Scaling limits of some random interface models

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Scaling limits of some random interface models

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Dedicated

To

Baba, Maa and Dada

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Contents

Chapter 1

The models and main results

1.1 Introduction

In this thesis, we study some probabilistic models of random interfaces. Interfaces between different phases have been topic of considerable interest in statistical physics. These interfaces are described by a family of random variables, indexed by the d-dimensional integer lattice, which are considered as a height configuration, namely they indicate the height of the interface above a reference hyperplane. The models are defined in terms of an energy function (Hamiltonian), which defines a Gibbs measure on the set of height configurations. More formally, let

$$\varphi = \{\varphi_x\}_{x \in \mathbb{Z}^d}$$

be a collection of real numbers indexed by the d-dimensional integer lattice $\mathbb{Z}^d$. Such a collection can be interpreted as a d-dimensional interface in $d+1$-dimensional Euclidean space $\mathbb{R}^{d+1}$ in the following manner: we think of φ_x as height variable, indicating the height of the interface above the point x in the d-dimensional reference hyperplane. We obtain a d-dimensional surface in $\mathbb{R}^{d+1}$ by interpolating the heights linearly between the integer points. We will in general forget about the interpolation, and call any configuration $\{\varphi_x\}_{x \in \mathbb{Z}^d}$ an interface. We identify the family $\{\varphi_x\}_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ with the (graph of the) mapping

$$\varphi : \mathbb{Z}^d \to \mathbb{R}$$

such that $\varphi(x) = \varphi_x$. We now introduce a probability measure on the set of interface configurations. Let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ be endowed with the product topology. We consider the product σ-field on Ω. Let Λ be a finite subset of $\mathbb{Z}^d$. We fix a configuration $\{\psi_x\}_{x\in\mathbb{Z}^d\setminus\Lambda}$ which plays the role of a boundary condition. The probability of a configuration φ depends on its energy which is given by a Hamiltonian $H^{\psi}_{\Lambda}(\varphi)$. The probability measure on Ω is given (formally) by

$$\mathbf{P}^{\psi,\beta}_{\Lambda}(\mathrm{d}\varphi) := \frac{1}{Z^{\psi,\beta}_{\Lambda}} \exp\left(-\beta H^{\psi}_{\Lambda}(\varphi)\right) \prod_{x\in\Lambda} \mathrm{d}\varphi_x \prod_{x\notin\Lambda} \delta_{\psi_x}(\mathrm{d}\varphi_x). \tag{1.1.1}$$

Here, $\beta \geq 0$ is called the inverse temperature, $\mathrm{d}\varphi_x$ is the one dimensional Lebesgue measure, δ_{ψ_x} is the Dirac mass at ψ_x and $Z^{\psi,\beta}_{\Lambda}$ is the constant which normalizes $\mathbf{P}^{\psi,\beta}_{\Lambda}$ to a probability measure (if it is finite). In other words, if $\mathbf{P}^{\psi,\beta}_{\Lambda}$ exists, it is the probability measure on the set of configurations restricted to be equal to ψ outside Λ and has density $(Z^{\psi,\beta}_{\Lambda})^{-1}\exp(-\beta H^{\psi}_{\Lambda}(\varphi))$ with respect to the product Lebesgue measure on $\mathbb{R}^{\Lambda}$.

Let us first see a concrete example of random interface models. The *gradient model (or ∇-model)* is a random interface model, where the Hamiltonian is given by

$$H^{\psi}_{\Lambda}(\varphi) = \frac{1}{2}\sum_{x,y\in\Lambda} p_{x,y} V(\varphi_x - \varphi_y) + \sum_{x\in\Lambda, y\notin\Lambda} p_{x,y} V(\varphi_x - \varphi_y).$$

Here $V : \mathbb{R} \to \mathbb{R}$ is an even convex function with $V(0) = 0$ and $p_{x,y}$ is the transition matrix of a random walk on the lattice $\mathbb{Z}^d$. If we assume that the random walk has finite range, that is, the step distributions have finite support (there are more general conditions under which the measure is well defined), then (1.1.1) defines a probability measure on $\mathbb{R}^{\Lambda}$. There is much literature available on this class of random interface models, for an overview see for example the lecture notes by Funaki [38], Giacomin et al. [40], Velenik [71]. All the models considered in this thesis are Gaussian. Due to Gaussianness , the parameter β of (1.1.1) is of no importance. So we set it to be equal to 1. Also from now we consider $\psi \equiv 0$. We shall say the model has 0-boundary conditions.

The discrete Gaussian free field:

An important example of the gradient model is the *Discrete Gaussian free field* (DGFF), also called *harmonic crystal*, where one considers $V(x) = x^2/2$ and

$$p_{x,y} = (2d)^{-1} 1_{\{|x-y|=1\}}.$$

In this case, the Hamiltonian can be written in the following form:

$$H(\varphi) = \frac{1}{4d} \sum_x \|\nabla \varphi_x\|^2$$

where ∇ is the discrete gradient defined by

$$\nabla \varphi_x := (\varphi_{x+e_1} - \varphi_x, \ldots, \varphi_{x+e_d} - \varphi_x),$$

$\|\cdot\|$ denotes the Euclidean norm and e_i denotes the canonical basis of $\mathbb{R}^d$. Let $\Gamma_\Lambda(x,y) := \mathbf{Cov}_\Lambda(\varphi_x, \varphi_y)$. The field $(\varphi_x)_{x\in\Lambda}$ is Gaussian, and its covariance matrix is given by the Green's function of the random walk $(S_n)_{n\geq 0}$ with the transition matrix $p_{x,y}$ which is killed at the exit of Λ, that is, for x, y in Λ

$$\Gamma_\Lambda(x,y) = (I-P)_\Lambda^{-1}(x,y) = \mathbf{E}^x \left(\sum_{n=0}^{\tau_\Lambda - 1} 1_{\{S_n = y\}} \right),$$

where $(I-P)_\Lambda = (\delta(x,y) - p_{x,y})_{x,y\in\Lambda}$, $\mathbf{E}^x$ is the law of the random walk started at x and $\tau_\Lambda = \inf\{n \geq 0 : S_n \notin \Lambda\}$. Note that in this case, $(I-P) = -\Delta$, where Δ is the discrete Laplacian matrix given by

$$\Delta(x,y) = \begin{cases} -1 & \text{if } x = y, \\ \frac{1}{2d} & \text{if } |x-y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

One can also, alternatively define Δ as a discrete differential operator acting on functions $f : \mathbb{Z}^d \to \mathbb{R}$ at a point $x \in \mathbb{Z}^d$

$$\Delta f(x) = \frac{1}{2d} \sum_{i=1}^{d} \left(f(x+e_i) + f(x-e_i) - 2f(x) \right).$$

The Green's function Γ_Λ thus satisfies the following discrete Dirichlet problem: for $x \in \Lambda$,

$$\begin{cases} -\Delta \Gamma_\Lambda(x,y) = \delta_x(y) & y \in \Lambda \\ \Gamma_\Lambda(x,y) = 0 & y \in \partial_1 \Lambda \end{cases},$$

where for $k \geq 1$,

$$\partial_k \Lambda := \{x \in \mathbb{Z}^d \setminus \Lambda : \mathrm{dist}(x, \Lambda) \leq k\} \tag{1.1.2}$$

with $\mathrm{dist}(\cdot,\cdot)$ being the graph distance. In $d=1$ the DGFF can be seen as the random walk bridge. More precisely, if $\Lambda = V_N = \{1, \cdots, N\}$ for $N \in \mathbb{N}$, then $(\varphi_1, \cdots, \varphi_N)$ has the same joint distribution as $(S_1, \cdots, S_N)$ conditionally on $S_{N+1} = 0$, where $(S_n)_{n\geq 0}$ is a random walk with $\mathcal{N}(0,2)$ increments started at 0. In $d=2$, DGFF belongs to the family of log-correlated Gaussian fields (see [7]).

DGFF has been studied extensively for its connections to the SLE processes, branching random walk and branching Brownian motion. In a breakthrough result Schramm and Sheffield [63] showed that the level lines of DGFF converges in distribution to SLE(4). The entropic repulsion, namely the estimates for the probability that the field is positive on a subset of V_N was studied by Bolthausen et al. [10, 11]. The behaviour of the maximum in two dimension was studied by Biskup and Louidor [8], Bolthausen et al. [11, 12], Bramson and Zeitouni [16], Bramson et al. [17], Daviaud [31] and the limiting distribution is given by a randomly shifted Gumbel. In higher dimensions $d \geq 3$ the behaviour of the maximum was studied by Chiarini et al. [22, 23]. They proved that the rescaled maximum is in the maximal domain of attraction of the Gumbel distribution.

We now see what happens to the scaling limit of DGFF. In $d=1$, we pointed out that the DGFF is the random walk bridge. Hence after appropriate scaling the interpolated field converges to the Brownian bridge in the space of continuous functions. More explicitly, let $(B_t : 0 \leq t \leq 1)$ be the standard Brownian motion on $[0,1]$. The Brownian bridge, which is the one dimensional Gaussian free field, is defined to be the process $(B_t^\circ : 0 \leq t \leq 1)$ where

$$B_t^\circ := B_t - tB_1, \ t \in [0,1].$$

Now let us consider the DGFF on $\Lambda = \{1, \ldots, N-1\}$ and define a continuous interpolation ψ_N for each N as follows:

$$\psi_N(t) = (2N)^{-\frac{1}{2}} \left[\varphi_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)(\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor})\right], \quad t \in [0,1].$$

Then one can show that ψ_N converges in distribution to $(B_t^\circ : 0 \leq t \leq 1)$ in the space of continuous functions $C[0,1]$. From the above convergence one can obtain the convergence of the maximum using continuous mapping theorem. In $d=2$, if we try to obtain convergence similar to the above with a scaling by $\sqrt{\log N}$ then the limit is nothing but a collection of independent normal random variables (see [7]). Hence it fails to retain any useful information from the model. This suggests one to take limit is some other suitable sense. Indeed, without any scaling one can obtain such limit in $d = 2$ and also in $d \geq 3$ with suitable scaling. Unlike $d = 1$, where the limiting field is a random function, namely, Brownian bridge, in $d \geq 2$, one does not have a random function, instead it becomes a random distribution. This random distribution is called the Gaussian free field (GFF). The importance of two dimensional Gaussian free field comes from conformal invariance and connection with other stochastic processes like SLE, CLE, Louville quantum gravity etc. We refer to [3, 5, 34, 65] for details and references on Gaussian free field. For this model quadratic potential allows one to have various tools at one's disposal, like the random walk representation of covariances and inequalities like FKG. These tools can be generalised to convex potentials in the form of the Brascamp–Lieb inequality and the Helffer–Sjöstrand random walk representation of the covariance. We refer to [38, 40, 55, 71] for an overview of the existing results. Outside the convex regime, the non-convex regime was recently studied for example in [9, 30].

The membrane model:

The membrane model(MM) is the Gaussian interface model where the Hamiltonian is given by

$$H(\varphi) := \frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\Delta \varphi_x|^2.$$

This model arises as model for (tensionless) semi-flexible membrane in statistical physics. Its mathematical treatment was first taken up by Sakagawa [60, 61] and then by Cipriani [25], Kurt [46, 47, 48]. Unlike the DGFF, the covariance function of this model does not have any random walk representation. For $\Lambda \Subset \mathbb{Z}^d$, define

$$G_\Lambda(x, y) := \mathbf{Cov}_\Lambda(\varphi_x, \varphi_y), \ x, y \in \Lambda.$$

Consider $\Lambda = V_N = [-N, N]^d \cap \mathbb{Z}^d$ and define for $x, y \in V_N$

$$\overline{G}_N(x, y) := \sum_{z \in V_N} \Gamma_{V_N}(x, z) \Gamma_{V_N}(z, y),$$

where Γ_{V_N} is the covariance function of the DGFF on V_N. Then $\overline{G}_N = (\Delta_\Lambda)^{-2}$ where $\Delta_\Lambda := (\Delta(x,y))_{x,y\in\Lambda}$. It is easy to see that $(\Delta_\Lambda)^2 \neq \Delta^2_\Lambda := (\Delta^2(x,y))_{x,y\in\Lambda}$. This difference is due to the restrictions of the operators to Λ. When we see their actions on a function at a point which is far away from the boundary, then they are roughly the same. In fact, one can show that in higher dimensions in the bulk the inverses of these two operators are close. For that we extend $\overline{G}_N(x,.)$ as a function on $V_N \cup \partial_2 V_N$ by requiring

$$\overline{G}_N(x,y) = 0 \qquad y \in V_{N+1} \setminus V_N$$
$$\Delta\overline{G}_N(x,y) = 0 \qquad y \in \partial_1 V_N.$$

It was proved in [47, Corollary 2.5.5] that for $d \geq 4$ and $\delta > 0$, there exists a constant $c_d = c_d(\delta)$ such that for any $x \in V_N^\delta := \{z \in V_N : \mathrm{dist}(z, V_N^c) \geq \delta N\}$,

$$\sup_{y\in V_N^\delta} |G_{V_N}(x,y) - \overline{G}_N(x,y)| \leq c_d N^{4-d} \text{ as } N \to \infty.$$

As the MM exhibits no random walk representation, and several correlation inequalities are lacking, the study of this model becomes difficult compared to the DGFF. Nonetheless it is possible, via analytic and numerical methods, to obtain sharp results on its behaviour. But like the DGFF, the covariance function of this model satisfies the following Dirichlet problem: for $x \in \Lambda$,

$$\begin{cases} \Delta^2 G_\Lambda(x,y) = \delta_x(y) & y \in \Lambda \\ G_\Lambda(x,y) = 0 & y \in \partial_2\Lambda, \end{cases}$$

where $\partial_2\Lambda$ is defined as in (1.1.2). Also in $d = 1$, the MM can be seen as an integrated random walk as follows: consider the model $(\varphi_x)_{x\in V_N}$ on $V_N = \{1, \ldots, N-1\}$ with zero boundary conditions outside V_N. Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. standard Gaussian random variables. We define $\{Y_i\}_{i\in\mathbb{Z}^+}$ to be the associated random walk starting at 0, that is,

$$Y_0 = 0,\ Y_n = \sum_{i=1}^n X_i,\ n \in \mathbb{N},$$

and $\{Z_i\}_{i\in\mathbb{Z}^+}$ to be the integrated random walk starting at 0, that is, $Z_0 = 0$ and for $n \in \mathbb{N}$

$$Z_n = \sum_{i=1}^n Y_i.$$

Then one can show that $\mathbf{P}_{V_N}$ is the law of the vector $(Z_1, \ldots, Z_{N-1})$ conditionally on $Z_N = Z_{N+1} = 0$ (see [20, Proposition 2.2]). Also, like the DGFF, the MM is log-correlated in $d = 4$.

For this model there are some results on the entropic repulsion and pinning effects [2, 13, 20, 46, 48], extreme value theory [24]. The entropic repulsion in higher dimensions ($d \geq 4$) was studied by Kurt [46, 48], Sakagawa [60]. We know that in $d = 1$ the model corresponds to an integrated Gaussian random walk. In [32] it was proved that for such processes with zero mean and finite variances the probability to be positive on an interval of side length N is of order $N^{-1/4}$, extending a result by Sinai [66] for the integrated simple random walk. Recently in the remaining cases, that is, in dimensions 2 and 3 the entropic repulsion was studied by Buchholz et al. [19]. The maximum of MM in $d = 4$ falls under the study of extreme value for log-correlated models. The extremes were first studied by Cipriani [25], Kurt [48]. The tightness of the recentered maximum follows from [33]. The full scaling limit was finally solved by Schweiger et al. [64] and it is a randomly shifted Gumbel, similar to the DGFF case in $d = 2$. In the higher dimensions the maximum was studied by Chiarini et al. [24]. Just like DGFF, for this model also they proved the rescaled maximum to be in the maximal domain of attraction of the Gumbel distribution.

The scaling limit of this model in $d = 1$ was studied by Caravenna and Deuschel [21]. They studied scaling limit for more general potentials than the quadratic one and also look at the situation in which a pinning force is added to the model. We briefly discuss their result for the MM. Consider the model on $V_N = [1\,,\, N-1] \cap \mathbb{Z}$ and define a continuous interpolation ψ_N by

$$\psi_N(t) := \frac{\varphi_{\lfloor Nt \rfloor}}{N^{\frac{3}{2}}} + \frac{Nt - \lfloor Nt \rfloor}{N^{\frac{3}{2}}}(\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}),\ t \in [0, 1].$$

Then ψ_N converges in distribution to the process $\{\hat{I}_t\}_{t\in[0\,,\,1]}$ in $C[0, 1]$, where the limiting process is defined as the marginal of the process $\{(\hat{B}_t\,,\, \hat{I}_t)\}_{t\in[0\,,\,1]} := \{(B_t\,,\, I_t)\}_{t\in[0\,,\,1]}$ conditionally on $(B_1\,,\, I_1) = (0\,,\, 0)$, where $\{B_t\}_{t\in[0\,,\,1]}$ be the standard Brownian motion on $[0\,,\, 1]$ and $I_t := \int_0^t B_s \mathrm{d}s$. We also mention that Hryniv and Velenik [44] considered general semiflexible membranes as well with a different scaling approach. Their results are derived using an integrated random walk representation which is difficult to adapt in higher dimensions. This thesis aims at complementing their work by determining the scaling limit in all $d \geq 2$. We shall prove that the convergence in $d = 2$ and 3 occurs in the space of continuous functions and hence one can derive the limiting maxima in $d \leq 3$.

The $(\nabla + \Delta)$-model:
The $(\nabla + \Delta)$-model is another Gaussian interface model where the Hamiltonian is given by the sum of the Hamiltonians of DGFF and MM, that is

$$H(\varphi) := \sum_{x \in \mathbb{Z}^d} \left(\frac{1}{4d} \|\nabla \varphi_x\|^2 + \frac{1}{2} |\Delta \varphi_x|^2 \right).$$

This model was first considered by Borecki [14], Borecki and Caravenna [15] in a more general set up with pinning. For this model also no random walk representation for the covariance function is known. Like the DGFF and MM, the covariance function $G_\Lambda(x, y) := \mathbf{Cov}_\Lambda(\varphi_x, \varphi_y)$ of this model satisfies the following Dirichlet problem: for $x \in \Lambda$,

$$\begin{cases} (-\Delta + \Delta^2) G_\Lambda(x, y) = \delta_x(y) & y \in \Lambda \\ G_\Lambda(x, y) = 0 & y \in \partial_2 \Lambda, \end{cases}$$

where $\partial_2 \Lambda$ is defined as in (1.1.2). The application of Gibbs measures, in particular the $(\nabla + \Delta)$-model, to the theory of biological membranes can be found in [49, 50, 59]. In the works of Borecki [14], Borecki and Caravenna [15] this model was studied in $d = 1$ under the influence of pinning in order to understand the localization behavior of the polymer. In higher dimensions the localization behavior was studied by Sakagawa [62].

1.2 Definition and basic properties of the models

In this thesis we consider some special instances of random interface models, namely where the Hamiltonian is given by

$$H(\varphi) = \sum_{x \in \mathbb{Z}^d} \left(\kappa_1 \|\nabla \varphi_x\|^2 + \kappa_2 (\Delta \varphi_x)^2 \right) \tag{1.2.1}$$

where κ_1 and κ_2 are two non-negative parameters. In the model of a membrane such as a lipid bilayer, the energy of the surface separating the water phase and the lipid phase is given by this $H(\varphi)$ where κ_1 and κ_2 are the lateral tension and the bending rigidity, respectively (see [49, 50, 59] etc.).

Define

$$J := -4d\kappa_1\Delta + 2\kappa_2\Delta^2.$$

The following result shows that the Gibbs measure (1.1.1) with Hamiltonian (1.2.1) exists. It follows by arguments similar to Lemma 1.2.2 in [47].

Lemma 1.2.1. *The Gibbs measure on $\mathbb{R}^\Lambda$ with boundary conditions ψ outside Λ and Hamiltonian (1.2.1) exists. It is the Gaussian field on Λ with mean*

$$m_x = -\sum_{y\in\Lambda} J_\Lambda^{-1}(x,y) \sum_{z\in\mathbb{Z}^d\setminus\Lambda} J(y,z)\psi_z, \ \ x \in \Lambda$$

and covariance matrix

$$\mathbf{Cov}_\Lambda(\varphi_x, \varphi_y) = J_\Lambda^{-1}(x,y)$$

where J_Λ is the matrix $(J(x,y))_{x,y\in\Lambda}$.

Let $G_\Lambda(x,y) := J_\Lambda^{-1}(x,y)$, $x, y \in \Lambda$. Then G_Λ is the unique solution to the following discrete boundary value problem: for $x \in \Lambda$

$$\begin{cases} JG_\Lambda(x,y) = \delta_x(y) & y \in \Lambda \\ G_\Lambda(x,y) = 0 & y \in \partial_2\Lambda \end{cases}, \tag{1.2.2}$$

where $\partial_2\Lambda$ is defined as in (1.1.2). In case $\Lambda = [-N,N]^d \cap \mathbb{Z}^d$, we denote the measure in (1.1.1) by $\mathbf{P}_N$. The following proposition answers a very basic question, namely the existence of the infinite volume measure or the thermodynamic limit.

Proposition 1.2.2 ([47, Proposition 1.2.3])**.** *Suppose κ_1, κ_2 are constants. The infinite volume measure*

$$\mathbf{P} := \lim_{N\to\infty} \mathbf{P}_N$$

exists if and only if

$$d \geq \begin{cases} 3 \textit{ when } \kappa_1 > 0 \\ 5 \textit{ when } \kappa_1 = 0 \end{cases}.$$

In these cases it is the centered Gaussian field on $\mathbb{Z}^d$ *with covariance matrix* J^{-1}. *Furthermore, for* $x \in \mathbb{Z}^d$,

$$J^{-1}(0,x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left(4d\kappa_1\mu(\theta) + 2\kappa_2\mu(\theta)^2\right)^{-1} \mathrm{e}^{-\iota\langle x,\theta\rangle} \,\mathrm{d}\,\theta \tag{1.2.3}$$

where

$$\mu(\theta) = \frac{1}{d}\sum_{i=1}^{d}(1-\cos(\theta_i)).$$

When $\kappa_1 > 0$, we call $d = 2$ the critical dimension, $d = 1$ the subcritical dimension and $d \geq 3$ the super critical dimensions. Similarly, when $\kappa_1 = 0$, we call $d = 4$ the critical dimension, $1 \leq d \leq 3$ the subcritical dimensions and $d \geq 5$ the super critical dimensions. We denote the infinite volume covariance by G, that is, $G(x,y) := J^{-1}(x,y)$. G has the following random walk representation: Let $\mathbf{E}^x$ be the law of the simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$ started at x.

- When $\kappa_1 = 1/4d$ and $\kappa_2 = 0$, that is when the model is the DGFF, then G has the representation

$$G(x,y) := \Gamma(x,y) = \mathbf{E}^x\left(\sum_{n=0}^{\infty} 1_{\{S_n=y\}}\right).$$

- When $\kappa_1 = 0$ and $\kappa_2 = 1/2$, that is the model is the MM, then G can be represented as (see [47, Proposition 1.2.4])

$$G(x,y) = \mathbf{E}^{x,y}\left(\sum_{n,m=0}^{\infty} 1_{\{S_n=\tilde{S}_m\}}\right), \; x,y \in \mathbb{Z}^d,$$

where $(S_n)_{n\geq 0}$ and $(\tilde{S}_n)_{n\geq 0}$ are two independent simple random walk on $\mathbb{Z}^d$ starting at x and y respectively.

- When κ_1 and κ_2 are both non-zero constants, we assume for simplicity $\kappa_1 = \kappa/4d$ and $\kappa_2 = 1/2$. Then

$$G(x, y) = (-\kappa\Delta + \Delta^2)^{-1}(x,\, y).$$

Let $\Gamma_\kappa(\cdot,\, \cdot)$ be the massive Green's function with mass $\sqrt{\kappa}$, that is,

$$\Gamma_\kappa(x, y) = \mathbf{E}^x \left(\sum_{m=0}^{\infty} \frac{1}{(1+\kappa)^{m+1}} 1_{\{S_m = y\}} \right).$$

Then one can show easily

$$G(x, y) = \sum_{z \in \mathbb{Z}^d} \Gamma(x, z)\Gamma_\kappa(z, y).$$

Also, the infinite volume covariance G satisfies the following property:

Lemma 1.2.3 ([60, Lemma 5.1]). *Let $d \geq 2\ell + 1$, where*

$$\ell = 1,\ q_\ell = \kappa_1 \ \textit{when}\ \kappa_1 > 0$$
$$\ell = 2,\ q_\ell = \kappa_2 \ \textit{when}\ \kappa_1 = 0.$$

Then

$$\lim_{\|x\| \to +\infty} \frac{G(x, 0)}{\|x\|^{2\ell - d}} = \frac{1}{q_\ell}\eta_\ell \tag{1.2.4}$$

where

$$\eta_\ell = (2\pi)^{-d} \int_0^{+\infty} \int_{\mathbb{R}^d} \exp\left(\iota\langle \zeta,\, \theta\rangle - \frac{1}{(2d)^\ell}\|\theta\|^{2\ell} t \right) \mathrm{d}\,\theta\, \mathrm{d}\,t$$

for any $\zeta \in \mathbb{S}^{d-1}$.

In case of the DGFF and MM, the maximum of the infinite volume model also was studied by Chiarini et al. [23, 24] and the results are same as those of the finite volume case.

Notation

In the following $C > 0$ always denotes a universal constant whose value however may change in each occurence. We will use $\xrightarrow{d}$ to denote convergence in distribution. We denote, for any $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, $d \geq 1$, the "integer part" of y as $\lfloor y \rfloor =$

$(\lfloor y_1 \rfloor, \ldots, \lfloor y_d \rfloor)$ and similarly $\{y\} = y - \lfloor y \rfloor$ is the "fractional part" of y. For real-valued functions $f(\cdot)$, $g(\cdot)$ we write $f \gg g$, $f \sim g$, $f \approx g$, $f \ll g$ when $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ equals ∞, 1, c and 0, respectively, where c is a non zero constant which may be 1 also. Also we write $f \asymp g$ if there exist two positive constants C_ℓ, C_r such that $C_\ell g(n) \le f(n) \le C_r g(n)$ for all n. We will use round brackets $(\cdot, \cdot)$ to denote the action of a dual space on the original space, and $\langle \cdot, \cdot \rangle$ for inner products.

1.3 Main results

As mentioned earlier, in this thesis we consider the model where the Hamiltonian is given by (1.2.1) and the boundary configuration $\psi \equiv 0$. We investigate a very natural probabilistic question:

"What happens to a random interface when one rescales it suitably?".

We study this scaling limit problem for the model for different values of κ_1 and κ_2. We study the following three models: The membrane model ($\kappa_1 = 0, \kappa_2 = 1/2$), the $(\nabla+\Delta)$-model (κ_1, κ_2 constants, for simplicity we take $\kappa_1 = 1/4d, \kappa_2 = 1/2$) and the model with scaling-dependent κ_1 and κ_2. For the first two models we obtain the scaling limit for the finite volume case in all dimensions and for the infinite volume case in the supercritical dimensions. And for the third model, that is, when κ_1 and κ_2 are scaling-dependent, we consider the different convergence rates of the ratio κ_2/κ_1 and obtain the scaling limit in such cases in all dimensions. In the subcritical dimensions we show convergence in the space of continuous functions and the proofs are completed by showing finite dimensional convergence and tightness. In the finite volume cases, in the critical and supercritical dimensions we show convergence in the space of distributions. In this case we need to look for appropriate spaces of distributions in which we can prove the convergence. As we will see in the proofs that it is the tightness which put restrictions on the choice of such spaces. We give precise description of such spaces where the limiting fields exist and the convergences hold.

In the finite volume case we always assume that our discrete models live well inside the discretization of a suitably chosen bounded domain (open, connected set) in $\mathbb{R}^d$. More precisely, we consider the models in the following set up. Let $d \ge 1$. Let D be a bounded domain in $\mathbb{R}^d$. For $N \in \mathbb{N}$, let $D_N = N\overline{D} \cap \mathbb{Z}^d$. Let us denote by Λ_N the set of points x in D_N such that, for every direction i, j, the points $x \pm e_i$, $x \pm (e_i \pm e_j)$ are all in D_N. In other words, $\Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$ is the largest set satisfying $\partial_2\Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$. We consider the model with $\Lambda = \Lambda_N$ and want to study what happens when we scale it suitably and let N tends to infinity. In the study we crucially use the property (1.2.2) which in our case takes the simple form: for $x \in \Lambda_N$

$$\begin{cases} (-4d\kappa_1\Delta + 2\kappa_2\Delta^2)G_{\Lambda_N}(x,y) = \delta_x(y) & y \in \Lambda_N \\ G_{\Lambda_N}(x,y) = 0 & y \in \partial_k\Lambda_N, \end{cases} \tag{1.3.1}$$

where $k = 1$ if $\kappa_2 = 0$ and $k = 2$ if $\kappa_2 > 0$. One might expect form here that the limiting fields should have connections with the corresponding continuum elliptic operator $(-4d\kappa_1\Delta_c + 2\kappa_2\Delta_c^2)$, where Δ_c is the continuum Laplacian defined by

$$\Delta_c := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

This is indeed the case as we will see in the next subsections. To prove the results there we use either the convergence of the Green's function or the convergence of the solution of the Dirichlet problems of the discrete approximation operator to the corresponding continuum counter part. In the infinite volume cases also the limiting fields are defined using the continuum elliptic operators. In these cases we show convergence using (1.2.3) and Fourier analysis.

The membrane model ($\kappa_1 = 0, \kappa_2 = 1/2$)

In Chapter 2, which is based on the article [28], we consider the membrane model and the main results are as follows. We study the scaling limit of this model for both the finite volume and the infinite volume measures. Depending on the dimension we have two different types of results.

