for the case where $r_1(P_d) \in (\overline{x}, b]$ follows from a symmetric argument. This shows f is group strategy-proof, and hence completes the proof of the theorem. ■

4.4 Applications

In this section, we present a couple of examples of our main result.

4.4.1 Multi-peaked Domains

In Section 4.1, we have discussed the importance of multi-peaked domains in modeling preferences of individuals in certain economic and political scenarios. In this subsection, we formally define this notion and show that these are special cases of partially single-peaked domains.

Definition 4.4.1 *A preference P is called multi-peaked if there are $d_0, p_1, d_1, p_2, d_2, \ldots, d_{k-1}, p_k, d_k$ with $a = d_0 \leq p_1 < d_1 < \ldots < p_k \leq d_k = b$ such that for all $i = 0, \ldots, k-1$ and all $x, y \in [d_i, d_{i+1}]$, $[x < y \leq p_{i+1}$ or $p_{i+1} \leq y < x]$ implies yPx. For such a preference P the alternatives $p_1, \ldots, p_k$ are called its peaks.*

We present a multi-peaked domain in Figure 4.4.1.

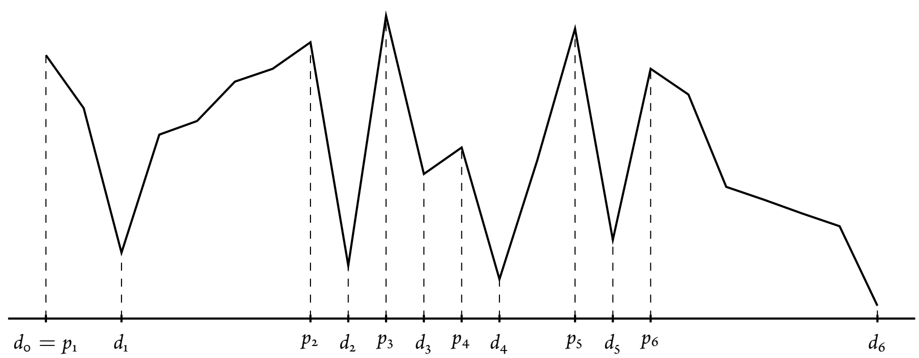


Figure 4.4.1 A multi-peaked preference

Definition 4.4.2 *Let c_1 and c_2 be such that $a \leq c_1 < c_2 - 1 \leq b$. Then, a domain $\mathcal{D}$ is called multi-peaked with critical values c_1, c_2 if each preference in $\mathcal{D}$ is either single-peaked or multi-peaked with all its peaks in the interval $[c_1, c_2]$.*

It is easy to verify that a multi-peaked domain with critical values $\underline{x}$ and $\overline{x}$ is a partially single-peaked domain. Thus, we have the following corollary.

Corollary 4.4.1 *Let $\mathcal{S}$ be a multi-peaked domain with critical values c_1 and c_2. Then, an SCF $f : \mathcal{S}^n \to X$ is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

4.4.2 Single-peaked Domains with respect to Partial Orders

As discussed in Section 4.1, expecting individuals to have a complete prior order over the alternatives is a strong prerequisite. In view of this, we relax this condition by requiring the individuals to have a partial prior order over the alternatives and to derive preferences based on such a partial order. In this subsection, we argue that such a domain is partially single-peaked.

Definition 4.4.3 *A binary relation is called a partial order if it is reflexive, antisymmetric, and transitive.*

Note that a partial order need not be complete. We denote a partial order by $\lhd$.[5] Also, we write $a \unlhd b$ to mean $a \lhd b$ or $a = b$.

Definition 4.4.4 *A preference P is said to be single-peaked with respect to a partial order $\lhd$ over X if for all distinct $x, y \in X$,*

$$[x \lhd y \unlhd r_1(P) \text{ or } r_1(P) \unlhd y \lhd x] \text{ implies } yPx.$$

A domain is called single-peaked with respect to a partial order $\lhd$ if it contains all single-peaked preferences with respect to $\lhd$.

Since every partial order can be thought of a subset of a linear order (as a binary relation), it can be shown that a single-peaked domain with respect to a partial order is partially single-peaked. However, we do not provide a concrete proof of this since that is a bit technical.[6] Nevertheless, in what follows we provide a few examples of single-peaked domains with respect to partial orders and show that those domains are partially single-peaked.

Example 4.4.1 *Suppose that the set of alternatives is partitioned into a number of subsets such that the designer knows how agents order (a priori) the alternatives in each*

[5]To be precise, the antisymmetric part of a partial order.
[6]A proof of this fact is available on request.

of those subsets, but does not know how agents compare alternatives in two different subsets.

More formally, suppose that X is partitioned into the subsets $X_1, \ldots, X_k$. For all $i = 1, \ldots, k$, let $\prec_i \in \mathbb{L}(X_i)$ be a linear order over X_i. Consider the partial order $\lhd$ over X given by the union of $\prec_i$s, that is, $x \lhd y$ if and only if there is $i = 1, \ldots, k$ such that $x, y \in X_i$ and $x \prec_i y$. In what follows, we consider a simple such partial order and present the single-peaked domain with respect to the same.

Let the set of alternatives be $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Suppose that X is partitioned into the sets $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$. Consider the partial order $\lhd$ given by $x_1 \lhd x_2 \lhd x_3$ and $x_4 \lhd x_5 \lhd x_6$. In Table 4.4.1, we present the single-peaked domain with respect to $\lhd$. Note that the domain has the property that its restriction on $\{x_1, x_2, x_3\}$ is single-peaked with respected to the prior order $x_1 \lhd x_2 \lhd x_3$ and on $\{x_4, x_5, x_6\}$ is single-peaked with respected to the prior order $x_4 \lhd x_5 \lhd x_6$. Since this domain is large, we provide only a few preferences that are significant for our purpose. Clearly, this domain is partially single-peaked with $\underline{x} = x_1$ and $\overline{x} = x_6$. Therefore, it follows from Theorem 4.3.1 that it is a dictatorial domain.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$\mathbf{x}_1$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_6$
$\mathbf{x}_2$	$\mathbf{x}_4$	$\mathbf{x}_1$	$\mathbf{x}_3$	$\mathbf{x}_2$	$\mathbf{x}_4$	$\mathbf{x}_3$	$\mathbf{x}_5$	$\mathbf{x}_4$	$\mathbf{x}_6$	$\mathbf{x}_5$	$\mathbf{x}_3$
x_3	x_2	x_3	x_1	x_1	x_5	x_2	x_6	x_6	x_4	x_4	x_5
x_4	x_5	x_4	x_4	x_4	x_6	x_1	x_3	x_3	x_3	x_3	x_2
x_5	x_3	x_5	x_5	x_5	x_2	x_5	x_2	x_2	x_2	x_2	x_4
x_6	x_6	x_6	x_6	x_6	x_1	x_6	x_1	x_1	x_1	x_1	x_1

Table 4.4.1 A single-peaked domain with respect to the partial order $\lhd$

Example 4.4.2 *In political science, it is often assumed that the parties can be ordered from left to right on the policy spectrum based on whether they are more liberal (left) or more conservative (right) in their policies. Deriving such an ordering can be done unambiguously over the parties who are clearly identifiable as more left or more right. However, ordering parties who are moderate in their policies (i.e., having policies around* *the center of the spectrum) may not be possible. To model such a situation, one needs to assume that the prior ordering of the parties (on the political spectrum) is not complete around the center of the spectrum. In what follows, we consider a simple such partial order and present the single-peaked domain with respect to the same.*

Suppose that the set of alternatives is given by $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$*. Consider the partial order* $\lhd$ *obtained from the linear order* $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5 \prec x_6$ *by making* x_3 *and* x_4 *incomparable, that is,* $\lhd$ *is given by* $x_1 \lhd x_2 \lhd x_3 \lhd x_5 \lhd x_6$ *and* $x_1 \lhd x_2 \lhd x_4 \lhd x_5 \lhd x_6$*. The single-peaked domain with respect to* $\lhd$ *is given in Table 4.4.2. Note that this domain is partially single-peaked with* $\underline{x} = x_2$ *and* $\overline{x} = x_5$*. Therefore, it follows from Theorem 4.3.1 and Theorem 4.3.2 that any unanimous and (group) strategy-proof SCF on this domain is a PDMMR.*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
$\mathbf{x}_1$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_5$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_6$
$\mathbf{x}_2$	$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_2$	$\mathbf{x}_4$	$\mathbf{x}_3$	$\mathbf{x}_5$	$\mathbf{x}_4$	$\mathbf{x}_6$	$\mathbf{x}_3$	$\mathbf{x}_5$	$\mathbf{x}_5$
x_3	x_4	x_3	x_1	x_3	x_1	x_5	x_5	x_3	x_6	x_4	x_4	x_4	x_3
x_4	x_3	x_4	x_4	x_1	x_4	x_6	x_6	x_6	x_3	x_3	x_2	x_3	x_4
x_5	x_5	x_5	x_5	x_5	x_5	x_2	x_2	x_2	x_2	x_2	x_6	x_2	x_2
x_6	x_6	x_6	x_6	x_6	x_6	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1

Table 4.4.2 A single-peaked domain with respect to the partial order $\lhd$

The following corollary summarizes the above discussion on single-peaked domains with respect to a partial order.

Corollary 4.4.2 *Let* $\lhd$ *be a partial order over* X *and let* $\mathcal{S}$ *be the single-peaked domain with respect to* $\lhd$*. Then, an SCF* $f : \mathcal{S}^n \to X$ *is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

4.4.3 Multiple Single-peaked Domain

In this subsection, we consider a well-known class of domains called multiple single-peaked domains and show that they are special cases of partially single-peaked domains.

We begin with introducing the notion of a single-peaked domain with respect to an arbitrary order over X.

Definition 4.4.5 *Let $\prec \in \mathbb{L}(X)$ be a prior order over X. Then, a preference $P \in \mathbb{L}(X)$ is single-peaked with respect to $\prec$ if for all $x, y \in X$, $[x \prec y \preceq r_1(P)$ or $r_1(P) \preceq y \prec x]$ implies yPx. A domain $\mathcal{S}_\prec$ is called a single-peaked domain with respect to $\prec$ if each preference in it is single-peaked with respect to $\prec$, and a domain $\bar{\mathcal{S}}_\prec$ is called maximal single-peaked with respect to $\prec$ if it contains all single-peaked preferences with respect to $\prec$.*

Definition 4.4.6 *Let $\mathcal{L} = \{\prec_1, \dots, \prec_q\}$, where $\prec_k \in \mathbb{L}(X)$ for all $1 \leq k \leq q$, be a set of q prior orders over X. Then, a domain is called a multiple single-peaked domain with respect to $\mathcal{L}$, denoted by $\mathcal{S}_\mathcal{L}$, if $\mathcal{S}_\mathcal{L} = \bigcup\limits_{k \in \{1,\dots,q\}} \bar{\mathcal{S}}_{\prec_k}$, where $\bar{\mathcal{S}}_{\prec_k}$ is the maximal single-peaked domain with respect to the prior order $\prec_k$. A multiple single-peaked domain with respect to $\mathcal{L}$ is called trivial if $\bar{\mathcal{S}}_\prec = \bar{\mathcal{S}}_{\prec'}$ for all $\prec, \prec' \in \mathcal{L}$.*

For ease of presentation, for any multiple single-peaked domain with respect to $\mathcal{L}$, we assume without loss of generality that the integer ordering $<$ is in the set $\mathcal{L}$.

Definition 4.4.7 *Let $\mathcal{S}_\mathcal{L}$ be a non-trivial multiple single-peaked domain with respect to a set of prior orders $\mathcal{L}$. Then, alternatives $u, v \in X$ with $u < v - 1$ are called break-points of $\mathcal{S}_\mathcal{L}$ if*

(i) for all preferences $P \in \mathcal{S}_\mathcal{L}$ and all $c, d \in X \setminus (u, v)$, $\left[d < c \leq r_1(P)\right.$ or $\left. r_1(P) \leq c < d\right]$ implies cPd, and

(ii) there exist $P, P' \in \mathcal{S}_\mathcal{L}$ such that $r_1(P) = u$, $r_2(P) \in (u+1, v]$, $r_1(P') = v$, and $r_2(P') \in [u, v-1)$.

Remark 4.4.1 *The break points, say u, v, of a non-trivial multiple single-peaked domain $\mathcal{S}_\mathcal{L}$ induce the partition $\{X_L, X_M, X_R\}$ of X, where $X_L = [a, u)$, $X_M = [u, v]$, and $X_R = (v, b]$. [67] calls such a partition the maximal common decomposition of X and the sets X_L, X_M, and X_R as the left component, the middle component, and the right component of alternatives, respectively.*

In the following, we illustrate the notion of break-points of a non-trivial multiple single-peaked domain by means of an example.

Example 4.4.3 *Let* $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ *be the set of alternatives. Consider the set of prior orders* $\mathcal{L} = \{<, \prec_1, \prec_2, \prec_3\}$, *where* $<= x_1x_2x_3x_4x_5x_6x_7$, $\prec_1= x_1x_2x_3x_5x_4x_6x_7$, $\prec_2= x_1x_2x_5x_4x_3x_6x_7$, *and* $\prec_3= x_1x_2x_4x_3x_5x_6x_7$. *Let* $\mathcal{S}_{\mathcal{L}}$ *be the multiple single-peaked domain with respect to* $\mathcal{L}$. *Clearly,* $\mathcal{S}_{\mathcal{L}}$ *is non-trivial since* $\bar{\mathcal{S}}_{\prec_1} \neq \bar{\mathcal{S}}_{\prec_2}$. *We claim* $u = x_2$ *and* $v = x_6$ *are the break points of* $\mathcal{S}_{\mathcal{L}}$. *It is easy to verify that* $\mathcal{S}_{\mathcal{L}}$ *satisfies Condition (i) in Definition 4.4.7. For Condition (ii), note that we have preferences* $P, P' \in \bar{\mathcal{S}}_{\prec_2} \subseteq \mathcal{S}_{\mathcal{L}}$ *where* $r_1(P) = x_2$, $r_2(P) = x_5$, $r_1(P') = x_6$, *and* $r_2(P') = x_3$. *Further, note that the maximal common decomposition of* X *is given by* $X_L = \{x_1\}$, $X_M = \{x_2, x_3, x_4, x_5, x_6\}$, *and* $X_R = \{x_7\}$.

It can be easily verified that every non-trivial multiple single-peaked domain is a partially single-peaked domain where $\underline{x}$ and $\overline{x}$ are the break-points. Thus, we have the following corollary.

Corollary 4.4.3 **([67])** *Let* $\mathcal{S}_{\mathcal{L}}$ *be a non-trivial multiple single-peaked domain with break-points* $\underline{x}$ *and* $\overline{x}$. *Then, an SCF* $f : \mathcal{S}^n_{\mathcal{L}} \to X$ *is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

4.4.4 Single-peaked Domains on Graphs

In this subsection, we introduce the notion of single-peaked domains on graphs and show that such a domain is partially single-peaked if the underlying graph satisfies some condition. All the graphs we consider in this subsection are undirected.