(i) Convergence in subcritical dimension ($d = 2, 3$): in this case we obtain convergence in the space of continuous functions. For simplicity we consider $D = (-1, 1)^d$ and $D_N = N\overline{D} \cap \mathbb{Z}^d$, where $N \in \mathbb{N}$. Let $(\varphi_x)_{x \in D_{N-1}}$ be the membrane model on D_{N-1}. First we want to define a *continuous* interpolation ψ_N of the discrete field to have convergence in the space of continuous functions. In $d = 2$, the interpolated field

$(\psi_N(t))_{t\in\overline{D}}$ is defined by

$$\psi_N(t) = \frac{1}{2dN}\left(\varphi_{\lfloor Nt\rfloor} + \{Nt_i\}(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}) + \{Nt_j\}(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i})\right), \quad \text{if}\,\{Nt_i\} \geq \{Nt_j\}$$

where $t = (t_1\,,\, t_2) \in \overline{D}$ and $i,\, j \in \{1,\, 2\}$, $i \neq j$. And in $d = 3$, $(\psi_N(t))_{t\in\overline{D}}$ is defined by

$$\begin{aligned}\psi_N(t) = \frac{1}{2d\sqrt{N}}&\left(\varphi_{\lfloor Nt\rfloor} + \{Nt_i\}\left(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}\right)\right.\\ &+\{Nt_j\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i}\right)\\ &\left.+\{Nt_k\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j+e_k} - \varphi_{\lfloor Nt\rfloor+e_i+e_j}\right)\right), \quad \{Nt_i\} \geq \{Nt_j\} \geq \{Nt_k\}\end{aligned}$$

where $t = (t_1\,,\, t_2\,,\, t_3) \in \overline{D}$ and $i,\, j,\, k \in \{1,\, 2,\, 3\}$ are pairwise different. We show that there exists a centered continuous Gaussian process $\psi_D^{\Delta^2}$ on $\overline{D}$ with covariance $G_D(\cdot,\, \cdot)$, the Green's function for the following continuum Dirichlet problem:

$$\begin{cases}\Delta_c^2 u(x) = f(x), & x \in D\\ D^\alpha u(x) = 0, & |\alpha| \leq 1,\, x \in \partial D,\end{cases}$$

where ∂D is the boundary of the domain D and for $\alpha = (\alpha_1,\, \ldots,\, \alpha_d)$ a multi-index with α_i being non-negative integers

$$D^\alpha u := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u,$$

$$|\alpha| := \sum_{i=1}^{d} \alpha_i.$$

Then ψ_N converges in distribution to $\psi_D^{\Delta^2}$ in the space of all continuous functions on $\overline{D}$. Furthermore the process $\psi_D^{\Delta^2}$ is almost surely Hölder continuous with exponent η, for every $\eta \in (0,\, 1)$ resp. $\eta \in (0,\, 1/2)$ in $d = 2$ resp. $d = 3$. As a consequence we obtain the scaling limit of the discrete maximum. Let $M_N := \max_{x\in D_N} \varphi_x$. Then as $N \uparrow \infty$

$$(2d)^{-1} N^{\frac{d-4}{2}} M_N \xrightarrow{d} \sup_{x\in\overline{D}} \psi_D^{\Delta^2}(x).$$

(ii) Convergence in critical and supercritical dimension ($d \geq 4$): in this case we obtain convergence in the space of distributions. Let D be a bounded domain in $\mathbb{R}^d$ with smooth boundary[1]. We briefly give the definition of the Sobolev space $\mathcal{H}^{-s}_{\Delta^2}(D)$ and the continuum membrane model. For a more detailed discussion see Chapter 2. By the spectral theorem for compact self-adjoint operators and elliptic regularity one can show that there exist smooth eigenfunctions $\{u_j\}_{j\in\mathbb{N}}$ of Δ_c^2 corresponding to the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ such that $\{u_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $L^2(D)$. Now for any $s > 0$ we define the following inner product on $C_c^\infty(D)$:

$$\langle f\,,\, g\rangle_{s,\,\Delta^2} := \sum_{j\in\mathbb{N}} \lambda_j^{s/2} \langle f\,,\, u_j\rangle_{L^2} \langle u_j\,,\, g\rangle_{L^2}\,.$$

Then $\mathcal{H}^s_{\Delta^2,0}(D)$ is defined to be the Hilbert space completion of $C_c^\infty(D)$ with respect to this inner product. We define $\mathcal{H}^{-s}_{\Delta^2}(D)$ to be its dual and the dual norm is denoted by $\|\cdot\|_{-s,\,\Delta^2}$. The following definition is from Chapter 2 and provides a description of the continuum membrane model $\Psi_D^{\Delta^2}$.

Definition 1.3.1 (Continuum membrane model). *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{\Delta^2} := \sum_{j\in\mathbb{N}} \lambda_j^{-1/2} \xi_j u_j.$$

Then $\Psi_D^{\Delta^2} \in \mathcal{H}^{-s}_{\Delta^2}(D)$ a.s. for all $s > (d-4)/2$ and is called the continuum membrane model.

Consider Λ_N as defined before. Let $(\varphi_x)_{x\in\Lambda_N}$ be the membrane model on Λ_N. Define Ψ_N by

$$(\Psi_N,\, f) := (2d)^{-1} \sum_{x\in\frac{1}{N}\Lambda_N} N^{-\frac{d+4}{2}} \varphi_{Nx} f(x)\,,\;\; f \in \mathcal{H}^s_{\Delta^2,0}(D).$$

[1]By a bounded domain D in $\mathbb{R}^d$ with smooth boundary we mean that at each point x on the boundary there is an open ball $B = B(x)$ centering the point x and a one-to-one smooth map ζ from B onto the unit ball in $\mathbb{R}^d$ such that $\zeta(B\cap D) \subset \mathbb{R}^d_+$, $\zeta(B\cap\partial D) \subset \partial\,\mathbb{R}^d_+$ and ζ^{-1} is smooth. Here $\mathbb{R}^d_+$ is the half space $\{y \in \mathbb{R}^d : y_d > 0\}$.

We show that, as $N \to \infty$, the field Ψ_N converges in distribution to $\Psi_D^{\Delta^2}$ in the topology of $\mathcal{H}_{\Delta^2}^{-s}(D)$ for $s > s_d$, where

$$s_d := \frac{d}{2} + 2\left(\left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2}\right\rfloor + 1\right)\right\rceil + \left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2}\right\rfloor + 6\right)\right\rceil - 1\right). \tag{1.3.2}$$

(iii) Infinite volume ($d \geq 5$): we also obtain the scaling limit in the infinite volume membrane model defined on the whole of $\mathbb{Z}^d$ and show that the rescaled field converges to the continuum bilaplacian field on $\mathbb{R}^d$. Let us first define the limiting field. For $f \in \mathcal{S}$, the Schwarz space, we define $\widehat{f}$ by

$$\widehat{f}(\theta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\iota\langle x,\theta\rangle} f(x) \,\mathrm{d}\, x.$$

Let us define an operator $(-\Delta_c)^{-1} : \mathcal{S} \to L^2(\mathbb{R}^d)$ as follows [1, Section 1.2.2]:

$$(-\Delta_c)^{-1} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\iota\langle x,\xi\rangle} \|\xi\|^{-2} \widehat{f}(\xi) \,\mathrm{d}\, \xi.$$

We use now the operator $(-\Delta_c)^{-1}$ to define the limiting field Ψ^{Δ^2}. The limiting field Ψ^{Δ^2} is a random variable taking values in $\mathcal{S}^*$ whose characteristic functional $\mathcal{L}_{\Psi^{\Delta^2}}$ is given by

$$\mathcal{L}_{\Psi^{\Delta^2}}(f) = \exp\left(-\frac{1}{2}\|(-\Delta_c)^{-1} f\|^2_{L^2(\mathbb{R}^d)}\right), \quad f \in \mathcal{S}.$$

To study scaling limit we consider $(\varphi_x)_{x\in\mathbb{Z}^d}$ to be the membrane model in $d \geq 5$ and define

$$\psi_N(x) := (2d)^{-1} N^{\frac{d-4}{2}} \varphi_{Nx}, \quad x \in \frac{1}{N}\mathbb{Z}^d.$$

For $f \in \mathcal{S}$ we define

$$(\Psi_N, f) := N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \psi_N(x) f(x).$$

Then $\Psi_N \in \mathcal{S}^*$ and the characteristic functional of Ψ_N is given by

$$\mathcal{L}_{\Psi_N}(f) = \exp(-\mathbf{Var}\,(\Psi_N, f)\,/2).$$

We show that $\Psi_N \xrightarrow{d} \Psi^{\Delta^2}$ in the strong topology of $\mathcal{S}^*$.

The $(\nabla + \Delta)$-model ($\kappa_1 = 1/4d, \kappa_2 = 1/2$)

In Chapter 3, we consider the model with $\kappa_1 = 1/4d$ and $\kappa_2 = 1/2$. We call this model the $(\nabla + \Delta)$-model. The details of Chapter 3 are based on the article [27].

This model interpolates between two well-known random interfaces, namely the discrete Gaussian free field and the membrane model. In [15, Remark 9] it was conjectured that, in the case of pinning for the one-dimensional $(\nabla+\Delta)$-model, the behaviour of the free energy should resemble the purely gradient case. In view of this remark it is natural to ask if the scaling limit of the mixed model is dominated by the gradient interaction, that is, the limit is the Gaussian free field (GFF). The main focus is to show that such a guess is true and indeed in any dimension the mixed model approximates the Gaussian free field.

From Proposition 1.2.2 it follows that the infinite volume limit exists if and only if $d \geq 3$. In this case also we study the scaling limit for both the finite volume and the infinite volume model. We first discuss the results in the finite volume case. We have two different types of results depending on the dimension as follows.

(i) Convergence in $d = 1$: in the subcritical case we obtain the convergence in the space of continuous functions. In this case for simplicity we consider $D = (0, 1)$ and the corresponding D_N and Λ_N as defined before, in particular $\Lambda_N = \{2, \ldots, N-2\}$. To study the scaling limit we define a continuous interpolation ψ_N for each N as follows:

$$\psi_N(t) = (2d)^{-\frac{1}{2}} N^{-\frac{1}{2}} \left[\varphi_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)(\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor})\right], \quad t \in \overline{D}.$$

We then show that ψ_N converges in distribution to the Brownian bridge on $[0, 1]$ in the space $C[0, 1]$. As a by-product of this result we obtain the convergence of the discrete maxima.

(ii) Convergence in $d \geq 2$: in this case the convergence is obtained in the space of distributions. We consider D to be a bounded domain in $\mathbb{R}^d$ with smooth boundary. We briefly give the definition of the Sobolev space $\mathcal{H}^{-s}_{-\Delta}(D)$ and the Gaussian free field. For a detail discussion see Chapter 3. By the spectral theorem for compact self-adjoint operators and elliptic regularity we know that there exist smooth eigenfunctions $(w_j)_{j\in\mathbb{N}}$ of $-\Delta_c$ corresponding to the eigenvalues $0<\nu_1\le\nu_2\le\cdots\to\infty$ such that $(w_j)_{j\ge 1}$ is an orthonormal basis of $L^2(D)$. Now for any $s>0$ we define the following inner product on $C_c^\infty(D)$:

$$\langle f\,,\,g\rangle_{s,\,-\Delta} := \sum_{j\in\mathbb{N}} \nu_j^s \langle f\,,\,w_j\rangle_{L^2}\langle w_j\,,\,g\rangle_{L^2}.$$

Then $\mathcal{H}^s_{-\Delta,0}(D)$ can be defined to be the completion of $C_c^\infty(D)$ with respect to this inner product. We define $\mathcal{H}^{-s}_{-\Delta}(D)$ to be its dual and the dual norm is denoted by $\|\cdot\|_{-s,\,-\Delta}$. We give the definition of the Gaussian free field whose well-definedness is proved in Proposition 3.4.4.

Definition 1.3.2 (Gaussian free field)**.** *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{-\Delta} := \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\xi_j w_j.$$

Then $\Psi_D^{-\Delta}\in\mathcal{H}^{-s}_{-\Delta}(D)$ a.s. for all $s>d/2-1$ and is called the Gaussian free field.

We define Λ_N as before and consider the model $(\varphi_x)_{x\in\Lambda_N}$ on Λ_N. We then define Ψ_N by

$$(\Psi_N,\,f) := (2d)^{-\frac{1}{2}} \sum_{x\in\frac{1}{N}\Lambda_N} N^{-\frac{d+2}{2}}\varphi_{Nx}f(x)\,,\quad f\in\mathcal{H}^s_{-\Delta,0}(D).$$

The result we show is that Ψ_N converges in distribution to the Gaussian free field $\Psi_D^{-\Delta}$ as $N\to\infty$ in the topology of $\mathcal{H}^{-s}_{-\Delta}(D)$ for $s>d$.

(iii) Infinite volume ($d\ge 3$): finally we study the scaling limit of the infinite volume model. We consider the infinite volume model $\varphi=(\varphi_x)_{x\in\mathbb{Z}^d}$ with law $\mathbf{P}$. We define for $N\in\mathbb{N}$

$$\psi_N(x) := (2d)^{-\frac{1}{2}} N^{\frac{d-2}{2}}\varphi_{Nx},\quad x\in\frac{1}{N}\mathbb{Z}^d\,.$$

For $f\in\mathcal{S}$ we define

$$(\Psi_N, f) := N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\psi_N(x)f(x).$$

The limiting field in this case is defined to be the field $\Psi^{-\Delta}$ on $\mathcal{S}^*$ whose characteristic functional $\mathcal{L}_{\Psi^{-\Delta}}$ is given by

$$\mathcal{L}_{\Psi^{-\Delta}}(f) = \exp\left(-\frac{1}{2}\|(-\Delta_c)^{-1/2} f\|^2_{L^2(\mathbb{R}^d)}\right), \quad f \in \mathcal{S},$$

where the operator $(-\Delta_c)^{-1/2} : \mathcal{S} \to L^2(\mathbb{R}^d)$ is defined by

$$(-\Delta_c)^{-1/2} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\iota\langle x,\xi\rangle} \|\xi\|^{-1} \widehat{f}(\xi)\, \mathrm{d}\,\xi.$$

We show in this case that $\Psi_N \xrightarrow{d} \Psi^{-\Delta}$ in the strong topology of $\mathcal{S}^*$.

The model with scaling-dependent κ_1 and κ_2

Notice that in the $(\nabla + \Delta)$-model in the limit the contribution of the part corresponding to the Laplacian gets dominated by the other term and we get Gaussian free field as the limit. Hence it is a very natural question to study what happens if we increase the strength of the Laplacian part. More specifically, let $d \geq 1$ and D be a bounded domain in $\mathbb{R}^d$. We define Λ_N as before and consider the model with $\Lambda = \Lambda_N, \kappa_1 = 1/4d, \kappa_2 = \kappa(N)/2$. We want to study what happens when we tune the parameter $\kappa(N)$ suitably as N tends to infinity. We study this question in Chapter 4, which is based on the article [29]. We assume κ_1 to be constant as it is easy to present the results in this format. Also for simplicity we write κ for $\kappa(N)$. The results for this model is split into two parts. In lower dimensions we have convergence in the space of continuous functions and in higher dimensions the convergence occurs in the space of distributions.

- *Lower dimensional results*

  In this case we consider $D = (0,1)^d$. Also here, according to the behaviour of κ as $N \to \infty$ we have three different limits. We define the continuous interpolation $\{\psi_N\}_{N\in\mathbb{N}}$ in the following fashion:

  - For $d = 1$ and $t \in \overline{D}$

$$\psi_N(t) = \mathbf{c}_N(1)\left[\varphi_{\lfloor Nt\rfloor} + (Nt - \lfloor Nt\rfloor)(\varphi_{\lfloor Nt\rfloor+1} - \varphi_{\lfloor Nt\rfloor})\right]. \tag{1.3.3}$$

– For $d = 2$ and $t = (t_1, t_2) \in \overline{D}$

$$\begin{aligned}\psi_N(t) = \mathbf{c}_N(2) \big[\varphi_{\lfloor Nt \rfloor} + \{Nt_i\} \left(\varphi_{\lfloor Nt \rfloor + e_i} - \varphi_{\lfloor Nt \rfloor}\right) \\ + \{Nt_j\} \left(\varphi_{\lfloor Nt \rfloor + e_i + e_j} - \varphi_{\lfloor Nt \rfloor + e_i}\right)\big], \quad \text{if } \{Nt_i\} \geq \{Nt_j\}\end{aligned} \tag{1.3.4}$$

where $i,\, j \in \{1,\, 2\}$, $i \neq j$.

– For $d = 3$ and $t = (t_1, t_2, t_3) \in \overline{D}$

$$\begin{aligned}\psi_N(t) = \mathbf{c}_N(3) \big[\varphi_{\lfloor Nt \rfloor} + \{Nt_i\} \left(\varphi_{\lfloor Nt \rfloor + e_i} - \varphi_{\lfloor Nt \rfloor}\right) \\ + \{Nt_j\} \left(\varphi_{\lfloor Nt \rfloor + e_i + e_j} - \varphi_{\lfloor Nt \rfloor + e_i}\right) \\ + \{Nt_k\} \left(\varphi_{\lfloor Nt \rfloor + e_i + e_j + e_k} - \varphi_{\lfloor Nt \rfloor + e_i + e_j}\right)\big], \quad \text{if } \{Nt_i\} \geq \{Nt_j\} \geq \{Nt_k\}\end{aligned} \tag{1.3.5}$$

where $i,\, j,\, k \in \{1,\, 2,\, 3\}$ and pairwise different. Here $\mathbf{c}_N(d)$, $d = 1,\, 2,\, 3$, are scaling factors which are specified in the following result.

We have the following convergence results.

(i) $\kappa \gg N^2$. Let $1 \leq d \leq 3$. Define a continuously interpolated field ψ_N as in (1.3.3), (1.3.4) and (1.3.5) with

$$\mathbf{c}_N(d) = (2d)^{-1} \sqrt{\kappa} N^{\frac{d-4}{2}}.$$

Then we have, as $N \to \infty$, that the field ψ_N converges in distribution to $\psi_D^{\Delta^2}$ in the space of continuous functions on $\overline{D}$, where $\psi_D^{\Delta^2}$ is defined to be the centered continuous Gaussian process on $\overline{D}$ with covariance $G_D(\cdot,\, \cdot)$, the Green's function for the biharmonic operator (as defined in Subsection 1.3).

(ii) $\kappa \sim 2dN^2$. Let $1 \leq d \leq 3$. Define a continuously interpolated field ψ_N as in (1.3.3), (1.3.4) and (1.3.5) with

$$\mathbf{c}_N(d) = (2d)^{-1} \sqrt{\kappa} N^{\frac{d-4}{2}}.$$

Define $\psi_D^{-\Delta+\Delta^2}$ to be the continuous Gaussian process in $\overline{D}$ with covariance $G_D(\cdot,\cdot)$, where G_D is the Green's function for the problem

$$\begin{cases}(-\Delta_c + \Delta_c^2)u(x) = f(x), & x \in D\\ D^\alpha u(x) = 0, & \forall\,|\alpha| \le 1,\, x \in \partial D.\end{cases}$$

Then ψ_N converges in distribution to the field $\psi_D^{-\Delta+\Delta^2}$ in the space of continuous functions on $\overline{D}$.

(iii) $\kappa \ll N^2$. Let $d = 1$. Define the continuously interpolated field ψ_N as in (1.3.3) with

$$\mathbf{c}_N(1) = (2)^{-\frac{1}{2}} N^{-\frac{1}{2}}.$$

Then as $N \to \infty$, ψ_N converges in distribution to the Brownian bridge, $\psi_D^{-\Delta}$, in the space of continuous functions on $\overline{D}$.

We remark here that in all the above three cases one can obtain the convergence of the discrete maximum.

- Higher dimensional results:

Assume that D has smooth boundary. We define the space $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ analogously to $\mathcal{H}^{-s}_{\Delta^2}(D)$. One can find smooth eigenfunctions $\{v_j\}_{j\in\mathbb{N}}$ of $-\Delta_c + \Delta_c^2$ corresponding to eigenvalues $0 < \mu_1 \le \mu_2 \le \cdots \to \infty$ such that $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(D)$. One can define, for $s > 0$, the following inner product for functions from $C_c^\infty(D)$:

$$\langle f, g\rangle_{s,-\Delta+\Delta^2} := \sum_{j\in\mathbb{N}} \mu_j^{s/2} \langle f, v_j\rangle_{L^2} \langle v_j, g\rangle_{L^2}.$$

Let $\mathcal{H}^s_{-\Delta+\Delta^2,0}(D)$ be the completion of $C_c^\infty(D)$ with the above inner product and $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ be its dual. The dual norm is denoted by $\|\cdot\|_{-s,-\Delta+\Delta^2}$. We describe the details on this space in Section 4.6. The well posedness of the series in the following definition is proved in Proposition 4.6.3 in Section 4.6.

Definition 1.3.3 (Continuum mixed model)**.** *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{-\Delta+\Delta^2} := \sum_{j\in\mathbb{N}} \mu_j^{-1/2} \xi_j v_j.$$

Then $\Psi_D^{-\Delta+\Delta^2} \in \mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ *a.s. for all* $s > (d-4)/2$ *and we call it the continuum mixed model.*

Depending on the behaviour of κ as $N \to \infty$ we have the following three convergence results.

(i) $\kappa \gg N^2$. Let $d \geq 4$. Define Ψ_N by

$$(\Psi_N, f) := (2d)^{-1}\sqrt{\kappa}N^{-\frac{d+4}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}^s_{\Delta^2,0}(D).$$

Then we have, as $N \to \infty$, that the field Ψ_N converges in distribution to the continuum membrane model $\Psi_D^{\Delta^2}$ in the topology of $\mathcal{H}^{-s}_{\Delta^2}(D)$ for $s > s_d$, where s_d is defined in (1.3.2).

(ii) $\kappa \sim 2dN^2$. Let $d \geq 4$. Define Ψ_N by

$$(\Psi_N, f) := (2d)^{-1}\sqrt{\kappa}N^{-\frac{d+4}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}^s_{-\Delta+\Delta^2,0}(D).$$

Then, as $N \to \infty$, the field Ψ_N converges in distribution to $\Psi_D^{-\Delta+\Delta^2}$ in the topology of $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ for $s > s_d$ where s_d is defined in (1.3.2).

(iii) $\kappa \ll N^2$. Let $d \geq 2$. Define Ψ_N by

$$(\Psi_N, f) := (2d)^{-\frac{1}{2}}N^{-\frac{d+2}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}^s_{-\Delta,0}(D).$$

Then, as $N \to \infty$, the field Ψ_N converges in distribution to the Gaussian free field $\Psi_D^{-\Delta}$ in the topology of $\mathcal{H}^{-s}_{-\Delta}(D)$ for $s > d/2 + \lfloor d/2 \rfloor + 2$.

Remark 1.3.4. *The choice* $\kappa \sim 2dN^2$ *is just for simplicity. Instead, one can work with* $\kappa \sim c_0 N^2$ *for a generic* $c_0 > 0$. *In that case the operator* $-\Delta_c + \Delta_c^2$ *needs to be replaced with* $-\Delta_c + \frac{c_0}{2d}\Delta_c^2$.

1.4 A main ingredient in the proofs

We prove all the above results by showing finite dimensional convergence and tightness. While doing so we heavily use property (1.2.2) of the covariance functions of the models.

More precisely, we estimate the error between the solutions of the Dirichlet problems involving the discrete elliptic operator and its continuous counterpart. Then we use those estimates to prove our results. In estimating the errors we use the idea of Thomée [69]. In some cases our set up falls under the more general set up in [69]. Unfortunately, the constants involved in those error bounds depends on the smoothness of solution. So even in those cases we need to improve his result quantitatively for our use. In other cases our set up becomes much more specific and hence requires more care in following his technique for the error estimation. For the model with scaling-dependent κ_1, κ_2 the idea of Thomée [69] falls short in the case $\kappa \ll N^2$. In this case we use a suitable cut-off function to estimate the error.

Let us see briefly how his idea works in case of the DGFF. Let $d \geq 2$ and let D be a bounded domain in $\mathbb{R}^d$ with smooth boundary. Let us consider the DGFF on the corresponding Λ_N and define Ψ_N by

$$(\Psi_N,\, f) := (2d)^{-\frac{1}{2}} N^{-\frac{d+2}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}^s_{-\Delta,0}(D).$$

We want to show that as $N \to \infty$, the field Ψ_N converges in distribution to the Gaussian free field $\Psi_D^{-\Delta}$ in the topology of $\mathcal{H}^{-s}_{-\Delta}(D)$ for $s > d/2 + \lfloor d/2 \rfloor + 3/2$. We first show the finite dimensonal convergence. For $f \in C_c^\infty(D)$ we have

$$\mathbf{Var}[(\Psi_N, f)] = N^{-d} \sum_{x \in \frac{1}{N}\Lambda_N} H_N(x) f(x),$$

where

$$H_N(x) := N^{-2} \sum_{y \in \frac{1}{N}\Lambda_N} G_{\Lambda_N}(Nx, Ny) f(y),\ x \in \frac{1}{N}\mathbb{Z}^d.$$

Now using (1.2.2) one can show that H_N satisfies the following discrete Dirichlet problem:

$$\begin{cases} -\Delta_{\frac{1}{N}} H_N(x) = f(x) & x \in \frac{1}{N}\Lambda_N \\ H_N(x) = 0 & x \notin \frac{1}{N}\Lambda_N. \end{cases}$$

Here, for any $h > 0$ the discrete approximation Δ_h of the Laplace operator is defined by

$$\Delta_h u(x) := \frac{1}{h^2} \sum_{i=1}^{d} (u(x + he_i) + u(x - he_i) - 2u(x)).$$

We now use Thomée [69]'s idea (in this case [69, Theorem 5.2]) to show that H_N in some sense is close to the solution u of the continuum Dirichlet problem

$$\begin{cases} -\Delta_c u(x) = f(x) & x \in D \\ u(x) = 0 & x \in \partial D. \end{cases}$$

More precisely, if we define $e_N(x) := H_N(x) - u(x)$ for x in the the set $\overline{D} \cap \frac{1}{N}\mathbb{Z}^d$, then

$$N^{-d} \sum_{x \in \frac{1}{N}\Lambda_N} e_N(x)^2 \leq C\frac{1}{N}.$$

Using this estimate one can show that

$$\mathbf{Var}[(\Psi_N, f)] \to_{N\to\infty} \int_D u(x) f(x) \mathrm{d}x = \mathbf{Var}[(\Psi_D^{-\Delta}, f)].$$

To complete the proof of finite dimensional convergence one just need to use the Gaussianity and the fact $C_c^\infty(D)$ is dense in $\mathcal{H}^s_{-\Delta,0}(D)$. In order to show tightness, one can obtain the constant C in the above estimation explicitly in terms of u and its partial derivatives and then show that

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,-\Delta}] < \infty \; \forall s > d/2 + \lfloor d/2 \rfloor + 3/2.$$

Note that here we are getting some restriction on s as far as the space of convergence is concerned. Tightness now follows as an application of Rellich's theorem. Thus Thomée [69]'s idea of error estimation indeed helps one to show that the scaling limit of the DGFF is the Gaussian free field. We prove our results in similar manner. The proofs are more involved and need lot more care in showing the estimates and different other things as we go beyond DGFF and the operator $-\Delta_c$. The thesis is mostly concerned with making the above ideas suitable for different interface models.

Chapter 2

The scaling limit of the membrane model

2.1 Introduction

In this chapter we study the scaling limit of the membrane model (MM), also known as discrete bilaplacian model. The membrane model is a special instance of interface models. It is the Gaussian interface for which

$$H(\varphi) := \frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\Delta \varphi_x|^2. \tag{2.1.1}$$

That is, MM is the field $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$, whose distribution is determined by the probability measure on $\mathbb{R}^{\mathbb{Z}^d}$, $d \geq 1$, with density

$$\mathbf{P}_\Lambda(\mathrm{d}\varphi) := \frac{1}{Z_\Lambda} \exp\left(-\frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\Delta \varphi_x|^2 \right) \prod_{x \in \Lambda} \mathrm{d}\varphi_x \prod_{x \in \mathbb{Z}^d \setminus \Lambda} \delta_0(\mathrm{d}\varphi_x),$$

where $\Lambda \Subset \mathbb{Z}^d$ is a finite subset, $\mathrm{d}\varphi_x$ is the 1-dimensional Lebesgue measure on $\mathbb{R}$, δ_0 is the Dirac measure at 0, and Z_Λ is a normalising constant. We are imposing zero boundary conditions i.e. almost surely $\varphi_x = 0$ for all $x \in \mathbb{Z}^d \setminus \Lambda$, but the definition holds for more general boundary conditions. In case $\Lambda = V_N := [-N, N]^d \cap \mathbb{Z}^d$, we will denote the measure $\mathbf{P}_{V_N}$ with Hamiltonian (2.1.1) by $\mathbf{P}_N$. Introduced by Sakagawa [60] in the probabilistic literature, the MM looks for certain aspects very similar to the DGFF: it is log-correlated in $d = 4$, is supercritical in $d \geq 5$ and is subcritical in $d \leq 3$. In particular in $d \leq 4$ there is no thermodynamic limit of the measures $\mathbf{P}_N$ as $N \uparrow \infty$. The MM displays however certain crucial difficulties, in that for example it exhibits no random walk representation, and several correlation inequalities are lacking. In this framework we present our work which aims at determining the scaling limit of the bilaplacian model. The answer in $d = 1$ was given by Caravenna and Deuschel [21], who also look at the situation in which a pinning force is added to the model. We complement their work by determining the scaling limit in all $d \geq 2$.

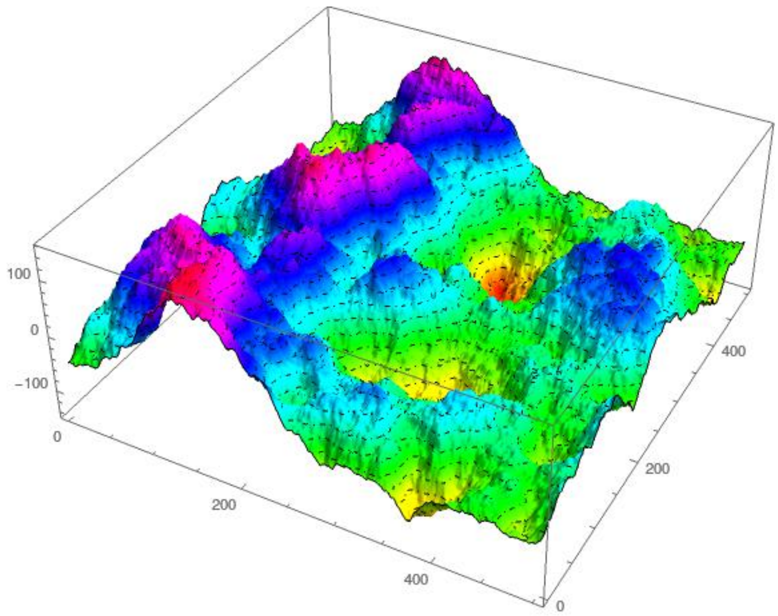

FIGURE 2.1: A sample of the MM in $d = 2$ on a box of side-length 500.

The main contributions are as follows:

♠ in $d = 2, 3$ we consider the discrete membrane model on a box of side-length $2N$ and interpolate it in a continuous way. We show that the process converges to a real-valued process with continuous trajectories and the convergence takes place in the space of continuous functions (see Theorem 2.2.1). The utility of this type of convergence is that it yields the scaling limit of the discrete maximum exploiting the continuous mapping theorem (Corollary 2.2.2). While the limiting maximum of the discrete membrane model was derived in $d \geq 5$ by Chiarini et al. [24] and in $d = 4$ by Schweiger et al. [64]. The limit field also turns to be Hölder continuous with exponent less than 1 in $d = 2$ and less than $1/2$ in $d = 3$.

The proof of the above facts is based on two basic steps: tightness and finite dimensional convergence. Tightness depends on the gradient estimates of the discrete Green's functions which were very recently derived in [54]; finite dimensional convergence follows from the convergence of the Green's function.

♠ In $d \geq 4$ the limiting process on a sufficiently nice domain D will be a fractional Gaussian field with Hurst parameter $H := s - d/2$ on D. The theory of fractional Gaussian fields was surveyed recently in [51]. The authors there construct the continuum membrane model using characteristic functionals. We take here a bit different route and give a representation using the eigenvalues of the biharmonic operator in the continuum. We remark however that these eigenvalues differ from the square of the Laplacian eigenvalues due to boundary conditions. The GFF theory which is based on $H_0^1(D)$ (the first order Sobolev space) needs to be replaced by $H_0^2(D)$ (second order Sobolev space).

Our main result is given in Theorem 2.3.11. Its proof is again split into two steps: finite dimensional convergence and tightness. Both steps crucially require an approximation result of PDEs given by Thomée [69]: there he gives quantitative estimates on the approximation of solutions of PDEs involving "nice" elliptic operators by their discrete counterparts. We believe that the techniques used in this chapter might have implications in the development of the theory of the membrane model, in particular the idea of tackling boundary values by rescaling the standard discrete Sobolev norm around the boundary. Especially in $d = 4$ this allows one to overcome the difficulty of extending estimates from the bulk up to the boundary, which is generally one stumbling block in the study of the MM.

♠ In $d \geq 5$ we also consider the infinite volume membrane model on $\mathbb{Z}^d$. We show in Lemma 2.4.2 that the limit is the fractional Gaussian field of Hurst parameter $H := 2 - d/2 < 0$ on $\mathbb{R}^d$ (see [51]) and we prove in Theorem 2.4.3 the convergence with the help of characteristic functionals. We utilise the classical result of Fernique [36] (recently extended in the tempered distribution setting by Biermé et al. [6]) stating that convergence of tempered distributions is equivalent to that of their characteristic functionals. Technical tools useful for this scope are the explicit Fourier transform of the infinite volume Green's function and the Poisson summation formula.

We stress that, regardless of the dimension, the field is always rescaled as $N^{(d-4)/2}\varphi_{Nx}$ for $x \in N^{-1}\mathbb{Z}^d$. Heuristically, the factor N^{4-d} corresponds to the order of growth of the variance of the model in a box, which we recall here for completeness.

i) In $d = 2, 3$ if $d(\cdot)$ denotes the distance to the boundary of V_N one has for some constant $C > 0$ [54, Theorem 1.1]

$$|\mathbf{Cov}_N(\varphi_x, \varphi_y)| \leq C \min\left(d(x)^{2-d/2} d(y)^{2-d/2}, \frac{d(x)^2 d(y)^2}{(\|x-y\|+1)^2}\right).$$

ii) In $d = 4$ let us denote the bulk of V_N by $V_N^\delta := \{x \in V_N : d(x) > \delta N\}$ for $\delta \in (0, 1)$. Then from [25, Lemma 2.1] we have: there exists a constant $C(\delta) > 0$ such that

$$\sup_{x,y \in V_N^\delta} \left|\mathbf{Cov}_N(\varphi_x, \varphi_y) - \frac{8}{\pi^2}(\log N - \log(\|x-y\|+1))\right| \leq C(\delta).$$

iii) In $d \geq 5$ the infinite volume covariance satisfies (see Lemma 1.2.3)

$$|\mathbf{Cov}(\varphi_x, \varphi_y)| \sim C_d \|x-y\|^{4-d} \text{ as } \|x-y\| \to \infty.$$

Interestingly this reflects the behavior of the characteristic singular solution (fundamental solution) of the biharmonic equation, which is

$$\begin{cases} C_d \|x\|^{4-d} & d \text{ odd or } d \text{ even and } d \geq 6 \\ C_d \|x\|^{4-d} \log\|x\| & d \text{ even and } d \leq 4. \end{cases}$$

The reader can consult [52], [53, Section 5] and references therein for sharp pointwise estimates of the Green's function of the bilaplacian in general domains and for regularity properties of the biharmonic Green's function. We fix a constant $\kappa := (2d)^{-1}$ throughout the whole chapter.

2.2 Convergence in $d = 2,\, 3$

2.2.1 Description of the limiting field

Let $D = (-1, 1)^d$ and $D_N = N\overline{D} \cap \mathbb{Z}^d$, where $N \in \mathbb{N}$. Let $(\varphi_x)_{x\in D_{N-1}}$ be the MM on D_{N-1} and let G_{N-1} be the covariance function for this model. From (1.2.2) it follows that G_{N-1} satisfies the following discrete boundary value problem for all $x \in D_{N-1}$:

$$\begin{cases} \Delta^2 G_{N-1}(x,\, y) = \delta_x(y), & y \in D_{N-1} \\ G_{N-1}(x,\, y) = 0, & y \notin D_{N-1} \end{cases}.$$

First we want to define a *continuous* interpolation ψ_N of the discrete field to have convergence in the space of continuous functions. There are many ways to define the field $(\psi_N(t))_{t\in\overline{D}}$. We take one of the simplest geometric ways which is akin to the interpolation of simple random walk trajectories in Donsker's invariance principle. Mind that we take the domain as a square since the recent gradient estimates and convergence of the Green's function of Müller and Schweiger [54] can be applied easily.

Interpolation in $d = 2$. Let $t = (t_1\,,\, t_2) \in \overline{D}$. Then $p := Nt$ lies in the square box with vertices $a = \lfloor Nt\rfloor, b = \lfloor Nt\rfloor + e_1, c = \lfloor Nt\rfloor + e_1 + e_2, d = \lfloor Nt\rfloor + e_2$, where $e_1,\, e_2$ are the standard basis vectors of $\mathbb{R}^2$. Suppose p is a point in the triangle abc. Then we can write $p = \alpha a + \beta b + \gamma c$ with $\alpha = 1 - \{Nt_1\}$, $\beta = \{Nt_1\} - \{Nt_2\}$, $\gamma = \{Nt_2\}$. And in this case we define

$$\psi_N(t) = \frac{\kappa}{N}[\alpha\varphi_{\lfloor Nt\rfloor} + \beta\varphi_{\lfloor Nt\rfloor+e_1} + \gamma\varphi_{\lfloor Nt\rfloor+e_1+e_2}].$$

Similarly, if $p \in \triangle acd$ then we define

$$\psi_N(t) = \frac{\kappa}{N}[\alpha'\varphi_{\lfloor Nt\rfloor} + \beta'\varphi_{\lfloor Nt\rfloor+e_2} + \gamma'\varphi_{\lfloor Nt\rfloor+e_1+e_2}]$$

where

$$\alpha' = 1 - \{Nt_2\},\ \beta' = \{Nt_2\} - \{Nt_1\},\ \gamma' = \{Nt_1\}.$$

Thus the interpolated field $(\psi_N(t))_{t\in\overline{D}}$ is defined by

$$\begin{aligned}\psi_N(t) = \frac{\kappa}{N}[&\varphi_{\lfloor Nt\rfloor} + \{Nt_i\}\left(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}\right)\\ &+ \{Nt_j\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i}\right)], \quad \text{if}\,\{Nt_i\} \geq \{Nt_j\}\end{aligned}$$

where $i,\, j \in \{1,\, 2\}$, $i \neq j$.

Interpolation in $d = 3$. In $d = 3$ the interpolated field can be defined in the same way as above. We use tetrahedrons to define the interpolated field as

$$\begin{aligned}\psi_N(t) &= \frac{\kappa}{\sqrt{N}}[\varphi_{\lfloor Nt\rfloor} + \{Nt_i\}\left(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}\right)\\ &+ \{Nt_j\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i}\right)\\ &+ \{Nt_k\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j+e_k} - \varphi_{\lfloor Nt\rfloor+e_i+e_j}\right)], \quad \{Nt_i\} \geq \{Nt_j\} \geq \{Nt_k\}\end{aligned}$$

where $t = (t_1\,,\, t_2\,,\, t_3) \in \overline{D}$ and $i,\, j,\, k \in \{1,\, 2,\, 3\}$ are pairwise different.

Note that in both $d = 2, 3$ we have

$$\psi_N(t) = \kappa N^{\frac{d-4}{2}}\varphi_{Nt}, \;\; t \in \frac{1}{N}\,\mathbb{Z}^d\,.$$

From the above construction it follows that, for each N, ψ_N is a continuous function on $\overline{D}$. This shows that ψ_N can be considered as a random variable taking values in $(C(\overline{D}), \mathcal{C}(\overline{D}))$ where $C(\overline{D})$ is the space of continuous functions on $\overline{D}$ and $\mathcal{C}(\overline{D})$ is its Borel σ-algebra. Also recall the definition of Green's function: the Green's function for the biharmonic operator is $G_D : D \times D \to \mathbb{R}$ such that for every fixed $x \in D$, it solves the equation

$$\Delta_c^2 G_D(x, y) = \delta_x(y), \qquad y \in D,$$

in the space $H_0^2(D)$, the completion of $C_c^\infty(D)$ with respect to the norm

$$\|f\|_{H_0^2(D)} := \|\nabla_c^2 f\|_{L^2(D)}.$$

In the above equations Δ_c^2, the continuum bilaplacian, acts on the y component, and ∇_c^2 is the Hessian. The detailed properties of such spaces are needed in $d \geq 4$ so we defer the discussions on them to Section 2.3. We denote the continuum Green's function by G_D to indicate the dependence on the domain D.

We are now ready to state our main result for the case $d = 2, 3$. It shows that the convergence of the above described process occurs in the space of continuous functions.

Theorem 2.2.1 (Scaling limit in $d = 2, 3$)**.** *Consider the interpolated membrane model $(\psi_N(t))_{t\in\overline{D}}$ in $d = 2$ and 3 as above. Then there exists a centered continuous Gaussian process $\psi_D^{\Delta^2}$ with covariance $G_D(\cdot, \cdot)$ on $\overline{D}$ such that ψ_N converges in distribution to $\psi_D^{\Delta^2}$ in the space of all continuous functions on $\overline{D}$. Furthermore the process $\psi_D^{\Delta^2}$ is almost surely Hölder continuous with exponent η, for every $\eta \in (0, 1)$ resp. $\eta \in (0, 1/2)$ in $d = 2$ resp. $d = 3$.*

An immediate consequence of the continuous mapping theorem is that, as $N \to \infty$,

$$\sup_{x\in\overline{D}} \psi_N(x) \xrightarrow{d} \sup_{x\in\overline{D}} \psi_D^{\Delta^2}(x).$$

It is easy to see that for any square or a cube A in the $\frac{1}{N}\mathbb{Z}^d$ lattice,

$$\sup_{x\in A} \psi_N(x) = \kappa N^{\frac{d-4}{2}} \max\{\varphi_{Nx} : x \text{ is a vertex of A}\}.$$

Hence $\sup_{x\in\overline{D}} \psi_N(x) = \kappa N^{\frac{d-4}{2}} \max_{x\in D_N} \varphi_x$. So combining these observations we obtain the scaling limit of the maximum of the discrete membrane model in lower dimensions.

Corollary 2.2.2. *Let $d \in \{2, 3\}$ and let $M_N = \max_{x\in D_N} \varphi_x$. Then as $N \uparrow \infty$*

$$\kappa N^{\frac{d-4}{2}} M_N \xrightarrow{d} \sup_{x\in\overline{D}} \psi_D^{\Delta^2}(x).$$

2.2.2 Proof of the scaling limit (Theorem 2.2.1)

The proof follows the general methodology of a functional CLT, namely, we first show the tightness of the interpolated field and secondly we show that the finite dimensional distributions converge. As a by-product of the proof, the limiting Gaussian process will be well-defined, that is, its covariance function will be positive definite. The finite dimensional convergence follows easily from the very recent work of Müller and Schweiger [54] where the convergence of the discrete Green's function to the continuum one is shown. Tightness also requires the crucial bounds on gradients which were derived in the same article. Since we have interpolated the field continuously and not piece-wise in boxes or cubes one of the main efforts is to deduce moment bounds from integer lattice points.

Tightness and Hölder continuity

To derive the tightness we need the following ingredients. The first one consists in the following bounds for the discrete Green's function and its gradients which follow from [54]. We define the directional derivative of a function $u : \mathbb{Z}^d \to \mathbb{R}$ as

$$D_i u(x) := u(x + e_i) - u(x),$$

and hence the discrete gradient becomes

$$\nabla u(x) = (D_i u(x))_{i=1}^d.$$

For functions of several variables we use a subscript to indicate the variable with respect to which a derivative is taken, for example in $D_{i,1} D_{j,2} u(x, y)$ we take the discrete derivative in the direction i in the variable x and in j in the variable y, and $\nabla_x G(x, y)$ means we are taking the gradient in the x variable. We now state some bounds on the covariance function and its gradient from [54], where they appear in a more general version.

Lemma 2.2.3 ([54, Theorem 1.1])**.** *Let $d \in \{2, 3\}$.*

(i) *For any $x, y \in \mathbb{Z}^d$*
$$|G_N(x, y)| \le C N^{4-d}.$$

(ii) *For any $x, y \in \mathbb{Z}^d$*
$$\|\nabla_x G_N(x, y)\| \le C N^{3-d}.$$

(iii) *For any $x, y \in \mathbb{Z}^d$*
$$\|\nabla_x \nabla_y G_N(x, y)\| \le \begin{cases} C \log\left(1 + \frac{N^2}{(\|x-y\|+1)^2}\right) & \text{if } d = 2 \\ C & \text{if } d = 3 \end{cases}. \tag{2.2.1}$$

Now from the estimate (2.2.1) and the fact that

$$\mathbf{E}\left[(\varphi_{z+e_i} - \varphi_z)^2\right] = D_{i,2}D_{i,1}G(z, z)$$

one can observe the following Fact.

Fact 2.2.4. *For $z \in \mathbb{Z}^d$*

$$\mathbf{E}\left[(\varphi_{z+e_i} - \varphi_z)^2\right] \leq \begin{cases} C \log N & \text{if } d = 2 \\ C & \text{if } d = 3 \end{cases}.$$

Next we want to show that the sequence $\{\psi_N\}_{N\in\mathbb{N}}$ is tight in $C(\overline{D})$. We use the following theorem, whose proof follows from that of Theorem 14.9 in [45].

Theorem 2.2.5. *Let $X^1, X^2, \ldots$ be continuous processes on $\overline{D}$ with values in a complete separable metric space (S, ρ). Assume that (X^n_0) is tight in S and that for constants $\alpha, \beta > 0$*

$$\mathbf{E}[\rho(X^n_s, X^n_t)^\alpha] \leq C\|s - t\|^{d+\beta}, \quad s,\, t \in \overline{D} \tag{2.2.2}$$

uniformly in n. Then (X^n) is tight in $C(\overline{D}, S)$ and for every $c \in (0, \beta/\alpha)$ the limiting processes are almost surely Hölder continuous with exponent c.

Observe that the process $(\psi_N(t))_{t\in\overline{D}}$ is Gaussian, and since from Lemma 2.2.3 it follows that $G_{N-1}(0, 0) \leq N^{4-d}$, it is easy to see that $(\psi_N(0))$ is tight. Again, using the properties of Gaussian laws, to show (2.2.2) it is enough to show the following the lemma.

Lemma 2.2.6. *Let $b \in (0,\, 1)$ in $d = 2$ and $b = 0$ in $d = 3$. Then there exists a constant $C > 0$ (which depends on b in $d = 2$) such that*

$$\mathbf{E}\left[|\psi_N(t) - \psi_N(s)|^2\right] \leq C\|t - s\|^{1+b} \tag{2.2.3}$$

for all $t, s \in \overline{D}$, uniformly in N.

This lemma will immediately give (2.2.2) and hence the Hölder continuity of the limiting field.

Corollary 2.2.7. *The field $\psi^{\Delta^2}_D$ is almost surely Hölder continuous with exponent η, where $\eta \in (0,\, 1)$ in $d = 2$ and $\eta \in (0,\, 1/2)$ in $d = 3$.*

Proof. We note that for $t,\, s \in \overline{D}$, the random variable $\psi_N(t) - \psi_N(s)$ is Gaussian. Therefore using Lemma 2.2.6 we have, for any α such that $(1+b)\alpha/2 > d$, that there is a constant C such that the following holds uniformly in N with $\beta := (1+b)\alpha/2 - d$:

$$\mathbf{E}[|\psi_N(t) - \psi_N(s)|^\alpha] \le C\|t-s\|^{d+\beta}, \quad s,\, t \in \overline{D}.$$

The conclusion follows then from Theorem 2.2.5. $\square$

Now we show the proof of the lemma.

Proof of Lemma 2.2.6. First we consider $d = 2$. We fix a $b \in (0,\, 1)$ and let $t,\, s \in \overline{D}$. We split the proof into a few cases.

Case 1: Suppose t, s belong to the same smallest square box in the lattice $\frac{1}{N}\mathbb{Z}^2$. First assume $\lfloor Nt \rfloor = \lfloor Ns \rfloor$, that is, the points are in the interior and not touching the top and right boundaries. In this case if we have $\{Nt_1\} \ge \{Nt_2\}$ and $\{Ns_1\} \ge \{Ns_2\}$. Then by definition of the interpolation we have

$$\begin{aligned}\psi_N(t) - \psi_N(s) = \kappa[&(t_1 - s_1)\left(\varphi_{\lfloor Nt \rfloor + e_1} - \varphi_{\lfloor Nt \rfloor}\right)\\ &+ (t_2 - s_2)\left(\varphi_{\lfloor Nt \rfloor + e_1 + e_2} - \varphi_{\lfloor Nt \rfloor + e_1}\right)].\end{aligned}$$

So from the above expression we have

$$\begin{aligned}\mathbf{E}\left[(\psi_N(t) - \psi_N(s))^2\right] &\le 2\kappa^2[(t_1 - s_1)^2\mathbf{E}[\left(\varphi_{\lfloor Nt \rfloor + e_1} - \varphi_{\lfloor Nt \rfloor}\right)^2]\\ &+ (t_2 - s_2)^2\mathbf{E}[\left(\varphi_{\lfloor Nt \rfloor + e_1 + e_2} - \varphi_{\lfloor Nt \rfloor + e_1}\right)^2]].\end{aligned}$$

Now from Fact 2.2.4 and $|t_1 - s_1|, |t_2 - s_2| < N^{-1}$ we obtain (2.2.3). The argument is similar if one has $\{Nt_1\} \le \{Nt_2\}$ and $\{Ns_1\} \le \{Ns_2\}$.

Again if $\{Nt_1\} \ge \{Nt_2\}$ and $\{Ns_1\} < \{Ns_2\}$, or if $\{Nt_1\} < \{Nt_2\}$ and $\{Ns_1\} \ge \{Ns_2\}$ then we consider the point u on the line segment joining t and s such that Nu is the point of intersection of the line segment joining Nt, Ns and the diagonal joining $\lfloor Nt \rfloor, \lfloor Nt \rfloor + e_1 + e_2$. Then we have using the above computations

$$\mathbf{E}\left[|\psi_N(t) - \psi_N(s)|^2\right] \le 2\mathbf{E}\left[|\psi_N(t) - \psi_N(u)|^2\right] + 2\mathbf{E}\left[|\psi_N(u) - \psi_N(s)|^2\right]$$

$$\leq C\left[\|t-u\|^{1+b} + \|u-s\|^{1+b}\right] \leq C\|t-s\|^{1+b}.$$

Now the other case, that is, when $\lfloor Nt \rfloor \neq \lfloor Ns \rfloor$ follows from above by continuity.

Case 2: Suppose t, s do not belong to the same smallest square box in the lattice $\frac{1}{N}\mathbb{Z}^2$. In this case if $\|t-s\| \leq 1/N$ then one can obtain (2.2.3) by the above case and a suitable point in between. So we assume $\|t-s\| > 1/N$. Depending on whether Nt and Ns belong to the discrete lattice we split the proof in two broad cases. We will use bounds on mixed discrete derivatives for a better control of finite differences of the Green's function.

<u>Sub-case 2 (a)</u> Suppose $t, s \in \frac{1}{N}\mathbb{Z}^2$. Then

$$\mathbf{E}\left[|\psi_N(t) - \psi_N(s)|^2\right] = \frac{\kappa^2}{N^2}\left[G_{N-1}(Nt, Nt) - G_{N-1}(Ns, Nt)\right.$$
$$\left. - G_{N-1}(Nt, Ns) + G_{N-1}(Ns, Ns)\right].$$

We assume without loss of generality $Ns_1 \leq Nt_1$, $Ns_2 \leq Nt_2$. Also denote $M := N(t_1 - s_1 + t_2 - s_2)$ and let $(u_i)_{i=0}^M$ be such that $u_i = s + i/Ne_1$ for $i \leq N(t_1 - s_1)$ and $u_i = s + (t_1 - s_1)e_1 + (i/N - (t_1 - s_1))e_2$ for $i > N(t_1 - s_1)$. Then

$$\mathbf{E}\left[|\psi_N(t) - \psi_N(s)|^2\right] = \frac{\kappa^2}{N^2}\sum_{i=0}^{M-1}\left[G_{N-1}(Nu_{i+1}, Nt) - G_{N-1}(Nu_i, Nt)\right]$$
$$- \left[G_{N-1}(Nu_{i+1}, Ns) - G_{N-1}(Nu_i, Ns)\right]$$
$$= \frac{\kappa^2}{N^2}\sum_{i,j=0}^{M-1}\left[G_{N-1}(Nu_{i+1}, Nu_{j+1}) - G_{N-1}(Nu_{i+1}, Nu_j)\right.$$
$$\left. - G_{N-1}(Nu_i, Nu_{j+1}) + G_{N-1}(Nu_i, Nu_j)\right] \leq \frac{C}{N^2}\sum_{i,j=0}^{M-1}\log\left(1 + \frac{N^2}{(\|Nu_i - Nu_j\| + 1)^2}\right)$$

where we have used (2.2.1) in the last inequality and we have absorbed the constant κ^2 in the generic constant C. Now using the definition of u_i, u_j the right-hand side above is bounded above by

$$\frac{C}{N^2}\sum_{i,j=0}^{M-1}\log\left(1 + \frac{N^2}{\left(\frac{|i-j|}{\sqrt{2}} + 1\right)^2}\right) \leq \frac{C}{N^2}\sum_{i,j=0}^{M-1}\log\left(1 + \frac{N}{(|i-j| + 1)}\right)$$

$$\leq \frac{CM}{N^2}\sum_{l=-M+1}^{M-1}\log\left(1+\frac{N}{(|l|+1)}\right) \leq \frac{CM}{N}\int_0^{\frac{M}{N}}\log\left(1+\frac{1}{x}\right)\mathrm{d}x$$
$$\leq C\Big(\frac{M}{N}\Big)^2\left[1+\log\left(1+\frac{N}{M}\right)\right] \leq C\|t-s\|^{1+b}.$$

Sub-case 2 (b) Suppose at least one between t, s does not belong to $\frac{1}{N}\mathbb{Z}^2$. Then

$$\begin{aligned}\mathbf{E}\left[|\psi_N(t)-\ \psi_N(s)|^2\right] &\leq 3\mathbf{E}\left[\left|\psi_N(t)-\psi_N\left(\frac{\lfloor Nt\rfloor}{N}\right)\right|^2\right]\\ &+3\mathbf{E}\left[\left|\psi_N\left(\frac{\lfloor Nt\rfloor}{N}\right)-\psi_N\left(\frac{\lfloor Ns\rfloor}{N}\right)\right|^2\right]+3\mathbf{E}\left[\left|\psi_N\left(\frac{\lfloor Ns\rfloor}{N}\right)-\psi_N(s)\right|^2\right]\\ &\leq C\left[\left\|t-\frac{\lfloor Nt\rfloor}{N}\right\|^{1+b}+\left\|\frac{\lfloor Nt\rfloor}{N}-\frac{\lfloor Ns\rfloor}{N}\right\|^{1+b}+\left\|\frac{\lfloor Ns\rfloor}{N}-s\right\|^{1+b}\right]\\ &\leq C\|t-s\|^{1+b}.\end{aligned}$$

Note that for the last inequality we have used our assumption $\|t-s\|>1/N$.

Now we consider $d=3$. Let $t,\ s\in\overline{D}$. We split the proof into cases similar to those of $d=2$. We give a brief description. For Case 1, suppose t, s belong to the same smallest cube in the lattice $\frac{1}{N}\mathbb{Z}^3$. First assume $\lfloor Nt\rfloor=\lfloor Ns\rfloor$. In this case if $\{Nt_1\}\geq\{Nt_2\}\geq\{Nt_3\}$ and $\{Ns_1\}\geq\{Ns_2\}\geq\{Ns_3\}$ then it follows from the definition of interpolation

$$\begin{aligned}\mathbf{E}\left[(\psi_N(t)-\psi_N(s))^2\right] \leq 3N\kappa^2[&(t_1-s_1)^2\mathbf{E}[(\varphi_{\lfloor Nt\rfloor+e_1}-\varphi_{\lfloor Nt\rfloor})^2]\\ &+(t_2-s_2)^2\mathbf{E}[(\varphi_{\lfloor Nt\rfloor+e_1+e_2}-\varphi_{\lfloor Nt\rfloor+e_1})^2]\\ &+(t_3-s_3)^2\mathbf{E}[(\varphi_{\lfloor Nt\rfloor+e_1+e_2+e_3}-\varphi_{\lfloor Nt\rfloor+e_1+e_2})^2]].\end{aligned}$$

Now from Fact 2.2.4 and the fact that $|t_1-s_1|,|t_2-s_2|,|t_3-s_3|<1/N$ we have (2.2.3). Note that this is a particular case of t, s lying in the same tetrahedral portion of the cube. Hence if $t,\ s$ lie in the same tetrahedral portion of the cube then by similar arguments (2.2.3) holds. If t, s do not lie in the same tetrahedral part then we consider points (at most 3) on the line segment joining them such that two consecutive between t, the selected points and s lie in the same tetrahedral part. Then applying the previous argument we can obtain (2.2.3). Now the case when $\lfloor Nt\rfloor\neq\lfloor Ns\rfloor$ follows by continuity. For Case 2, we describe Sub-case 2(a) which turns out to be simpler in $d=3$. The rest of the argument is similar to that in $d=2$. Suppose $t, s\in\frac{1}{N}\mathbb{Z}^3$ with

$\|t-s\| > 1/N$. Then

$$\mathbf{E}\left[|\psi_N(t)-\psi_N(s)|^2\right] = \frac{\kappa^2}{N}[G_{N-1}(Nt,Nt) - G_{N-1}(Ns,Nt) - G_{N-1}(Nt,Ns) + G_{N-1}(Ns,Ns)]$$

Without loss of generality assume $Ns_1 \le Nt_1$, $Ns_2 \le Nt_2$, $Ns_3 \le Nt_3$. Then

$$\begin{aligned} G_{N-1}(Nt,Nt) - G_{N-1}(Ns,Nt) &= \sum_{i=1}^{N(t_1-s_1)} D_{1,1}G_{N-1}(Ns+(i-1)e_1, Nt) \\ &+ \sum_{j=1}^{N(t_2-s_2)} D_{2,1}G_{N-1}(Ns+N(t_1-s_1)e_1+(j-1)e_2, Nt) \\ &+ \sum_{l=1}^{N(t_3-s_3)} D_{3,1}G_{N-1}(Ns+N(t_1-s_1)e_1+N(t_2-s_2)e_2+(l-1)e_3, Nt) \\ &\le C\left(N(t_1-s_1)+N(t_2-s_2)+N(t_3-s_3)\right) \le CN\|t-s\|. \end{aligned}$$

Hence (2.2.3) follows. □

Finite dimensional convergence

The content of this subsection is to show

Proposition 2.2.8. *With the notation of Theorem 2.2.1, for all $s,\, t \in \overline{D}$,*

$$\lim_{N\to\infty} \mathrm{Cov}(\psi_N(t),\psi_N(s)) = \mathrm{Cov}(\psi_D^{\Delta^2}(t), \psi_D^{\Delta^2}(s)).$$

Proof. To show the finite dimensional convergence we use [54, Corollary 1.4] (there the domain was $(0,\,1)^d$ but the result works for D as well). We observe that for $h := 1/N$, one has $G_{N-1}(x,y) = 4d^2h^{d-4}G_h(hx,hy)$ where G_h satisfies for $x \in \mathrm{int}(D_h)$ with $D_h = [-1,1]^d \cap h\,\mathbb{Z}^d$ the following boundary value problem

$$\begin{cases} \Delta_h^2 G_h(x,y) = \frac{1}{h^d}\delta_x(y) & y \in \mathrm{int}(D_h) \\ G_h(x,y) = 0 & y \notin \mathrm{int}(D_h) \end{cases},$$

where Δ_h is defined by

$$\Delta_h f(x) := \frac{1}{h^2}\sum_{i=1}^{d}(f(x+he_i)+f(x-he_i)-2f(x)) \tag{2.2.4}$$

and f is any function on $h\mathbb{Z}^d$. We call such a function a grid function. Let $\psi_D^{\Delta^2}$ be the Gaussian process on $\overline{D}$ such that $\mathbf{E}[\psi_D^{\Delta^2}(t)\psi_D^{\Delta^2}(s)] = G_D(t,s)$ for all t, $s \in \overline{D}$, where G_D is the Green's function for the biharmonic equation with homogeneous Dirichlet boundary conditions (it will be a by-product of this proof that such a process exists). First we consider $d = 2$. For $t \in \overline{D}$ we have

$$\psi_N(t) = \psi_{N,1}(t) + \psi_{N,2}(t),$$

where $\psi_{N,1}(t) = \frac{\kappa}{N}\varphi_{\lfloor Nt \rfloor}$ and

$$\psi_{N,2}(t) = \frac{\kappa}{N} \sum_{i,j\in\{1,2\}, i\neq j} \mathbb{1}_{(\{Nt_i\}\geq\{Nt_j\})}(t)[\{Nt_i\} \left(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}\right)$$
$$+ \{Nt_j\} \left(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i}\right)].$$

Then using Fact 2.2.4 we have $\mathbf{Var}(\psi_{N,2}(t)) \leq C(\log N)N^{-2}$ and hence $\psi_{N,2}(t)$ converges to zero in probability as N tends to infinity.

Again if $t \in D$ then

$$\mathbf{Var}(\psi_{N,1}(t)) = \frac{\kappa^2}{N^2} G_{N-1}(\lfloor Nt\rfloor, \lfloor Nt\rfloor) = G_h(h\lfloor Nt\rfloor, h\lfloor Nt\rfloor)$$

and $G_h(h\lfloor Nt\rfloor, h\lfloor Nt\rfloor)$ converges to $G_D(t,t)$ by [54, Corollary 1.4]. Also if $t \in \partial D$ then $\mathbf{Var}(\psi_{N,1}(t)) = 0 = G_D(t,t)$. Hence $\psi_N(t) \xrightarrow{d} \psi_D^{\Delta^2}(t)$.

Similarly one can show using Lemma 2.2.3, Fact 2.2.4 and [54, Corollary 1.4] that for any $t, s \in \overline{D}$,

$$\mathrm{Cov}(\psi_N(t), \psi_N(s)) \to \mathrm{Cov}(\psi_D^{\Delta^2}(t), \psi_D^{\Delta^2}(s)).$$

Since these variables under consideration are Gaussian, the finite dimensional follows from the convergence of the covariance.

In $d = 3$, for $t \in \overline{D}$ we have

$$\psi_N(t) = \frac{\kappa}{\sqrt{N}}\varphi_{\lfloor Nt\rfloor} + \frac{\kappa}{\sqrt{N}} \sum_{i,\,j,\,k\in\{1,\,2,\,3\},\,\text{pairwise different}} \mathbb{1}_{(\{Nt_i\}\geq\{Nt_j\}\geq\{Nt_k\})}(t)$$
$$[\{Nt_i\} \left(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}\right) + \{Nt_j\} \left(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i}\right)$$
$$+ \{Nt_k\} \left(\varphi_{\lfloor Nt\rfloor+e_i+e_j+e_k} - \varphi_{\lfloor Nt\rfloor+e_i+e_j}\right)]$$

$$=: \psi_{N,1}(t) + \psi_{N,2}(t).$$

By means of Fact 2.2.4 we have $\mathbf{Var}(\psi_{N,2}(t)) \leq C/N$ and hence $\psi_{N,2}(t)$ converges to zero in probability as $N \to \infty$. The rest of the proof is the same as $d = 2$ and follows from [54, Corollary 1.4]. □

2.3 Convergence of finite volume measure in $d \geq 4$

In this section D denotes a bounded domain in $\mathbb{R}^d$, $d \geq 4$, with smooth boundary.