Definition 4.4.8 *A path in an undirected graph* $G = \langle X, E \rangle$ *from a node* x *to a node* y, *denoted by* $\pi_G(x, y)$, *is defined as a sequence of nodes* $(x_1, \ldots, x_k)$ *such that* $\{x_i, x_{i+1}\} \in E$ *for all* $i = 1, \ldots, k-1$. *An undirected graph* $G = \langle X, E \rangle$ *is called connected if for all* $x, y \in X$, *there is a path from* x *to* y.

Definition 4.4.9 *An undirected graph $G = \langle X, E\rangle$ is called a tree if for every two distinct nodes $x, y \in X$, there is a unique path from x to y. A spanning tree of an undirected connected graph G is defined as a connected subgraph of G that is a tree. For an undirected connected graph G, we denote by $\mathcal{T}_G$ the set of all spanning trees of G.*

Definition 4.4.10 *Let $T = \langle X, E\rangle$ be a tree. Then, a domain is called single-peaked with respect to T, denoted by $\mathcal{S}_T$, if for all $P \in \mathcal{S}_T$ and all distinct $x, y \in X$,*

$$[x \in \pi_T(r_1(P), y)] \implies [xPy].$$

Definition 4.4.11 *Let $G = \langle X, E\rangle$ be an undirected connected graph. Then, a domain is called single-peaked with respect to G, denoted by $\mathcal{S}_G$, if $\mathcal{S}_G = \cup_{T\in\mathcal{T}_G}\mathcal{S}_T$.*

Note that if T is the undirected line graph on X, then $\mathcal{S}_T$ is the maximal single-peaked domain. In Lemma 4.4.1, we show that if a domain is single-peaked with respect to an undirected partial line graph as defined in Definition 4.2.4, then it is a partially single-peaked domain.

Lemma 4.4.1 *Let G be an undirected partial line graph. Then, $\mathcal{S}_G$ is a partially single-peaked domain.*

Proof: Let G be an undirected partial line graph. We show that $\mathcal{S}_G$ is a partially single-peaked domain. Let $G = G_1 \cup G_2$, where $G_1 = \langle X, E_1\rangle$ is the undirected line graph on X and $G_2 = \langle[\underline{x}, \overline{x}], E_2\rangle$ is an undirected graph such that $\{\underline{x}, y\}, \{\overline{x}, z\} \in E_2$ for some $y \in (\underline{x} + 1, \overline{x}]$ and $z \in [\underline{x}, \overline{x} - 1)$.

First, we show that $\mathcal{S}_G$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Take $P \in \mathcal{S}_G$ with $r_1(P) \in [\underline{x}, \overline{x}]$ and take $u, v \in X \setminus (\underline{x}, \overline{x})$. Suppose $[v < u \leq r_1(P)$ or $r_1(P) \leq u < v]$. Consider an arbitrary spanning tree T of G. Then, by the definition of G, $u \in \pi_T(r_1(P), v)$, and hence uPv. Therefore, P satisfies single-peakedness outside $[\underline{x}, \overline{x}]$. Using a similar argument, it can be shown that a preference P with $r_1(P) \notin [\underline{x}, \overline{x}]$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$.

Next, we show that $\mathcal{S}_G$ violates single-peakedness over $[\underline{x}, \overline{x}]$. Consider the tree $T = \langle X, E\rangle$ such that $E = (E_1 \setminus \{\underline{x}, \underline{x} + 1\}) \cup \{\underline{x}, y\}$. Since $G_1 = \langle X, E_1\rangle$ is the undirected line graph on X, T is a spanning tree of G. Because $\{\underline{x}, y\} \in E$, there is a preference $Q = \underline{x}y\ldots \in \mathcal{S}_T \subseteq \mathcal{S}_G$. Similarly, there is a preference $Q' = \overline{x}z\ldots \in \mathcal{S}_G$. If $y \neq \overline{x}$ and $z \neq \underline{x}$, then clearly Q and Q' satisfy Definition 4.2.9. On the other hand, if, for instance, $y = \overline{x}$, then that means there is an edge $\{\underline{x}, \overline{x}\}$ in G, and consequently, z can be chosen as $\underline{x}$. This shows $\mathcal{S}_G$ violates single-peakedness over $[\underline{x}, \overline{x}]$.

Now, we show that $\mathcal{S}_G$ contains a top-connected single-peaked domain. Since G_1 is the undirected line graph on X, $\mathcal{S}_{G_1}$ is the maximal single-peaked domain. Moreover, since G_1 is a spanning tree of G, $\mathcal{S}_{G_1} \subseteq \mathcal{S}_G$. This completes the proof of the lemma. ■

Combining Theorem 4.3.1 and Theorem 4.3.2 with Lemma 4.4.1, we obtain the following characterization of the unanimous and strategy-proof SCFs on a single-peaked domain with respect to an undirected partial line graph.

Corollary 4.4.4 *Let $G = \langle X, E \rangle$ be an undirected partial line graph. Suppose $\mathcal{S}_G$ is the single-peaked domain with respect to G. Then, an SCF $f : \mathcal{S}_G^n \to X$ is unanimous and (group) strategy-proof if and only if it is a PDMMR.*

4.5 Conclusion

In this chapter, we have considered non-single-peaked domains that arise in the literature of economics and political science. We have modelled them as partially single-peaked domains and have characterized all unanimous and (group) strategy-proof rules on those as PDMMR.

4.6 Proof of Theorem 4.3.1

We use the following theorem in Chapter 3 in the proof of Theorem 4.3.1. It characterizes the unanimous and strategy-proof SCFs on a top-connected single-peaked domain as min-max rules.

Theorem 4.6.1 (Theorem 3.4.1) *Let $\mathcal{S}$ be a top-connected single-peaked domain. Then, an SCF $f : \mathcal{S}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

Proof: [Proof of Theorem 4.3.1] (If part) Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Suppose f^β be a PDMMR on $\tilde{\mathcal{S}}^n$. Then, f^β is unanimous by definition. We show that f^β is strategy-proof. Let d be the partial dictator of f^β. If $r_1(P_d) \in [\underline{x}, \overline{x}]$, then $f^\beta(P_N) = r_1(P_d)$, and hence f^β cannot be manipulated at a preference profile $P_N \in \tilde{\mathcal{S}}^n$. Take $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_d) \in [a, \underline{x})$. Then, by Lemma 4.3.1, $f^\beta(P_N) \in [a, \underline{x}]$. Take $i \in N$ such that $r_1(P_i) \leq f^\beta(P_N)$. By the definition of f^β, $f^\beta(P'_i, P_{N\setminus i}) \geq f^\beta(P_N)$ for all $P'_i \in \tilde{\mathcal{S}}$. Since $f^\beta(P_N) \leq \underline{x}$, by the definition of a partially single-peaked domain, $r_1(P_i) \leq f^\beta(P_N)$ means $f^\beta(P_N) P_i u$ for all $u > f^\beta(P_N)$. Therefore, agent i cannot manipulate f^β at P_N. By a symmetric argument, agent i cannot manipulate f^β at a preference profile where $r_1(P_i) \geq f^\beta(P_N)$. Using a similar argument, it follows that f^β cannot be manipulated at a preference profile P_N with $r_1(P_d) \in (\overline{x}, b]$. This completes the proof of the if part.

(Only-if part) Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a unanimous and strategy-proof SCF. We show that f is a PDMMR. Let $\mathcal{S}$ be a top-connected single-peaked domain contained in $\tilde{\mathcal{S}}$. Such a domain must exist by Definition 4.2.10. By Theorem 4.6.1, f restricted to $\mathcal{S}^n$ must be a min-max rule. We establish a few properties of f in the following sequence of lemmas. As mentioned earlier, we use the notation $\mathcal{S}$ in all these lemmas to denote a (fixed) top-connected single-peaked domain contained in $\tilde{\mathcal{S}}$.

Our next lemma and its corollary show that f satisfies tops-onlyness for a particular type of preference profiles. It says the following. Let c be an arbitrary alternative. Consider a preference profile P_N such that for all $i \in N$, P_i is single-peaked and $r_1(P_i) \in \{\underline{x}, c\}$. Suppose the outcome of f at P_N is c. Consider a tops-equivalent preference profile P'_N where the agents with top-ranked alternative c in P_N do not change their preferences in P'_N. Then, the outcome of f at P'_N must be c.

Lemma 4.6.1 *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(P_S, P_{N\setminus S}) \in \mathcal{S}^n$ and*

$(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ *are two tops-equivalent preference profiles such that* $r_1(P_i) = \underline{x}$ *for all* $i \in S$*, and* $r_1(P_j) = c$ *for all* $j \in N \setminus S$*. Then,* $f(P_S, P_{N\setminus S}) = c$ *implies* $f(P'_S, P_{N\setminus S}) = c$*.*

Proof: Take S such that $\emptyset \subsetneq S \subsetneq N$. We prove the lemma using induction on $|c - \underline{x}|$. By unanimity, the lemma holds for $c = \underline{x}$. Suppose the lemma holds for all c such that $|c - \underline{x}| \leq k$. We prove the lemma for all c such that $|c - \underline{x}| = k+1$. Take c such that $|c - \underline{x}| = k + 1$. Let $(P_S, P_{N\setminus S}) \in \mathcal{S}^n$ and $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ be two tops-equivalent preference profiles such that $r_1(P_i) = \underline{x}$ for all $i \in S$, and $r_1(P_j) = c$ for all $j \in N \setminus S$. Suppose $f(P_S, P_{N\setminus S}) = c$. We show $f(P'_S, P_{N\setminus S}) = c$. We show this for $\underline{x} < c$, the proof for the case $\underline{x} > c$ is similar. Since $\underline{x} < c$ and $|c - \underline{x}| = k + 1$, we have $c = \underline{x} + k + 1$. Let $(P_S, \hat{P}_{N\setminus S}) \in \mathcal{S}^n$ be such that $\hat{P}_j = (\underline{x} + k)(\underline{x} + k + 1)\dots$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = \underline{x} + k + 1$, we have $f(P_S, \hat{P}_{N\setminus S}) = \underline{x} + k$. Since $(P_S, \hat{P}_{N\setminus S})$ and $(P'_S, \hat{P}_{N\setminus S})$ are tops-equivalent and $r_1(\hat{P}_j) = \underline{x} + k$ for all $j \in N \setminus S$, we have by the induction hypothesis, $f(P'_S, \hat{P}_{N\setminus S}) = \underline{x}+k$. For all $j \in N\setminus S$, let $\bar{P}_j = (\underline{x}+k+1)(\underline{x}+k)\dots \in \mathcal{S}$. Since $f(P'_S, \hat{P}_{N\setminus S}) = \underline{x}+k$, by moving the agents $j \in N \setminus S$ from $\hat{P}_j$ to $\bar{P}_j$ one-by-one and applying strategy-proofness at every step, we have $f(P'_S, \bar{P}_{N\setminus S}) \in \{\underline{x} + k, \underline{x} + k + 1\}$. We claim $f(P'_S, \bar{P}_{N\setminus S}) = \underline{x} + k + 1$. Assume for contradiction that $f(P'_S, \bar{P}_{N\setminus S}) = \underline{x} + k$. Recall that $P_i \in \mathcal{S}$ for all $i \in S$. Since $(\underline{x} + k)P_i(\underline{x} + k + 1)$ for all $i \in S$, by moving the agents $i \in S$ from P'_i to P_i one-by-one and applying strategy-proofness at every step, we have $f(P_S, \bar{P}_{N\setminus S}) \leq \underline{x} + k$. Since $r_1(P_j) = r_1(\bar{P}_j) = \underline{x} + k + 1$ for all $j \in N \setminus S$, by strategy-proofness, $f(P_S, P_{N\setminus S}) \neq \underline{x} + k + 1$. This contradicts our assumption that $f(P_S, P_{N\setminus S}) = \underline{x} + k + 1$. Therefore, $f(P'_S, \bar{P}_{N\setminus S}) = \underline{x} + k + 1$. Since $r_1(P_j) = r_1(\bar{P}_j) = \underline{x} + k + 1$ for all $j \in N \setminus S$, we have by strategy-proofness, $f(P'_S, P_{N\setminus S}) = \underline{x} + k + 1$. This completes the proof of the lemma. ■

Corollary 4.6.1 *Let* $\emptyset \subsetneq S \subsetneq N$ *and let* $c \in X$*. Suppose* $(P_S, P_{N\setminus S}) \in \mathcal{S}^n$ *and* $(P'_S, P_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ *are two tops-equivalent preference profiles such that* $r_1(P_i) = \bar{x}$ *for all* $i \in S$*, and* $r_1(P_j) = c$ *for all* $j \in N \setminus S$*. Then,* $f(P_S, P_{N\setminus S}) = c$ *implies* $f(P'_S, P_{N\setminus S}) = c$*.*

Our next lemma shows that the outcome of f at a boundary preference profile cannot be strictly in-between $\underline{x}$ and $\overline{x}$.[7]

Lemma 4.6.2 *Let $P_N \in \tilde{\mathcal{S}}^n$ be such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Then, $f(P_N) \notin (\underline{x}, \overline{x})$.*

Proof: Assume for contradiction that $f(P_N) = u \in (\underline{x}, \overline{x})$ for some $P_N \in \tilde{\mathcal{S}}^n$ such that $r_1(P_i) \in \{a, b\}$ for all $i \in N$. Let $S = \{i \in N \mid r_1(P_i) = a\}$. Then, it must be that $\emptyset \subsetneq S \subsetneq N$ as otherwise we are done by unanimity. Let $r_2(Q) = y$ and $r_2(Q') = z$, where $Q, Q' \in \tilde{\mathcal{S}}$ are as given in Definition 4.2.9. We distinguish three cases based on the relative positions of y, z, and u.

Case 1. Suppose $y \in (\underline{x} + 1, \overline{x})$, $z \in (\underline{x}, \overline{x} - 1)$, and $u \in (\underline{x}, z] \cup [y, \overline{x})$.