Remark 2.3.1 (Regularity of the boundary of the domain)**.** *The assumption of smoothness of the boundary is required to obtain asymptotics of the eigenvalues of the biharmonic operator (cf. Proposition 2.3.9).*

2.3.1 Description of the limiting field

Spectral theory for the biharmonic operator

Let $C_c^\infty(D)$ denote the space of infinitely differentiable functions $u : D \to \mathbb{R}$ with compact support contained in the interior of D. Recall that for $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multi-index

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u.$$

Suppose $f, g \in L^1_{loc}(D)$. One says that g is the α-th weak partial derivative of f (written $D^\alpha f = g$) if

$$\int_D f D^\alpha u \, \mathrm{d}\, x = (-1)^{|\alpha|} \int_D g u \, \mathrm{d}\, x \quad \forall\, u \in C_c^\infty(D).$$

The Sobolev space $W^{k,p}$ is defined in the usual way as

$$W^{k,p} = \{f \in L^1_{loc}(D) : \; D^\alpha f \in L^p(D), \, |\alpha| \leq k\}.$$

Denote by $H^k(D) := W^{k,2}(D)$, $k = 0, 1, \ldots$, which is a Hilbert space with norm

$$\|f\|_{H^k(D)} = \left(\sum_{|\alpha| \leq k} \int_D |D^\alpha f|^2 \, \mathrm{d}\, x \right)^{1/2}.$$

It is true that if $a > b$ then $H^a(D) \subset H^b(D)$. Let us define another Hilbert space,

$$H_0^k(D) := \overline{C_c^\infty(D)}^{\|\cdot\|_{H^k(D)}}$$

and let $H^{-k}(D) = [H_0^k(D)]^*$ be its dual. In this section we will use round brackets $(\cdot,\, \cdot)$ to denote the action of a dual Hilbert space on the original space, and $\langle \cdot,\, \cdot\rangle$ for inner products. We consider the inner product

$$\langle u, v\rangle_{H_0^2} = \int_D \Delta_c u \Delta_c v \,\mathrm{d}\, x$$

which induces a norm on $H_0^2(D)$ equivalent to the standard Sobolev norm [39, Corollary 2.29]. We always consider $H_0^2(D)$ with this norm.

We review briefly the spectral theory for the biharmonic operator as it helps us to give an explicit construction of the continuum bilaplacian field. We have the following theorem, which basically says that we can construct an operator B being the inverse of the bilaplacian (see also Remark 2.3.8).

Theorem 2.3.2. *There exists a bounded linear isometry*

$$B_0 : H^{-2}(D) \to H_0^2(D)$$

such that, for all $f \in H^{-2}(D)$ and for all $v \in H_0^2(D)$,

$$(f,\, v) = \langle v,\, B_0 f\rangle_{H_0^2}\,.$$

Moreover, the restriction B on $L^2(D)$ of the operator $i \circ B_0 : H^{-2}(D) \to L^2(D)$ is a compact and self-adjoint operator, where $i : H_0^2(D) \hookrightarrow L^2(D)$ is the inclusion map.

Proof. Fix $f \in H^{-2}(D)$. By the Riesz representation theorem there exists a unique $u_f \in H_0^2(D)$ such that for all $v \in H_0^2(D)$

$$(f,\, v) = \langle v, u_f\rangle_{H_0^2}\,.$$

We define $B_0 f := u_f$. Then by definition B_0 is a bounded linear isometry and for all $v \in H_0^2(D)$

$$(f,\, v) = \langle v, B_0 f\rangle_{H_0^2}\,.$$

We have $H_0^2(D) \hookrightarrow H_0^1(D) \hookrightarrow L^2(D)$ and the second embedding is compact. So $i : H_0^2(D) \hookrightarrow L^2(D)$ is compact and hence the operator $i \circ B_0 : H^{-2}(D) \to L^2(D)$ is compact. This implies that the restriction B is compact. B is self-adjoint as for any $f, g \in L^2(D)$,

$$\langle Bf, g\rangle_{L^2} = (g,\, Bf) = \langle Bf, Bg\rangle_{H_0^2} = (f,\, Bg) = \langle f, Bg\rangle_{L^2}\,. \qquad \square$$

Consequently we can find now an orthonormal basis of elements of $H_0^2(D)$, as the next theorem shows.

Theorem 2.3.3. *There exist $u_1,\, u_2,\, \dots$ in $H_0^2(D)$ and numbers*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

such that

- *$\{u_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $L^2(D)$,*
- *$Bu_j = \lambda_j^{-1} u_j$, where B is as in Theorem 2.3.2,*
- *$(u_j, v)_{H_0^2} = \lambda_j \langle u_j, v\rangle_{L^2}$ for all $v \in H_0^2(D)$,*
- *$\{\lambda_j^{-1/2} u_j\}$ is an orthonormal basis for $H_0^2(D)$.*

Proof. By the spectral theorem for compact self-adjoint operators we get an orthonormal basis of $L^2(D)$ consisting of eigenvectors of B with $Bu_j = \widetilde{\lambda}_j u_j$ and eigenvalues $\widetilde{\lambda}_j \to 0$. Note that for any $f \in L^2(D)$, $Bf = 0$ implies that

$$\langle v,\, f\rangle_{L^2} = \langle v,\, Bf\rangle_{H_0^2} = 0 \quad \forall v \in H_0^2(D)$$

and hence $\langle g, f\rangle_{L^2} = 0$ for all $g \in L^2(D)$ (since $H_0^2(D)$ is dense in $L^2(D)$) and so $f \equiv 0$. Thus 0 is not an eigenvalue of B and we have for any $j \in \mathbb{N}$

$$u_j = \frac{1}{\widetilde{\lambda}_j} Bu_j = B\frac{u_j}{\widetilde{\lambda}_j} \in \mathrm{Range}(B) \subset H_0^2(D).$$

Hence $u_j \in H_0^2(D)$. Now observe that, for any $j \in \mathbb{N}$, $\widetilde{\lambda}_j \langle u_j, v\rangle_{H_0^2} = \langle Bu_j, v\rangle_{H_0^2} = \langle u_j, v\rangle_{L^2}$ for all $v \in H_0^2(D)$. So this gives

$$\widetilde{\lambda}_j \langle u_j, u_j\rangle_{H_0^2} = \|u_j\|_{L^2} = 1.$$

But $\langle u_j, u_j\rangle_{H_0^2} > 0$ and hence $\tilde{\lambda}_j > 0$ for all $j \in \mathbb{N}$. We define $\lambda_j := 1/\widetilde{\lambda}_j$. So we can conclude

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty.$$

Moreover $Bu_j = \lambda_j^{-1} u_j$ and

$$\langle u_j, v\rangle_{H_0^2} = \lambda_j \langle u_j, v\rangle_{L^2} \quad \forall\, v \in H_0^2(D). \tag{2.3.1}$$

We now show that $\{\lambda_j^{-1/2} u_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $H_0^2(D)$. Indeed we have

$$\begin{aligned}\left\langle \lambda_j^{-\frac{1}{2}} u_j, \lambda_k^{-\frac{1}{2}} u_k \right\rangle_{H_0^2} &= \lambda_j^{-\frac{1}{2}} \lambda_k^{-\frac{1}{2}} \langle u_j, u_k\rangle_{H_0^2} \\ &= \lambda_j^{\frac{1}{2}} \lambda_k^{-\frac{1}{2}} \langle u_j, u_k\rangle_{L^2} = \delta_{jk}.\end{aligned}$$

So $\{\lambda_j^{-1/2} u_j\}$ is an orthonormal system. But for any $v \in H_0^2(D)$, $\langle u_j, v\rangle_{H_0^2} = 0$ for all j implies that $\langle u_j, v\rangle_{L_2} = 0$ for all j which in turn implies $v = 0$. This completes the proof. □

Corollary 2.3.4. *For each $j \in \mathbb{N}$ one has $u_j \in C^\infty(D)$. Moreover u_j is an eigenfunction of Δ_c^2 with eigenvalue λ_j.*

Proof. We have for all $v \in H_0^2(D)$:

$$\left\langle \Delta_c^2 u_j, v\right\rangle_{L^2} \overset{GI}{=} \langle u_j, v\rangle_{H_0^2} \overset{\text{Theorem 2.3.3}}{=} \lambda_j \langle u_j, v\rangle_{L^2}$$

where "GI" stands for Green's first identity

$$\int_D u\Delta_c v \,\mathrm{d}\, V = -\int_D \nabla_c u \cdot \nabla_c v \,\mathrm{d}\, V + \int_{\partial D} u \nabla_c v \cdot \mathbf{n} \,\mathrm{d}\, S.$$

Thus u_j is an eigenfunction of Δ_c^2 with eigenvalue λ_j in the weak sense. The smoothness of u_j follows from the fact that Δ_c^2 is an elliptic operator with smooth coefficients and the elliptic regularity theorem [37, Theorem 9.26]. Hence u_j is an eigenfunction of Δ_c^2 with eigenvalue λ_j. □

Remark 2.3.5. *As a consequence of the above, one easily has that*

$$\|f\|_{H_0^2}^2 = \sum_{j\geq 1} \lambda_j \langle f, u_j \rangle_{L^2}^2$$

for any $f \in H_0^2(D)$.

We conclude this subsection with some bounds for the derivatives of the eigenfunctions u_j of Theorem 2.3.3.

Lemma 2.3.6. *The following bounds hold:*

$$\sup_{x\in D}|u_j(x)| \leq C\lambda_j^{l_0}, \tag{2.3.2}$$

$$\sum_{|\alpha|\leq 2} \sup_{x\in D}|D^\alpha u_j(x)| \leq C\lambda_j^{l_2}, \tag{2.3.3}$$

$$\sum_{|\alpha|\leq 5} \sup_{x\in D}|D^\alpha u_j(x)| \leq C\lambda_j^{l_5} \tag{2.3.4}$$

where

$$l_m := \left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + m + 1\right)\right\rceil, \qquad m = 0, 2, 5.$$

Proof. Taking $l_0 = \lceil 1/4(\lfloor d/2 \rfloor + 1)\rceil$ we obtain from [35, Chapter 5, Theorem 6 (ii)] that $\sup_{x\in D}|u_j(x)| \leq C\|u_j\|_{H^{4l_0}(D)}$. Now a repeated application of [39, Corollary 2.21] gives

$$\sup_{x\in D}|u_j(x)| \leq C\|u_j\|_{H^{4l_0}(D)} \leq C\lambda_j\|u_j\|_{H^{4l_0-4}(D)} \leq \cdots \leq C\lambda_j^{l_0}.$$

The other two bounds are obtained similarly. We make a passing remark that the smoothness of the boundary is needed in the results quoted above. □

Definition of the limiting field via Wiener series

For any $v \in C_c^\infty(D)$ and for any $s > 0$ we define

$$\|v\|_{s,\Delta^2}^2 := \sum_{j\in\mathbb{N}} \lambda_j^{s/2} \langle v, u_j \rangle_{L^2}^2 .$$

We define $\mathcal{H}^s_{\Delta^2,0}(D)$ to be the Hilbert space completion of $C_c^\infty(D)$ with respect to the norm $\|\cdot\|_{s,\Delta^2}$. Then $\left(\mathcal{H}^s_{\Delta^2,0}(D)\,,\,\|\cdot\|_{s,\Delta^2}\right)$ is a Hilbert space for all $s>0$.

Remark 2.3.7.

- *Note that for $s=2$ we have $\mathcal{H}^2_{\Delta^2,0}(D)=H^2_0(D)$ by Remark 2.3.5.*
- *$i:\mathcal{H}^s_{\Delta^2,0}(D)\hookrightarrow L^2(D)$ is a continuous embedding.*

Dual spaces. For $s>0$ we define $\mathcal{H}^{-s}_{\Delta^2}(D)=(\mathcal{H}^s_{\Delta^2,0}(D))^*$, the dual space of $\mathcal{H}^s_{\Delta^2,0}(D)$. Then we have

$$\mathcal{H}^s_{\Delta^2,0}(D)\subseteq L^2(D)\subseteq\mathcal{H}^{-s}_{\Delta^2}(D).$$

We denote the dual norm by $\|\cdot\|_{-s,\Delta^2}$. For any $v\in L^2(D)$ we have

$$\|v\|^2_{-s,\Delta^2}=\sum_{j\in\mathbb{N}}\lambda_j^{-s/2}\langle v,\,u_j\rangle^2_{L^2}\,.$$

Remark 2.3.8. *By means of integration by parts we obtain, for every $f\in C_c^\infty(D)$, that the solution u_f of the boundary value problem*

$$\begin{cases}\Delta_c^2u(x)=f(x), & x\in D\\ D^\alpha u(x)=0, & |\alpha|\le 1,\,x\in\partial D.\end{cases}\tag{2.3.5}$$

where $\alpha=(\alpha_1,\ldots,\alpha_d)$ is a multi-index with α_i's being non-negative integers, is such that for all $v\in C_c^\infty(D)$

$$\int_D v(x)f(x)\,\mathrm{d}\,x=\int_D v(x)\Delta_c^2u_f(x)\,\mathrm{d}\,x=\langle v,\,u_f\rangle_{H_0^2}\,.$$

Using the denseness of $C_c^\infty(D)$ in $H^2_0(D)$ we conclude from Theorem 2.3.2 that $B_0f=u_f$. Thus we have

$$\|f\|^2_{-2,\Delta^2}=\int_D u_f(x)f(x)\,\mathrm{d}\,x=\|u_f\|^2_{H_0^2}.$$

Before we show the definition of the continuum membrane model, we need an analog of Weyl's law for the eigenvalues of the biharmonic operator.

Proposition 2.3.9 ([4, Theorem 5.1], [57])**.** *There exists an explicit constant c such that, as $j\uparrow+\infty$,*

$$\lambda_j\sim c^{-d/4}j^{4/d}.$$

The result we will prove now shows the well-posedness of the series expansion for $\Psi_D^{\Delta^2}$.

Proposition 2.3.10. *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{\Delta^2} := \sum_{j\in\mathbb{N}} \lambda_j^{-1/2} \xi_j u_j.$$

Then $\Psi_D^{\Delta^2} \in \mathcal{H}_{\Delta^2}^{-\frac{s}{2}}(D)$ a.s. for all $s > (d-4)/2$.

Proof. Fix $s > (d-4)/2$. Clearly $u_j \in L^2(D) \subseteq \mathcal{H}_{\Delta^2}^{-\frac{s}{2}}(D)$. We need to show that $\|\Psi_D^{\Delta^2}\|_{-s,\Delta^2} < +\infty$ almost surely. Now this boils down to showing the finiteness of the random series

$$\|\Psi_D^{\Delta^2}\|_{-s,\Delta^2}^2 = \sum_{j\geq 1} \lambda_j^{-s/2} \left(\sum_{k\geq 1} \lambda_k^{-1/2} u_k \xi_k \, , u_j \right)^2 = \sum_{j\geq 1} \lambda_j^{-\frac{s}{2}-1} \xi_j^2$$

where the last equality is true since $(u_j)_{j\geq 1}$ form an orthonormal basis of $L^2(D)$. Observe that the assumptions of Kolmogorov's two-series theorem are satisfied: indeed using Proposition 2.3.9 one has

$$\sum_{j\geq 1} \mathbf{E}\left(\lambda_j^{-\frac{s}{2}-1} \xi_j^2 \right) \asymp \sum_{j\geq 1} j^{-\frac{4}{d}\left(\frac{s}{2}+1\right)} < +\infty$$

for $s > (d-4)/2$ and

$$\sum_{j\geq 1} \mathbf{Var}\left(\lambda_j^{-\frac{s}{2}-1} \xi_j^2 \right) \asymp \sum_{j\geq 1} j^{-\frac{4}{d}(s+2)} < +\infty$$

for $s > (d-8)/4$. The result then follows. □

Definition of the limiting field via abstract Wiener spaces

We want now to connect the series representation given in Proposition 2.3.10 with an equivalent characterisation of $\Psi_D^{\Delta^2}$. This alternative definition can be given through the theory of abstract Wiener space (AWS). For a comprehensive overview of the theory we refer the readers to [68] for example. For our purposes it will suffice to recall that an abstract Wiener space is a triple $(\Theta, H, \mathcal{W})$, where

- Θ is a separable Banach space,
- H is a Hilbert space which is continuously embedded as a dense subspace of Θ, equipped with the scalar product $\langle\cdot,\,\cdot\rangle_H$,
- $\mathcal{W}$ is a Gaussian probability measure on Θ defined as follows.

Let Θ^* be the dual space of Θ. Given any $x^* \in \Theta^*$ there exists a unique $h_{x^*} \in H$ such that for all $h \in H$, $(h, x^*) = \langle h, h_{x^*}\rangle_H$ where $(\cdot,\, x^*)$ denotes the action of x^* on Θ. The σ-algebra $\mathcal{B}(\Theta)$ on Θ is such that all the maps $\theta \mapsto (\theta,\, x^*)$ are measurable. $\mathcal{W}$ is a probability measure such that, for all $x^* \in \Theta^*$,

$$\mathsf{E}_{\mathcal{W}}\left[\exp\left(\iota(\cdot, x^*)\right)\right] = \exp\left(-\frac{\|h_{x^*}\|_H^2}{2}\right). \tag{2.3.6}$$

In other words, the variable $(\cdot, x^*)$ under $\mathcal{W}$ is a centered Gaussian with variance $\|h_{x^*}\|_H^2$. Next, we introduce the *Paley–Wiener map* $\mathcal{I}$. $\mathcal{I}$ is viewed as a mapping

$$\begin{aligned} \mathcal{I}:\ h_{x^*} \in H \quad &\mapsto \quad \mathcal{I}(h_{x^*}) \in L^2(\mathcal{W}) \\ &\qquad\quad \theta \in \Theta \mapsto [\mathcal{I}(h_{x^*})](\theta) := (\theta,\, x^*). \end{aligned}$$

Since $\{h_{x^*} : x^* \in \Theta^*\}$ is dense in H (see [68, Lemma 8.2.3]), the map $h_{x^*} \mapsto I(h_{x^*})$ can be uniquely extended as a linear isometry from H to $L^2(\mathcal{W})$. [68, Theorem 8.2.6] yields that the family of Paley–Wiener integrals $\{\mathcal{I}\,(h) :\ h \in H\}$ is Gaussian, where each $\mathcal{I}(h)$ has mean zero and variance $\|h\|_H^2$. Given (2.3.6) the family $\{\mathcal{I}(u_j) :\ \{u_j\}_{j\in\mathbb{N}}$ orthonormal basis of $H\}$ is formed by i.i.d. standard Gaussians.

In our setting, by combining [68, §8.3.2] and the Wiener series given in Proposition 2.3.10, we can take $H := H_0^2(D)$ and $\mathcal{W}$ to be the law of $\Psi_D^{\Delta^2}$ on $\Theta := \mathcal{H}_{\Delta^2}^{-s}(D)$, for an arbitrary $s > (d-4)/2$. (the choice of Θ is not unique as explained in [68, Corollary 8.3.2]). Also by theorem 2.3.2 we can index the Paley–Wiener integrals $\mathcal{I}(u)$ over $u \in \mathcal{H}_{\Delta^2,0}^2(D)$ or take the maps $\mathcal{I}(B_0(f))$ over $f \in \mathcal{H}_{\Delta^2}^{-2}(D)$.

2.3.2 Discretisation set-up

We will use the parameter $h := 1/N$ for $N \in \mathbb{N}$. Let $D_h := \overline{D} \cap h\mathbb{Z}^d$. Let us denote by R_h the set of points ξ in D_h such that for every $i,\, j \in \{1, \ldots d\}$, the points $\xi \pm$

$h(e_i \pm e_j)$, $\xi \pm he_i$ are all in D_h. Let $\Lambda_N = \frac{1}{h} R_h \subset \mathbb{Z}^d$ be the "blow-up" of R_h. In other words, $\Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$ is the largest set satisfying $\partial_2 \Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$ where $\partial_2 \Lambda_N := \{y \in \mathbb{Z}^d \setminus \Lambda_N : \mathrm{dist}(y, \Lambda_N) \leq 2\}$ is the double (outer) boundary of Λ_N of points at ℓ^1 distance at most 2 from it. Let $(\varphi_z)_{z \in \Lambda_N}$ be the membrane model on Λ_N whose covariance is denoted by G_{Λ_N}. It satisfies the following boundary value problem: for all $x \in \Lambda_N$,

$$\begin{cases} \Delta^2 G_{\Lambda_N}(x, y) = \delta_x(y), & y \in \Lambda_N \\ G_{\Lambda_N}(x, y) = 0, & y \notin \Lambda_N \end{cases} . \tag{2.3.7}$$

Define Ψ_h by

$$(\Psi_h, f) := \kappa \sum_{x \in R_h} h^{\frac{d+4}{2}} \varphi_{x/h} f(x), \quad f \in \mathcal{H}^s_{\Delta^2,0}(D). \tag{2.3.8}$$

We first show that $\Psi_h \in \mathcal{H}^{-s}_{\Delta^2}(D)$ for all $s > d/2 + \lfloor d/2 \rfloor + 1$. Clearly Ψ_h is a linear functional on $\mathcal{H}^s_{\Delta^2,0}(D)$. Also we have for any $f \in \mathcal{H}^s_{\Delta^2,0}(D)$

$$\begin{aligned} |(\Psi_h, f)| &= \left| \kappa \sum_{x \in R_h} h^{\frac{d+4}{2}} \varphi_{x/h} \sum_{j \geq 1} \langle f\,,\, u_j \rangle_{L^2} u_j(x) \right| \\ &= \left| \kappa h^{\frac{d+4}{2}} \sum_{j \geq 1} \lambda_j^{-\frac{s}{2}} \sum_{x \in R_h} \varphi_{x/h} u_j(x) \lambda_j^{\frac{s}{2}} \langle f\,,\, u_j \rangle_{L^2} \right| \end{aligned}$$

where in the first equality we have used the fact that $f \in L^2(D)$ and therefore $f = \sum_{j \geq 1} \langle f\,,\, u_j \rangle_{L^2} u_j$. Thus

$$|(\Psi_h, f)| \leq \left(\sum_{j \geq 1} \lambda_j^{-\frac{s}{2}} (\Psi_h, u_j)^2 \right)^{\frac{1}{2}} \|f\|_{s, \Delta^2},$$

where (Ψ_h, u_j) can be defined analogous to (2.3.8) as

$$(\Psi_h, u_j) = \kappa \sum_{x \in R_h} h^{\frac{d+4}{2}} \varphi_{x/h} u_j(x).$$

To show Ψ_h is bounded, with the aid of Lemma 2.3.6 we observe that

$$\begin{aligned} \sum_{j \geq 1} \lambda_j^{-\frac{s}{2}} (\Psi_h, u_j)^2 &= \kappa^2 h^{d+4} \sum_{j \geq 1} \lambda_j^{-\frac{s}{2}} \left(\sum_{x \in R_h} \varphi_{x/h} u_j(x) \right)^2 \\ &\overset{(2.3.2)}{\leq} \kappa^2 h^{d+4} \left(\sum_{x \in R_h} |\varphi_{x/h}| \right)^2 \sum_{j \geq 1} \lambda_j^{-\frac{s}{2} + 2l_0} \end{aligned}$$

Now using Proposition 2.3.9 we conclude that the sum in the right hand side in finite whenever $s > d/2 + \lfloor d/2 \rfloor + 1$. Thus we have shown that $\Psi_h \in \mathcal{H}^{-s}_{\Delta^2}(D)$ for all $s > d/2 + \lfloor d/2 \rfloor + 1$ and we have

$$\|\Psi_h\|^2_{-s,\Delta^2} \leq \sum_{j\geq 1} \lambda_j^{-\frac{s}{2}} (\Psi_h, u_j)^2. \tag{2.3.9}$$

The result we want to show is

Theorem 2.3.11 (Scaling limit in $d \geq 4$)**.** *One has that, as $h \to 0$, the field Ψ_h converges in distribution to $\Psi^{\Delta^2}_D$ of Proposition 2.3.10 in the topology of $\mathcal{H}^{-s}_{\Delta^2}(D)$ for $s > s_d$, where*

$$s_d := \frac{d}{2} + 2\left(\left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right)\right\rceil + \left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 6\right)\right\rceil - 1\right).$$

Remark 2.3.12. *An analogous result holds in $d = 2,\ 3$, but we will not discuss it here as it is superseded by Theorem 2.2.1.*

2.3.3 Proof of the scaling limit (Theorem 2.3.11)

Once again we need to prove tightness and "convergence of marginal laws". In $d \geq 4$ however we are concerned with a field which is not defined pointwise, so that "marginal" from now takes on the meaning of the law of $(\Psi_h,\ f)$, namely the action of Ψ_h, seen as a distribution, on the test function f. The results are built on the approximation of the continuum Dirichlet problem for the bilaplacian by Thomée [69], combined with classical embeddings for Sobolev spaces.

Convergence of the marginals

To prove that the scaling limit is indeed $\Psi^{\Delta^2}_D$ we first have to find the marginal limiting laws. The set $C^\infty_c(D)$ is dense in $\mathcal{H}^s_{\Delta^2,0}(D)$, so we can use only smooth and compactly supported functions to test the convergence.

Proposition 2.3.13. *$(\Psi_h,\ f)$ converges in law to $(\Psi^{\Delta^2}_D,\ f)$ as $h \to 0$ for any f smooth and compactly supported in D.*

Proof. Since the Gaussian field φ is centered, we shall focus on the convergence of the variance only. Note that $\mathbf{Var}\left(\Psi^{\Delta^2}_D,\ f\right) = \|f\|^2_{-2,\Delta^2}$. Remark 2.3.8 tells us that we can limit ourselves to showing that

$$\lim_{h \to 0} \mathbf{Var}(\Psi_h,\, f) = \int_D u(x) f(x) \,\mathrm{d}\, x$$

where u is the solution of (2.3.5). We define

$$G_{R_h}(x, y) := \mathbf{E}[\varphi_{x/h}\varphi_{y/h}]\,,\ x, y \in D_h.$$

By (2.3.7) we have, for all $x \in R_h$,

$$\begin{cases} \Delta_h^2 G_{R_h}(x,y) = \frac{4d^2}{h^4}\delta_x(y), & y \in R_h \\ G_{R_h}(x,y) = 0, & y \notin R_h \end{cases}.$$

We have

$$\begin{aligned} \mathbf{Var}[(\Psi_h, f)] &= \kappa^2 \sum_{x,y \in R_h} h^{d+4} G_{R_h}(x,y) f(x) f(y) \\ &= \sum_{x \in R_h} h^d H_h(x) f(x) \end{aligned}$$

where $H_h(x) = \kappa^2 \sum_{y \in R_h} h^4 G_{R_h}(x,y) f(y)$, $x \in D_h$. It is immediate that H_h is the solution of the following Dirichlet problem,

$$\begin{cases} \Delta_h^2 H_h(x) = f(x), & x \in R_h \\ H_h(x) = 0, & x \notin R_h. \end{cases}$$

It is known that the above discrete solution is close to the continuum solution. The details of the result are described in Section 2.5; here we only recall that if we define $e_h(x) := u(x) - H_h(x)$ for $x \in D_h$ and $R_h f$ is the restriction of a function f to the set R_h as in (2.5.2), then from Theorem 2.5.5 we have

$$\|R_h e_h\|_{h,\,grid} \leq C h^{1/2}. \tag{2.3.10}$$

We have defined $\|f\|_{h,\,grid}^2 := h^d \sum_{\xi \in h\mathbb{Z}^d} f(\xi)^2$, where f is any grid function with finite support. Hence we get that

$$\mathbf{Var}\left[(\Psi_h, f)\right] = -\sum_{x \in R_h} e_h(x) f(x) h^d + \sum_{x \in R_h} u(x) f(x) h^d.$$

By Cauchy–Schwarz the first term in absolute value is bounded by $\|R_h e_h\|_{h,\,grid}\|f\|_{h,\,grid}$ and it goes to zero by (2.3.10) as $h \to 0$. For the second term we have

$$\lim_{h\to 0}\sum_{x\in R_h} u(x)f(x)h^d = \int_D u(x)f(x)\,\mathrm{d}\,x. \tag{2.3.11}$$

□

Tightness

We next prove the following lemma.

Lemma 2.3.14.

$$\limsup_{h\to 0}\mathbf{E}[\|\Psi_h\|^2_{-s,\Delta^2}] < \infty \quad \forall\, s > s_d.$$

Proof. From (2.3.9) we have

$$\mathbf{E}\left[\|\Psi_h\|^2_{-s,\Delta^2}\right] \le \sum_{j\in\mathbb{N}} \lambda_j^{-s/2}\mathbf{E}[(\Psi_h\,,\,u_j)^2].$$

Note that $u = \lambda_j^{-1}u_j$ is the unique solution of (2.3.5) for $f = u_j$. We therefore obtain as in the proof of proposition 2.3.13 by defining $e_{h,j}$ to be the error corresponding to $f = u_j$

$$\begin{aligned}\mathbf{E}[(\Psi_h\,,\,u_j)^2] &= -\sum_{x\in R_h} e_{h,j}(x)u_j(x)h^d + \sum_{x\in R_h}\lambda_j^{-1}u_j(x)u_j(x)h^d\\ &\le C\sup_{x\in D}|u_j(x)|\left(h^d\sum_{x\in R_h}e_{h,j}(x)^2\right)^{1/2} + C\lambda_j^{-1}\left(\sup_{x\in D}|u_j(x)|\right)^2.\end{aligned}$$

Using Theorem 2.5.5 along with the bounds (2.3.2)-(2.3.3)-(2.3.4) we obtain

$$\begin{aligned}\mathbf{E}[(\Psi_h\,,\,u_j)^2] &\le C\lambda_j^{l_0}[\lambda_j^{2l_5-2}h^2 + h\left(\lambda_j^{2l_5-2}h^6 + \lambda_j^{2l_2-2}\right)]^{\frac{1}{2}} + C\lambda_j^{2l_0-1}\\ &\le C\lambda_j^{l_0+l_5-1}.\end{aligned}$$

Therefore we have

$$\mathbf{E}\left[\|\Psi_h\|^2_{-s,\Delta^2}\right] \le C\sum_{j\in\mathbb{N}}\lambda_j^{-\frac{s}{2}}\lambda_j^{l_0+l_5-1}.$$

Thus

$$\limsup_{h\to 0} \mathbf{E}[\|\Psi_h\|^2_{-s,\Delta^2}] < \infty \qquad \text{if} \qquad \sum_{j\in\mathbb{N}} \lambda_j^{-\frac{s}{2}+l_0+l_5-1} < \infty.$$

And from proposition 2.3.9 we obtain that $\sum_{j\in\mathbb{N}} \lambda_j^{-\frac{s}{2}+l_0+l_5-1} < \infty$ whenever $s > s_d$. $\square$

To show tightness of Ψ_h we need the following theorem:

Theorem 2.3.15. *For $0 \leq s_1 < s_2$, $\mathcal{H}^{-s_1}_{\Delta^2}(D)$ is compactly embedded in $\mathcal{H}^{-s_2}_{\Delta^2}(D)$.*

Proof. It is enough to prove that $\mathcal{H}^{s_2}_{\Delta^2,0}(D)$ is compactly embedded in $\mathcal{H}^{s_1}_{\Delta^2,0}(D)$. The inclusion $\mathcal{H}^{s_2}_{\Delta^2,0}(D) \hookrightarrow \mathcal{H}^{s_1}_{\Delta^2,0}(D)$ is linear and continuous. To prove the inclusion to be compact let B be the unit ball of $\mathcal{H}^{s_2}_{\Delta^2,0}(D)$. Given $\epsilon > 0$ we choose $N \in \mathbb{N}$ large enough so that $N^{s_1-s_2} < \epsilon^4$. Now we consider the subspace Z of $\mathcal{H}^{s_2}_{\Delta^2,0}(D)$ defined by $Z := \left\{f \in \mathcal{H}^{s_2}_{\Delta^2,0}(D) : \langle f, u_j\rangle_{L^2} = 0 \;\forall\, j < N\right\}$. Then for any $f \in B \cap Z$ we have

$$\begin{aligned}\|f\|^2_{s_1,\Delta^2} &= \sum_{j\in\mathbb{N}} \lambda_j^{s_1/2} \langle f, u_j\rangle^2_{L^2} = \sum_{j\geq N} \lambda_j^{s_1/2} \langle f, u_j\rangle^2_{L^2} = \sum_{j\geq N} \lambda_j^{s_1/2-s_2/2}\lambda_j^{s_2/2} \langle f, u_j\rangle^2_{L^2} \\ &\leq N^{(s_1-s_2)/2} \sum_{j\geq N} \lambda_j^{s_2/2} \langle f, u_j\rangle^2_{L^2} = N^{(s_1-s_2)/2}\|f\|^2_{s_2,\Delta^2} < \epsilon^2.\end{aligned}$$

Also note that the dimension of $\mathcal{H}^{s_2}_{\Delta^2,0}(D)/Z$ is finite, so the unit ball of $\mathcal{H}^{s_2}_{\Delta^2,0}(D)/Z$ is compact and hence can be covered by finitely many balls of radius ϵ. Hence B can be covered by finitely many balls of radius 2ϵ in the $\|\cdot\|_{s_1,\Delta^2}$-norm. Since ϵ is arbitrary, B is precompact in $\mathcal{H}^{s_1}_{\Delta^2,0}(D)$. Therefore the inclusion map is compact. $\square$

Corollary 2.3.16. *The sequence $(\Psi_h)_{h=\frac{1}{N},N\in\mathbb{N}}$ is tight in $\mathcal{H}^{-s}_{\Delta^2}(D)$ for all $s > s_d$.*

Proof. Fix $s_0 > s_d$ and let $s_d < s_1 < s_0$. By Theorem 2.3.15, for any $R > 0$, $\overline{B_{\mathcal{H}^{-s_1}_{\Delta^2}(D)}}(0,\, R)$ is compact in $\mathcal{H}^{-s_0}_{\Delta^2}(D)$. By Lemma 2.3.14 we have for some $M > 0$

$$\mathbf{E}[\|\Psi_h\|^2_{-s_1,\Delta^2}] \leq M \quad \forall\, h.$$

Given $\epsilon > 0$, we take $R = \sqrt{2M\epsilon^{-1}}$ so that $MR^{-2} < \epsilon$. Now for all h

$$\mathbf{P}\left(\Psi_h \notin \overline{B_{\mathcal{H}^{-s_1}_{\Delta^2}(D)}(0\,,\, R)}\right) = \mathbf{P}\left(\|\Psi_h\|^2_{-s_1,\Delta^2} > R\right) \leq \frac{\mathbf{E}[\|\Psi_h\|^2_{-s_1,\Delta^2}]}{R^2} < \epsilon.$$

Thus $(\Psi_h)_h$ is tight in $\mathcal{H}^{-s_0}_{\Delta^2}(D)$. $\square$

Having obtained tightness and convergence of the marginals, all is left to do is to combine these ideas together to show the scaling limit.

Proof of Theorem 2.3.11. As (Ψ_h) is tight in $\mathcal{H}^{-s}_{\Delta^2}(D)$, it is enough to prove that every converging subsequence (Ψ_{h_i}) converges in distribution to $\Psi^{\Delta^2}_D$. Let (Ψ_{h_i}) be a subsequence of (Ψ_h) converging in distribution to Ψ in $\mathcal{H}^{-s}_{\Delta^2}(D)$. Then (Ψ_{h_i}, f) converges in distribution to (Ψ, f) for any $f \in \mathcal{H}^s_{\Delta^2,0}(D)$. But since (Ψ_h, f) converges in distribution to $(\Psi^{\Delta^2}_D, f)$ for all $f \in C^\infty_c(D)$, we must have $(\Psi^{\Delta^2}_D, f) \overset{d}{=} (\Psi, f)$ for all $f \in C^\infty_c(D)$. Now let $g \in \mathcal{H}^s_{\Delta^2,0}(D)$. Since $C^\infty_c(D)$ is dense in $\mathcal{H}^s_{\Delta^2,0}(D)$ we have a sequence (f_k) in $C^\infty_c(D)$ such that $f_k \to g$ in $\mathcal{H}^s_{\Delta^2,0}(D)$. Therefore $(\Psi^{\Delta^2}_D, f_k)$ and (Ψ, f_k) converge to $(\Psi^{\Delta^2}_D, g)$ and (Ψ, g) respectively. And hence $(\Psi^{\Delta^2}_D, f_k)$ and (Ψ, f_k) converge in distribution to $(\Psi^{\Delta^2}_D, g)$ and (Ψ, g) respectively. But since $(\Psi^{\Delta^2}_D, f_k) \overset{d}{=} (\Psi, f_k)$ for all k, we have $(\Psi^{\Delta^2}_D, g) \overset{d}{=} (\Psi, g)$. Thus we have $(\Psi^{\Delta^2}_D, f) \overset{d}{=} (\Psi, f)$ for all $f \in \mathcal{H}^s_{\Delta^2,0}(D)$. Hence $\Psi^{\Delta^2}_D \overset{d}{=} \Psi$, since the fields under considerations are linear. □

2.4 Convergence in infinite volume in $d \geq 5$

In this section we deal with the infinite volume membrane model defined on the whole of $\mathbb{Z}^d$ and show that the rescaled field converges to the continuum bilaplacian field on $\mathbb{R}^d$. Let $\mathbf{P}_N$ be the finite volume MM measure defined on V_N as mentioned in the Introduction. From Proposition 1.2.2 it follows that in $d \geq 5$ there exists $\mathbf{P}$ on $\mathbb{R}^{\mathbb{Z}^d}$ such that $\mathbf{P}_N \to \mathbf{P}$ in the weak topology of probability measures. Under $\mathbf{P}$, the canonical coordinates $(\varphi_x)_{x\in\mathbb{Z}^d}$ form a centered Gaussian process with covariance given by

$$G(x, y) = \Delta^{-2}(x, y) = \sum_{z\in\mathbb{Z}^d} \Delta^{-1}(x, z)\Delta^{-1}(z, y) = \sum_{z\in\mathbb{Z}^d} \Gamma(x, z)\Gamma(z, y),$$

where Γ denotes the covariance of the DGFF. Γ has an easy representation in terms of the simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$ given by

$$\Gamma(x, y) = \sum_{m\geq 0} \mathrm{P}_x[S_m = y]$$

(P_x is the law of S starting at x). This entails that

$$G(x,\, y) = \sum_{m \geq 0} (m+1) \mathrm{P}_x[S_m = y] = \mathrm{E}_{x,y}\left[\sum_{\ell,\, m=0}^{+\infty} \mathbb{1}_{\{S_m = \tilde{S}_\ell\}}\right] \tag{2.4.1}$$

where S and $\tilde{S}$ are two independent simple random walks started at x and y respectively. First one can note from this representation that $G(\cdot,\, \cdot)$ is translation invariant. The existence of the infinite volume measure in $d \geq 5$ gives that $G(0,0) < +\infty$. It is convenient to consider the convergence in the space of tempered distribution (dual of the Schwartz space on $\mathbb{R}^d$). For this we are giving some preliminary theoretical results.

2.4.1 Description of the limiting field

We consider $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ to be the Schwartz space that consists of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that, for all $m \in \mathbb{N} \cup \{0\}$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$,

$$\|f\|_{m,\alpha} = \sup_{x \in \mathbb{R}^d} (1 + \|x\|^m) |D^\alpha f(x)| < \infty.$$

$\mathcal{S}$ is a linear vector space and it is equipped with the topology generated by the family of semi-norms $\|\cdot\|_{m,\alpha}$, $m \in \mathbb{N} \cup \{0\}$ and $\alpha \in (\mathbb{N} \cup \{0\})^d$. The topological dual $\mathcal{S}^*$ of $\mathcal{S}$ is called the space of tempered distributions. For $F \in \mathcal{S}^*$ and $f \in \mathcal{S}$ we denote $F(f)$ by (F, f). We shall work with two topologies on $\mathcal{S}^*$, the strong topology τ_s and the weak topology τ_w. The strong topology τ_s is generated by the family of semi-norms $\{e_B : B \text{ is a bounded subset of } \mathcal{S}\}$ where $e_B(F) = \sup_{f \in B}(F, f), F \in \mathcal{S}^*$. τ_w is induced by the family of semi-norms $\{|(\cdot\,, f)| : f \in \mathcal{S}\}$. In particular F_n converges to F in $\mathcal{S}^*$ with respect to the weak topology when $\lim_n (F_n, f) = (F, f)$ for all $f \in \mathcal{S}$. It can be shown that the Borel σ-fields corresponding to both topologies coincide. Therefore we shall talk about the Borel σ-field $\mathcal{B}(\mathcal{S}^*)$ of $\mathcal{S}^*$ without specifying the topology.

Let $(\Omega\,, \mathcal{A},\, \mathrm{P})$ be a probability space. By a generalized random field defined on $(\Omega, \mathcal{A},\, \mathrm{P})$, we refer to a random variable X with values in $(\mathcal{S}^*, \mathcal{B}(\mathcal{S}^*))$. For $(X_n)_{n \geq 1}$ and X generalized random fields with laws $(\mathbf{P}_{X_n})_{n \geq 1}$ and $\mathbf{P}_X$ respectively, we say that X_n converges in distribution to X (and write $X_n \overset{d}{\to} X$) with respect to the strong topology if

$$\lim_{n \to \infty} \int_{\mathcal{S}^*} \varphi(F) \mathrm{d}\mathbf{P}_{X_n}(F) = \int_{\mathcal{S}^*} \varphi(F) \mathrm{d}\mathbf{P}_X(F) \quad \forall \varphi \in C_b(\mathcal{S}^*, \tau_s)$$

where $C_b(\mathcal{S}^*, \tau_s)$ is the space of bounded continuous functions on $\mathcal{S}^*$ given the strong topology. The convergence in distribution with respect to the weak topology is defined similarly with test functions in $C_b(\mathcal{S}^*, \tau_w)$. For a generalized random field X with law $\mathbf{P}_X$, we define its characteristic functional by

$$\mathcal{L}_X(f) = \mathbf{E}(\mathrm{e}^{\iota(X,f)}) = \int_{\mathcal{S}^*} \mathrm{e}^{\iota(F,f)} \mathrm{d}\mathbf{P}_X(F)$$

for $f \in \mathcal{S}$. Note that $\mathcal{L}_X$ is positive definite, continuous, and $\mathcal{L}_X(0) = 1$. The Bochner–Minlos theorem says that the converse is also true: if a functional $\mathcal{L} : \mathcal{S} \to \mathbb{C}$ is positive definite, continuous at 0 and satisfies $\mathcal{L}(0) = 1$ then there exists a generalized random field X defined on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ such that $\mathcal{L}_X = \mathcal{L}$. For a proof of this theorem see for instance [42, Appendix 1]. Another important feature of characteristic functions is that their convergence determines convergence of generalised random fields. This is classical result of Lévy which was generalized and proved in the nuclear space setting first by Fernique [36]. We use the version for tempered distributions which was recently proved in [6].

Fact 2.4.1 ([6, Corollary 2.4])**.** *Let $(X_n)_{n\geq 1}$, X be generalized random fields. The following conditions are equivalent:*

(i) $X_n \xrightarrow{d} X$ *in the strong topology.*

(ii) $X_n \xrightarrow{d} X$ *in the weak topology.*

(iii) $\mathcal{L}_{X_n}(f) \to \mathcal{L}_X(f)$ *for all* $f \in \mathcal{S}$.

(iv) $(X_n, f) \xrightarrow{d} (X, f)$ *in* $\mathbb{R}$ *for all* $f \in \mathcal{S}$.

For $f \in \mathcal{S}$ we define $\widehat{f}$ by

$$\widehat{f}(\theta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\iota\langle x,\theta\rangle} f(x)\, \mathrm{d}\, x.$$

Let us define an operator $(-\Delta_c)^{-1} : \mathcal{S} \to L^2(\mathbb{R}^d)$ as follows [1, Section 1.2.2]:

$$(-\Delta_c)^{-1} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\iota\langle x,\xi\rangle} \|\xi\|^{-2} \widehat{f}(\xi)\, \mathrm{d}\, \xi.$$

We use now the operator $(-\Delta_c)^{-1}$ to define the limiting field Ψ^{Δ^2}. It is the fractional Gaussian field of parameter $s := 2$ described in [51, Section 3.1], to which we refer for a proof of the following fact, relying on the Bochner–Minlos theorem.

Lemma 2.4.2. *There exists a generalized random field Ψ^{Δ^2} on $\mathcal{S}^*$ whose characteristic functional $\mathcal{L}_{\Psi^{\Delta^2}}$ is given by*

$$\mathcal{L}_{\Psi^{\Delta^2}}(f) = \exp\left(-\frac{1}{2}\|(-\Delta_c)^{-1} f\|^2_{L^2(\mathbb{R}^d)}\right), \quad f \in \mathcal{S}.$$

Consider $(\varphi_x)_{x\in\mathbb{Z}^d}$ to be the membrane model in $d \geq 5$. We define

$$\psi_N(x) := \kappa N^{\frac{d-4}{2}} \varphi_{Nx}, \quad x \in \frac{1}{N}\mathbb{Z}^d.$$

For $f \in \mathcal{S}$ we define

$$(\Psi_N, f) := N^{-d} \sum_{x\in\frac{1}{N}\mathbb{Z}^d} \psi_N(x) f(x). \tag{2.4.2}$$

The above definition makes sense since, using Mill's ratio and the uniform boundedness of $G(\cdot,\cdot)$, one can show that, as $\|x\| \to \infty$,

$$|\psi_N(x)| = O\big(N^{\frac{d-4}{2}}\sqrt{\log(1+\|x\|)}\big) \quad a.s.$$

via a Borell-Cantelli argument. This justifies (2.4.2) using the fast decay of f at infinity. Also it follows that $\Psi_N \in \mathcal{S}^*$ and the characteristic functional of Ψ_N is given by

$$\mathcal{L}_{\Psi_N}(f) := \exp(-\mathbf{Var}\,(\Psi_N, f)\,/2).$$

The following theorem shows that the field Ψ_N constructed above converges to Ψ^{Δ^2} defined in Lemma 2.4.2.

Theorem 2.4.3 (Scaling limit in $d \geq 5$)**.** *Let $d \geq 5$ and Ψ_N be the field on $\mathcal{S}^*$ defined by (2.4.2). Then $\Psi_N \xrightarrow{d} \Psi^{\Delta^2}$ in the strong topology where Ψ^{Δ^2} is defined in Lemma 2.4.2.*

2.4.2 Proof of the scaling limit (Theorem 2.4.3)

The proof of our last theorem of this chapter relies on the result recalled in Fact 2.4.1, therefore unlike the two previous theorems it is not divided into tightness and finite dimensional convergence. The argument is based on Fourier analysis, and will be a consequence of two claims which we will show after the main proof.

Proof of Theorem 2.4.3. We first show that for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathbf{E}\left[(\Psi_N, f)^2\right] \to \|(-\Delta_c)^{-1} f\|^2_{L^2(\mathbb{R}^d)}.$$

By our definition we have for $f,\, g \in \mathcal{S}$

$$\mathrm{Cov}((\Psi_N, f)\,, (\Psi_N, g)) = \kappa^2 N^{-(d+4)} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} G(0, N(y-x)) f(x) g(y).$$

Hence

$$\mathbf{E}\left[(\Psi_N, f)^2\right] = \kappa^2 N^{-(d+4)} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} G(0, N(y-x)) f(x) f(y).$$

From (1.2.3) we have

$$G(0,x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} (\mu(\theta))^{-2} \,\mathrm{e}^{-\iota\langle x,\theta\rangle} \,\mathrm{d}\,\theta$$

where $\mu(\theta) = \frac{1}{d}\sum_{i=1}^d (1-\cos(\theta_i)) = \frac{2}{d}\sum_{i=1}^d \sin^2(\frac{\theta_i}{2})$. Hence we have

$$\begin{aligned}
\mathbf{E}\left[(\Psi_N, f)^2\right] &= \frac{\kappa^2 N^{-(d+4)}}{(2\pi)^d} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} \int_{[-\pi,\pi]^d} (\mu(\theta))^{-2} \,\mathrm{e}^{-\iota\langle N(y-x),\theta\rangle} f(x) f(y) \,\mathrm{d}\,\theta \\
&= \frac{\kappa^2 N^{-(d+4)}}{(2\pi)^d} \sum_{x,y \in \frac{1}{N}\mathbb{Z}^d} \int_{[-\pi,\pi]^d} (\mu(\theta))^{-2} \,\mathrm{e}^{-\iota\langle (y-x),N\theta\rangle} f(x) f(y) \,\mathrm{d}\,\theta \\
&= \frac{\kappa^2 N^{-4}}{(2\pi)^d} \int_{[-N\pi,N\pi]^d} \left(\mu\left(\frac{\theta}{N}\right)\right)^{-2} \left| N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \mathrm{e}^{-\iota\langle x,\theta\rangle} f(x)\right|^2 \mathrm{d}\,\theta. \qquad (2.4.3)
\end{aligned}$$

We have used in the above Fubini's theorem, justified by the following bound [26, Lemma 7]: there exists $C > 0$ such that for all $N \in \mathbb{N}$ and $w \in [-N\pi/2, N\pi/2]^d \setminus \{0\}$ we have

$$\frac{1}{\|w\|^4} \le N^{-4} \left(\sum_{i=1}^d \sin^2\left(\frac{w_i}{N}\right)\right)^{-2} \le \left(\frac{1}{\|w\|^2} + \frac{C}{N^2}\right)^2. \qquad (2.4.4)$$

We make two claims which will prove the convergence of variance.

Claim 2.4.4.

$$\lim_{N\to+\infty}\left|\frac{\kappa^2 N^{-4}}{(2\pi)^d}\int_{[-N\pi,N\pi]^d}\left(\mu\left(\frac{\theta}{N}\right)\right)^{-2}\left|N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta\right.$$
$$\left.-\frac{1}{(2\pi)^d}\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}\left|N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta\right|=0.$$

Next we claim the convergence of the following term:

Claim 2.4.5.

$$\lim_{N\to+\infty}\frac{1}{(2\pi)^d}\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}\left|N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta=\|(-\Delta_c)^{-1}f\|^2_{L^2(\mathbb{R}^d)}.$$

Claims 2.4.4-2.4.5 entail that

$$\lim_{N\to\infty}\mathcal{L}_{\Psi_N}(f)=\exp\left(-\frac{1}{2}\|(-\Delta_c)^{-1}f\|^2_{L^2(\mathbb{R}^d)}\right).$$

Thus we have for all $f\in\mathcal{S}$

$$\mathcal{L}_{\Psi_N}(f)\to\mathcal{L}_{\Psi_{\Delta^2}}(f).$$

Hence the conclusion follows from Fact 2.4.1. □

To prove the above two claims we use crucially the following estimate for approximating Riemann sums for Schwartz functions. Since we could not find a reference we provide a short proof of the following fact:

Lemma 2.4.6. *For any $N\geq 1$ and $s>0$ we have*

$$\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\iota\left\langle\frac{x}{N},\theta\right\rangle}f\left(\frac{x}{N}\right)-\widehat{f}(\theta)\right|\leq CN^{-s}\tag{2.4.5}$$

where C may depend on f.

Proof. To show the above result we use the Poisson summation formula [67, Chapter 7]. Let us define $g(x):=(2\pi)^{-d/2}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)$. Using the Poisson summation formula we get

$$N^{-d}\sum_{x\in\mathbb{Z}^d} g\left(\frac{x}{N}\right) = \sum_{x\in\mathbb{Z}^d}\widehat{f}(\theta+2\pi xN).$$

Hence we have

$$\begin{aligned}\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\iota\left\langle\frac{x}{N},\theta\right\rangle}f\left(\frac{x}{N}\right)-\widehat{f}(\theta)\right| &\le \sum_{x\neq 0,x\in\mathbb{Z}^d}|\widehat{f}(\theta+2\pi xN)|\\ &\le \sum_{x\neq 0,x\in\mathbb{Z}^d}\frac{C}{\|\theta+2\pi xN\|_\infty^s}\end{aligned}$$

where the last inequality holds for any $s \ge 0$ because $f \in \mathcal{S}$. But

$$\|2\pi xN\|_\infty \le \|\theta+2\pi xN\|_\infty + \|\theta\|_\infty \le \|\theta+2\pi xN\|_\infty + N\pi$$

and hence, for $s > 1$, $\|2\pi xN\|_\infty^s \le 2^{s-1}\left(\|\theta+2\pi xN\|_\infty^s + (N\pi)^s\right)$. Thus for any $s \ge d_0 > d$, we have

$$\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\iota\left\langle\frac{x}{N},\theta\right\rangle}f\left(\frac{x}{N}\right)-\widehat{f}(\theta)\right| \le \sum_{x\neq 0,x\in\mathbb{Z}^d}\frac{C}{(N\pi)^s(2\|x\|_\infty^s-1)} \le CN^{-s}. \quad\square$$

where the constant C depends on d_0 but not on s. Hence the result follows.

We can now begin with the proof of the two claims.

Proof of Claim 2.4.4. Recall that $\kappa = 1/(2d)$. Using the bound (2.4.4) for $w_i = \theta_i/2$ we have

$$\begin{aligned}&\left|\frac{\kappa^2N^{-4}}{(2\pi)^d}\int_{[-N\pi,N\pi]^d}\left(\mu\left(\frac{\theta}{N}\right)\right)^{-2}|N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)|^2\,\mathrm{d}\,\theta\right.\\ &\qquad\left.-\frac{1}{(2\pi)^d}\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}\left|N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\,\mathrm{d}\,\theta\right|\\ &\le\int_{[-N\pi,N\pi]^d}\left(2\|\theta\|^{-2}\frac{C}{N^2}+\frac{C}{N^4}\right)\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\,\mathrm{d}\,\theta.\end{aligned}$$

Using $(a+b)^2 \leq 2(a^2+b^2)$ after adding and subtracting $\widehat{f}(\theta)$ in the modulus above we have the bound

$$\int_{[-N\pi,N\pi]^d} \left(2\|\theta\|^{-2}\frac{C}{N^2}+\frac{C}{N^4}\right)(CN^{-s}+|\widehat{f}(\theta)|)^2 \,\mathrm{d}\,\theta$$
$$\leq \int_{[-N\pi,N\pi]^d} 2\left(2\|\theta\|^{-2}\frac{C}{N^2}+\frac{C}{N^4}\right)(CN^{-2s}+|\widehat{f}(\theta)|^2) \,\mathrm{d}\,\theta.$$

Again the last term amounts to estimating

$$CN^{-2s-2}O(N^{d-2})+CN^{-2s-4}O(N^d)$$
$$+CN^{-2}\int_{[-N\pi,N\pi]^d}\|\theta\|^{-2}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta+CN^{-4}\int_{[-N\pi,N\pi]^d}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta$$

which goes to 0 due to the fact that $f \in \mathcal{S}$. □

Proof of Claim 2.4.5. We have

$$\|(-\Delta_c)^{-1}f\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\|\theta\|^{-4}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta$$

and

$$\left|\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)|^2-\int_{\mathbb{R}^d}\|\theta\|^{-4}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta\right|$$
$$\leq\left|\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)|^2-\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta\right|$$
$$+\left|\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta-\int_{\mathbb{R}^d}\|\theta\|^{-4}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta\right|.$$

Clearly the second term goes to zero as N tends to infinity. As for the first term we have the following bound

$$\left|\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)|^2-\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}|\widehat{f}(\theta)|^2\,\mathrm{d}\,\theta\right|$$
$$\leq\int_{[-N\pi,N\pi]^d}\|\theta\|^{-4}\left||(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)|^2-|\widehat{f}(\theta)|^2\right|\mathrm{d}\,\theta$$

$$\leq (2\|f\|_{L^1} + C) \int_{[-N\pi,N\pi]^d} \|\theta\|^{-4} \left| (2\pi)^{-d/2} N^{-d} \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\iota\left\langle \frac{x}{N},\theta\right\rangle} f\left(\frac{x}{N}\right) - \widehat{f}(\theta) \right| \mathrm{d}\,\theta$$
$$= O(N^{d-4-s}).$$

where the bound in the second inequality is obtained using the formula $(a^2 - b^2) = (a+b)(a-b)$ and (2.4.5). Thus the first term also goes to zero as N tends to infinity. □

2.5 Quantitative estimate on the discrete approximation in [69]

This section is devoted to obtaining quantitative estimates on approximation of solutions of PDEs. The building block of our analysis is the paper [69]. Let D be any bounded domain in $\mathbb{R}^d$ with C^2 boundary. We denote $L := \Delta_c^2$ and consider the following continuum Dirichlet problem:

$$\begin{cases} Lu(x) = f(x), & x \in D \\ D^\alpha u(x) = 0, & |\alpha| \leq 1,\, x \in \partial D. \end{cases} \tag{2.5.1}$$

Let $h > 0$. We will call the points in $h\mathbb{Z}^d$ the grid points in $\mathbb{R}^d$. We consider $L_h u := \Delta_h^2 u$ to be the discrete approximation of Lu. Thus we have, for $x \in h\mathbb{Z}^d$,

$$\begin{aligned} L_h u(x) &= \frac{1}{h^2}\sum_{i=1}^{d}(\Delta_h u(x+he_i) + \Delta_h u(x-he_i) - 2\Delta_h u(x)) \\ &= \frac{1}{h^4}\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\{u(x+h(e_i+e_j)) + u(x-h(e_i+e_j)) + u(x+h(e_i-e_j))\right. \\ &\quad \left. + u(x-h(e_i-e_j))\} - 4d\sum_{i=1}^{d}\{u(x+he_i)+u(x-he_i)\} + 4d^2u(x)\right]. \end{aligned}$$

Let D_h be the set of grid points in $\overline{D}$ i.e. $D_h = \overline{D} \cap h\mathbb{Z}^d$. We say that ξ is an interior grid point in D_h or $\xi \in R_h$ if for every i, j, the points $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ are all in D_h. We denote B_h to be $D_h \setminus R_h$. We will denote by $\mathcal{D}_h$ the set of grid functions vanishing outside R_h. For a grid function f we define $R_h f \in \mathcal{D}_h$ by

$$R_h f(\xi) = \begin{cases} f(\xi) & \xi \in R_h \\ 0 & \xi \notin R_h. \end{cases} \tag{2.5.2}$$

In [69] it is crucially used that the discrete approximation of the elliptic operator is consistent. In our case it is easy to see this using Taylor's expansion.

Lemma 2.5.1. *The operator L_h is consistent with the operator L, that is, if W is a neighborhood of the origin in $\mathbb{R}^d$ and $u \in C^4(W)$ then*

$$L_h u(0) = Lu(0) + o(1) \ \text{ as } h \to 0.$$

We will divide R_h further into R_h^* and B_h^* where R_h^* is the set of ξ in R_h such that for every i, j, the points $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ are all in R_h and B_h^* is the set of remaining points in R_h. Thus we have

$$D_h = B_h \cup R_h = B_h \cup B_h^* \cup R_h^*.$$

We say that the domain D has property $\mathcal{B}_2^*$ if there is a natural number K such that for all sufficiently small h, the following is valid: consider for any $\xi \in B_h^*$ all half-rays through ξ. At least one of them contains within the distance Kh from ξ two consecutive grid-points in B_h.

The following proposition shows that if the boundary of the domain is regular enough then the property $\mathcal{B}_2^*$ is true. Namely, recall the uniform exterior ball condition (UEBC) for a domain D, which states that there exists $\delta > 0$ such that for any $z \in \partial D$ there is a ball $B_\delta(c)$ of radius δ with center at some point c satisfying $\overline{B_\delta(c)} \cap \overline{D} = \{z\}$ [41, page 27]. We show that the UEBC is a sufficient condition for $\mathcal{B}_2^*$ to hold. In particular, any domain with C^2 boundary satisfy the UEBC and hence possesses $\mathcal{B}_2^*$.

Proposition 2.5.2. *If a bounded domain D satisfies the UEBC then the property $\mathcal{B}_2^*$ holds.*

Since the proof of this result is purely geometric and combinatorial in nature we discuss it in Section 2.6. We would like to remark that property $\mathcal{B}_2^*$ is a crucial requirement in the proof of Theorem 2.5.4. In fact, it allows us to use Thomée's result [69, Lemma 3.4] which compares the standard discrete Sobolev norm with a modified Sobolev norm weighted on boundary points.

We now define the finite difference analogue of the Dirichlet's problem (2.5.1). For given h, we look for a function $u(\xi)$ defined on D_h such that

$$L_h u_h(\xi) = f(\xi), \quad \xi \in R_h \tag{2.5.3}$$

and

$$u_h(\xi) = 0, \quad \xi \in B_h. \tag{2.5.4}$$

It follows from Lemma 2.5.1 and [69, Theorem 5.1] that the finite difference Dirichlet problem (2.5.3) and (2.5.4) has exactly one solution for arbitrary f. Recall also the norm $\|f\|^2_{h,\,grid} := h^d \sum_{\xi \in h\mathbb{Z}^d} f(\xi)^2$. Before we prove the approximation theorem, let us cite two results from [69] (stated, in the original article, in a slightly more general way). The first lemma is the discrete analogue of the Poincaré inequality.

Lemma 2.5.3 ([69, Lemma 3.1])**.** *There are constants $C > 0$ independent of f and h such that*

$$\|f\|_{h,\,grid} \le C\|\partial_j f\|_{h,\,grid}, \quad j = 1, \ldots, d$$

and

$$\|f\|_{h,\,grid} \le C\|f\|_{h,\,2} := \left(\sum_{|\beta|\le 2} \|\partial^\beta f\|^2_{h,\,grid}\right)^{1/2}$$

for any grid function f vanishing outside R_h, where

$$\partial_j f(x) := \frac{1}{h}(f(x+he_j) - f(x)),\ j = 1, \ldots, d$$

and

$$\partial^\beta f := \partial_1^{\beta_1} \cdots \partial_d^{\beta_d} f,\ \beta = (\beta_1, \ldots, \beta_d),\ \beta_i \ge 0.$$

Our aim is to estimate the error while approximating the solution of the boundary value problem involving the continuum operator L by its discrete counter part. More precisely, we want a bound of the $\|\cdot\|_{h,\,grid}$-norm of the error restricted to R_h. If we denote this restriction of the error by g, then by the above lemma we have $\|g\|_{h,\,grid} \le C\|g\|_{h,\,2}$. Also one can show that $\|g\|_{h,\,2} \le C\|L_h g\|_{h,\,grid}$. To estimate $\|L_h g\|_{h,\,grid}$ one can use Taylor expansion for a point which is well inside the domain. But near the boundary this is no longer possible and for those points the estimate obtained using the boundary condition is not useful. To overcome this obstacle we define a new operator $L_{h,2}$, where we suitably truncate and modify the operator L_h near the boundary. Also we use the result $\|g\|_{h,\,2} \leq C\|L_{h,2}g\|_{h,\,grid}$ to estimate the error. We give the definition of the operator $L_{h,2}$ from [69] as follows:

$$L_{h,2}f(x) = \begin{cases} L_hf(x) & x \in R_h^* \\ h^2L_hf(x) & x \in B_h^* \\ 0 & x \notin R_h. \end{cases}$$

Theorem 2.5.4 ([69, Theorem 4.2])**.** *There exists a constant $C > 0$ such that for all grid functions f vanishing outside R_h*

$$\|f\|_{h,\,2} \leq C\|L_{h,\,2}f\|_{h,\,grid},$$

where C is independent of h as well.