We consider the case where $u \in (\underline{x}, z]$, the proof for the case where $u \in [y, \overline{x})$ follows from a symmetric argument. Let $P'_N \in \mathcal{S}^n$ be such that $r_1(P'_i) = z$ for all $i \in S$, and $P'_j = (\overline{x} - 1)(\overline{x}) \ldots$ for all $j \in N \setminus S$. Further, let $\hat{P}_N \in \mathcal{S}^n$ be such that $r_1(\hat{P}_i) = \underline{x}$ for all $i \in S$ and $r_1(\hat{P}_j) = \underline{x} + 1$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = u$, we have $f(P'_S, P'_{N\setminus S}) = z$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = \underline{x} + 1$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = \underline{x} + 1$, by Lemma 4.6.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = \underline{x} + 1$, where $Q_i = Q$ for all $i \in S$. Consider the preference profile $(Q'_S, P'_{N\setminus S})$, where $Q'_i = Q'$ for all $i \in S$. Note that $f(P'_S, P'_{N\setminus S}) = z$ and $Q' = \overline{x}z \ldots$. Therefore, by moving the agents $i \in S$ from P'_i to Q' one-by-one and using strategy-proofness at every step, we have $f(Q'_S, P'_{N\setminus S}) \in \{\overline{x}, z\}$. We claim $f(Q'_S, P'_{N\setminus S}) = \overline{x}$. Assume for contradiction that $f(Q'_S, P'_{N\setminus S}) = z$. Since $\overline{x} P'_j z$ for all $j \in N \setminus S$, by moving the agents $j \in N \setminus S$ from P'_j to Q' one-by-one and applying strategy-proofness at every step, we have $f(Q'_S, Q'_{N\setminus S}) \neq \overline{x}$. However, this contradicts unanimity. So, $f(Q'_S, P'_{N\setminus S}) = \overline{x}$. For all $i \in S$, let $\tilde{P}_i \in \mathcal{S}$ be such that $r_1(\tilde{P}_i) = \overline{x}$. By strategy-proofness, $f(\tilde{P}_S, P'_{N\setminus S}) = \overline{x}$. Since f is a min-max rule on $\mathcal{S}^n$, this means $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = \overline{x}$. For all $i \in S$, let $\tilde{P}'_i \in \mathcal{S}$ be such that $r_1(\tilde{P}'_i) = y$. Because $(\tilde{P}_S, \hat{P}_{N\setminus S}), (\tilde{P}'_S, \hat{P}_{N\setminus S}) \in \mathcal{S}^n$ and f is a min-max rule on $\mathcal{S}^n$, $f(\tilde{P}_S, \hat{P}_{N\setminus S}) = \overline{x}$ implies $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = y$. Because $f(\tilde{P}'_S, \hat{P}_{N\setminus S}) = y$ and $Q = \underline{x}y \ldots$, by moving the agents $i \in S$ from $\tilde{P}'_i$ to Q one-by-one and applying strategy-proofness at every step, we

[7] A *boundary preference profile* is one where the top-ranked alternative of each agent is either a or b.

have $f(Q_S, \hat{P}_{N\setminus S}) \in \{\underline{x}, y\}$. Since $\{\underline{x}+1\} \cap \{\underline{x}, y\} = \emptyset$ by our assumption, this is a contradiction to our earlier finding $f(Q_S, \hat{P}_{N\setminus S}) = \underline{x}+1$. This completes the proof of the lemma for Case 1.

CASE 2. Suppose $y \in (\underline{x}+1, \overline{x})$, $z \in (\underline{x}, \overline{x}-1)$, $z < y-1$, and $u \in (z, y)$.

Let $P'_N, \hat{P}_N \in \mathcal{S}^n$ be such that $r_1(P'_i) = y$ and $r_1(\hat{P}_i) = \underline{x}$ for all $i \in S$, and $r_1(P'_j) = \overline{x}$ and $r_1(\hat{P}_j) = z$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = u$, we have $f(P'_S, P'_{N\setminus S}) = y$ and $f(\hat{P}_S, \hat{P}_{N\setminus S}) = z$. As $f(\hat{P}_S, \hat{P}_{N\setminus S}) = z$, by Lemma 4.6.1, we have $f(Q_S, \hat{P}_{N\setminus S}) = z$, where $Q_i = Q$ for all $i \in S$. Again, as $f(P'_S, P'_{N\setminus S}) = y$, by Corollary 4.6.1, we have $f(P'_S, Q'_{N\setminus S}) = y$, where $Q'_j = Q'$ for all $j \in N \setminus S$. Because $f(Q_S, \hat{P}_{N\setminus S}) = z$ and $Q' = \overline{x}z\ldots$, by moving the agents $j \in N \setminus S$ from $\hat{P}_j$ to Q' one-by-one and using strategy-proofness at every step, we have $f(Q_S, Q'_{N\setminus S}) \in \{\overline{x}, z\}$. Again, because $f(P'_S, Q'_{N\setminus S}) = y$, $Q = \underline{x}y\ldots$, by moving the agents $i \in S$ from P'_i to Q one-by-one and using strategy-proofness at every step, we have $f(Q_S, Q'_{N\setminus S}) \in \{\underline{x}, y\}$. Since $\{\underline{x}, y\} \cap \{\overline{x}, z\} = \emptyset$ by our assumption, this is a contradiction. This completes the proof of the lemma for Case 2.

CASE 3. Suppose $y = \overline{x}$, $z = \underline{x}$, and $u \in (z, y)$.

Let $P'_N \in \mathcal{S}^n$ be such that $r_1(P'_i) = \underline{x}$ for all $i \in S$ and $r_1(P'_j) = \overline{x}$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(P_S, P_{N\setminus S}) = u$, we have $f(P'_S, P'_{N\setminus S}) = u$. Take $i \in N$ and consider the preference profile $(Q_i, P'_{S\setminus i}, P'_{N\setminus S})$, where $Q_i = Q$. Since $r_1(P'_i) = r_1(Q_i) = \underline{x}$ and $f(P'_S, P'_{N\setminus S}) \neq \underline{x}$, by strategy-proofness, $f(Q_i, P'_{S\setminus i}, P'_{N\setminus S}) \neq \underline{x}$. Continuing in this manner, it follows that $f(Q_S, P'_{N\setminus S}) \neq \underline{x}$, where $Q_i = Q$ for all $i \in S$. Moreover, since $r_2(Q_i) = \overline{x}$ for all $i \in S$ and $r_1(P'_j) = \overline{x}$ for all $j \in N \setminus S$, by unanimity and strategy-proofness, $f(Q_S, P'_{N\setminus S}) \in \{\underline{x}, \overline{x}\}$. Since $f(Q_S, P'_{N\setminus S}) \neq \underline{x}$, this means $f(Q_S, P'_{N\setminus S}) = \overline{x}$. Let $Q'_j = Q'$ for all $j \in N \setminus S$. As $f(Q_S, P'_{N\setminus S}) = \overline{x}$ and $r_1(Q') = \overline{x}$, by strategy-proofness, $f(Q_S, Q'_{N\setminus S}) = \overline{x}$. Now, if we first move the agents $j \in N \setminus S$ from P'_j to Q' and then move the agents $i \in S$ from P'_i to Q, then it follows from a similar argument that $f(Q_S, Q'_{N\setminus S}) = \underline{x}$. Since $\underline{x} \neq \overline{x}$, this is a contradiction to our earlier finding that $f(Q_S, Q'_{N\setminus S}) = \overline{x}$. This completes the proof of the lemma for Case 3.

Since Cases 1, 2 and 3 are exhaustive, this completes the proof of the lemma. ■

Let $(\beta_S)_{S \subseteq N}$ be the parameters of f restricted to $\mathcal{S}^n$. In Lemma 4.6.3 and Lemma 4.6.4, we establish a few properties of these parameters.

Lemma 4.6.3 *For all $S \subseteq N$, $\beta_S \in [a, \underline{x}]$ if and only if $\beta_{N \setminus S} \in [\overline{x}, b]$.*

Proof: Take $S \subseteq N$. It is enough to show that $\beta_S \in [a, \underline{x}]$ implies $\beta_{N \setminus S} \in [\overline{x}, b]$. Assume for contradiction that $\beta_S, \beta_{N \setminus S} \in [a, \underline{x}]$. Let $Q' \in \tilde{\mathcal{S}}$ with $r_1(Q') = \overline{x}$ be as given in Definition 4.2.9. Suppose $r_2(Q') = z$. Take $u \in (z, \overline{x})$. Let $(P_S, P_{N \setminus S}) \in \mathcal{S}^n$ be such that $r_1(P_i) = a$ for all $i \in S$ and $r_1(P_j) = b$ for all $j \in N \setminus S$. Since f restricted to $\mathcal{S}^n$ is a min-max rule, $f(P_S, P_{N \setminus S}) = \beta_S \in [a, \underline{x}]$. Let $(P'_S, P'_{N \setminus S}) \in \mathcal{S}^n$ be such that $r_1(P'_i) = z$ for all $i \in S$ and $r_1(P'_j) = u$ for all $j \in N \setminus S$. Since $f(P_S, P_{N \setminus S}) \in [a, \underline{x}]$, by uncompromisingness of f restricted to $\mathcal{S}^n$, we have $f(P'_S, P'_{N \setminus S}) = z$. Because $Q' = \overline{x}z\ldots$, by moving the agents $i \in S$ one-by-one from P'_i to Q' and applying strategy-proofness at every step, we have $f(Q'_S, P'_{N \setminus S}) \in \{\overline{x}, z\}$, where $Q'_i = Q'$ for all $i \in S$.

Now, let $(\bar{P}_S, \bar{P}_{N \setminus S}) \in \mathcal{S}^n$ be such that $r_1(\bar{P}_i) = b$ for all $i \in S$ and $r_1(\bar{P}_j) = a$ for all $j \in N \setminus S$. Again, since f restricted to $\mathcal{S}^n$ is a min-max rule, $f(\bar{P}_S, \bar{P}_{N \setminus S}) = \beta_{N \setminus S} \in [a, \underline{x}]$. Recall that for $j \in N \setminus S$, $P'_j \in \mathcal{S}$ with $r_1(P'_j) = u$. Consider $(P''_S, P'_{N \setminus S}) \in \mathcal{S}^n$ such that $r_1(P''_i) = \overline{x}$ for all $i \in S$. Since $f(\bar{P}_S, \bar{P}_{N \setminus S}) \in [a, \underline{x}]$, by uncompromisingness of f restricted to $\mathcal{S}^n$, we have $f(P''_S, P'_{N \setminus S}) = u$. Because $r_1(P''_i) = \overline{x} = r_1(Q')$ for all $i \in S$, by Corollary 4.6.1, it follows that $f(Q'_S, P'_{N \setminus S}) = u$. However, as $u \notin \{\overline{x}, z\}$, this is a contradiction to our earlier finding that $f(Q'_S, P'_{N \setminus S}) \in \{\overline{x}, z\}$. This completes the proof of the lemma. ■

The following lemma says that there is exactly one agent i such that $\beta_i \in [a, \underline{x}]$.

Lemma 4.6.4 *It must be that $|\{i \in N \mid \beta_i \in [a, \underline{x}]\}| = 1$.*

Proof: Suppose there are $i \neq j \in N$ such that $\beta_i, \beta_j \in [a, \underline{x}]$. By Lemma 4.6.3, $\beta_i \in [a, \underline{x}]$ implies $\beta_{N \setminus i} \in [\overline{x}, b]$. Since $j \in N \setminus i$ and $\beta_T \leq \beta_S$ for all $S \subseteq T$, $\beta_{N \setminus i} \in [\overline{x}, b]$ implies $\beta_j \in [\overline{x}, b]$, a contradiction. Hence, there can be at most one agent $i \in N$ such that $\beta_i \in [a, \underline{x}]$.

Now, suppose $\beta_i \in [\overline{x}, b]$ for all $i \in N$. By Lemma 4.6.3, this means $\beta_{N\setminus i} \in [a, \underline{x}]$ for all $i \in N$. Therefore, there must be $S \subseteq N$ such that $\beta_S \in [a, \underline{x}]$ and for all $S' \subsetneq S$, $\beta_{S'} \in [\overline{x}, b]$. By unanimity, $S \neq \emptyset$. If S is singleton, say $\{i\}$ for some $i \in N$, then $\beta_i \in [a, \underline{x}]$ and we are done. So assume that there are $j \neq k \in S$.

Consider the preference profile $P_N \in \mathcal{S}^n$ such that $r_1(P_j) = \underline{x} + 1$, $r_2(P_j) = \underline{x}$, $r_1(P_i) = y$ for all $i \notin S$, and $r_1(P_i) = \underline{x}$ for all $i \in S \setminus j$. Since $\beta_S \in [a, \underline{x}]$ and $\beta_{S'} \in [\overline{x}, b]$ for all $S' \subsetneq S$, it follows from the definition of a min-max rule that $f(P_N) = \underline{x} + 1$. Let $P'_k \in \mathcal{S}$ be such that $r_1(P'_k) = y$. Since $\beta_{S\setminus k} \in [\overline{x}, b]$ and f restricted to $\mathcal{S}^n$ is a min-max rule, it follows that $f(P'_k, P_{N\setminus k}) = y$. Consider the preference profile $(Q_k, P_{N\setminus k})$, where $Q_k = Q$. Because $f(P'_k, P_{N\setminus k}) = y$ and $Q_k = \underline{x}y\ldots$, by strategy-proofness, $f(Q_k, P_{N\setminus k}) \in \{\underline{x}, y\}$. Suppose $f(Q_k, P_{N\setminus k}) = \underline{x}$. Because $f(P_N) = \underline{x} + 1$ and $r_1(P_k) = \underline{x}$, this means agent k manipulates at P_N via Q_k. So, $f(Q_k, P_{N\setminus k}) = y$. Let $P'_j \in \mathcal{S}$ be such that $r_1(P'_j) = \underline{x}$. Since $\beta_S \in [a, \underline{x}]$ and $\underline{x}$ is the top-ranked alternative of the agents in S at preference profile $(P'_j, P_{N\setminus j})$, we have $f(P'_j, P_{N\setminus j}) = \underline{x}$. As $r_1(P_k) = r_1(Q_k) = \underline{x}$, this means $f(P'_j, Q_k, P_{N\setminus\{j,k\}}) = \underline{x}$. Because $f(Q_k, P_{N\setminus k}) = y$, $r_1(P_j) = \underline{x} + 1$, and $r_2(P_j) = \underline{x}$, agent j manipulates at $(Q_k, P_{N\setminus k})$ via P'_j. This completes the proof of the lemma. ■

Remark 4.6.1 *By Lemma 4.6.3 and Lemma 4.6.4, it follows that f restricted to $\mathcal{S}^n$ is a PDMMR.*

Our next lemma establishes that f is uncompromising.[8] First, we introduce few notations that we use in the proof of the lemma. For $P_N \in \tilde{\mathcal{S}}^n$, let $\tilde{N}(P_N) = \{i \in N \mid P_i \notin \mathcal{S}\}$ be the set of agents who do not have single-peaked preferences at P_N. Moreover, for $0 \leq l \leq n$, let $\tilde{\mathcal{S}}^n_l = \{P_N \in \tilde{\mathcal{S}}^n \mid |\tilde{N}(P_N)| \leq l\}$ be the set of preference profiles where at most l agents have non-single-peaked preferences. Note that $\tilde{\mathcal{S}}^n_0 = \mathcal{S}^n$ and $\tilde{\mathcal{S}}^n_n = \tilde{\mathcal{S}}^n$.

Lemma 4.6.5 *The SCF f is uncompromising.*

[8]Since every SCF satisfying uncompromisingness is tops-only, Lemma 4.6.5 shows that a partially single-peaked domain is a tops-only domain. It can be easily verified that partially single-peaked domains fail to satisfy the sufficient conditions for a domain to be tops-only identified in [20] and [22].

Proof: Since $\tilde{\mathcal{S}}^n_0 = \mathcal{S}^n$, f restricted to $\tilde{\mathcal{S}}^n_0$ is uncompromising. Suppose f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising for some $k < n$. We show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is uncompromising. It is enough to show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only. To see this, note that if f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only, then f is uniquely determined on $\tilde{\mathcal{S}}^n_{k+1}$ by its outcomes on $\mathcal{S}^n$. Therefore, since f restricted to $\mathcal{S}^n$ is uncompromising, f is uncompromising on $\tilde{\mathcal{S}}^n_{k+1}$.