We have now all the ingredients to show the following.

Theorem 2.5.5. *Let $u \in C^5(\overline{D})$ be the solution of the Dirichlet's problem 2.5.1 and u_h be the solution of the discrete problem (2.5.3)-(2.5.4). If $e_h := u - u_h$ then we have for all sufficiently small h*

$$\|R_he_h\|^2_{h,\,grid} \leq C\left[M_5^2h^2 + h(M_5^2h^6 + M_2^2)\right]$$

where $M_k = \sum_{|\alpha|\leq k}\sup_{x\in D}|D^\alpha u(x)|$.

Proof. We denote all constants by C and they do not depend on u, f. First we use Lemma 2.5.3 and Theorem 2.5.4 to obtain

$$\|R_he_h\|^2_{h,\,grid} \leq C\|L_{h,\,2}R_he_h\|^2_{h,\,grid}.$$

The proof now boils down to bounding $\|L_{h,\,2}R_he_h\|^2_{h,\,grid}$. Using Taylor's expansion we have for all $x \in R_h$ and for small h

$$L_hu(x) = Lu(x) + h^{-4}\mathcal{R}_5(x)$$

where $|\mathcal{R}_5(x)| \le CM_5h^5$. We obtain for $\xi \in R_h$,

$$\begin{aligned}L_he_h(\xi) &= L_hu(\xi) - L_hu_h(\xi)\\ &= Lu(\xi) + h^{-4}\mathcal{R}_5(\xi) - L_hu_h(\xi) = h^{-4}\mathcal{R}_5(\xi).\end{aligned}$$

For $\xi \in R_h^*$ we have

$$L_{h,2}R_he_h(\xi) = L_hR_he_h(\xi) = L_he_h(\xi) = h^{-4}\mathcal{R}_5(\xi).$$

For $\xi \in B_h^*$ at least one among $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ is in B_h. For any $\eta \in B_h \setminus \partial D$ we consider a point $b(\eta)$ on ∂D of minimal distance to η. Note that this distance is at most $2h$. Also note that $u_h(\eta) = 0$ by definition. Now using Taylor expansion and the fact that the value of u and all its first order derivatives are zero at $b(\eta)$ one sees that

$$u(\eta) = u_h(\eta) + \mathcal{R}_2(\eta)$$

where $|\mathcal{R}_2(\eta)| \le CM_2h^2$. For $\xi \in B_h^*$ denote by

$$S_{i,j}(\xi) = \{\eta : \eta \in B_h \setminus (B_h \cap \partial D) \cap \{\xi \pm he_i, \xi \pm h(e_i \pm e_j)\}\}.$$

Therefore, for $\xi \in B_h^*$,

$$\begin{aligned}L_{h,2}R_he_h(\xi) &= h^2L_hR_he_h(\xi)\\ &= h^2\left\{L_he_h(\xi) - h^{-4}\sum_{i,j=1}^{d}\sum_{\eta\in S_{i,j}(\xi)} C(\eta)e_h(\eta)\right\}\\ &= h^{-2}\mathcal{R}_5(\xi) + h^{-2}C\mathcal{R}_2^{'}(\xi)\end{aligned}$$

where $C(\eta)$ is a constant depending on η and $|\mathcal{R}_2^{'}(\xi)| \le CM_2h^2$. Hence

$$\begin{aligned}\|L_{h,2}R_he_h\|^2_{h,grid} &= h^d\sum_{x\in R_h}(L_{h,2}R_he_h(x))^2\\ &= h^d\left[\sum_{x\in R_h^*}(L_{h,2}R_he_h(x))^2 + \sum_{x\in B_h^*}(L_{h,2}R_he_h(x))^2\right]\\ &= h^d\left[\sum_{x\in R_h^*}(h^{-4}\mathcal{R}_5(x))^2 + \sum_{x\in B_h^*}(h^{-2}\mathcal{R}_5(x) + h^{-2}C\mathcal{R}_2^{'}(x))^2\right]\end{aligned}$$

$$\leq h^d \left[\sum_{x \in R_h^*} CM_5^2 h^2 + \sum_{x \in B_h^*} (CM_5^2 h^6 + CM_2^2) \right]$$
$$\leq C \left[M_5^2 h^2 + h(M_5^2 h^6 + M_2^2) \right]$$

where in the last inequality we have used that the number of points in B_h^* is $O(h^{-(d-1)})$ following from [56, Lemma 5.4] and the assumption of a C^2 boundary. This concludes the proof. □

2.6 Proof of Proposition 2.5.2

We start with some heuristic explanation on the proof. The assumption of C^2 boundary ensures that there is no cone like structure in the domain, that is, locally the boundary surface is like sphere. So if we consider a grid point in the domain which is close to the boundary and consider all the $2d$ half-rays through the point, then at least one of them meets the boundary within a short distance depending on the closeness of the point to the boundary. In the following proof we make this idea rigorous.

Proof of Proposition 2.5.2. If $d = 1$ then it is easy to see from the definition that $\mathcal{B}_2^*$ holds. So we assume $d \geq 2$. For any $y \in D_h$ we denote by $N(y)$ the neighbourhood of y, that is,

$$N(y) := \{y \pm he_i, y \pm he_i \pm he_j : 1 \leq i, j \leq d\}.$$

We consider in fact a second-nearest neighbourhood in the graph distance, due to the interaction of the discrete bilaplacian and Thomée's definition of neighbour. Let us now recall the definitions:

$$D_h = \overline{D} \cap h\,\mathbb{Z}^d,$$
$$R_h = \{x \in D_h : N(x) \subseteq D_h\},$$
$$B_h = D_h \setminus R_h,$$
$$R_h^* = \{x \in R_h : N(x) \subseteq R_h\},$$
$$B_h^* = R_h \setminus R_h^*.$$

Thus $D_h = B_h \cup B_h^* \cup R_h^*$. We want to show that for sufficiently small h the following holds: for any $x \in B_h^*$ there exists $i \in \{1, \ldots, d\}$ such that any two consecutive points of either $\{x + he_i,\, x + 2he_i,\, x + 3he_i,\, x + 4he_i\}$ or $\{x - he_i,\, x - 2he_i,\, x - 3he_i,\, x - 4he_i\}$ belong to B_h. The proof is done on a case-by-case basis. We prove the existence of two consecutive points by broadly considering the following two possibilities:

- suppose $x \in B_h^*$ is such that $\operatorname{dist}(x, B_h) = 1$, then we get an $i_0 \in \{1, \ldots, d\}$ so that either $x + he_{i_0},\, x + 2he_{i_0} \in B_h$ or $x - he_{i_0},\, x - 2he_{i_0} \in B_h$.
- Now suppose $x \in B_h^*$ is such that $\operatorname{dist}(x, B_h) = 2$. In this case if $\{x \pm 2he_i : 1 \le i \le d\} \cap B_h$ is non-empty then we get an $i_0 \in \{1, \ldots, d\}$ so that either $x+2he_{i_0}, x+3he_{i_0} \in B_h$ or $x - 2he_{i_0}, x - 3he_{i_0} \in B_h$. Otherwise, $\{x \pm 2he_i : 1 \le i \le d\} \cap B_h$ is empty and $\{x \pm he_i \pm he_j : 1 \le i, j \le d, i \ne j\} \cap B_h$ is non-empty. And then we extract an i_0 so that either $x + 3he_{i_0}, x + 4he_{i_0} \in B_h$ or $x - 3he_{i_0}, x - 4he_{i_0} \in B_h$.

In the process of obtaining these suitable points, we rule out some of the cases which do not arise due to the regularity of the boundary.

Fix $x \in B_h^*$. Then $N(x) \subset D_h$ and $N(x) \cap B_h \neq \emptyset$.

1. Suppose $\{x \pm he_i : 1 \le i \le d\} \cap B_h \neq \emptyset$. We assume for simplicity that $x + he_1 \in B_h$ as the argument will be similar for other directions. If $x + 2he_1 \in B_h$, then there is nothing to prove. More elaborate is the case when $x + 2he_1 \in R_h$. Then we have

$$\begin{aligned}
&N(x) \subseteq \overline{D},\\
&N(x + he_1) \nsubseteq \overline{D},\\
&N(x + 2he_1) \subseteq \overline{D}.
\end{aligned}$$

Observe that from the preceding inclusions we must have

$$\{x + he_1 \pm he_i \pm he_j : 2 \le i, j \le d\} \nsubseteq \overline{D}. \tag{2.6.1}$$

We now partition this set into 2 subsets and argue separately.

1.1. Suppose $\{x+he_1 \pm 2he_i : 2 \le i \le d\} \nsubseteq \overline{D}$. Let us assume that $x+he_1+2he_2 \notin \overline{D}$. Then by definition of B_h we have $x+he_2,\, x+2he_2 \in B_h$ and we are done. Similar is the case for other points.

1.2. We are left with the situation where $\{x+he_1 \pm 2he_i : 2 \le i \le d\} \subset \overline{D}$ and $\{x+he_1 \pm he_i \pm he_j : 2 \le i,j \le d, i \ne j\} \nsubseteq \overline{D}$. Note that this situation is not possible in $d=2$ and hence from now we consider $d \ge 3$ for this subcase.

Again we continue with a particular choice $x+he_1+he_2+he_3 \notin \overline{D}$. The other occurrences can be handled similarly. Note that with this choice we have $x+he_2$, $x+he_3 \in B_h$. So if at least one between $x+2he_2$ and $x+2he_3$ belongs to B_h then we are done. Otherwise we have the following situation:

$$\begin{aligned}\{x,\, x+2he_1,\, x+2he_2,\, x+2he_3\} &\subset R_h,\\ \{x+he_1,\, x+he_2,\, x+he_3\} &\subset B_h,\\ \{x+he_1 \pm 2he_i : 2 \le i \le d\} &\subseteq \overline{D}\end{aligned}$$

and $x+he_1+he_2+he_3 \notin \overline{D}$. Note here that the point $x+he_1+he_2+he_3$, which is at graph distance 3 from x, is not in $\overline{D}$. However its nearby points $\{x+2he_1+he_2+he_3,\, x+he_2+he_3,\, x+he_1+2he_2+he_3,\, x+he_1+he_3,\, x+he_1+he_2+2he_3,\, x+he_1+he_2\}$ stay inside $\overline{D}$. We show that such a situation cannot happen due to the UEBC. Indeed, since the domain satisfies UEBC, we can find for small h a ball $B_\delta(c)$ for some $c \in \mathbb{R}^d$ such that $x+he_1+he_2+he_3 \in B_\delta(c)$ and $\overline{B_\delta(c)} \cap \overline{D} = \{y\}$ for some $y \in \partial D$. Clearly, if $x = (x_1, \ldots, x_d)$ and $c = (c_1, \ldots, c_d)$ then

$$\sum_{i=1}^{d} (c_i - x_i)^2 > \delta^2 \tag{2.6.2}$$

and

$$\sum_{i=1}^{3} (c_i - x_i - h)^2 + \sum_{i=4}^{d} (c_i - x_i)^2 < \delta^2. \tag{2.6.3}$$

Since $x+2he_1+he_2+he_3$, $x+he_2+he_3 \in \overline{D}$ we have

$$(c_1 - x_1 - 2h)^2 + (c_2 - x_2 - h)^2 + (c_3 - x_3 - h)^2 + \sum_{i=4}^{d} (c_i - x_i)^2 \ge \delta^2, \tag{2.6.4}$$

$$(c_1 - x_1)^2 + (c_2 - x_2 - h)^2 + (c_3 - x_3 - h)^2 + \sum_{i=4}^{d} (c_i - x_i)^2 \ge \delta^2. \tag{2.6.5}$$

Now subtracting (2.6.4), respectively (2.6.5), from (2.6.3) we get, respectively,

$$(2c_1 - 2x_1 - 3h)h \le 0,$$

$$(2c_1 - 2x_1 - h)(-h) \leq 0.$$

Hence

$$(c_1 - x_1)^2 \leq \frac{9h^2}{4}. \tag{2.6.6}$$

Similarly using the points $x + he_1 + 2he_2 + he_3$, $x + h_1 + h_3$, $x + he_1 + he_2 + 2he_3$, $x + he_1 + he_2$ in $\overline{D}$ we obtain

$$(c_2 - x_2)^2 \leq \frac{9h^2}{4}, \tag{2.6.7}$$

$$(c_3 - x_3)^2 \leq \frac{9h^2}{4}. \tag{2.6.8}$$

We now observe that

$$x + he_1 \pm he_4 \in N(x) \subseteq \overline{D}. \tag{2.6.9}$$

Consequently

$$\begin{aligned}(c_1 - x_1 - h)^2 + (c_2 - x_2)^2 + (c_3 - x_3)^2 + (c_4 - x_4 - h)^2 \\ + \sum_{i=5}^{d} (c_i - x_i)^2 \geq \delta^2\end{aligned} \tag{2.6.10}$$

and

$$\begin{aligned}(c_1 - x_1 - h)^2 + (c_2 - x_2)^2 + (c_3 - x_3)^2 + (c_4 - x_4 + h)^2 \\ + \sum_{i=5}^{d} (c_i - x_i)^2 \geq \delta^2.\end{aligned} \tag{2.6.11}$$

Subtracting (2.6.10) from (2.6.3) we derive, after a few simple manipulations,

$$(c_4 - x_4) \leq \frac{11h}{4}.$$

Similarly subtracting (2.6.11) from (2.6.3) we obtain

$$(c_4 - x_4) \geq -\frac{11h}{4}.$$

Thus

$$(c_4 - x_4)^2 \leq \frac{121h^2}{16}.$$

Re-running the above argument considering $x+he_1 \pm he_i \in \overline{D}$, $5 \leq i \leq d$, in place of $x + he_1 \pm he_4$ in (2.6.9), and using equations similar to (2.6.10) and (2.6.11) we obtain all in all that

$$(c_i - x_i)^2 \leq \frac{121h^2}{16}, \quad i = 4, \dots, d. \tag{2.6.12}$$

Finally we observe that, for small enough h, (2.6.7), (2.6.8) and (2.6.12) together contradict (2.6.2). This completes Case 1.

2. For this case we have $\{x \pm he_i : 1 \leq i \leq d\} \cap B_h = \emptyset$ but $\{x \pm he_i \pm he_j : 1 \leq i, j \leq d\} \cap B_h \neq \emptyset$. Here also we consider two subcases.

   2.1. First we consider the subcase when $\{x \pm 2he_i : 1 \leq i \leq d\} \cap B_h \neq \emptyset$. For simplicity we continue with a particular choice $x + 2he_1 \in B_h$. In this case if $x + 3he_1 \in B_h$ then we are done. So we assume $x + 3he_1 \in R_h$. Observe that

$$N(x + 2he_1) \nsubseteq \overline{D}$$
$$N(x \pm he_i),\ N(x + 3he_1) \subseteq \overline{D}$$

   which imply that we must have

$$\{x + 2he_1 \pm he_i \pm he_j : 1 < i, j \leq d\} \nsubseteq \overline{D}. \tag{2.6.13}$$

   We consider two different situations.

   2.1.1. Let us first consider the situation when $\{x+2he_1 \pm 2he_i : 1 < i \leq d\} \nsubseteq \overline{D}$. In particular we consider without loss of generality $x + 2he_1 + 2he_2 \notin \overline{D}$. Note that this implies $x + 2he_2 \in B_h$. So if $x + 3he_2 \in B_h$ then we are done. Otherwise we have $x + 3he_2 \in R_h$. But in this case we see that $x + 2he_1 + 2he_2 \notin \overline{D}$ and its nearby points $\{x + 3he_1 + 2he_2, x + he_1 + 2he_2, x + 2he_1 + 3he_2, x + 2he_1 + he_2, x + 2he_1 + he_2 \pm he_i : 3 \leq i \leq d\}$ stay inside $\overline{D}$. It can be shown that this case is impossible by UEBC with a similar argument as in Case 1.2.

   2.1.2. We now consider the other situation (note the such a situation does not appear in $d = 2$) when $\{x+2he_1 \pm 2he_i : 1 < i \leq d\} \subseteq \overline{D}$. So using (2.6.13) without loss of generality we choose a particular element, say $x + 2he_1 + he_2 + he_3 \notin \overline{D}$. One can show that this situation is not possible for small enough h by arguments similar to Case 1.2. with the observation $\{x + 3he_1 + he_2 + he_3, x + he_1 + he_2 + he_3, x + 2he_2 + he_3, x + he_3, x + he_2 + 2he_3, x + he_2, x + h_2 + h_3 \pm he_i : 4 \le i \le d\} \subseteq \overline{D}$.

2.2. We are left with the subcase when

$$\begin{aligned} &\{x \pm 2he_i : 1 \le i \le d\} \cap B_h = \emptyset, \\ &\{x \pm he_i \pm he_j : 1 \le i, j \le d, i \neq j\} \cap B_h \neq \emptyset. \end{aligned} \tag{2.6.14}$$

Now consider points which are of the form $\{x \pm 3he_i : 1 \le i \le d\}$ and depending on whether they have non-empty intersection with B_h one can split the argument into two further cases. We use points of the above form as their neighbourhoods contain points which are at graph distance 5 from x in certain directions.

2.2.1. First we consider the case when $\{x \pm 3he_i : 1 \le i \le d\} \cap B_h \neq \emptyset$. If say, $x + 3he_1 \in B_h$ then it must be that $x + 4he_1 \in B_h$ too. Indeed, were this not true one would have

$$\begin{aligned} N(x + 3he_1) &\nsubseteq \overline{D}, \\ N(x + 4he_1) &\subseteq \overline{D}, \\ N(x + 2he_1) &\subseteq \overline{D}. \end{aligned}$$

From these equations we observe that one would have $\{x+3he_1 \pm he_i \pm he_j : 1 < i, j \le d\} \nsubseteq \overline{D}$. Now this would give rise to a contradiction by similar argument used in Case 1.2.

2.2.2. We now focus on the case when

$$\{x \pm 3he_i : 1 \le i \le d\} \cap B_h = \emptyset. \tag{2.6.15}$$

We show that this situation can not arise. To keep the argument simple, using (2.6.14), we assume without loss of generality $x + he_1 + he_2 \in B_h$. Then

$$\{x + he_1 + he_2 \pm he_i,\ x + he_1 + he_2 \pm he_i \pm he_j : 1 \le i, j \le d\} \nsubseteq \overline{D}.$$

Since we are in Case 2 and (2.6.14)-(2.6.15) hold we have

$$N(x \pm he_i),\ N(x \pm 2he_i),\ N(x \pm 3he_i) \subseteq \overline{D} \text{ for all } i,$$

so it must be that

$$\{x + he_1 + he_2 \pm he_i \pm he_j : 3 \leq i, j \leq d, i \neq j\} \nsubseteq \overline{D}. \tag{2.6.16}$$

Notice that such a situation cannot arise in $d = 3$ and hence we concentrate on $d \geq 4$. To analyse the situation arising out of (2.6.16), we suppose

$$x + he_1 + he_2 + he_3 + he_4 \notin \overline{D}.$$

Note that here we cannot follow the steps of Case 1.2. because we do not know if any of the points $x + 2he_1 + he_2 + he_3 + he_4$, $x + he_1 + 2he_2 + he_3 + he_4$, $x + he_1 + he_2 + 2he_3 + he_4$, $x + he_1 + he_2 + he_3 + 2he_4$ are in $\overline{D}$. So we argue in a slightly different way.
By UEBC for h small enough we can find a ball $B_\delta(c)$ for some $c \in \mathbb{R}^d$ such that $x + he_1 + he_2 + he_3 + he_4 \in B_\delta(c)$ and $\overline{B_\delta(c)} \cap \overline{D} = \{y\}$ for some $y \in \partial D$. Clearly, if $x = (x_1, \ldots, x_d)$ and $c = (c_1, \ldots, c_d)$ then

$$\sum_{i=1}^{d} (c_i - x_i)^2 > \delta^2 \tag{2.6.17}$$

and

$$\sum_{i=1}^{4} (c_i - x_i - h)^2 + \sum_{i=5}^{d} (c_i - x_i)^2 < \delta^2. \tag{2.6.18}$$

Also $x + he_2 + he_3 + he_4 \in \overline{D}$ gives

$$(c_1 - x_1)^2 + \sum_{i=2}^{4} (c_i - x_i - h)^2 + \sum_{i=5}^{d} (c_i - x_i)^2 \geq \delta^2. \tag{2.6.19}$$

Subtracting (2.6.19) from (2.6.18) we get

$$c_1 - x_1 \geq \frac{h}{2}.$$

Similarly we obtain

$$c_i - x_i \geq \frac{h}{2}, \quad i = 2, 3, 4.$$

Now we impose a condition on the maximum value of $\{c_i - x_i : i = 1, 2, 3, 4\}$ and see that when it is bounded by a factor of h one gets a contradiction. Let $c_k - x_k = \max\{c_i - x_i : i = 1, 2, 3, 4\}$. First suppose $c_k - x_k \leq 7h/2$. Then we have

$$(c_i - x_i)^2 \leq \frac{49h^2}{4}, \quad i = 1, 2, 3, 4.$$

Now using $\{x + he_1 + he_2 \pm he_j : 5 \leq j \leq d\} \subseteq \overline{D}$ we deduce

$$(c_j - x_j)^2 \leq Ch^2, \quad 5 \leq j \leq d.$$

where C is a constant depending on d. Thus we obtain

$$\sum_{i=1}^{d} (c_i - x_i)^2 \leq Ch^2$$

for some constant C. This contradicts (2.6.17) for small enough h. Now suppose we are not in the above situation, that is, $c_k - x_k > 7h/2$. For simplicity let $k = 4$. Then we find a contradiction by observing that the point $x + he_2 + he_3 + 3he_4$ can not lie in $\overline{D}$. Indeed, we have

$$\begin{aligned}
&(c_1 - x_1)^2 + (c_2 - x_2 - h)^2 + (c_3 - x_3 - h)^2 + (c_4 - x_4 - 3h)^2 \\
&\qquad + \sum_{i=5}^{d} (c_i - x_i)^2 - \sum_{i=1}^{4} (c_i - x_i - h)^2 - \sum_{i=5}^{d} (c_i - x_i)^2 \\
&= (c_1 - x_1)^2 - (c_1 - x_1 - h)^2 + (c_4 - x_4 - 3h)^2 - (c_4 - x_4 - h)^2 \\
&= -h[2(c_4 - x_4) - 7h + 2((c_4 - x_4) - (c_1 - x_1))] < 0.
\end{aligned}$$

Thus

$$\begin{aligned}
(c_1 - x_1)^2 + (c_2 - x_2 - h)^2 + (c_3 - x_3 - h)^2 + (c_4 - x_4 - 3h)^2& \\
+ \sum_{i=5}^{d} (c_i - x_i)^2 < \delta^2.&
\end{aligned}$$

This implies that $x + he_2 + he_3 + 3he_4 \in B_\delta(c)$ which is impossible as $x + he_2 + he_3 + 3he_4 \in N(x + 3he_4) \subseteq \overline{D}$. This completes the proof. $\square$

Chapter 3

The scaling limit of the $(\nabla + \Delta)$-model

3.1 Introduction

The $(\nabla + \Delta)$-model is another special instance of the more general class of random interfaces. For this model the Hamiltonian is given by

$$H(\varphi) = \sum_{x \in \mathbb{Z}^d} \big(\kappa_1 \|\nabla \varphi_x\|^2 + \kappa_2 (\Delta \varphi_x)^2\big) \tag{3.1.1}$$

where κ_1, κ_2 are two positive constants. Thus $(\nabla + \Delta)$-model is the field $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$, whose distribution is determined by a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$, $d \geq 1$. The probability measure is given by

$$\mathbf{P}_\Lambda(\mathrm{d}\varphi) := \frac{1}{Z_\Lambda} \exp\left(- \sum_{x \in \mathbb{Z}^d} \big(\kappa_1 \|\nabla \varphi_x\|^2 + \kappa_2 (\Delta \varphi_x)^2\big) \right) \prod_{x \in \Lambda} \mathrm{d}\varphi_x \prod_{x \in \mathbb{Z}^d \setminus \Lambda} \delta_0(\mathrm{d}\varphi_x), \tag{3.1.2}$$

where $\Lambda \Subset \mathbb{Z}^d$ is a finite subset, $\mathrm{d}\varphi_x$ is the Lebesgue measure on $\mathbb{R}$, δ_0 is the Dirac measure at 0, and Z_Λ is a normalizing constant. We are imposing zero boundary conditions: almost surely $\varphi_x = 0$ for all $x \in \mathbb{Z}^d \setminus \Lambda$, but the definition holds for more general boundary conditions. In the physics literature, the above Hamiltonian is considered to be the energy of a semiflexible membrane (or semiflexible polymer if $d = 1$) where the parameters κ_1 and κ_2 are the lateral tension and the bending rigidity, respectively. In the works of Borecki [14], Borecki and Caravenna [15] this model was studied in $d = 1$ under the influence of pinning in order to understand the localization behavior of the polymer.

This model interpolates between two well-known random interfaces namely the discrete Gaussian free field and the membrane model. In [15, Remark 9] it was conjectured that, in the case of pinning for the one-dimensional $(\nabla+\Delta)$-model, the behaviour of the free energy should resemble the purely gradient case. In view of this remark it is natural to ask if the scaling limit of the model is dominated by the gradient interaction, that is, the limit is Gaussian free field (GFF). The main focus of this chapter is to show that such a guess is true and indeed in any dimension the model approximates the Gaussian free field.

We will consider the lattice approximation of both domains and $\mathbb{R}^d$ and investigate the behavior of the rescaled interface when the lattice size decreases to zero. We will use techniques coming from discrete PDEs which were already employed in Chapter 2 to derive the scaling limit of the membrane model. We show that in $d = 1$ convergence occurs in the space of continuous functions whilst in higher dimensions the limit is no longer a function, but a random distribution, and convergence takes place in a Sobolev space of negative index. In this sense one can also think of this model as a perturbation of the discrete Gaussian free field.

3.2 Main results

3.2.1 The model

Let Λ be a finite subset of $\mathbb{Z}^d$ and $\mathbf{P}_\Lambda$ and $H(\varphi)$ be as in (3.1.2) and (3.1.1) respectively. It follows from Lemma 1.2.1 that the Gibbs measure (3.1.2) on $\mathbb{R}^\Lambda$ with Hamiltonian (3.1.1) exists. Note that (3.1.1) can be written as

$$H(\varphi) = \frac{1}{2}\langle \varphi, (-4d\kappa_1\Delta + 2\kappa_2\Delta^2)\varphi\rangle_{\ell^2(\mathbb{Z}^d)}. \tag{3.2.1}$$

We are interested in the "truly" mixed case, that is when κ_1 and κ_2 are strictly positive. Therefore using the fact that the measure induced by (3.2.1) is Gaussian without any loss of generality we will work with the following Hamiltonian:

$$H(\varphi) = \frac{1}{2}\langle \varphi, (-\kappa\Delta + \Delta^2)\varphi \rangle_{\ell^2(\mathbb{Z}^d)} \tag{3.2.2}$$

where $\kappa > 0$ is a constant. Thus if we write $G_\Lambda(x, y) := \mathbf{E}_\Lambda(\varphi_x \varphi_y)$, it follows from Lemma 1.2.1 that G_Λ solves the following discrete boundary value problem: for $x \in \Lambda$

$$\begin{cases} (-\kappa\Delta + \Delta^2)G_\Lambda(x,y) = \delta_x(y) & y \in \Lambda \\ G_\Lambda(x,y) = 0 & y \notin \Lambda \end{cases}. \tag{3.2.3}$$

In the case $\Lambda = [-N, N]^d \cap \mathbb{Z}^d$ we will denote the measure (3.1.2) by $\mathbf{P}_N$. It follows from Proposition 1.2.2 that in $d \geq 3$ there exists a thermodynamic limit $\mathbf{P}$ of the measures $\mathbf{P}_N$ as $N \uparrow \infty$. Under $\mathbf{P}$, the field $(\varphi_x)_{x \in \mathbb{Z}^d}$ is a centered Gaussian process with covariance given by

$$G(x, y) = (-\kappa\Delta + \Delta^2)^{-1}(x, y).$$

Since κ is a fixed constant, in order to simplify the exposition we will fix it to be 1 throughout. This would not change the nature of the limit except for a scaling constant.

3.2.2 Main results

Since the infinite volume measure of the mixed model exists in $d \geq 3$, we split the scaling limit convergence into two parts: the infinite volume case, in which we study the $(\nabla + \Delta)$-model under $\mathbf{P}$, and the finite volume case in which our object of interest is the scaling limit of measures $\mathbf{P}_{\Lambda_N}$, for some chosen $\Lambda_N \Subset \mathbb{Z}^d$. We fix for the whole chapter the constant $k := 1/\sqrt{2d}$. The main results are as follows.

In $d \geq 3$ (Section 3.3) we consider the infinite volume model $\varphi = (\varphi_x)_{x \in \mathbb{Z}^d}$ with law $\mathbf{P}$. We define for $N \in \mathbb{N}$

$$\psi_N(x) := kN^{\frac{d-2}{2}} \varphi_{Nx}, \quad x \in \frac{1}{N}\mathbb{Z}^d.$$

For $f \in \mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ (the Schwartz space) we define

$$(\Psi_N, f) := N^{-d} \sum_{x \in \frac{1}{N}\mathbb{Z}^d} \psi_N(x) f(x). \tag{3.2.4}$$

This definition makes sense since, using Mill's ratio and the uniform bound on G one can show that as $\|x\| \to \infty$

$$|\psi_N(x)| = O(N^{\frac{d-2}{2}}\sqrt{\log(1+\|x\|)}) \quad \text{a.s.}$$

via a Borel-Cantelli argument. Also it follows that with this definition $\Psi_N \in \mathcal{S}^*$ and the characteristic functional of Ψ_N is given by

$$\mathcal{L}_{\Psi_N}(f) := \exp(-\mathbf{Var}\,(\Psi_N, f)\,/2).$$

As for the limiting field, we have by an application of the Bochner–Minlos theorem that there exists a generalized random field $\Psi^{-\Delta}$ on $\mathcal{S}^*$ whose characteristic functional $\mathcal{L}_{\Psi^{-\Delta}}$ is given by

$$\mathcal{L}_{\Psi^{-\Delta}}(f) = \exp\left(-\frac{1}{2}\|(-\Delta_c)^{-1/2}f\|^2_{L^2(\mathbb{R}^d)}\right), \quad f \in \mathcal{S}, \tag{3.2.5}$$

where the operator $(-\Delta_c)^{-1/2} : \mathcal{S} \to L^2(\mathbb{R}^d)$ is defined by

$$(-\Delta_c)^{-1/2}f(x) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{\iota\langle x,\xi\rangle}\|\xi\|^{-1}\widehat{f}(\xi)\,\mathrm{d}\,\xi.$$

Here $\widehat{f}$ is the Fourier transform of f. For properties of the field $\Psi^{-\Delta}$ see also [51, Section 3]. We are now ready to state our main result for the infinite volume case.

Theorem 3.2.1 (Scaling limit in $d \geq 3$)**.** *One has that $\Psi_N \overset{d}{\to} \Psi^{-\Delta}$ in the strong topology of $\mathcal{S}^*$.*

In the finite volume case in $d \geq 2$ (Section 3.4) we take D to be a bounded domain in $\mathbb{R}^d$ with smooth boundary. We discretise D appropriately and "blow it up": this discretisation will be called $\Lambda = \Lambda_N$ (it will be defined properly in Section 3.4). On Λ we define the mixed model φ with law (3.1.2) and Hamiltonian (3.2.2) and define Ψ_N by

$$\Psi_N := k\sum_{x\in\frac{1}{N}\Lambda_N} N^{-\frac{d+2}{2}}\varphi_{Nx}\delta_x.$$

One can show Ψ_N is a distribution living in the negative Sobolev space $\mathcal{H}^{-s}_{-\Delta}(D)$ for all $s > d$. To describe the limiting field, there are many equivalent ways to define the

Gaussian free field $\Psi_D^{-\Delta}$ on a domain. One of them is to think of it as a collection of centered Gaussian variables $(\Psi_D^{-\Delta}, f)$ indexed by $C_c^\infty(D)$ with covariance structure given by

$$\mathbf{E}[(\Psi_D^{-\Delta}, f)(\Psi_D^{-\Delta}, g)] = \iint_{D\times D} f(x)g(y)G_D(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y, \quad f,\, g \in C_c^\infty(D)$$

where G_D is the Green's function of the continuum Dirichlet problem with zero boundary conditions. We now state the main result for the finite volume $(\nabla + \Delta)$-interaction.

Theorem 3.2.2 (Scaling limit in $d \geq 2$ under finite volume)**.** Ψ_N *converges in distribution to the zero boundary Gaussian free field* $\Psi_D^{-\Delta}$ *as* $N \to \infty$ *in the topology of* $\mathcal{H}_{-\Delta}^{-s}(D)$ *for* $s > d$*.*

A special case for finite volume measures is $d = 1$ (Subsection 3.4.4). In this example, the GFF becomes a Brownian bridge, and the type of convergence we obtain is different from all other dimensions (convergence occurs in the space of continuous functions). In this case we consider the mixed model on the "blow up" $\Lambda = \Lambda_N$ of an appropriate discretisation of [0, 1]. We define a continuous interpolation ψ_N of the rescaled interface and obtain the following theorem:

Theorem 3.2.3 (Scaling limit in $d = 1$)**.** ψ_N *converges in distribution to the Brownian bridge on* $[0,1]$ *in the space* $C[0,1]$*.*

As a by-product of this theorem we obtain the convergence of the discrete maximum in $d = 1$.

3.2.3 Idea of the proofs

We begin by explaining the ideas behind the proofs of the infinite volume case (Section 3.3). For the whole space GFF the variance of $(\Psi_D^{-\Delta}, f)$ can be expressed as

$$\|(-\Delta_c)^{-1/2} f\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \|\theta\|^{-2} |\widehat{f}(\theta)|^2 \,\mathrm{d}\,\theta. \tag{3.2.6}$$

Given the appearance of the Fourier transforms in the limit, we write the discrete Green's function in terms of the inverse Fourier transform. We see that a scaling factor appears in such a way the contribution from the Δ^2 factor in the Hamiltonian vanishes, ensuring convergence to a purely gradient model.

In the finite volume case we show first finite dimensional convergence and secondly tightness. Since the measures are Gaussian the finite dimensional convergence follows from the convergence of the covariance function. However, the behaviour of the covariance of the mixed model is not known explicitly in finite volume (for example, it lacks the classical random walk representation of Ginzburg-Landau models). So we use the expedient of finite difference scheme in proving the convergence. The key fact which is used is that the Green's function satisfies the Dirichlet problem (3.2.3). We show that the discrete solution is equal to that of the continuum Dirichlet problem with a negligible error. This approximation is obtained from the interesting approach of Thomée [69]. His idea, adapted to our setting, is the following: if we write the operator $(-\Delta + \Delta^2)$ in the rescaled lattice $h\,\mathbb{Z}^d$ for h small, then due to the scaling we end up dealing with $(-\Delta_h + h^2/(2d)\Delta_h^2)$. To quantify how negligible the presence of Δ_h^2 is, we use some discrete Sobolev inequalities. In Section 3.5 we derive precise estimate of the error. This section is of independent interest, as it concerns the approximation of PDEs. We remark that our methodology seems to be robust enough to deal with different interface models whenever the interaction is given in terms of a discrete elliptic operator.

3.3 Infinite volume case

In this section we prove Theorem 3.2.1.

3.3.1 Proof of Theorem 3.2.1

By Corollary 2.4.1 to prove the convergence in distribution it is enough to show that $\mathcal{L}_{\Psi_N}(f) \to \mathcal{L}_{\Psi^{-\Delta}}(f)$ for all $f \in \mathcal{S}$. Given the Gaussian nature of the variables we consider, and the fact that they are centered, it suffices to show that for any $f \in \mathcal{S}$

$$\mathbf{E}\left[(\Psi_N, f)^2\right] \to \|(-\Delta_c)^{-1/2} f\|^2_{L^2(\mathbb{R}^d)}.$$

By definition of the field and translation invariance we have that

$$\begin{aligned}\mathbf{E}\left[(\Psi_N, f)^2\right] &= k^2 N^{-(d+2)} \sum_{x,y\in\frac{1}{N}\mathbb{Z}^d} \mathbf{E}[\varphi_{Nx}\varphi_{Ny}] f(x)f(y)\\ &= k^2 N^{-(d+2)} \sum_{x,y\in\frac{1}{N}\mathbb{Z}^d} G(0, N(y-x)) f(x) f(y). \end{aligned} \tag{3.3.1}$$

Now our goal is to shift these expression to Fourier coordinates. We have from (1.2.3)

$$G(0,x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \left(\mu(\theta)+\mu(\theta)^2\right)^{-1} \mathrm{e}^{-\iota\langle x,\theta\rangle}\,\mathrm{d}\,\theta \tag{3.3.2}$$

where $\mu(\theta) = \frac{1}{d}\sum_{i=1}^d (1-\cos(\theta_i))$. We estimate the integrand in (3.3.2) by the following lemma, whose proof is deferred to page 82:

Lemma 3.3.1. *There exists a constant $C>0$ such that for all $\theta\in[-N\pi, N\pi]^d\setminus\{0\}$*

$$N^{-2}\left(\frac{\|\theta\|^2}{2dN^2}+\frac{\|\theta\|^4}{4d^2N^4}\right)^{-1} \le N^{-2}\left(\mu\left(\frac{\theta}{N}\right)+\mu\left(\frac{\theta}{N}\right)^2\right)^{-1} \le \frac{2d}{\|\theta\|^2}+\frac{Cd}{2N^2}.$$

Returning to the expression (3.3.1) and plugging in (3.3.2) we have

$$\begin{aligned}&\mathbf{E}\left[(\Psi_N, f)^2\right]\\ &= \frac{k^2N^{-(d+2)}}{(2\pi)^d}\sum_{x,y\in\frac{1}{N}\mathbb{Z}^d}\int_{[-\pi,\pi]^d}\left(\mu(\theta)+\mu(\theta)^2\right)^{-1}\mathrm{e}^{-\iota\langle N(y-x),\theta\rangle} f(x)f(y)\,\mathrm{d}\,\theta\\ &= \frac{k^2N^{-2}}{(2\pi)^d}\int_{[-N\pi,N\pi]^d}\left(\mu\left(\frac{\theta}{N}\right)+\mu\left(\frac{\theta}{N}\right)^2\right)^{-1}\left|N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta.\end{aligned} \tag{3.3.3}$$

Here we exchange sum and integral due to Lemma 3.3.1. We make two claims which will immediately prove the convergence of variance.

Claim 3.3.2.

$$\begin{aligned}\lim_{N\to+\infty}\int_{[-N\pi,N\pi]^d}&\left[N^{-2}\left(\mu\left(\frac{\theta}{N}\right)+\mu\left(\frac{\theta}{N}\right)^2\right)^{-1}-2d\|\theta\|^{-2}\right]\times\\ &\times\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta = 0.\end{aligned}$$

Next we claim the following convergence which is immediate from the estimates (3.3.7) and (3.2.6).

Claim 3.3.3.

$$\lim_{N\to+\infty}\frac{1}{(2\pi)^d}\int_{[-N\pi,N\pi]^d}\|\theta\|^{-2}\left|N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta=\|(-\Delta_c)^{-1/2}f\|^2_{L^2(\mathbb{R}^d)}.$$

Claims 3.3.2-3.3.3 entail that

$$\lim_{N\to\infty}\mathcal{L}_{\Psi_N}(f)=\exp\left(-\frac{1}{2}\|(-\Delta_c)^{-1/2}f\|^2_{L^2(\mathbb{R}^d)}\right).$$

Thus we have for all $f\in\mathcal{S}$, $\mathcal{L}_{\Psi_N}(f)\to\mathcal{L}_{\Psi^{-\Delta}}(f)$ and hence convergence in distribution follows. □

This completes the proof of convergence in $d\geq 3$ modulo Lemma 3.3.1 and Claim 3.3.2 which we still owe the reader. We proceed to fill this gap.

Proof of Lemma 3.3.1. We know from [26, Lemma 7] that there exists $C>0$ such that for all $N\in\mathbb{N}$ and $w\in[-N\pi/2,N\pi/2]^d\setminus\{0\}$

$$\frac{1}{\|w\|^4}\leq N^{-4}\left(\sum_{i=1}^d\sin^2\left(\frac{w_i}{N}\right)\right)^{-2}\leq\left(\frac{1}{\|w\|^2}+\frac{C}{N^2}\right)^2. \tag{3.3.4}$$

Therefore

$$\left(\frac{2dN^2}{\|\theta\|^2}+\frac{Cd}{2}\right)^{-1}\leq\mu\Big(\frac{\theta}{N}\Big)\leq\frac{\|\theta\|^2}{2dN^2}$$

and hence

$$\begin{aligned}N^{-2}\left(\frac{\|\theta\|^2}{2dN^2}+\frac{\|\theta\|^4}{4d^2N^4}\right)^{-1}&\leq N^{-2}\left(\mu\Big(\frac{\theta}{N}\Big)+\mu\Big(\frac{\theta}{N}\Big)^2\right)^{-1}\\&\leq N^{-2}\left(\left(\frac{2dN^2}{\|\theta\|^2}+\frac{Cd}{2}\right)^{-1}+\left(\frac{2dN^2}{\|\theta\|^2}+\frac{Cd}{2}\right)^{-2}\right)^{-1}\\&\leq\frac{2d}{\|\theta\|^2}+\frac{Cd}{2N^2}.\end{aligned}$$

□

Proof of Claim 3.3.2. By Lemma 3.3.1 we can sandwich the expression in the statement of the Claim between two infinitesimal quantities. The lower bound is given by

$$\int_{[-N\pi,N\pi]^d}\left[N^{-2}\left(\frac{\|\theta\|^2}{2dN^2}+\frac{\|\theta\|^4}{4d^2N^4}\right)^{-1}-2d\|\theta\|^{-2}\right]\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta \tag{3.3.5}$$

and the upper bound is given by

$$\int_{[-N\pi,N\pi]^d}\frac{Cd}{2N^2}\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\mathrm{d}\,\theta. \tag{3.3.6}$$

We show that both the limit of (3.3.5) and (3.3.6) are zero as $N\to\infty$. Recall that from (2.4.5) we have that for any N and $s>0$ large enough

$$\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\iota\left\langle\frac{x}{N},\theta\right\rangle}f\left(\frac{x}{N}\right)-\widehat{f}(\theta)\right|\le CN^{-s}. \tag{3.3.7}$$

Using (3.3.7) it follows that (3.3.6) converges to zero. For (3.3.5) observe that the integrand goes to zero and we can apply the dominated convergence theorem due to the following integrable bound:

$$\begin{aligned}
&\left|\left[N^{-2}\left(\frac{\|\theta\|^2}{2dN^2}+\frac{\|\theta\|^4}{4d^2N^4}\right)^{-1}-2d\|\theta\|^{-2}\right]\left|(2\pi)^{-d/2}N^{-d}\sum_{x\in\frac{1}{N}\mathbb{Z}^d}\mathrm{e}^{-\iota\langle x,\theta\rangle}f(x)\right|^2\right.\\
&\le\left|\left[N^{-2}\left(\frac{\|\theta\|^2}{2dN^2}+\frac{\|\theta\|^4}{4d^2N^4}\right)^{-1}-2d\|\theta\|^{-2}\right]2\left(CN^{-2s}+|\widehat{f}(\theta)|^2\right)\right|\\
&\le\frac{8d}{\|\theta\|^2}\left(CN^{-2s}+|\widehat{f}(\theta)|^2\right).
\end{aligned}$$

□

3.4 Finite volume case

3.4.1 Set-up

We begin by deriving a useful upper bound on the variance of the mixed model. Let $d\ge 1$ and for any $\Lambda\Subset\mathbb{Z}^d$ let $\mathbf{P}^{GFF}_\Lambda$ denote the probability measure on $\mathbb{R}^{\mathbb{Z}^d}$ of the discrete

Gaussian free field with zero boundary conditions outside Λ. Then the following bound holds:

Lemma 3.4.1. *For all $x \in \mathbb{Z}^d$*

$$G_\Lambda(x,\, x) \leq \mathbf{E}_\Lambda^{GFF}(\varphi_x^2). \tag{3.4.1}$$

Proof. Note that we actually have

$$H(\varphi)|_{\varphi \equiv 0 \text{ on } \Lambda^c} = \frac{1}{2}\langle \varphi, (-\Delta_\Lambda + \Delta_\Lambda^2)\varphi\rangle_{\ell^2(\Lambda)}$$

where Δ_Λ and Δ_Λ^2 denote the restriction of the operators Δ and Δ^2 to functions which are zero outside Λ, respectively. The bound is thus obtained for any $x \in \Lambda$ by applying [18, Theorem 5.1] with $F((\varphi_x)_{x\in\Lambda}) := \exp[-\frac{1}{2}\langle\varphi, \Delta_\Lambda^2\varphi\rangle_{\ell^2(\Lambda)}]$ on $\mathbb{R}^\Lambda$, $A := -1/2\,\Delta_\Lambda$ and $\alpha := 2$. Here we note that F is log-concave. The case for $x \in \mathbb{Z}^d \setminus \Lambda$ follows easily by the boundary conditions imposed on the interface. □

We must set up now the right discretisation of domains to be able to obtain an interface converging to GFF. Let D be any bounded domain in $\mathbb{R}^d$ with smooth boundary. For $N \in \mathbb{N}$, let $D_N = N\overline{D} \cap \mathbb{Z}^d$. Let us denote by Λ_N the set of points x in D_N such that $x \pm (e_i \pm e_j)$, $x \pm e_i$ are all in D_N for all $i,\, j = 1, \ldots, d$. Let us now consider the mixed model with $\Lambda = \Lambda_N$ and zero boundary conditions outside Λ_N. The key result of this Subsection is to show that the variance of $(\Psi_N,\, f)$ converges to that of $(\Psi_D^{-\Delta},\, f)$, that is, to the norm of the solution of a suitable Dirichlet problem.

Remark 3.4.2. *The reduction from smooth boundary to piece-wise smooth boundaries can perhaps be achieved but we will not aim for such a generalization.*

Proposition 3.4.3. *Let f be a smooth and compactly supported function on D and consider*

$$(\Psi_N,\, f) = k \sum_{x \in \frac{1}{N}\Lambda_N} N^{-\frac{d+2}{2}} \varphi_{Nx} f(x).$$

Then

$$\lim_{N\to\infty} \mathbf{Var}[(\Psi_N,\, f)] = \int_D u(x) f(x)\,\mathrm{d}\,x,$$

where u is the solution of the Dirichlet problem

$$\begin{cases} -\Delta_c u(x) = f(x) & x \in D \\ u(x) = 0 & x \in \partial D. \end{cases} \tag{3.4.2}$$

Proof. We denote $G_{\frac{1}{N}}(x,y) := \mathbf{E}_{\Lambda_N}[\varphi_{Nx}\varphi_{Ny}]$ for $x, y \in N^{-1}D_N$. Note that if $\Delta_{\frac{1}{N}}$ (defined in (2.2.4)) is the discrete Laplacian on $N^{-1}\mathbb{Z}^d$ then by (3.2.3) we have, for all $x \in N^{-1}\Lambda_N$,

$$\begin{cases} \Big(-\frac{1}{2dN^2}\Delta_{\frac{1}{N}} + \frac{1}{4d^2N^4}\Delta^2_{\frac{1}{N}} \Big) G_{\frac{1}{N}}(x,y) = \delta_x(y) & y \in \frac{1}{N}\Lambda_N \\ G_{\frac{1}{N}}(x,y) = 0 & y \notin \frac{1}{N}\Lambda_N. \end{cases} \tag{3.4.3}$$

We have

$$\begin{aligned} \mathbf{Var}[(\Psi_N,\, f)] &= k^2 \sum_{x,y \in \frac{1}{N}\Lambda_N} N^{-d-2} G_{\frac{1}{N}}(x,y) f(x) f(y) \\ &= \sum_{x \in \frac{1}{N}\Lambda_N} N^{-d} H_N(x) f(x) \end{aligned}$$

where $H_N(x) = k^2 \sum_{y \in \frac{1}{N}\Lambda_N} N^{-2} G_{\frac{1}{N}}(x,y) f(y)$ for $x \in N^{-1}D_N$. It is immediate from (3.4.3) that H_N is the solution of the following Dirichlet problem:

$$\begin{cases} \Big(-\Delta_{\frac{1}{N}} + \frac{1}{2dN^2}\Delta^2_{\frac{1}{N}} \Big) H_N(x) = f(x) & x \in \frac{1}{N}\Lambda_N \\ H_N(x) = 0 & x \notin \frac{1}{N}\Lambda_N. \end{cases} \tag{3.4.4}$$

Define the error between the solutions of (3.4.4) and (3.4.2) by $e_N(x) := H_N(x) - u(x)$ for $x \in N^{-1}D_N$. Then using Theorem 3.5.1 we have

$$N^{-d} \sum_{x \in \frac{1}{N}\Lambda_N} e_N(x)^2 \leq C N^{-1}. \tag{3.4.5}$$

Rewriting the variance we deduce

$$\mathbf{Var}[(\Psi_N,\, f)] = \sum_{x \in \frac{1}{N}\Lambda_N} e_N(x) f(x) N^{-d} + \sum_{x \in \frac{1}{N}\Lambda_N} u(x) f(x) N^{-d}.$$

Note that by Cauchy-Schwarz inequality and (3.4.5) the first summand goes to zero as $N \to \infty$. The second term is a Riemann sum and converges to $\int_D u(x) f(x)\, \mathrm{d}\, x$. $\square$

3.4.2 The Gaussian free field

In this case we consider $d \geq 2$ and D and Λ_N as in the previous Subsection. First we discuss briefly some definitions about the GFF. In $d = 2$ the results can be found already in the literature, see for example [5, Section 1.3].

By the spectral theorem for compact self-adjoint operators we know that there exist eigenfunctions $(w_j)_{j\in\mathbb{N}}$ of $-\Delta_c$ corresponding to the eigenvalues $0 < \nu_1 \leq \nu_2 \leq \ldots \to \infty$ such that $(w_j)_{j\geq 1}$ is an orthonormal basis of $L^2(D)$. By elliptic regularity, we have that w_j is smooth for all j. Let $s > 0$ and we define the following inner product on $C_c^\infty(D)$:

$$\langle f, g\rangle_{s,-\Delta} := \sum_{j\in\mathbb{N}} \nu_j^s \langle f\,,\, w_j\rangle_{L^2} \langle w_j\,,\, g\rangle_{L^2}.$$

Then $\mathcal{H}^s_{-\Delta,0}(D)$ can be defined to be the completion of $C_c^\infty(D)$ with respect to this inner product and $\mathcal{H}^{-s}_{-\Delta}(D)$ is defined to be its dual. We denote the dual norm by $\|\cdot\|_{-s,-\Delta}$. Here we note that $\mathcal{H}^s_{-\Delta,0}(D) \subset L^2(D) \subset \mathcal{H}^{-s}_{-\Delta}(D)$ for any $s > 0$.

In case $f \in L^2(D)$ then we have

$$\|f\|^2_{-s,-\Delta} = \sum_{j\in\mathbb{N}} \nu_j^{-s} \langle f\,,\, w_j\rangle^2_{L^2}\,.$$

Also observe that $(\nu_j^{-1/2} w_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}^1_{-\Delta,0}(D)$. In the following proposition we give the definition of the Gaussian free field $\Psi_D^{-\Delta}$ via its Wiener series, generalising the two-dimensional result in [34, Subsection 4.2].

Proposition 3.4.4. *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set the Gaussian free field to be*

$$\Psi_D^{-\Delta} := \sum_{j\in\mathbb{N}} \nu_j^{-1/2} \xi_j w_j.$$

Then $\Psi_D^{-\Delta} \in \mathcal{H}^{-s}_{-\Delta}(D)$ a.s. for all $s > d/2 - 1$.

Proof. Fix $s > d/2 - 1$. Clearly $w_j \in L^2(D) \subseteq \mathcal{H}^{-s}_{-\Delta}(D)$. We want to show that $\|\Psi^{-\Delta}_D\|_{-s,-\Delta} < \infty$ with probability one. We have

$$\|\Psi^{-\Delta}_D\|^2_{-s} = \sum_{j\in\mathbb{N}} \nu_j^{-1-s}\xi_j^2.$$

The last sum is finite a.s. by Kolmogorov's two series theorem as we have

$$\sum_{j\in\mathbb{N}} \mathbf{E}[\nu_j^{-1-s}\xi_j^2] \asymp \sum_{j\in\mathbb{N}} j^{-\frac{2}{d}(1+s)} < \infty$$

and

$$\sum_{j\in\mathbb{N}} \mathbf{Var}[\nu_j^{-1-s}\xi_j^2] \asymp \sum_{j\in\mathbb{N}} j^{-\frac{4}{d}(1+s)} < \infty.$$

Here we have used the Weyl's asymptotic $\nu_j \sim C j^{\frac{2}{d}}$ for some explicit constant C. Thus we have $\Psi^{-\Delta}_D \in \mathcal{H}^{-s}_{-\Delta}(D)$ a.s. □

3.4.3 Proof of Theorem 3.2.2

We are now ready to show the main result on the scaling limit in the finite volume case. All notations are borrowed from Subsections 3.4.1-3.4.2.

Proof of Theorem 3.2.2. We first show that for $f \in C_c^\infty(D)$

$$(\Psi_N, f) \stackrel{d}{\to} (\Psi^{-\Delta}_D, f). \tag{3.4.6}$$

This follows from the following two observations: on the one hand by Proposition 3.4.3 and integration by parts we obtain

$$\mathbf{Var}[(\Psi_N, f)] \to \int_D u(x)f(x)\,\mathrm{d}\,x = \|f\|^2_{-1,-\Delta}.$$

On the other hand from the definition of GFF it follows that

$$\mathbf{Var}[(\Psi^{-\Delta}_D, f)] = \sum_{j\in\mathbb{N}} \nu_j^{-1}\langle w_j, f\rangle^2_{L^2} = \|f\|^2_{-1,-\Delta}.$$

Consequently we obtain (3.4.6) since both (Ψ_N, f) and $(\Psi^{-\Delta}_D, f)$ are centered Gaussians.

Next we want to show that the sequence $(\Psi_N)_{N\in\mathbb{N}}$ is tight in $\mathcal{H}^{-s}_{-\Delta}(D)$ for all $s>d$. It is enough to show that

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,-\Delta}] < \infty \quad \forall\, s>d. \tag{3.4.7}$$

The tightness of $(\Psi_N)_{N\in\mathbb{N}}$ would then follow immediately from (3.4.7) and the fact that, for $0\le s_1<s_2$, $\mathcal{H}^{-s_1}_{-\Delta}(D)$ is compactly embedded in $\mathcal{H}^{-s_2}_{-\Delta}(D)$. In order to show (3.4.7) we first observe that for any $f\in\mathcal{H}^{s}_{-\Delta,0}(D)$

$$\begin{aligned}
|(\Psi_N, f)| &= \Big|k \sum_{x\in\frac{1}{N}\Lambda_N} N^{-\frac{d+2}{2}}\varphi_{Nx}\sum_{j\ge1}\langle f\,,\,w_j\rangle_{L^2} w_j(x)\Big| \\
&= kN^{-\frac{d+2}{2}}\Big|\sum_{j\ge1}\nu_j^{-\frac{s}{2}}\sum_{x\in\frac{1}{N}\Lambda_N}\varphi_{Nx}w_j(x)\nu_j^{\frac{s}{2}}\langle f\,,\,w_j\rangle_{L^2}\Big| \\
&\le kN^{-\frac{d+2}{2}}\left(\sum_{j\ge1}\nu_j^{-s}\left(\sum_{x\in\frac{1}{N}\Lambda_N}\varphi_{Nx}w_j(x)\right)^2\right)^{\frac{1}{2}}\|f\|_{s,-\Delta}
\end{aligned}$$

where in the first equality we have used the fact that $f\in L^2(D)$ and therefore $f=\sum_{j\ge1}\langle f\,,\,w_j\rangle_{L^2}w_j$. Thus we have, using the definition of dual norm,

$$\|\Psi_N\|^2_{-s,-\Delta}\le\sum_{j\ge1}\nu_j^{-s}k^2N^{-(d+2)}\left(\sum_{x\in\frac{1}{N}\Lambda_N}\varphi_{Nx}w_j(x)\right)^2.$$

Therefore

$$\begin{aligned}
\mathbf{E}_{\Lambda_N}\|\Psi_N\|^2_{-s,-\Delta} &\le \sum_{j\ge1}\nu_j^{-s}k^2N^{-(d+2)}\sum_{x,y\in\frac{1}{N}\Lambda_N}G_{\frac{1}{N}}(x,y)w_j(x)w_j(y) \\
&\le \sum_{j\ge1}\nu_j^{-s}k^2N^{-2}\|G_{\frac{1}{N}}w_j\|_{\ell^2(\frac{1}{N}\Lambda_N)}\|w_j\|_{\ell^2(\frac{1}{N}\Lambda_N)}
\end{aligned} \tag{3.4.8}$$

where for any grid function f we define

$$\|f\|^2_{\ell^2(\frac{1}{N}\Lambda_N)} := N^{-d}\sum_{x\in\frac{1}{N}\Lambda_N} f(x)^2.$$

From (3.4.3) it follows that $G_{\frac{1}{N}}$ is the Green's function for $-\frac{1}{2dN^2}\Delta_{\frac{1}{N}}+\frac{1}{4d^2N^4}\Delta^2_{\frac{1}{N}}$. Let $\nu_1^{(N)},\nu_2^{(N)},\ldots$ be the eigenvalues of $G_{\frac{1}{N}}$. Define P_i to be the projection on the i-th eigenspace. Then using orthogonality we have

$$\|G_{\frac{1}{N}} w_j\|^2_{\ell^2(\frac{1}{N}\Lambda_N)} = \sum_i \left(\nu_i^{(N)}\right)^2 \|P_i w_j\|^2_{\ell^2(\frac{1}{N}\Lambda_N)} \le \left(\nu_{max}^{(N)}\right)^2 \|w_j\|^2_{\ell^2(\frac{1}{N}\Lambda_N)} \tag{3.4.9}$$

where $\nu_{max}^{(N)}$ is the largest eigenvalue of $G_{\frac{1}{N}}$. Using (3.4.9) in (3.4.8) we obtain

$$\begin{aligned}\mathbf{E}_{\Lambda_N}\|\Psi_N\|^2_{-s,-\Delta} &\le \sum_{j\ge 1} \nu_j^{-s} k^2 N^{-2} \nu_{max}^{(N)} \|w_j\|^2_{\ell^2(\frac{1}{N}\Lambda_N)} \\ &\le C \sum_{j\ge 1} \nu_j^{-s} k^2 N^{-2} \nu_{max}^{(N)} \left(\sup_{x\in D} w_j(x)\right)^2.\end{aligned}$$

From [70, Theorem 1.4] we know that for any $x \in D$, $|w_j(x)| \le \nu_j^{d/4}$. Theorem 3.5.3 on the other hand gives that $N^{-2}\nu_{max}^{(N)}$ is bounded above (as ν_1 is bounded away from zero). Using these observations we have

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}\|\psi_N\|^2_{-s,-\Delta} \le C \sum_{j\ge 1} \nu_j^{-s+\frac{d}{2}}.$$

The last sum is finite whenever $s > d$.

Thus we have proved (3.4.7). A standard uniqueness argument using the facts that $\mathcal{H}^{-s}_{-\Delta}(D)$ is the topological dual of $\mathcal{H}^{s}_{-\Delta,0}(D)$ and $C_c^\infty(D)$ is dense in $\mathcal{H}^{s}_{-\Delta,0}(D)$ (see proof of Theorem 2.3.11) completes the proof of Theorem 3.2.2. □

3.4.4 One-dimensional case

Set-up

In this case for simplicity we consider $D = (0,1)$ and the corresponding D_N and Λ_N as defined in Subsection 3.4.1, in particular $\Lambda_N = \{2, \ldots, N-2\}$. To study the scaling limit we define a continuous interpolation ψ_N for each N as follows:

$$\psi_N(t) = kN^{-\frac{1}{2}} \left[\varphi_{\lfloor Nt\rfloor} + (Nt - \lfloor Nt\rfloor)(\varphi_{\lfloor Nt\rfloor+1} - \varphi_{\lfloor Nt\rfloor})\right], \quad t \in \overline{D}.$$

In the proof of Theorem 3.2.3 we use Theorem 2.2.5. Another bound we will need is the following:

Lemma 3.4.5. *There exists $C > 0$ such that for all $x, y \in \mathbb{Z}$*

$$\mathbf{E}_{\Lambda_N}[(\varphi_x - \varphi_y)^2] \leq C|y - x|. \tag{3.4.10}$$

Proof. Note that it is enough to show the inequality for $x, y \in \{1, \dots, N-1\}$. The Brascamp-Lieb inequality as in the proof of Lemma 3.4.1 yields

$$\mathbf{E}_{\Lambda_N}[(\varphi_x - \varphi_y)^2] \leq \mathbf{E}^{DGFF}_{\Lambda_N}[(\varphi_x - \varphi_y)^2].$$

Let $(X_m)_{m=2}^{N-1}$ be a collection of i.i.d. $\mathcal{N}(0,2)$ random variables and let $S = (S_i)_{i=1}^{N-1}$ be the simple random walk on $\mathbb{Z}$ with X_m's as increments. We have that the field $(\varphi_1, \dots, \varphi_{N-2}, \varphi_{N-1})$ under $\mathbf{P}^{DGFF}_{\Lambda_N}$ has the same law of S conditionally on $S_1 = S_{N-1} = 0$. Now we define the process $(S'_1, \dots, S'_{N-1})$ by

$$S'_i := S_i - \frac{i-1}{N-2} S_{N-1}.$$

As a consequence

$$(S_1, \dots, S_{N-1} | S_1 = S_{N-1} = 0) \overset{d}{=} (S'_1, \dots, S'_{N-1}).$$

Then for $1 \leq i < j \leq N-1$ we have

$$\begin{aligned}
\mathbf{E}[(S'_j - S'_i)^2] &= \mathbf{E}\left[\left(\sum_{m=i+1}^{j} X_m - \frac{j-i}{N-2} S_{N-1}\right)^2\right] \\
&= 2(j-i) + 2\frac{(j-i)^2}{N-2} - 2\frac{(j-i)^2}{N-2} 2 \\
&= 2(j-i)\left[1 - \frac{j-i}{N-2}\right].
\end{aligned}$$

This shows the statement. □

Proof of Theorem 3.2.3

To prove weak convergence we show tightness and finite dimensional convergence. It is easy to see that $(\psi_N(0))_{N \geq 1}$ is tight. Therefore tightness will follow from Theorem 2.2.5 if we show that (2.2.2) is satisfied. Using the properties of Gaussian laws, to show (2.2.2) it is enough to prove the following: there exists $C > 0$ such that

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t) - \psi_N(s)|^2\right] \le C|t-s| \tag{3.4.11}$$

for all t, $s \in \overline{D}$ uniformly in N. To show (3.4.11) we consider the following two cases.