Take $P_N \in \tilde{\mathcal{S}}^n_{k+1}$ and $j \in \tilde{N}(P_N)$. Let $\hat{P}_j \in \mathcal{S}$ be such that $r_1(\hat{P}_j) = r_1(P_j)$. Then, P_N and $(\hat{P}_j, P_{N\setminus j})$ are tops-equivalent and $(\hat{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$. It is sufficient to show that $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$. Assume for contradiction that $f(P_N) \neq f(\hat{P}_j, P_{N\setminus j})$. Assume, without loss of generality, that the partial dictator of f restricted to $\mathcal{S}^n$ is agent 1. Then, by the induction hypothesis, agent 1 is the partial dictator of f restricted to $\tilde{\mathcal{S}}^n_k$, i.e., for all $P_N \in \tilde{\mathcal{S}}^n_k$, if $r_1(P_1) \in [a, \underline{x})$ then $f(P_N) \in [a, \underline{x}]$, if $r_1(P_1) \in (\overline{x}, b]$ then $f(P_N) \in [\overline{x}, b]$, and if $r_1(P_1) \in [\underline{x}, \overline{x}]$ then $f(P_N) = r_1(P_1)$. We distinguish two cases based on the position of the top-ranked alternative of agent 1.

Case 1. Suppose $r_1(P_1) \in [a, \underline{x}) \cup (\overline{x}, b]$.

We consider the case where $r_1(P_1) \in [a, \underline{x})$, the proof for the case where $r_1(P_1) \in (\overline{x}, b]$ follows from symmetric arguments. Since $r_1(P_1) \in [a, \underline{x})$, we have $f(\hat{P}_j, P_{N\setminus j}) \in [a, \underline{x}]$. Because $\hat{P}_j$ is single-peaked, if $f(\hat{P}_j, P_{N\setminus j}) < f(P_N) \leq r_1(\hat{P}_j)$ or $r_1(\hat{P}_j) \leq f(P_N) < f(\hat{P}_j, P_{N\setminus j})$, then agent j manipulates at $(\hat{P}_j, P_{N\setminus j})$ via P_j. Moreover, since $f(\hat{P}_j, P_{N\setminus j}) \in [a, \underline{x}]$, if $f(P_N) < f(\hat{P}_j, P_{N\setminus j}) \leq r_1(\hat{P}_j)$ or $r_1(P_j) \leq f(\hat{P}_j, P_{N\setminus j}) < f(P_N)$, then by the definition of a partially single-peaked domain, agent j manipulates at $(P_j, P_{N\setminus j})$ via $\hat{P}_j$. Now, suppose $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < f(P_N)$. Let $\bar{P}_j \in \mathcal{S}$ be such that $r_1(\bar{P}_j) = f(P_N)$. Since f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising and $f(\hat{P}_j, P_{N\setminus j}) < r_1(\hat{P}_j) < r_1(\bar{P}_j)$, we have $f(\bar{P}_j, P_{N\setminus j}) = f(\hat{P}_j, P_{N\setminus j})$. Because $r_1(\bar{P}_j) = f(P_N)$, it follows that agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Using a similar argument, it can be shown that $f(P_N) < r_1(\hat{P}_j) < f(\hat{P}_j, P_{N\setminus j})$ leads to a manipulation by agent j. Therefore, $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [a, \underline{x})$. This completes the proof of the lemma for Case 1.

Case 2. Suppose $r_1(P_1) \in [\underline{x}, \overline{x}]$.

Since agent 1 is the partial dictator, $f(\hat{P}_j, P_{N\setminus j}) = r_1(P_1)$. Consider $\bar{P}_j \in \mathcal{S}$ such that $r_1(\bar{P}_j) = f(P_N)$. Since $(\bar{P}_j, P_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$, by the induction hypothesis, we have $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1)$. Because $r_1(\bar{P}_j) = f(P_N)$ and $f(\bar{P}_j, P_{N\setminus j}) = r_1(P_1) \neq f(P_N)$, agent j manipulates at $(\bar{P}_j, P_{N\setminus j})$ via P_j. Therefore, $f(P_N) = f(\hat{P}_j, P_{N\setminus j})$ when $r_1(P_1) \in [\underline{x}, \overline{x}]$. This completes the proof of the lemma for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of the lemma by induction. ■

Now, we complete the proof of the only-if part of Theorem 4.3.1. Since f is uncompromising on $\tilde{\mathcal{S}}^n$ and uncompromisingness implies tops-onlyness, the fact that f restricted to $\mathcal{S}^n$ is a min-max rule with parameters $(\beta_S)_{S\subseteq N}$ satisfying the properties as stated in Lemma 4.6.3 and Lemma 4.6.4 implies that f is a PDMMR on $\tilde{\mathcal{S}}^n$. ■

5

On Strategy-proofness and Uncompromisingness

5.1 Introduction

5.1.1 Background of the problem

The single-peaked restriction, introduced in [15], is by far the most enduring theme in the literature on domain restrictions in strategy-proof social choice. Loosely put, a preference over a set of alternatives, given a prior ordering over this set, is called *single-peaked* if it decreases as one moves away (according to the prior order) from its top-ranked alternative. Such preference restrictions arise naturally in many economic and political applications such as in the models of locating a firm in a unidimensional spatial market ([46]), setting the rate of carbon dioxide emissions ([15]), and setting the level of public expenditure ([69]).

[54] and [86] characterize the unanimous and strategy-proof social choice functions (SCF) on these domains as *min-max rules*. [86] shows that min-max rules satisfy a well-known property called uncompromisingness ([17]). An SCF satisfies *uncompromisingness* if no agent can take an extreme position to influence the social outcome to their advantage. In Chapter 3, it is shown that top-connected single-peaked domains are the only domains where the set of unanimous and strategy-proof SCFs coincide with that of *all* min-max rules. In other words, top-connected single-peaked domains are the only domains on which unanimity and strategy-proofness are equivalent to uncompromisingness.

5.1.2 Our motivation

The characterization result in [54] and [86] makes the implicit assumption that the underlying domain is *maximal single-peaked*, i.e., it admits *all* single-peaked preferences. The maximality assumption in [54] and [86] is quite unrealistic as is seen in the models of voting ([84], [5]), taxation and redistribution ([38]), determining the levels of income redistribution ([44], [79]), and measuring tax reforms in the presence of horizontal inequity ([45]). Chapter 3 relaxes this maximality assumption by considering top-connected single-peaked domains. However, the following examples suggest that the domain restrictions in several practical scenarios are *not* top-connected single-peaked.

(i) **Directional theories of issue voting**: [64], [66], and [65] consider models of electoral competition where voters react to policies or ideologies in a symbolic way. Such a reaction will have two components: (i) direction, and (ii) intensity. For example, consider a situation where the government is setting the level of public expenditure in the education sector. Let us represent the intensity of such an expenditure using integers between -5 to 5. Negative (or positive) values represent decreasing (or increasing) the existing level of public expenditure whereas the integer zero represents the neutral position, i.e., maintain the status quo level of expenditure. In such a setting, it is expected that if an individual's most preferred level of expenditure is negative then he would prefer a level of expenditure to its left to one to its right. Such preferences are known as *left single-peaked preferences* ([61]) where every alternative to the left of its top-ranked alternative is preferred to the ones to the right. Similarly, one can introduce the notion of *right single-peaked preferences*. Thus, the relevant domain restriction in this example would be that the preferences are left single-peaked when the top-ranked alternative is a negative value, they are right single-peaked when the top-ranked alternative is a positive value, and they are single-peaked when the top-ranked alternative is zero. It is straightforward to see that such domains are neither single-crossing nor top-connected single-peaked.

(ii) **One-dimensional Euclidean domains**: [26] introduces the notion of one-dimensional Euclidean domains. They arise in situations where individuals in a society collectively choose the location of a new facility, such as a bus stop or a library, and want it as close to their own locations as possible. For example, consider the situation where a social planner is proposing to locate a bus stop. In such situations, it is natural that an individual will prefer a location to another if the physical distance of the former is lesser than the latter. Thus, the crucial property of the preferences that arise here is that they are determined by the Euclidean distance. Clearly, top-connectedness is not guaranteed in such situations.

In view of this, our primary motivation is to provide a general characterization of the unanimous and strategy-proof SCFs on arbitrary single-peaked domains.

5.1.3 Our contribution

First, we show that an SCF is unanimous and strategy-proof on an arbitrary single-peaked domain if and only if it is weakly uncompromising. Weak uncompromisingness implies that whenever an agent's top-ranked alternative moves closer to the outcome, the outcome does not change. Moreover, if an agent moves his top-ranked alternative away from the outcome, the outcome can change only in a restricted way. We show that if a domain is top-connected, then weak uncompromisingness boils down to uncompromisingness, and consequently, the set of unanimous and strategy-proof SCFs coincide with that of min-max rules.

Next, we provide a parametric characterization of the unanimous and strategy-proof SCFs. We observe that parametrically characterizing strategy-proof rules under the requirement of unanimity is a hard problem as it involves a huge set of parameters. Hence, we restrict our attention to *anonymous* SCFs. A social choice function is called *anonymous* if it is outcome equivalent at any two anonymous profiles. A pair of preference profiles is called *anonymous* if one profile can be obtained from the other by permuting the set of individuals. We introduce a class of SCFs, called *sequentially median rules*, and show that these rules are unanimous, anonymous and strategy-proof on arbitrary single-peaked domains. Further, in a setting with atmost three players, we show that sequentially median rules characterize the set of unanimous, anonymous, and strategy-proof SCFs on arbitrary single-peaked domains.

Lastly, by means of examples, we provide a general algorithm to construct unanimous, anonymous and strategy-proof SCFs using weak uncompromisingness. We also observe that the algorithm can be extended to the case of non-anonymous SCFs in a straightforward manner.

5.1.4 Remainder

The rest of the chapter is organized as follows. We describe the usual social choice framework in Section 5.2. Section 5.3 studies the unanimous and strategy-proof SCFs on single-peaked domains and Section 5.5 provides a parametric characterization of such anonymous SCFs. Section 5.6 provides a general algorithm to construct unanimous and strategy-proof rules using the notion of weak uncompromisingness and Section 5.7 concludes the chapter. All the omitted proofs are collected in the Appendix.

5.2 Preliminaries

Let $N = \{1, ..., n\}$ be a set of at least two agents, who collectively choose an element from a finite set $A = \{a, a+1, \ldots, b-1, b\}$ of at least three alternatives, where a is an integer. For $x, y \in A$, we define the intervals $[x, y] = \{z \in A \mid x \leq z \leq y \text{ or } y \leq z \leq x\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *preference* P over A is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on A. The *upper-contour set* (*lower-contour set*) of P at an alternative $x \in A$, denoted by $U(P, x)$ $(L(P, x))$, is given by $U(P, x) = \{y \in A \mid yPx\} \cup \{x\}$ $(L(P, x) = \{y \in A \mid xPy\} \cup \{x\})$. We denote by $\mathbb{L}(A)$ the set of all preferences over A. An alternative $x \in A$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(A)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k-1$. A domain of admissible preferences, denoted by $\mathcal{D}$, is a subset of $\mathbb{L}(A)$. For $B \subseteq A$ and a domain $\mathcal{D}$, $\mathcal{D}^B = \{P \in \mathcal{D} \mid r_1(P) \in B\}$. An element $P_N = (P_1, \ldots, P_n) \in \mathcal{D}^n$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \{x \in A \mid r_1(P_i) = x \text{ for some } i \in N\}$. A domain $\mathcal{D}$ of preferences is *regular* if for all $x \in X$, there exists a preference $P \in \mathcal{D}$ such that $r_1(P) = x$. All the domains we consider in this chapter are assumed to be regular.

Definition 5.2.1 *A social choice function (SCF) f on $\mathcal{D}^n$ is a mapping $f : \mathcal{D}^n \to A$.*

Definition 5.2.2 *An SCF $f : \mathcal{D}^n \to A$ is unanimous if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in A$, we have $f(P_N) = x$.*

Definition 5.2.3 *An SCF $f : \mathcal{D}^n \to A$ is manipulable if there exists $i \in N$, $P_N \in \mathcal{D}^n$, and $P_i^{'} \in \mathcal{D}$ such that $f(P_i^{'}, P_{N \setminus i}) P_i f(P_N)$. An SCF f is strategy-proof if it is not manipulable.*

Definition 5.2.4 *An SCF $f : \mathcal{D}^n \to A$ is called dictatorial if there exists $i \in N$ such that for all $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.*

Definition 5.2.5 *A domain $\mathcal{D}$ is called dictatorial if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to A$ is dictatorial.*

Definition 5.2.6 *Two preference profiles P_N, P'_N are called tops-equivalent if $r_1(P_i) = r_1(P'_i)$ for all agents $i \in N$.*

Definition 5.2.7 *An SCF $f : \mathcal{D}^n \to A$ is called tops-only if for any two tops-equivalent $P_N, P'_N \in \mathcal{D}^n$, $f(P_N) = f(P'_N)$.*

Definition 5.2.8 *A domain $\mathcal{D}$ is called tops-only if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to A$ is tops-only.*

5.3 SCFs on arbitrary single-peaked domains

In this section, we provide a characterization of the SCFs on arbitrary single-peaked domains.

Definition 5.3.1 *A preference $P \in \mathbb{L}(A)$ is called single-peaked if for all $x, y \in A$, $[x < y \leq r_1(P)$ or $r_1(P) \leq y < x]$ implies yPx. A domain $\mathcal{S}$ is called a single-peaked domain if each preference in it is single-peaked, and a domain $\bar{\mathcal{S}}$ is called a maximal single-peaked domain if it contains all single-peaked preferences.*

Definition 5.3.2 *Let $\mathcal{S}$ be a single-peaked domain. An SCF $f : \mathcal{S}^n \to A$ satisfies the Pareto property if for all $P_N \in \mathcal{S}^n$, $\min(\tau(P_N)) \leq f(P_N) \leq \max(\tau(P_N))$.*

Definition 5.3.3 *Let $\mathcal{S}$ be a single-peaked domain. Let $\beta = (\beta_S)_{S \subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in A$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{S}^n \to A$ is called a min-max rule with respect to β if*

$$f^\beta(P_N) = \min_{S \subseteq N}\{\max_{i \in S}\{r_1(P_i), \beta_S\}\}.$$

Next, we introduce the notion of uncompromisingness.

Definition 5.3.4 *Let $\mathcal{S}$ be a single-peaked domain. An SCF $f : \mathcal{S}^n \to A$ is called uncompromising if for all $P_N \in \mathcal{S}^n$, all $i \in N$, and all $P'_i \in \mathcal{S}$:*

(i) if $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$, then $f(P_N) = f(P'_i, P_{-i})$, and

(ii) if $f(P_N) < r_1(P_i)$ and $f(P_N) \leq r_1(P'_i)$, then $f(P_N) = f(P'_i, P_{-i})$.

Remark 5.3.1 *Every min-max rule is uncompromising.*[1]

Remark 5.3.2 *If an SCF satisfies uncompromisingness, then by definition, it is tops-only.*

Now, we introduce the notion of weak uncompromisingness. We begin with defining the notion of left-right interval.

Definition 5.3.5 *Let $\mathcal{S}$ be a single-peaked domain. Then, an interval $[x, y]$ is called a left-right interval on $\mathcal{S}$ with cut-off $z \in [x, y]$ if for all $P \in \mathcal{S}^{[x,z]}$, we have $xP(z+1)$ and for all $P \in \mathcal{S}^{(z,y]}$, we have $yP(z-1)$.*

Definition 5.3.6 *Let $\mathcal{S}$ be a single-peaked domain. Then an SCF $f : \mathcal{S}^n \to A$ satisfies weak uncompromisingness with respect to $\mathcal{S}$ if f is unanimous and for all $P_N \in \mathcal{S}^n$, all $i \in N$, and all $P'_i \in \mathcal{S}$ with $|r_1(P_i) - r_1(P'_i)| \leq 1$*

(i) if $r_1(P'_i) \in [r_1(P_i), f(P_N)]$, then $f(P'_i, P_{-i}) = f(P_N)$, and

(ii) if $r_1(P_i) \in [f(P_N), r_1(P'_i)]$ and $f(P_N) \neq f(P'_i, P_{-i})$, then $[f(P_N), f(P'_i, P_{-i})]$ is a left-right interval on $\mathcal{S}$ with cut-off $r_1(P_i)$.

Remark 5.3.3 *Note that if an SCF satisfies weak uncompromisingness, then by definition, it is tops-only.*

5.4 Results

Theorem 5.4.1 *Let $\mathcal{S}$ be a single-peaked domain. Then, an SCF $f : \mathcal{S}^n \to A$ is unanimous and strategy-proof if and only if it is a weak uncompromising rule.*

The proof of this theorem is relegated to Appendix 5.8.1.

[1] For details, see [86].

5.5 A parametric characterization

In this section, we show how Theorem 5.4.1 can be used to construct a parametric (functional form) characterization of the unanimous and strategy-proof SCFs. We provide this characterization for three agents, it will be clear from the presentation of the same for more number of agents will be complicated.

Definition 5.5.1 (Sequentially median parameters) *Let $\mathcal{S}$ be a single-peaked domain. Then, a collection of parameters $\beta_1, \ldots, \beta_k$ with $a = \beta_1 < \ldots < \beta_k = b$ is called sequentially median parameters on $\mathcal{S}$ if for all $j = 1, \ldots k-1$, there exists $y_j \in (\beta_j, \beta_{j+1})$ such that for all $P \in \mathcal{S}^{(\beta_j, y_j]}$, we have $\beta_j Pr_2(P)$ and for all $P \in \mathcal{S}^{(y_j, \beta_j)}$, we have $\beta_{j+1} Pr_2(P)$.*

Note that for a collection of sequentially median parameters $\beta_1, \ldots, \beta_k$, for all $y \in (\beta_j, \beta_{j+1})$ and for all $P, P' \in \mathcal{S}^y$, we have $\beta_j P \beta_{j+1}$ if and only if $\beta_j P' \beta_{j+1}$. In view of this, for such alternative y, we write $\beta_j y \beta_{j+1}$ (or $\beta_j y \beta_{j+1}$) to mean $\beta_j P \beta_{j+1}$ (or $\beta_{j+1} y \beta_j$) for all $P \in \mathcal{S}^y$.

For a collection of median parameters $\beta^*_1, \ldots, \beta^*_{n-1}$, we denote by f^{β^*}, the median rule with respect to $\beta^*_1, \ldots, \beta^*_{n-1}$.

Definition 5.5.2 (Sequentially median rules) *An SCF $f : \mathcal{S}^n \to X$ is called sequentially median if there are sequentially median parameters $\beta_1, \ldots, \beta_k$ and median parameters $\beta^*_1, \ldots, \beta^*_{n-1}$ where $\beta^*_1, \ldots, \beta^*_{n-1} \in \{\beta_1, \ldots, \beta_k\}$ such that*

$$f(P_N) = \begin{cases} f^{\beta^*}(P_N) \text{ if } f^{\beta^*}(P_N) \in \{\beta_1, \ldots, \beta_k\}, \\ \min\{\beta_{j+1}, \max(\tau(P_N))\} \text{ if } f^{\beta^*}(P_N) \in (\beta_j, \beta_{j+1}) \text{ and } \beta_{j+1} f^{\beta^*}(P_N) \beta_j, \\ \max\{\beta_j, \min(\tau(P_N))\} \text{ if } f^{\beta^*}(P_N) \in (\beta_j, \beta_{j+1}) \text{ and } \beta_j f^{\beta^*}(P_N) \beta_{j+1}. \end{cases}$$

Lemma 5.5.1 *Let $f : \mathcal{S}^n \to X$ be a sequentially median rule with respect to sequentially median parameters $\beta_1, \ldots, \beta_k$ and $f^{\beta^*} : \mathcal{S}^n \to X$ be a median rule with respect to median parameters $\beta^*_1, \ldots, \beta^*_{n-1}$For all $P_N, P'_N \in \mathcal{S}^n$, $f^{\beta^*}(P_N) \geq f^{\beta^*}(P'_N)$ implies $f(P_N) \geq f(P'_N)$.*

The proof of this lemma is left to the reader.

Theorem 5.5.1 *Let $\mathcal{S}$ be a single-peaked domain. Then, every sequentially median rule $f : \mathcal{S}^n \to X$ is unanimous, anonymous and strategy-proof.*

The proof of this theorem is relegated to Appendix 5.8.2.

Theorem 5.5.2 *Let $\mathcal{S}$ be a single-peaked domain and let $n \leq 3$. Then, an SCF $f : \mathcal{S}^n \to X$ is unanimous, anonymous and strategy-proof if and only if it is a sequentially median rule.*

The proof of this theorem is relegated to Appendix 5.8.3.

5.6 Discussion

In this section, we provide two examples to illustrate how the weak uncompromisingness property helps in constructing a unanimous and strategy-proof SCF. For both these examples, anonymity is assumed for simplicity. One can use the same procedure to construct SCFs that are not anonymous.

5.6.1 Left single-peaked domain

In this subsection, we first present a formal definition of left (right) single-peaked domains and construct a unanimous, anonymous and strategy-proof SCF on such domains using weak uncompromisingness.

Definition 5.6.1 *A single-peaked preference P is called left single-peaked (right single-peaked) if for all $u < r_1(P) < v$, we have uPv (vPu). A set of single-peaked preferences $\mathcal{S}$ is called left single-peaked (right single-peaked) if it contains all left single-peaked (all right single-peaked) preferences.*

Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$. Suppose that the admissible domain of preferences is left single-peaked over these alternatives.

In Table 5.6.1, we construct an anonymous SCF that satisfies weak uncompromisingness. In each row of Table 5.6.1, one agent's top-ranked alternative is moved from a_2 to a_8. In the first row, agent 4's top-ranked alternative is moved from a_2 to a_8. Note that the outcome 'jumps' from a_1 to a_3 while the top-ranked alternative of agent 4 moves from a_2 to a_3. Further, note that the interval $[a_1, a_3]$ is a left-right interval with cut-off a_2 as required by weak uncompromisingness. Similarly, in the second row, the outcomes jumps from a_3 to a_5 while the top-ranked alternative of agent 3 moves from a_4 to a_5. Also, the interval $[a_3, a_5]$ is a left-right interval with cut-off a_4. In this fashion, it can be checked that this SCF satisfies weak uncompromisingness, and hence, by Theorem 5.4.1, it is strategy-proof.

$(r_1(P_i))_{i\in N}$	(a_1, a_1, a_1, a_2)	(a_1, a_1, a_1, a_3)	(a_1, a_1, a_1, a_4)	(a_1, a_1, a_1, a_5)	(a_1, a_1, a_1, a_6)	(a_1, a_1, a_1, a_7)	(a_1, a_1, a_1, a_8)
$f(P_N)$	a_1	a_3	a_3	a_3	a_3	a_3	a_3
$(r_1(P_i))_{i\in N}$	(a_1, a_1, a_2, a_8)	(a_1, a_1, a_3, a_8)	(a_1, a_1, a_4, a_8)	(a_1, a_1, a_5, a_8)	(a_1, a_1, a_6, a_8)	(a_1, a_1, a_7, a_8)	(a_1, a_1, a_8, a_8)
$f(P_N)$	a_3	a_3	a_3	a_5	a_5	a_7	a_7
$(r_1(P_i))_{i\in N}$	(a_1, a_2, a_8, a_8)	(a_1, a_3, a_8, a_8)	(a_1, a_4, a_8, a_8)	(a_1, a_5, a_8, a_8)	(a_1, a_6, a_8, a_8)	(a_1, a_7, a_8, a_8)	(a_1, a_8, a_8, a_8)
$f(P_N)$	a_7	a_7	a_7	a_7	a_7	a_7	a_8
$(r_1(P_i))_{i\in N}$	(a_2, a_8, a_8, a_8)	(a_3, a_8, a_8, a_8)	(a_4, a_8, a_8, a_8)	(a_5, a_8, a_8, a_8)	(a_6, a_8, a_8, a_8)	(a_7, a_8, a_8, a_8)	(a_8, a_8, a_8, a_8)
$f(P_N)$	a_8	a_8	a_8	a_8	a_8	a_8	a_8

Table 5.6.1 Construction of unanimous, anonymous, and strategy-proof SCFs using weak uncompromisingness

5.6.2 Euclidean domains

In this subsection, we first present a formal definition of Euclidean domains and construct a unanimous, anonymous and strategy-proof SCF on such domains using weak uncompromisingness.

For ease of presentation, we assume that the set of alternatives are (finitely many) elements of the interval $[0, 1]$.[2] Let $0 = a_1 < \ldots < a_m = 1$ be the alternatives. Assume that the individuals are located at arbitrary locations in $[0, 1]$ and derive their preferences using Euclidean distances of the alternatives from their own location.

[2] With abuse of notation, we denote by $[0, 1]$ the set of real numbers in-between 0 and 1.

We call such preferences Euclidean. Below, we provide formal definitions of these.

Definition 5.6.2 *A preference P is called Euclidean if there is $x \in [0,1]$, called the location of P, such that for all alternatives $a, b \in A$, $|x-a| < |x-b|$ implies aPb. A domain is called Euclidean if it contains all Euclidean preferences.*

Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}$ with the distance function as given in Figure 5.6.1. Suppose that the admissible domain of preferences is Euclidean over these alternatives.

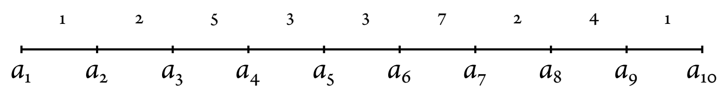


Figure 5.6.1 Distance function of a Euclidean domain

In Table 5.6.1, we construct an anonymous SCF that satisfies weak uncompromisingness. In each row of Table 5.6.1, one agent's top-ranked alternative is moved from a_2 to a_8. In the first row, agent 4's top-ranked alternative is moved from a_2 to a_8. Note that the outcome 'jumps' from a_1 to a_5 while the top-ranked alternative of agent 4 moves from a_2 to a_3. Further, note that the interval $[a_1, a_5]$ is a left-right interval with cut-off a_2 as required by weak uncompromisingness. Similarly, in the second row, the outcomes jumps from a_5 to a_7 while the top-ranked alternative of agent 3 moves from a_5 to a_6. Also, the interval $[a_5, a_7]$ is a left-right interval with cut-off a_5. In this fashion, it can be checked that this SCF satisfies weak uncompromisingness, and hence, by Theorem 5.4.1, it is strategy-proof.

5.7 Conclusion

This chapter studies the structure of the unanimous and strategy-proof SCFs on arbitrary single-peaked domains. It characterizes such SCFs by means of weak uncompromisingness. Further, it provides a parametric characterization of such SCFs for the case of 2 and 3 agents under the assumption of anonymity.

$(r_1(P_i))_{i\in N}$	(a_1,a_1,a_1,a_2)	(a_1,a_1,a_1,a_3)	(a_1,a_1,a_1,a_4)	(a_1,a_1,a_1,a_5)	(a_1,a_1,a_1,a_6)	(a_1,a_1,a_1,a_7)	(a_1,a_1,a_1,a_8)	(a_1,a_1,a_1,a_9)	(a_1,a_1,a_1,a_{10})
$f(P_N)$	a_1	a_5	a_5	a_5	a_5	a_5	a_5	a_5	a_5
$(r_1(P_i))_{i\in N}$	(a_1,a_1,a_2,a_{10})	(a_1,a_1,a_3,a_{10})	(a_1,a_1,a_4,a_{10})	(a_1,a_1,a_5,a_{10})	(a_1,a_1,a_6,a_{10})	(a_1,a_1,a_7,a_{10})	(a_1,a_1,a_8,a_{10})	(a_1,a_1,a_9,a_{10})	(a_1,a_1,a_{10},a_{10})
$f(P_N)$	a_5	a_5	a_5	a_5	a_7	a_7	a_7	a_7	a_7
$(r_1(P_i))_{i\in N}$	(a_1,a_2,a_{10},a_{10})	(a_1,a_3,a_{10},a_{10})	(a_1,a_4,a_{10},a_{10})	(a_1,a_5,a_{10},a_{10})	(a_1,a_6,a_{10},a_{10})	(a_1,a_7,a_{10},a_{10})	(a_1,a_8,a_{10},a_{10})	(a_1,a_1,a_{10},a_{10})	(a_1,a_1,a_{10},a_{10})
$f(P_N)$	a_7	a_7	a_7	a_7	a_7	a_7	a_9	a_9	a_9
$(r_1(P_i))_{i\in N}$	$(a_2,a_{10},a_{10},a_{10})$	$(a_3,a_{10},a_{10},a_{10})$	$(a_4,a_{10},a_{10},a_{10})$	$(a_5,a_{10},a_{10},a_{10})$	$(a_6,a_{10},a_{10},a_{10})$	$(a_7,a_{10},a_{10},a_{10})$	$(a_8,a_{10},a_{10},a_{10})$	$(a_9,a_{10},a_{10},a_{10})$	$(a_{10},a_{10},a_{10},a_{10})$
$f(P_N)$	a_9	a_9	a_9	a_9	a_9	a_9	a_9	a_9	a_{10}

Table 5.6.2 Construction of unanimous, anonymous, and strategy-proof SCFs using weak uncompromisingness

5.8 Appendix

5.8.1 Proof of Theorem 5.4.1

Proof: (If part) f is unanimous by definition. To show that f is strategy-proof, assume for contradiction that f is manipulable. Thus there exists $P_N \in \mathcal{S}^n$, $P'_i \in \mathcal{S}$ such that $f(P'_i, P_{-i})P_i f(P_N)$. Without loss of generality we can assume $|r_1(P_i) - r_1(P'_i)| \leq 1$. Note that if $r_1(P'_i) \in [r_1(P_i), f(P_N)]$ then by Condition (i) in Definition 5.3.6, $f(P_N) = f(P'_i, P_{-i})$. This means $r_1(P'_i) \notin [r_1(P_i), f(P_N)]$. Without loss of generality we can assume $f(P_N) \leq r_1(P_i) < r_1(P'_i)$. But by Condition (ii) in Definition 5.3.6, this means $[f(P_N), f(P'_i, P_{-i})]$ is a left-right interval on $\mathcal{S}$ with cutoff $r_1(P_i)$. This, in particular, implies $f(P_N)P_i r_1(P'_i)$ and hence by single-peakedness $f(P_N)P_i f(P'_i, P_{-i})$.

(Only-if part) Let $f : \mathcal{S}^n \to A$ be unanimous and strategy-proof SCF. We show it is weak uncompromising. Since f is unanimous by assumption we proceed to show that f satisfies Condition (i) and (ii) in Definition 5.3.6.

Condition (i): Let $P_N \in \mathcal{S}^n$ and $P'_i \in \mathcal{S}$ such that $|r_1(P_i) - r_1(P'_i)| \leq 1$ and $r_1(P'_i) \in [r_1(P_i), f(P_N)]$. We show $f(P'_i, P_{-i}) = f(P_N)$. If $|r_1(P_i) - r_1(P'_i)| < 1$, i.e, $r_1(P_i) = r_1(P'_i)$ then, by tops-onlyness $f(P_N) = f(P'_i, P_{-i})$. So, we consider the case $|r_1(P_i) - r_1(P'_i)| = 1$. Assume for contradiction $f(P_N) \neq f(P'_i, P_{-i})$. By strategy-proofness $f(P'_i, P_{-i})P'_i f(P_N)$. Since $r_1(P'_i) \in [r_1(P_i), f(P_N)]$, if $f(P'_i, P_{-i}) \in [r_1(P_i), f(P_N))$ then, agent i manipulates at P_N via P'_i. On the other hand if $f(P_N) \in [r_1(P_i), f(P'_i, P_{-i}))$ then, agent i manipulates at (P'_i, P_{-i}) via P_i. Thus $r_1(P_i) \in [f(P_N),$

$f(P_i', P_{-i})]$. Without loss of generality we assume $f(P_i', P_{-i}) < f(P_N)$. Let $T = \{j \in N | r_1(P_j) \leq r_1(P_i')\}$. Consider a profile $\bar{P}_N$ such that $\bar{P}_j = P_i'$ if $j \in T$, otherwise $\bar{P}_j = P_j$. Note that since $r_1(P_i') < f(P_N)$, by group strategy-proofness and the Pareto property $f(\bar{P}_N) = f(P_N)$. But as $f(P_i', P_{-i})P_i'f(P_N)$, the set of agents in T will manipulate at $\bar{P}_N$ via (P_i', P_{-i}), a contradiction. Thus $f(P_N) = f(P_i', P_{-i})$.

Condition(ii): Note that if $f(P_N) = f(P_i', P_{-i})$ then there is nothing to show. So, without loss of generality, we assume $f(P_N) \leq r_1(P_i) < r_1(P_i') \leq f(P_i', P_{-i})$. Again if $f(P_N) = r_1(P_i) < r_1(P_i') = f(P_i', P_{-i})$ then $|f(P_N) - f(P_i', P_{-i})| = 1$ and by definition $[f(P_N) - f(P_i', P_{-i})]$ is a left-right interval on $\mathcal{S}$ w.r.t. $r_1(P_i)$. So we further assume $f(P_N) \leq r_1(P_i) < r_1(P_i') < f(P_i', P_{-i})$. Take $a \in [r_1(P_i'), f(P_i', P_{-i})]$. Then by Condition (i), $f(P^a, P_{-i}) = f(P_i', P_{-i})$. Now if $f(P_N) = r_1(P_i)$, then by strategy-proofness for all $P^a \in \mathcal{S}$ $f(P_i', P_{-i})P^a f(P_N)$, which shows $[f(P_N), f(P_i', P_{-i})]$ is a left right interval on $\mathcal{S}$ with cut-off $r_1(P_i)$. So, we assume $f(P_N) < r_1(P_i)$. It is enough to show that for all $a \in [r_1(P_i'), f(P_i', P_{-i})]$, and all $P^a \in \mathcal{S}$, $f(P_i', P_{-i})P^a r_1(P_i)$. Suppose not, and for some $a \in [r_1(P_i'), f(P_i', P_{-i})]$ and $P^a \in \mathcal{S}$, $r_1(P_i)P^a f(P_i', P_{-i})$. Let $T = \{j \in N | r_1(P_j) \leq r_1(P_i')\}$. Consider the profile $\bar{P}_N$ where $\bar{P}_j = P_i'$ if $j \in T$ and $\bar{P}_j = P_j$ if $j \notin T$. Then, by condition (i), $f(\bar{P}_N) = f(P_i', P_{-i})$ as $r_1(P_i') < f(P_i', P_{-i})$. Let $T' = \{j \in N | r_1(P_j) \leq r_1(P_i)\}$. Consider the profile $\hat{P}_N$ such that $\hat{P}_j = P_i$ if $j \in T'$ and $\hat{P}_j = P_j$ if $j \notin T'$.

Claim: $f(\hat{P}_N) = r_1(P_i)$.

Note that by the Pareto property $f(\hat{P}_N) \geq r_1(P_i)$, as by construction $r_1(\hat{P}_j) \geq r_1(P_i)$ for all $j \in N$. Since by strategy-proofness $f(P_N)P_i f(P_i', P_{-i})$, $f(\hat{P}_N) \neq f(P_i', P_{-i})$. If $f(\hat{P}_N) = f(P_i', P_{-i})$ then, agents in T' will manipulate at $\hat{P}_N$ via (P_N). Now suppose $f(\hat{P}_N) > r_1(P_i)$ then by condition (i) $f(\hat{P}_N) = f(\bar{P}_N)$ but that means $f(\hat{P}_N) = f(P_i', P_{-i})$ as $f(\bar{P}_N) = f(P_i', P_{-i})$. Thus $f(\hat{P}_N) = r_1(P_i)$, which completes the proof of the claim.

Let $T'' = \{j \in N | r_1(P_j) \leq a\}$, and $\tilde{P}_N$ be such that $\tilde{P}_j = P^a$ if $j \in T''$ and $\hat{P}_j = P_j$ if $j \notin T''$. Using a similar argument as in the case of $f(\bar{P}_N)$, we can show $f(\tilde{P}_N) = f(P_i', P_{-i})$. But since by our assumption $r_1(P_i)P^a f(P_i', P_{-i})$, agents in T'' will manipulate at $\tilde{P}_N$ via $\hat{P}_N$. This is a contradiction to group strategy-proofness and hence $f(P_i', P_{-i})P^a r_1(P_i)$. This completes the proof of condition (ii). ■

5.8.2 Proof of Theorem 5.5.1

Proof: Let $\mathcal{S}$ be a single-peaked domain and let $f : \mathcal{S}^n \to A$ be a sequentially median rule.

Anonymity of f follows from the definition. We show f is unanimous. Take a unanimous profile $P_N \in \mathcal{S}^n$. Let $\min \tau(P_N) = \max \tau(P_N) = x$. Since f^{β^*} is unanimous, $f^{\beta^*} = x$. If $x \in \{\beta_1, \dots, \beta_k\}$, then $f(P_N) = f^{\beta^*}(P_N) = x$. Suppose $f^{\beta^*}(P_N) \in (\beta_j, \beta_{j+1})$ for some $j = 1, \dots, k-1$. Then, $\min\{\max(\tau(P_N), \beta_j)\} = \max\{\min(\tau(P_N), \beta_{j+1})\} = x$, and hence $f(P_N) = x$. This shows f is unanimous.

Now, we show that f is strategy-proof. Take $P_N \in \mathcal{S}^n$, $i \in N$, and $P'_i \in \mathcal{S}$. Note that if $r_1(P_i) < f^{\beta^*}(P_N)$ and $r_1(P'_i) \leq f^{\beta^*}(P_N)$, then $f^{\beta^*}(P_N) = f^{\beta^*}(P'_i, P_{N\setminus i})$, and hence, by the definition of a sequentially median rule, $f(P_N) = f(P'_i, P_{N\setminus i})$. Similarly, if $r_1(P_i) > f^{\beta^*}(P_N)$ and $r_1(P'_i) \geq f^{\beta^*}(P_N)$, then $f(P_N) = f(P'_i, P_{N\setminus i})$.

So, suppose $r_1(P_i) < f^{\beta^*}(P_N)$ and $r_1(P_i) > f^{\beta^*}(P_N)$. Then, by the property of a median rule, $f^{\beta^*}(P'_i, P_{N\setminus i}) \geq f^{\beta^*}(P_N)$. If $f^{\beta^*}(P'_i, P_{N\setminus i}) = f^{\beta^*}(P_N)$, then by the definition of a sequentially median rule, $f(P_N) = f(P'_i, P_{N\setminus i})$. So, assume $f^{\beta^*}(P'_i, P_{N\setminus i}) > f^{\beta^*}(P_N)$ and $f(P_N) > f(P'_i, P_{N\setminus i})$. Suppose $j < k$ is such that $f^{\beta^*}(P_N) \in [\beta_j, \beta_{j+1})$. This, together with the fact that $f(P_N) > f(P'_i, P_{N\setminus i})$, implies either $f(P_N) = \beta_j$ and $f(P'_i, P_{N\setminus i}) \geq \beta_{j+1}$ or $f(P_N) = \beta_{j+1}$ and $f(P'_i, P_{N\setminus i}) \geq \beta_{j+2}$. If $f(P_N) = \beta_j$ and $f(P'_i, P_{N\setminus i}) \geq \beta_{j+1}$, then by the definition of a sequentially median rule, it must be that $\beta_j f^{\beta^*}(P_N)\beta_{j+1}$. Since $r_1(P_i) \leq f^{\beta^*}(P_N)$, by the property of sequential parameters, we have $\beta_j P_i x$ for all $x \geq \beta_{j+1}$, and hence i does not manipulate. On the other hand, if $f(P_N) = \beta_{j+1}$ and $f(P'_i, P_{N\setminus i}) \geq \beta_{j+2}$, then since $r_1(P_i) \leq f^{\beta^*}(P_N)$ and $f^{\beta^*}(P_N) < \beta_{j+1}$, we have $\beta_{j+1} P_i x$ for all $x \geq \beta_{j+1}$, and hence i does not manipulate. It can be shown by similar logic that i cannot manipulate when $r_1(P_i) > f^{\beta^*}(P_N)$ and $r_1(P_i) < f^{\beta^*}(P_N)$.

Suppose $r_1(P_i) = f^{\beta^*}(P_N)$. If $f^{\beta^*}(P_N) \in \{\beta_1, \dots, \beta_k\}$ then $f(P_N) = f^{\beta^*}(P_N) = r_1(P_i)$, and hence, i cannot manipulate. So, assume $f^{\beta^*}(P_N) \in (\beta_j, \beta_{j+1})$ for some $j < k$. Assume without loss of generality that $f(P_N) = \beta_j$. Then, by the definition of a sequentially median rule, $\beta_j P_i \beta_{j+1}$. Again, by the definition of a sequentially median rule, $f(P'_i, P_{N\setminus i}) \notin (\beta_j, \beta_{j+1})$. Also, by the single-peakedness of P_i, $\beta_i P_i x$ for all $x \notin [\beta_j, \beta_{j+1})$. This means i cannot manipulate at P_N. This completes the proof of the theorem. ■

5.8.3 Proof of Theorem 5.5.2

Proof: The proof of the if-part follows from Theorem 5.5.1. We proceed to prove the only-if part of the theorem. Let $f : \mathcal{S}^n \to A$ be a unanimous, anonymous, and strategy-proof SCF. By Theorem 5.4.1, f satisfies weak uncompromisingness. Define $\beta^*_0 = a$, $\beta^*_n = b$, and for all $k \in \{1, \ldots, n-1\}$, define $\beta^*_k = f(P_N)$ where P_N is such that $r_1(P_i) = b$ for all $i = 1, \ldots, k$ and $r_1(P_i) = a$ for all $i = k+1, \ldots, n$.

Take $k \in \{0, \ldots, n-1\}$ such that $\beta^*_k < \beta^*_{k+1}$. Let us denote β^*_k by β_{k1}. We construct β_{k2} as follows. Consider a(ny) profile where the top-ranked alternatives of agents $1, \ldots, k$ is b, of agents $k+1, \ldots, n-1$ is a, and of agent n is β_{k1}. We first argue that the outcome at such a profile must be β_{k1}. To see that, consider the profile where the top-ranked alternatives of all agents except n remain the same as above and that of n is a. By the definition of β_{k1}, the outcome at this profile is β_{k1}. Since at the former profile, the top-ranked alternative of agent n is β_{k1}, by a straight-forward application of strategy-proofness it follows that the outcome at that profile must be β_{k1}. Now, keeping the top-ranked alternatives of all agents except n unchanged, we keep moving that of agent n 'continuously' (i.e., each time one step) towards the right direction from β_{k1}. We do this till the outcome changes from β_{k1} for the first time. Since $\beta^*_{k+1} > \beta_{k1}$, outcome must change at some time point by this procedure. Note that by weak uncompromisingness, this changed outcome must lie on the right of β_{k1}. We define β_{k2} as the new outcome whenever this first-time-change happens. We follow this method recursively to define β_{k3}, β_{k4}, and so on. Below, we describe this method formally.

Let $P_{N\setminus n}$ be such that $r_1(P_i) = b$ for all $i = 1, \ldots, k$ and $r_1(P_i) = a$ for all $i = k+1, \ldots, n-1$. Suppose P^2_n is such that $f(P_{N\setminus n}, P_n) = \beta_{k1}$ for all P_n with $r_1(P_n) \in [\beta_{k1}, r_1(P^2_n) - 1]$ and $f(P_{N\setminus n}, P^2_n) \neq \beta_{k1}$. As we have argued in the preceding paragraph, weak uncompromisingness implies $\beta_{k1} < r_1(P^2_n) \leq f(P_{N\setminus n}, P^2_n)$. Define $\beta_{k2} = f(P_{N\setminus n}, P^2_n)$. Having defined β_{kl}, if $\beta_{kl} < \beta^*_{k+1}$, then define $\beta_{k(l+1)} =$

$f(P_{N\setminus n}, P_n^{l+1})$, where P_n^{l+1} is such that $f(P_{N\setminus n}, P_n) = \beta_{kl}$ for all P_n with $r_1(P_n) \in [\beta_{kl}, r_1(P_n^{l+1}) - 1]$ and $f(P_{N\setminus n}, P_n^{l+1}) > \beta_{kl}$. As we have argued before, $\beta_{kl} < \beta_{k(l+1)}$. Since $f(P_{N\setminus n}, P_n) = \beta^*_{k+1}$ for all P_n such that $r_1(P_n) \geq \beta^*_{k+1}$, it follows that $\beta_{k(l+1)} \leq \beta^*_{k+1}$. Moreover, for the same reason, there must be l_k such that $\beta_{l_k+1} = \beta^*_{k+1}$. Thus, we obtain a collection of alternatives $(\beta_{k1}, \beta_{k2}, \ldots, \beta_{kl_k})$.

Following this procedure for each $k \in \{0, \ldots, n-1\}$, we obtain the following collection of alternatives $(\beta_{01}, \ldots, \beta_{0l_0}, \ldots, \beta_{k1}, \ldots, \beta_{kl_k}, \ldots, \beta_{(n-1)1}, \ldots, \beta_{(n-1)l_{n-1}}, \beta_n)$. By construction, these collection of alternatives satisfy the properties of sequentially median parameters. In what follows, we show that the SCF f is a sequentially median rule with respect to these parameters.

Take $P_N \in \mathcal{S}^n$. Let f^{β^*} be the median rule with respect to the parameters $\beta^*_0, \beta^*_1, \ldots, \beta^*_n$. Suppose $f^{\beta^*}(P_N) = \beta_{kl}$ for some $k = 0, 1, \ldots, n-1$ and some $l = 1, \ldots, l_k$. By the definition of the median rule, this means $\beta_{kl} = \text{median}\{r_1(P_1), \ldots, r_1(P_n), \beta^*_0, \ldots, \beta^*_n\}$. Suppose $l = 1$. That is, $\beta_{kl} = \beta^*_k$. Then, by the definition of the median rule, $f^{\beta^*}(P_N) = \beta^*_k$ implies that P_N is such that $r_1(P_i) = b$ for all $i = 1, \ldots, k$ and $r_1(P_i) = a$ for all $i = k+1, \ldots, n$. Now, by weak uncompromisingness, $f(P_N) = \beta^*_k$ for all $P_N \in \mathcal{S}^n$ such that $r_1(P_i) \geq \beta^*_k$ for all $i = 1, \ldots, k$ and $r_1(P_i) \leq \beta^*_k$ for all $i = k+1, \ldots, n$. Thus, for all profiles P_N such that $\text{median}\{r_1(P_1), \ldots, r_1(P_n), \beta^*_0, \ldots, \beta^*_n\} = \beta^*_k$, we have $f(P_N) = \beta^*_k$.

Now, suppose $l \neq 1$. Then, it must be that $\beta_{kl} = r_1(P_i)$ for some $i \in N$. By the definition of the median rule, this means $|\{i \in N \mid r_1(P_i) \geq \beta_{kl}\}| \geq k+1$ and $|\{i \in N \mid r_1(P_i) \geq \beta^*_{kl}\}| \geq n-k$. We show $f(P_N) = \beta_{kl}$ for all such profiles. Consider P_N such that $r_1(P_i) = b$ for all $i = 1, \ldots, k$, $r_1(P_i) = a$ for all $i = k+1, \ldots, n-1$, and $r_1(P_n) = \beta_{kl}$. By the construction of β_{kl}, $f(P_N) = \beta_{kl}$. Now, by weak uncompromisingness, we conclude that $f(P_N) = \beta_{kl}$ for all P_N as described above.

Suppose $f^{\beta^*}(P_N) \in (\beta_{kl}, \beta_{k(l+1)})$ for some $k = 0, \ldots, n-1$ and some $l = 1, \ldots, l_k - 1$. Assume without loss of generality, $\beta_{kl} f^{\beta^*}(P_N) \beta_{k(l+1)}$. This, in particular, means $f(P_N) = \beta_{kl}$. By the definition of f^{β^*}, $|\{i \in N \mid r_1(P_i) \geq \beta_{kl}\}| \geq k+1$ and $|\{i \in N \mid r_1(P_i) \leq \beta_{kl}\}| \geq n-k$. Since $n \leq 3$, it must be that there exists at most one agent with top-ranked alternative strictly less than $f^{\beta^*}(P_N)$ and at most one agent with top-ranked alternative strictly greater than $f^{\beta^*}(P_N)$. Consider the profile $\bar{P}_N \in \mathcal{S}^n$ where $r_1(\bar{P}_i) = b$ for all $i = 1, \ldots, k$, $r_1(\bar{P}_i) = a$ for all $i = k+1, \ldots, n-1$, and $r_1(\bar{P}_n) = f^{\beta^*}(P_N)$. Since $n \leq 3$, by weak uncompromisingness, $f(\bar{P}_N) = f(P_N)$. By the construction of β_{kl} and $\beta_{k(l+1)}$, $f(\bar{P}_N) = \beta_{kl}$. Combining, we have $f(P_N) = \beta_{kl}$ which completes the proof of the theorem. ■

6

Social Choice on Domains based on Trees

6.1 Introduction

6.1.1 Background and motivation

The incompatibility of strategy-proofness and non-dictatorship as shown in the Gibbard-Satterthwaite theorem ([43], [75]) has led researchers to weaken their assumption that the domain of admissible preferences is unrestricted. Among the domain restrictions, the single-peaked restriction has attracted special interest due to its practical appeal. The seminal works of [15], [54], and [86] consider single-peaked preferences when the set of alternatives are arranged over the real line. A rich literature has developed around the single-peaked restriction when the set of alternatives have much more general structural properties (see [9], [29], [76],

[55], and [56]).

[29] and [27] consider single-peaked preferences when alternatives are arranged on a *tree*.[1] When alternatives are arranged on a tree, a preference is called *single-peaked* if it falls when one moves farther away from its top-ranked alternative along any path. Both these articles consider *maximal* single-peaked domains on a tree. [29] shows that such domains guarantee the existence of a majority winner and [27] characterizes the non-manipulable SCFs on such domains as medians of dictatorial and constant rules. [55] and [56] consider single-peaked domains based on a general notion of betweenness. They consider *rich single-peaked domains* which we find very demanding in the context of trees. A single-peaked domain is *rich* if (i) it is top-connected, and (ii) for every path from a junction node to a leaf, there exists a preference which places the alternatives along the path consecutively at the top.[2] Hence, our main motivation is to study the structure of unanimous and strategy-proof SCFs on arbitrary single-peaked domains on trees so as to widen the applicability of this framework.

6.1.2 Our contribution

We introduce the notion of top-connected single-peaked domains. Loosely speaking, it requires that for any two adjacent alternatives, there exists a preference which places one at the top and the other at the second rank. We show that every unanimous and strategy-proof SCF on top-connected single-peaked domains on a tree satisfies the Pareto property and tops-onlyness. It is worth noting that the tops-onlyness result does not follow from the sufficient conditions provided in [20] for a domain to be tops-only. Further, we characterize the unanimous and strategy-proof SCFs on single-peaked domains that satisfy a stronger requirement called *strong connectedness*. A single-peaked domain is called *strongly connected* if for any two adjacent alternatives x and y, there exist two preferences such that (i) one places x at the top and y at the second rank, and the other places y at the top and x

[1] An undirected graph with the alternatives as nodes is a *tree* if there is a *unique path* connecting any two alternatives.

[2] A node is called a *junction* if its degree is at least 3, and called a *leaf* if its degree is equal to 1.

at the second rank, and (ii) the relative ranking of all the other alternatives remains the same in both of them.

Lastly, we contrast our result with the related result in [55] and [56]. They characterize the unanimous and strategy-proof SCFs on a *rich single-peaked domain* with respect to their general notion on betweenness. It is straightforward to see that both top-connectedness and strong connectedness are weaker than their richness condition. Therefore, in the context of trees, their results follow as a corollary of ours.

6.1.3 Remainder

The rest of the chapter is organized as follows. We describe the usual social choice framework in Section 6.2. Section 6.3 establish a few properties of unanimous and strategy-proof SCFs on top-connected single-peaked domains and Section 6.4 characterizes such SCFs on strongly connected single-peaked domains. Section 6.5 concludes the chapter.

6.2 Preliminaries

Let $N = \{1, ..., n\}$ be a set of at least two agents, who collectively choose an element from a finite set $A = \{a, a+1, \dots, b-1, b\}$ of at least three alternatives. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *preference* P over A is a complete, transitive, and antisymmetric binary relation (also called a linear order) defined on A. The *upper-contour set* (*lower-contour set*) of P at an alternative $x \in A$, denoted by $U(P,x)$ ($L(P,x)$), is given by $U(P,x) = \{y \in A \mid yPx\} \cup \{x\}$ ($L(P,x) = \{y \in A \mid xPy\} \cup \{x\}$). We denote by $\mathbb{L}(A)$ the set of all preferences over A. An alternative $x \in A$ is called the k^{th} *ranked alternative* in a preference $P \in \mathbb{L}(A)$, denoted by $r_k(P)$, if $|\{a \in X \mid aPx\}| = k-1$. A domain of admissible preferences, denoted by $\mathcal{D}$, is a subset of $\mathbb{L}(A)$. For $B \subseteq A$ and a domain $\mathcal{D}$, $\mathcal{D}^B = \{P \in \mathcal{D} \mid r_1(P) \in B\}$. An element $P_N = (P_1, \dots, P_n) \in \mathcal{D}^n$ is called a *preference profile*. The *top-set* of a preference profile P_N, denoted by $\tau(P_N)$, is defined as $\tau(P_N) = \{x \in A \mid r_1(P_i) = x \text{ for some } i \in N\}$. A domain $\mathcal{D}$ of preferences is *regular* if for all $x \in X$, there exists a preference $P \in \mathcal{D}$ such that $r_1(P) = x$. All the domains we consider in this chapter are assumed to be regular.

Definition 6.2.1 *A social choice function (SCF) f on $\mathcal{D}^n$ is a mapping $f : \mathcal{D}^n \to A$.*

Definition 6.2.2 *An SCF $f : \mathcal{D}^n \to A$ is unanimous if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in A$, we have $f(P_N) = x$.*

Definition 6.2.3 *An SCF $f : \mathcal{D}^n \to A$ is manipulable if there exists $i \in N$, $P_N \in \mathcal{D}^n$, and $P_i^{'} \in \mathcal{D}$ such that $f(P_i^{'}, P_{N\setminus i})P_i f(P_N)$. An SCF f is strategy-proof if it is not manipulable.*

Definition 6.2.4 *An SCF $f : \mathcal{D}^n \to A$ is called group manipulable if there is a preference profile P_N, a non-empty coalition $C \subseteq N$, and a preference profile $P'_C \in \mathcal{D}^{|C|}$ of the agents in C such that $f(P'_C, P_{N\setminus C})P_i f(P_N)$ for all $i \in C$. An SCF $f : \mathcal{D}^n \to A$ is called group strategy-proof if it is not group manipulable.*

Definition 6.2.5 *An SCF $f : \mathcal{D}^n \to A$ is called dictatorial if there exists $i \in N$ such that for all $P_N \in \mathcal{D}^n$, $f(P_N) = r_1(P_i)$.*

Definition 6.2.6 *A domain $\mathcal{D}$ is called dictatorial if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to A$ is dictatorial.*

Definition 6.2.7 *Two preference profiles $P_N, P'_N \in \mathcal{D}^n$ are called tops-equivalent if $r_1(P_i) = r_1(P'_i)$ for all agents $i \in N$.*

Definition 6.2.8 *An SCF $f : \mathcal{D}^n \to A$ is called tops-only if for any two tops-equivalent $P_N, P'_N \in \mathcal{D}^n$, $f(P_N) = f(P'_N)$.*

Definition 6.2.9 *A domain $\mathcal{D}$ is called tops-only if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to A$ is tops-only.*

Definition 6.2.10 *An SCF $f : \mathcal{S}^n \to A$ satisfies the Pareto property if for all $P_N \in \mathcal{S}_N$ such that xP_iy for all $i \in N$ and some $x, y \in X$, we have $f(P_N) \neq y$.*

Definition 6.2.11 *An SCF $f : \mathcal{S}^n \to A$ satisfies uncompromisingness if for all $P_N \in \mathcal{S}^n$, all $i \in N$, and all $P'_i \in \mathcal{S}$ such that $\{r_1(P_i), r_1(P'_i)\} \in E$,*

(i) $r_1(P'_i) \in (r_1(P_i), f(P_N)]$ *or* $r_1(P_i) \in (r_1(P'_i), f(P_N)]$ *implies* $f(P'_i, P_{N\setminus i}) = f(P_N)$*, and*

(ii) $r_1(P'_i) \in (f(P_N), r_1(P_i)]$ *or* $r_1(P_i) \in (f(P_N), r_1(P'_i)]$ *implies* $f(P'_i, P_{N\setminus i}) = f(P_N)$.

Next, we introduce a graph structure on the set of alternatives. A collection $E \subseteq \{\{a, b\} \mid a, b \in A, \ a \neq b\}$ is an *undirected graph*. The elements of E are called *edges*. The *degree* of $a \in A$ is the number of edges to which it belongs, i.e., the number $|\{\{x, y\} \in E \mid a \in \{x, y\}\}|$. For $a, b \in A$ a *path* $[a, b]$ is a sequence of nodes $(a_1, \dots, a_k)$ such that $a_1 = a$, $a_k = b$, and $(a_i, a_{i+1}) \in E$ for all $i = 1, \dots, k-1$. In this case, by $(a, b]$ we denote the sequence $(a_2, \dots, a_k)$, and by (a, b) the sequence $(a_2, \dots, a_{k-1})$. Whenever it is clear from the context, the notations $[a, b]$, $(a, b]$, and (a, b) will also be used to denote the sets of nodes (instead of the sequences) that appear in the path.

A graph E is a *tree* if for all $a, b \in A$ there is a unique path $[a, b]$. Throughout this chapter, we assume that E is an arbitrary but fixed tree. By $A_L \subseteq A$, we denote the set of alternatives with degree 1 (also called *leafs*), and by $A_J \subseteq A$, we denote the set of alternatives with degree more than two (also called *junction nodes*). Thus, the alternatives in A are partitioned into three sets: the leafs, the junction nodes, and the nodes with degree exactly two.

Definition 6.2.12 *A preference P is single-peaked if for all distinct $x, y \in A$ with $y \neq r_1(P)$,*

$$x \in [r_1(P), y] \implies xPy.$$

A domain $\mathcal{S}$ is single-peaked if each preference in it is single-peaked.

Example 6.2.1 *Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of alternatives arranged over the tree in Figure 6.2.1. The alternatives a_1,a_2, and a_3 form the leaves of this tree and the alternative a_4 form the only junction node. Therefore, $A_L = \{a_1, a_2, a_3\}$ and $A_J = \{a_4\}$. A* *preference P over A such that $a_1Pa_4Pa_2Pa_3$ is single-peaked whereas preference P' over A such that $a_1P'a_2P'a_4P'a_3$ is not. This is because the alternative a_4 comes before a_2 in the path $[a_1, a_2]$ and hence, in any single-peaked preference which places a_1 at the top, a_4 must be preferred to a_2. Note that single-peakedness does not place any restriction on a preference which places a_4 at the top as the alternatives a_1,a_2, and a_3 are on different paths from a_4. More generally, single-peakedness does not place any restriction in the preference over the alternatives belonging to two different paths from the top-ranked alternative.*

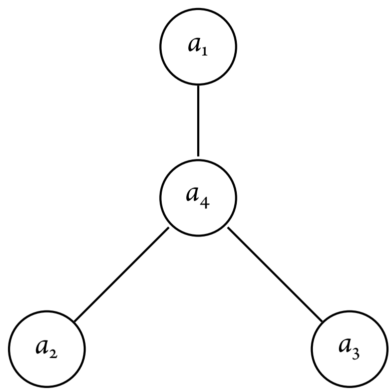


Figure 6.2.1 A tree graph

6.3 Pareto Property and Tops-onlyness

In this section, we introduce the notion of top-connected single-peaked domains on trees and establish a few important properties of unanimous and strategy-proof SCFs on such domains like the Pareto property and tops-onlyness.

Definition 6.3.1 *A single-peaked domain $\mathcal{S}$ is called top-connected if for all distinct $x, y \in A$ such that $\{x, y\} \in E$, there exists $P, P' \in \mathcal{S}$ such that $r_1(P) = r_2(P') = x$ and and $r_2(P) = r_1(P') = y$.*

Example 6.3.1 *Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of alternatives arranged over the tree in Figure 6.2.1. Then, the set of single-peaked preferences in Table 6.3.1 is top-connected.*

Notice that the maximal single-peaked domain based on this tree would have 6 preferences with the top-ranked alternative a_4 whereas the domain in Table 6.3.1 contains only three such preferences.

P_1	P_2	P_3	P_4	P_5	P_6	P_7
$\mathbf{a_4}$	$\mathbf{a_1}$	$\mathbf{a_1}$	$\mathbf{a_4}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_3}$
$\mathbf{a_1}$	$\mathbf{a_4}$	$\mathbf{a_4}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_3}$	$\mathbf{a_4}$
a_2	a_2	a_3	a_1	a_3	a_1	a_2
a_3	a_3	a_2	a_3	a_1	a_2	a_1

Table 6.3.1 A top-connected single-peaked domain based on the tree in Figure 6.2.1

The following theorem shows that unanimity and the Pareto property are equivalent under strategy-proofness on top-connected single-peaked domains.

Theorem 6.3.1 *Let $\mathcal{S}$ be a top-connected single-peaked domain. Every unanimous and strategy-proof SCF $f : \mathcal{S}^n \to A$ satisifies Pareto property.*

Proof: Assume for contradiction that $f(P_N) = y$ for some $P_N \in \mathcal{S}_N$ such that xP_iy for all $i \in N$ and some $x, y \in X$. Let $u \in \cap_{i\in N}[r_1(P_i), y]$ be such that $u, y \in E$. Such an alternative u exists because $\mathcal{S}$ is a single-peaked domain on a tree and xP_iy for all $i \in N$. Consider agent 1 and consider the preference $P'_1 \in \mathcal{S}$ such that $r_1(P'_1) = u$ and $r_2(P'_1) = y$. Since uP_1y, by strategy-proofness, $f(P'_1, P_{N\setminus 1}) = y$. By sequentially moving agents i from P_i to P'_i such that $r_1(P'_i) = u$ and $r_2(P'_i) = y$ and by applying strategy-proofness at each stage, it follows that $f(P'_N) = y$. However, since $r_1(P'_i) = u$ for all $i \in N$, this contradicts unanimity of f. This completes the proof of the theorem. ■

The following theorem shows that top-connected single-peaked domains are tops-only.

Theorem 6.3.2 *Let $\mathcal{S}$ be a top-connected single-peaked domain. Every unanimous and strategy-proof SCF $f : \mathcal{S}^n \to A$ is tops-only.*

Proof: Let $P_N \in \mathcal{S}^n$, $i \in N$, and $P_i' \in \mathcal{S}$ be such that $r_1(P_i) = r_1(P_i')$. It is enough to show that $f(P_N) = f(P_i', P_{N \setminus i})$. Suppose not. Let $f(P_N) = x \neq y = f(P_i', P_{N \setminus i})$. By strategy-proofness of f, xP_iy and $yP_i'x$. Therefore, $y \notin [r_1(P_i), x]$ and $x \notin [r_1(P_i), y]$.

Let $[x, y] = (x, u^1, \ldots, u^k, y)$. Let $S \subseteq N$ be such that $j \in S$ implies $r_1(P_j) \in [b, u^k]$ for all $b \in A_L$ such that $u^k \in [b, y]$. Similarly, let $T \subseteq N$ be such that $j \in T$ implies $r_1(P_j) \in [b, y]$ for all $b \in A_L$ such that $y \in [b, u^k]$.[3] Construct the profile $\bar{P}_N$ from P_N such that $r_1(\bar{P}_j) = u^k$ and $r_2(\bar{P}_j) = y$ if $j \in S$, $r_1(\bar{P}_j) = y$ and $r_2(\bar{P}_j) = u^k$ if $j \in T$, and $r_1(\bar{P}_j) = r_1(P_j)$ if $j \in N \setminus (S \cup T)$.

Claim 1. $f(P_N) = x$ implies $f(\bar{P}_N) = u^k$.

Proof:[Proof of Claim 1] Consider the profile $\tilde{P}^1_N \in \mathcal{S}^n$ such that $r_1(\tilde{P}^1_j) = x$ if $r_1(P_j) \in [b, x]$ for all $b \in A_L$ such that $x \in [b, u^1]$ and $\tilde{P}^1_j = P_j$ otherwise. Since $f(P_N) = x$, we have $f(\tilde{P}^1_N) = x$. Next, consider the profile $\tilde{P}^2_N \in \mathcal{S}^n$ such that $r_1(\tilde{P}^2_j) = u^1$ and $r_2(\tilde{P}^2_j) = x$ if $r_1(\tilde{P}^1_j) \in [b, u^1]$ for all $b \in A_L$ such that $u^1 \in [b, u^2]$ and $\tilde{P}^2_j = \tilde{P}^1_j$ otherwise. By moving agents one-by-one from $\tilde{P}^1_N$ to $\tilde{P}^2_N$ and applying unanimity and strategy-proofness at each step, we have $f(\tilde{P}^2_N) \in \{x, u^1\}$. By Theorem 6.3.1, $f(\tilde{P}^2_N) \neq x$, and therefore, $f(\tilde{P}^2_N) = u^1$. Next, consider the profile $\tilde{P}^3_N \in \mathcal{S}^n$ such that $r_1(\tilde{P}^3_j) = u^2$ and $r_2(\tilde{P}^3_j) = u^1$ if $r_1(\tilde{P}^2_j) \in [b, u^2]$ for all $b \in A_L$ such that $u^2 \in [b, u^3]$ and $\tilde{P}^3_j = \tilde{P}^2_j$ otherwise. By moving agents one-by-one from $\tilde{P}^2_N$ to $\tilde{P}^3_N$ and applying unanimity and strategy-proofness at each step, we have $f(\tilde{P}^3_N) \in \{u^1, u^2\}$ and by Theorem 6.3.1, $f(\tilde{P}^3_N) = u^2$.

Continuing in this manner, consider the profile $\tilde{P}^{k+1}_N \in \mathcal{S}^n$ such that $r_1(\tilde{P}^{k+1}_j) = u^k$ and $r_2(\tilde{P}^{k+1}_j) = u^{k-1}$ if $r_1(\tilde{P}^{k+1}_j) \in [b, u^k]$ for all $b \in A_L$ such that $u^{k-1} \in [b, u^k]$ and $\tilde{P}^3_j = \tilde{P}^2_j$ otherwise. Using the same arguments as before, we have $f(\tilde{P}^{k+1}_N) = u^k$. Observe that $y\tilde{P}^k_j u^k$ for all $j \in T$. Now, move agents from the profile $\tilde{P}^{k+1}_N$ to $\bar{P}_N$ one-by-one and by applying strategy-proofness at each stage, we have $f(\bar{P}_N) = u^k$. This completes proof of Claim 1. ■

Claim 2. $f(P_i', P_{N \setminus i}) = y$ implies $f(\bar{P}_N) = y$.

Proof:[Proof of Claim 2] Consider the profile $\hat{P}^1_N \in \mathcal{S}^n$ such that $r_1(\hat{P}^1_j) = y$ and

[3] The sets S and T are defined with respect to the profile P_N. However, the same could also be defined with respect to the profile (P_i', P_N) as the profiles P_N, (P_i', P_N) are tops-equivalent.

$r_2(\hat{P}^1_j) = u^k$ if $r_1(P_j) \in [b, y]$ for all $b \in A_L$ such that $y \in [b, u^k]$ and $\hat{P}^1_j = P_j$ otherwise. Since $f(P'_i, P_{N\setminus i}) = y$, we have $f(\hat{P}^1_N) = y$. Observe that $u^k \hat{P}^1_j y$ for all $j \in S$. Next, move agents from the profile $\hat{P}^1_N$ to $\bar{P}_N$ one-by-one and by applying strategy-proofness at each stage, we have $f(\bar{P}_N) = y$. This completes proof of Claim 2. ■

Since $u^k \neq y$, Claim 1 contradicts Claim 2. This completes proof of the theorem. ■

6.4 Main Result

In this section, we introduce the notion of strongly connected single-peaked domains on trees and characterize the unanimous and strategy-proof SCFs on it.

Definition 6.4.1 *A single-peaked domain $\mathcal{S}$ is called strongly connected if for all distinct $x, y \in A$ such that $\{x, y\} \in E$*

(i) there exists $P, P' \in \mathcal{S}$ such that $r_1(P) = r_2(P') = x$ and and $r_2(P) = r_1(P') = y$, and

(ii) $r_k(P) = r_k(P')$ for all $k \in \{3, \ldots, m\}$.

Example 6.4.1 *Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of alternatives arranged over the tree in Figure 6.2.1. Then, the set of single-peaked preferences in Table 6.4.1 is strongly connected. Strong connectedness requires that for any pair of alternatives, there exists two preferences which places these alternatives at the top two ranks and these preferences restricted to other alternatives must be the same. Therefore, strong connectedness imposes a stronger restriction on the domain as opposed to top-connectedness. To see this, observe that for the pair of alternatives a_2 and a_4, the restriction of the preferences P_4 and P_5 to the set of alternatives $\{a_1, a_3\}$ is not the same for the top-connected single-peaked domain in Example 6.3.1, and hence, it is not strongly connected.*

P_1	P_2	P_3	P_4	P_5	P_6	P_7
$\mathbf{a_4}$	$\mathbf{a_1}$	$\mathbf{a_1}$	$\mathbf{a_4}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_3}$
$\mathbf{a_1}$	$\mathbf{a_4}$	$\mathbf{a_4}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_3}$	$\mathbf{a_4}$
a_2	a_2	a_3	a_1	a_1	a_2	a_2
a_3	a_3	a_2	a_3	a_3	a_1	a_1

Table 6.4.1 A strongly connected single-peaked domain based on the tree in Figure 6.2.1

The following theorem shows that uncompromisingness is a necessary condition for an SCF to be unanimous and strategy-proof on a strongly connected single-peaked domain.

Theorem 6.4.1 *Let $\mathcal{S}$ be a strongly connected single-peaked domain. Then, every unanimous and strategy-proof SCF $f : \mathcal{S}^n \to A$ is uncompromising.*

Proof: Let $\mathcal{S}$ be a strongly connected single-peaked domain. Consider $P_N \in \mathcal{S}^n$, $i \in N$, and $P'_i \in \mathcal{S}$ such that $\{r_1(P_i), r_1(P'_i)\} \in E$ and $r_1(P'_i) \in (r_1(P_i), f(P_N)]$. It is sufficient to prove that $f(P'_i, P_{N\setminus i}) = f(P_N)$. Let $r_1(P_i) = x$, $r_1(P'_i) = y$, and $f(P_N) = z$.

Since f is tops-only (Theorem 6.3.2), we assume without loss of generality, $r_1(P_i) = x$ and $r_2(P_i) = y$. Let $\bar{P}_i \in \mathcal{S}$ such that $r_1(\bar{P}_i) = y$, $r_2(\bar{P}_i) = x$ and $r_k(\bar{P}_i) = r_k(P_i)$. If $z = y$, then by strategy-proofness, we have $f(\bar{P}_i, P_{N\setminus i}) = z$. Suppose $z \neq x, y$. Since $r_k(\bar{P}_i) = r_k(P_i)$, by strategy-proofness, $f(\bar{P}_i, P_{N\setminus i}) = z$. Since f is tops-only (Theorem 6.3.2), $f(P'_i, P_{N\setminus i}) = f(\bar{P}_i, P_{N\setminus i}) = z$. This completes the proof of the theorem. ∎

6.5 Concluding remarks

In this chapter, we have considered single-peaked domains when alternatives are arranged on a tree. We have shown that when such domains satisfy top-connectedness, then every unanimous and strategy-proof SCFs on these domains satisfy the Pareto property and tops-onlyness. Further, when such domains are strongly connected, we have characterized all unanimous and strategy-proof rules as uncompromising rules.

7
Epilogue

In a standard social choice setting, the present thesis is concerned with studying domain restrictions when designing unanimous and strategy-proof social choice functions. The thesis contains 7 chapters (including the present chapter). In what follows, we provide a brief summary of the main results in each chapter:

(i) Chapter 1 introduces the strategy-proof social choice literature, provides motivation to the problems studies in the thesis and provides a brief overview of the subsequent chapters.

(ii) Chapter 2 contributes to the literature on dictatorial domains where we introduce the notion of top-circular domains and provide two sufficient conditions for it to be dictatorial.

(iii) Chapter 3 considers arbitrary single-peaked domains and shows that, under mild conditions, every unanimous and strategy-proof social choice function defined on such domains satisfy the Pareto property and tops-onlyness. This chapter further provides a domain characterization of a special class of social choice functions called the min-max rules as top-connected single-peaked domains.

(iv) Chapter 4 considers partially single-peaked domains, domains where preferences violate single-peakedness over a subset of alternatives. This chapter characterizes the unanimous and strategy-proof social choice functions and provides (almost) necessary and sufficient conditions on the admissible domain of preferences for this characterization to hold.

(v) Chapter 5 provides a general characterization of the unanimous and strategy-proof social choice function on arbitrary (not necessarily top-connected domains) single-peaked domains as weak uncompromising rules.

(vi) Chapter 6 considers single-peaked preferences when the set of alternatives are arranged on a tree and characterizes the unanimous and strategy-proof social choice functions on strongly connected single-peaked domains as uncompromising rules.

We discuss a few interesting open problems for future research. A long standing open problem in the literature on domain restrictions in strategy-proof social choice is the characterization of dictatorial domains. A partial answer is provided in [70] who answers this question for the case of social choice functions satsfying the Pareto property.

A few other related open problems are: (i) characterizing domains where every strategy-proof rules are tops-only, and (ii) domains where the notions of strategy-proofness and group strategy-proofness are equivalent. [20] partially answers (i) by providing sufficient conditions on domains for it to be tops-only. However, the present thesis shows that several practical domain restrictions such as arbitrary single-peaked domains, partially single-peaked domains, etc. do not satisfy their conditions. [7] partially answers (i) by providing sufficient conditions on domains for it to be tops-only. However, partially single-peaked domains and single-peaked domains based on trees do not satisfy their sufficient conditions.

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