- Suppose $t, s \in [x, x+1/N]$ for some $x \in N^{-1}D_N$. Then we have

$$\psi_N(t) - \psi_N(s) = kN^{-\frac{1}{2}}\left[(Nt - Ns)(\varphi_{Nx+1} - \varphi_{Nx})\right].$$

  Now using (3.4.10) and the fact that $|t-s| \le 1/N$ we get (3.4.11).

- Next suppose $s \in [x, x+1/N)$ and $t \in [y, y+1/N)$ for some x, $y \in N^{-1}D_N$ and $t > x + 1/N$. In this case if $|t-s| \le 1/N$ then one can obtain (3.4.11) using the above case and a suitable point in between. So we assume $|t-s| > 1/N$. We first note that

$$\begin{aligned}
\mathbf{E}_{\Lambda_N}\left[|\psi_N(y) - \psi_N(x)|^2\right] &= k^2 N^{-1}\mathbf{E}_{\Lambda_N}[(\varphi_{Ny} - \varphi_{Nx})^2] \\
&\le CN^{-1}\mathbf{E}^{DGFF}_{\Lambda_N}[(\varphi_{Ny} - \varphi_{Nx})^2] \\
&\overset{(3.4.10)}{\le} C(y-x).
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{E}_{\Lambda_N}\left[|\psi_N(t) - \psi_N(s)|^2\right] \le C\Big(&\mathbf{E}_{\Lambda_N}\left[|\psi_N(t) - \psi_N(y)|^2\right] + \mathbf{E}_{\Lambda_N}\left[|\psi_N(y) - \psi_N(x)|^2\right] \\
&+ \mathbf{E}_{\Lambda_N}\left[|\psi_N(x) - \psi_N(s)|^2\right]\Big) \le C|t-s|.
\end{aligned}$$

Thus the sequence $(\psi_N)_N$ is tight in $C[0,1]$.

To conclude the finite dimensional convergence we first show the convergence of the covariance matrix. Let G_D be the Green's function for the problem

$$\begin{cases} -\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) = f(x) & x \in D \\ u(x) = 0 & x \in \partial D. \end{cases}$$

We note here that

$$G_D(x,y) = \min\{x,\, y\} - xy, \quad x, y \in \overline{D}$$

which also turns out to be the covariance function of the Brownian bridge, denoted by $(B^\circ_t : 0 \le t \le 1)$. For $x, y \in \overline{D} \cap N^{-1}\mathbb{Z}$ we define

$$G_{\frac{1}{N}}(x,y) := \frac{k^2}{N} G_{\Lambda_N}(Nx, Ny).$$

We now interpolate $G_{\frac{1}{N}}$ in a piece-wise constant fashion on small squares of $\overline{D}\times\overline{D}$ to get a new function $G^I_{\frac{1}{N}}$: we define the value of $G^I_{\frac{1}{N}}$ in the square $[x, x+1/N)\times[y, y+1/N)$ to be equal to $G_{\frac{1}{N}}(x,y)$ for all x, y in $\overline{D}\cap N^{-1}\mathbb{Z}$. We show that $G^I_{\frac{1}{N}}$ converges uniformly to G_D on $\overline{D}\times\overline{D}$. Indeed, let $F_N := G^I_{\frac{1}{N}} - G_D$. From the proof of Proposition 3.4.3 it follows that, for any f, $g \in C_c^\infty(D)$,

$$\lim_{N\to\infty} \sum_{x,y\in\frac{1}{N}D_N} N^{-2} G^I_{\frac{1}{N}}(x,y) f(x) g(y) = \iint_{D\times D} G_D(x,y) f(x) g(y)\,\mathrm{d}\,x\,\mathrm{d}\,y.$$

Again from Riemann sum convergence we have

$$\lim_{N\to\infty} \sum_{x,y\in\frac{1}{N}D_N} N^{-2} G_D(x,y) f(x) g(y) = \iint_{D\times D} G_D(x,y) f(x) g(y)\,\mathrm{d}\,x\,\mathrm{d}\,y.$$

Thus we get

$$\lim_{N\to\infty} \sum_{x,y\in\frac{1}{N}D_N} N^{-2} F_N(x,y) f(x) g(y) = 0. \tag{3.4.12}$$

Note that G_D is bounded and

$$\sup_{x,y\in\frac{1}{N}D_N} |G_{\Lambda_N}(Nx,Ny)| \overset{(3.4.1)}{\le} C \sup_{z\in D_N} \mathbf{E}^{GFF}_{\Lambda_N}[\varphi_z^2] \le CN.$$

These imply that

$$\sup_{x,y\in\overline{D}} |F_N(x,y)| \le C.$$

Now using (3.4.10) one can prove similarly as the diagonalization argument for the Arzelá-Ascoli theorem that F_N has a subsequence converging uniformly to some function F which is bounded by C. With abuse of notation we denote this subsequence by F_N. We then have

$$\lim_{N\to\infty} \sum_{x,y\in\frac{1}{N}D_N} N^{-2} F_N(x,y) f(x) g(y) = \iint_{D\times D} F(x,y) f(x) g(y)\,\mathrm{d}\,x\,\mathrm{d}\,y.$$

Uniqueness of the limit gives

$$\iint_{D\times D} F(x,y)f(x)g(y)\,\mathrm{d}\,x\,\mathrm{d}\,y = 0$$

by (3.4.12). From this we obtain that $F(x,y)=0$ for almost every x and almost every y. The definition by interpolation of $G^I_{\frac{1}{N}}$ ensures that F is pointwise equal to zero. Finally, the fact that the original sequence F_N converges uniformly to zero follows using the subsequence argument.

We now show the finite dimensional convergence. First let $t \in \overline{D}$. We write

$$\psi_N(t) = \psi_{N,1}(t) + \psi_{N,2}(t)$$

where $\psi_{N,1}(t) := kN^{-\frac{1}{2}}\varphi_{\lfloor Nt\rfloor}$ and $\psi_{N,2}(t) := kN^{-\frac{1}{2}}(Nt-\lfloor Nt\rfloor)(\varphi_{\lfloor Nt\rfloor+1}-\varphi_{\lfloor Nt\rfloor})$. From (3.4.10) it follows that $\mathbf{E}_{\Lambda_N}[\psi_{N,2}(t)^2]$ goes to zero as N tends to infinity. Therefore to show that $\psi_N(t) \xrightarrow{d} B^\circ_t$ it is enough to show that $\mathbf{Var}[\psi_{N,1}(t)] \to G_D(t,t)$. But we have

$$\mathbf{Var}[\psi_{N,1}(t)] = k^2N^{-1}G_{\Lambda_N}\left(\lfloor Nt\rfloor, \lfloor Nt\rfloor\right) = G^I_{\frac{1}{N}}(t,t) \to G_D(t,t)$$

since the sequence F_N converges to zero uniformly. Since the variables under consideration are Gaussian, one can show the finite dimensional convergence using the convergence of the Green's functions. □

Remark 3.4.6. *From* (3.4.11) *we have, for any $\alpha > 2$, that there exists a constant C such that the following holds uniformly in N with $\beta := \alpha/2 - d$:*

$$\mathbf{E}_{\Lambda_N}[|\psi_N(s)-\psi_N(t)|^\alpha] \le C|s-t|^{d+\beta}, \quad s,\, t \in \overline{D}.$$

Thus from Theorem 2.2.5 we recover the well-known Hölder continuity of the Brownian bridge with exponent η for any $\eta \in (0,1/2)$.

Remark 3.4.7. *In $d=1$, by the continuous mapping theorem together with Theorem 3.2.3 we have*

$$\sup_{t\in\overline{D}} \psi_N(t) \xrightarrow{d} \sup_{t\in\overline{D}} B^\circ_t$$

which gives the scaling limit for $M_N := \max_{x\in D_N}\varphi_x$:

$$\lim_{N\to\infty}\mathbf{P}_{\Lambda_N}\left(kN^{-\frac{1}{2}}M_N\le z\right)=\begin{cases}1-\mathrm{e}^{-2z^2} & \text{if } z>0\\ 0 & \text{otherwise.}\end{cases}$$

3.5 Error estimate in the discrete approximation of the Dirichlet problem

This section is devoted to showing that the solution of the continuum Dirichlet problem can be approximated well by the Green's function of the mixed model, and we will give a quantitative meaning to this statement. We shall use the ideas from [69], namely, to employ a truncated operator with which the problems of approximation around the boundary of the discretised domain can be ignored in a nice manner. We recall that the quantitative version of the results derived in [69] was essential to the proof of Theorem 3.2.2. We begin by introducing some definitions.

In this section we consider D to be any bounded domain in $\mathbb{R}^d$ with boundary ∂D which is C^2. We consider the following continuum Dirichlet problem

$$\begin{cases}Lu(x)=f(x) & x\in D\\ u(x)=0 & x\in\partial D.\end{cases}\tag{3.5.1}$$

where L is the elliptic differential operator $L:=-\Delta_c$.

Let $h>0$. We will call the points in $h\mathbb{Z}^d$ as the grid-points in $\mathbb{R}^d$. We consider

$$L_h=-\Delta_h+\frac{h^2}{2d}\Delta_h^2$$

to be an approximation of L. We have, for $x\in h\mathbb{Z}^d$, that

$$\begin{aligned}L_hf(x)=&-\frac{1}{h^2}\sum_{i=1}^{d}(f(x+he_i)+f(x-he_i)-2f(x))\\ &+\frac{1}{2dh^2}\sum_{i,j=1}^{d}\left[f(x+h(e_i+e_j))+f(x-h(e_i+e_j))+f(x+h(e_i-e_j))\right.\\ &+f(x-h(e_i-e_j))-2f(x+he_i)\\ &\left.-2f(x-he_i)-2f(x+he_j)-2f(x-he_j)+4f(x)\right].\end{aligned}$$

A concept crucially used in [69] is that the discrete approximation of an elliptic operator must be consistent with its continuum counterpart. In our case it is possible to see, using Taylor's expansion, that the operator L_h is consistent with the operator L, that is, if W is a neighborhood of the origin in $\mathbb{R}^d$ and $u \in C^2(W)$ then $L_h u(0) = Lu(0) + o(1)$ as $h \to 0$. Also from the definition of ellipticity of a difference operator given in [69, page 302] it follows that L_h is elliptic.

Now let D_h be the set of grid points in $\overline{D}$ i.e. $D_h = \overline{D} \cap h\mathbb{Z}^d$. We say that ξ is an interior grid point in D_h or $\xi \in R_h$ if ξ, $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ are all in D_h for every $i, j \in \{1, \ldots, d\}$. We denote B_h to be $D_h \setminus R_h$. For a grid function f we define by $R_h f$ a new grid function vanishing outside R_h as

$$R_h f(\xi) = \begin{cases} f(\xi) & \text{if } \xi \in R_h \\ 0 & \text{if } \xi \notin R_h. \end{cases}$$

We will divide R_h further into R_h^* and B_h^* where R_h^* is the set of ξ in R_h such that $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ are all in $R_h \cup (B_h \cap \partial D)$ for every $i, j \in \{1, \ldots, d\}$ and B_h^* is the set of remaining points in R_h. Thus we have

$$D_h = B_h \cup R_h = B_h \cup B_h^* \cup R_h^*.$$

We now define the finite difference analogue of the Dirichlet's problem (3.5.1). For given h, we look for a function $u_h(\xi)$ defined on D_h such that

$$L_h u_h(\xi) = f(\xi), \quad \xi \in R_h. \tag{3.5.2}$$

We consider furthermore the boundary conditions

$$u_h(\xi) = 0, \quad \xi \in B_h. \tag{3.5.3}$$

One can argue that the finite difference Dirichlet problem (3.5.2) and (3.5.3) has exactly one solution for arbitrary f [69, Theorem 5.1].

For grid functions vanishing outside D_h we define the norm $\|\cdot\|_{h,\,grid}$ by

$$\|f\|^2_{h,\,grid} := h^d \sum_{\xi \in D_h} f(\xi)^2.$$

Mind that we are using this norm only in the current section and thus there is no risk of confusion with the norm defined in Subsection 3.4.2. We now prove the main result of this section.

Theorem 3.5.1. *Let $u \in \mathcal{C}^3(\overline{D})$ be a solution of the Dirichlet's problem (3.5.1) and u_h be the solution of the discrete problem (3.5.2) and (3.5.3). If $e_h := u - u_h$ then for sufficiently small h we have*

$$\|R_h e_h\|^2_{h,\,grid} \le C\left[M_3^2 h^2 + h(M_3^2 h^4 + M_1^2)\right]$$

where $M_k = \sum_{|\alpha| \le k} \sup_{x \in D} |D^\alpha u(x)|$.

Proof. We denote by C all constants which do not depend on u, f. A standard Taylor's expansion gives for all $x \in D$ and for small h

$$L_h u(x) = Lu(x) + h^{-2}\mathcal{R}_3(x)$$

where

$$|\mathcal{R}_3(x)| \le C M_3 h^3. \tag{3.5.4}$$

So we obtain for $\xi \in R_h$

$$\begin{aligned} L_h e_h(\xi) &= L_h u(\xi) - L_h u_h(\xi) \\ &= Lu(\xi) + h^{-2}\mathcal{R}_3(\xi) - L_h u_h(\xi) \\ &= h^{-2}\mathcal{R}_3(\xi). \end{aligned}$$

The truncated operator $L_{h,1}$ is defined as follows:

$$L_{h,1} f(x) := \begin{cases} L_h f(x) & x \in R_h^* \\ h L_h f(x) & x \in B_h^* \\ 0 & x \notin R_h. \end{cases}$$

For $\xi \in R_h^*$ we have

$$L_{h,1}R_h e_h(\xi) = L_h R_h e_h(\xi) = L_h e_h(\xi) = h^{-2}\mathcal{R}_3(\xi). \tag{3.5.5}$$

For $\xi \in B_h^*$ at least one of $\xi \pm h(e_i \pm e_j)$, $\xi \pm he_i$ is in $B_h \setminus (B_h \cap \partial D)$. As the value of the solution of (3.5.1) is known to be zero on the boundary ∂D, we have for $\eta \in B_h$

$$u(\eta) = u_h(\eta) + \mathcal{R}_1(\eta)$$

where $|\mathcal{R}_1(\eta)| \leq CM_1 h$. For $\xi \in B_h^*$ denote by

$$S_{i,j}(\xi) = \{\eta : \eta \in B_h \setminus (B_h \cap \partial D) \cap \{\xi \pm he_i, \xi \pm h(e_i \pm e_j)\}\}.$$

Therefore, for $\xi \in B_h^*$,

$$\begin{aligned} L_{h,1}R_h e_h(\xi) &= hL_h R_h e_h(\xi) \\ &= h\{L_h e_h(\xi) - h^{-2}\sum_{i,j=1}^{d}\sum_{\eta \in S_{i,j}(\xi)} C(\eta)e_h(\eta)\} \\ &= h^{-1}\mathcal{R}_3(\xi) + h^{-1}\mathcal{R}_1'(\xi) \end{aligned} \tag{3.5.6}$$

where $C(\eta)$ is a constant depending on η and

$$|\mathcal{R}_1'(\xi)| \leq CM_1 h. \tag{3.5.7}$$

Hence

$$\begin{aligned} \|L_{h,1}R_h e_h\|_{h,\,grid}^2 &= h^d \sum_{x \in R_h} (L_{h,1}R_h e_h(x))^2 \\ &= h^d \left[\sum_{x \in R_h^*} (L_{h,1}R_h e_h(x))^2 + \sum_{x \in B_h^*} (L_{h,1}R_h e_h(x))^2\right] \\ &\overset{(3.5.5),(3.5.6)}{=} h^d \left[\sum_{x \in R_h^*} (h^{-2}\mathcal{R}_3(x))^2 + \sum_{x \in B_h^*} (h^{-1}\mathcal{R}_3(x) + h^{-1}\mathcal{R}_1'(x))^2\right] \\ &\overset{(3.5.4),(3.5.7)}{\leq} Ch^d \left[\sum_{x \in R_h^*} M_3^2 h^2 + \sum_{x \in B_h^*} (M_3^2 h^4 + M_1^2)\right] \\ &\leq C\left[M_3^2 h^2 + h(M_3^2 h^4 + M_1^2)\right] \end{aligned}$$

where the last inequality holds as the number of points in B_h^* is $O(h^{-(d-1)})$ which follows from [56, Lemma 5.4] due to assumption of a C^2 boundary. Finally using Theorem 4.2 and Lemma 3.1 of [69] we obtain

$$\|R_h e_h\|_{h,\,grid}^2 \le C\left[M_3^2 h^2 + h(M_3^2 h^4 + M_1^2)\right] \tag{3.5.8}$$

which completes the proof. $\square$

Remark 3.5.2. *Note that in the above proof we used Theorem 4.2 in [69] which requires the domain to satisfy a property called $\mathcal{B}_1^*$. In the same article it is pointed out that for any domain $\mathcal{B}_1^*$ holds by definition.*

Theorem 3.5.3. *Let A_h be the matrix $h^2 L_h$ and let $\mu_1^{(h)}$ be the smallest eigenvalue of A_h. Then*

$$\nu_1 = \lim_{h\to 0} h^{-2}\mu_1^{(h)},$$

where ν_1 is the smallest eigenvalue of $-\Delta_c$.

The proof of the above result follows by imitating the proof of Theorem 8.1 of Thomée [69] which we skip here.

Chapter 4

Scaling limit of semiflexible polymers: a phase transition

4.1 Introduction

In this chapter we study the model for which the Hamiltonian is given by

$$H(\varphi) = \sum_{x\in\mathbb{Z}^d} \big(\kappa_1\|\nabla\varphi_x\|^2 + \kappa_2(\Delta\varphi_x)^2\big) \tag{4.1.1}$$

κ_1, κ_2 are two non-negative parameters. More specifically, we consider the model $\varphi = (\varphi_x)_{x\in\mathbb{Z}^d}$, whose distribution is determined by a probability measure on $\mathbb{R}^{\mathbb{Z}^d}$, $d \geq 1$. The probability measure is given by

$$\mathbf{P}_\Lambda(\mathrm{d}\varphi) := \frac{1}{Z_\Lambda}\exp\left(-\sum_{x\in\mathbb{Z}^d}\big(\kappa_1\|\nabla\varphi_x\|^2 + \kappa_2(\Delta\varphi_x)^2\big)\right)\prod_{x\in\Lambda}\mathrm{d}\varphi_x \prod_{x\in\mathbb{Z}^d\setminus\Lambda}\delta_0(\mathrm{d}\varphi_x), \tag{4.1.2}$$

where $\Lambda \Subset \mathbb{Z}^d$ is a finite subset, $\mathrm{d}\varphi_x$ is the Lebesgue measure on $\mathbb{R}$, δ_0 is the Dirac measure at 0, and Z_Λ is a normalizing constant and the parameters κ_1, κ_2 depend on Λ. We are imposing zero boundary conditions: almost surely $\varphi_x = 0$ for all $x \in \mathbb{Z}^d \setminus \Lambda$, but the definition holds for more general boundary conditions. The main aim of this chapter is to show how the dependency on the size of the set Λ of κ_1 and κ_2 affects the scaling limit of $\mathbf{P}_\Lambda$.

In Chapter 3 the scaling limit of the $(\nabla + \Delta)$-model is studied. There it is shown that if one takes the lattice size to go to zero, under a suitable scaling the Laplacian term is dominated by the gradient and the limit becomes the Gaussian free field. A very natural question, which we aim at investigating in this chapter, is whether one can interpolate between the continuum Gaussian free field and the membrane model by tuning κ_2/κ_1 suitably. To the best of our knowledge, the influence of the length on the shape of the polymer through κ_1 and κ_2 has not been systematically addressed in the literature. In [59] a phase transition on the surface tension for mixed polymers has been investigated according to a suitable rescaling of $\sqrt{\kappa_2/\kappa_1}$ depending on the lattice size. However the model studied in [59] is integer-valued, so it differs from the one studied in the present chapter.

We now briefly describe the phase transition picture which appears in the scaling limit. We restrict our focus to $d = 1$ for heuristic explanations. Let us consider the Hamiltonian described in (4.1.1). We take $\Lambda = \{1, \ldots, N-1\}$ for $N \in \mathbb{N}$, $\kappa_1 = 1/4$ and $\kappa_2 = \kappa(N)/2$. In $d = 1$ in the DGFF case ($\kappa_2 = 0$) it is well-known that the finite volume measure can be given by a random walk bridge and in the membrane case ($\kappa_1 = 0$) by an integrated random walk bridge ([20]). Therefore the scaling limit for the DGFF and membrane turns out to be Brownian bridge and the integrated Brownian bridge, respectively. In $d = 1$, a representation for the $(\nabla + \Delta)-$model using random walks was obtained in [14]. The details of the representation are recalled in Section 4.6.

Let γ and σ be as in (4.6.3) and (4.6.4), respectively. Let $(\widetilde{\varepsilon}_i)_{i\in\mathbb{Z}^+}$ be i.i.d. normal random variables with mean zero and variance $\sigma^2/(1-\gamma)^2$. For $n \geq 1$, let $W_n = S_n - U_n$, where $S_n = \sum_{k=1}^{n} \widetilde{\varepsilon}_k$ and $U_n = \gamma^n \widetilde{\varepsilon}_1 + \gamma^{n-1}\widetilde{\varepsilon}_2 + \cdots + \gamma\widetilde{\varepsilon}_n$. From [14, Proposition 1.10] it is known that the finite volume measure of the model is given by the joint distribution of $(W_n)_{1\leq n\leq N-1}$ conditioned on $W_N = W_{N+1} = 0$. We look at the unconditional process and see how the parameter $\kappa(N)$ changes the variance. It follows from (4.6.3) and (4.6.4) that

$$\sigma^2 \approx \frac{1}{\kappa(N)} \quad \text{and} \quad (1-\gamma) \approx \frac{1}{\sqrt{\kappa(N)}}.$$

So for the case when $\kappa(N) \ll N^2$ we have

$$\mathbf{Var}(S_{N-1}) \approx N, \ \mathbf{Var}(U_{N-1}) \approx \sqrt{\kappa(N)} \ \text{ and } \ \mathbf{Cov}(S_{N-1}, U_{N-1}) \approx \sqrt{\kappa(N)}$$

which together imply that $\mathbf{Var}(W_{N-1}) \approx N$, thus the random walk dominates with its scaling $\sqrt{N}$.

When $\kappa(N) \gg N^2$ the situation is a bit more complicated and one can compute that (see Section 4.6)

$$\mathbf{Var}(W_{N-1}) \approx \frac{N^3}{\kappa(N)}.$$

It turns out that the Laplacian part dominates under this scaling. When $\kappa(N) \sim N^2$ then the contribution from S_{N-1} and U_{N-1} is similar and hence both the gradient and Laplacian interaction come into picture. The reader can see a simulation of the free boundary case that is, the trajectories of $(W_n)_{1\le n\le N}$, in Figure 4.1 and Figure 4.2. We plotted the two cases $\kappa \ll N^2$ and $\kappa \gg N^2$ in different pictures as the height scalings are different.

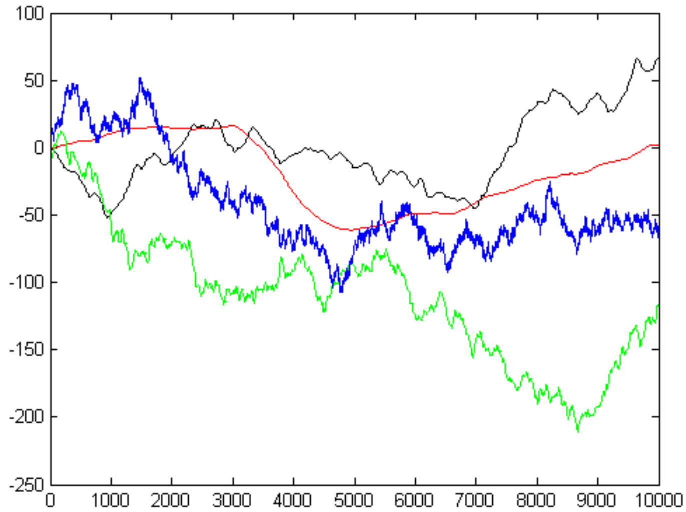

FIGURE 4.1: This is a simulation of some trajectories of $(W_n)_{1\le n\le N}$ with $N = 10^4$ and $\kappa = 0$, $\kappa = 2\times 10^2$, $\kappa = 2\times 10^4$, $\kappa = 2\times 10^6$.

We stress that in the above description we did not consider boundary effects which can cause considerable difficulty in understanding these processes explicitly. In Section 4.6 we have pointed out the conditional representation of W_{N-1}. One can see that it is not easy to determine whether the above transition can be pushed to the conditional processes and hence the finite volume measure. The aim of this chapter is to go beyond such representations and show the above transition holds true in general dimensions and get the explicit limits in each of the cases. In this respect, we also record that the integrated random walk representations of $d = 1$ cannot be extended to $d > 1$. In previous

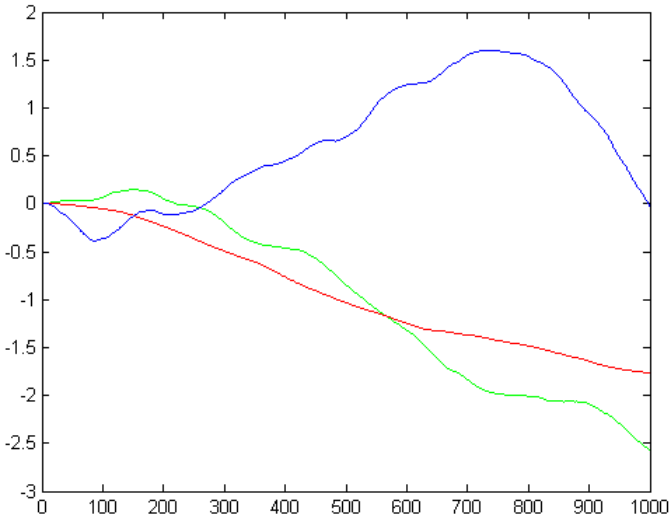

FIGURE 4.2: This is a simulation of some trajectories of $(W_n)_{1\leq n\leq N}$ with $N = 10^3$ and $\kappa = 2\times 10^{6.5}$, $\kappa = 2\times 10^7$, $\kappa = 2\times 10^8$.

chapters we introduced a finite difference method to approximate solutions of PDEs to successfully obtain the scaling limit of the membrane model and the $(\nabla+\Delta)$-model with fixed coefficients. The idea was inspired by the work of Thomée [69]. Finite difference methods was also employed in the works of Müller and Schweiger [54], Schweiger et al. [64] to obtain important estimates on the discrete Green's function of the membrane model.

The main results of the chapter are as follows. We consider the model on $\Lambda_N \Subset \mathbb{Z}^d$ for a suitable Λ_N defined later in Section 4.2. Also, we assume $\kappa_1 = 1/(4d)$, $\kappa_2 = \kappa(N)/2$ and distinguish three regimes for $\kappa = \kappa(N)$.

(a) Let $\kappa \gg N^2$. In $d \geq 1$, we show that the appropriately rescaled field converges to the continuum membrane model. The continuum membrane model is roughly a centered Gaussian process whose covariance is given by the Green's function of the Bilaplacian Dirichlet problem. For $d \geq 4$, in Theorem 4.2.7 we show the convergence takes place in a distributional space (more precisely a negatively-indexed Sobolev space). In $d = 1, 2$ and 3 we show in Theorem 4.2.1 that the limiting Gaussian process has continuous paths.

(b) Let $\kappa \sim 2dN^2$. In $d \geq 4$ we show (Theorem 4.2.7) that the rescaled field converges to a random distribution in an appropriate Sobolev space and the covariance of the limiting Gaussian field is given by the Dirichlet problem involving the elliptic operator $-\Delta_c + \Delta_c^2$. In $d = 1, 2$ and 3, again we show (in Theorem 4.2.1) the convergence takes place in the space of continuous functions.

(c) Let $\kappa \ll N^2$. In $d \geq 2$ we show (in Theorem 4.2.7) that the rescaled field converges in distribution to the Gaussian free field. Again, since the Gaussian free field is a random distribution the convergence takes place in a negatively-indexed Sobolev space. In $d = 1$, we show (in Theorem 4.2.1) that the limiting process is the Brownian bridge, confirming the heuristics presented above.

To derive the above results, the main technique we use is the approximation of the solution of a continuum Dirichlet problem with its discrete counterpart. Using Sobolev estimates it can be shown that the closeness of the solutions is related to the approximation of the discrete elliptic operator to the continuum one. This idea has been already employed in the previous two chapters.

But in the present scenario, the discrete elliptic operators have coefficients which depend on N and hence the estimates in [69] are not applicable directly. In addition, the rough behaviour around the boundary in the case of constant coefficients was dealt with by considering a truncation of the discrete elliptic operator. The operators were rescaled around the boundary and this helped in controlling their behaviour. The same technique becomes a bit more involved in the present case. This helps us to tackle with the cases $\kappa \gg N^2$ and $\kappa \sim 2dN^2$ but the method falls short when $\kappa \ll N^2$. In this case we take care the boundary effects and discretization separately, adjusting the boundary values with an appropriate cut-off function. We deal with these technical issues in Section 4.2.3. Let us mention in passing that we believe that the result in Section 4.2.3 is of independent interest and can be applied to discrete elliptic operators where coefficients depend on the scaling of the lattice.

4.2 Set-up and main results

Let Λ be a finite subset of $\mathbb{Z}^d$, $d \geq 1$, and $\mathbf{P}_\Lambda$ and $H(\varphi)$ be as in (4.1.2) and (4.1.1) respectively. It follows from Lemma 1.2.1 that the Gibbs measure (4.1.2) on $\mathbb{R}^\Lambda$ with

Hamiltonian (4.1.1) exists. Note that (4.1.1) can be written as

$$H(\varphi) = \frac{1}{2}\langle \varphi, (-4d\kappa_1\Delta + 2\kappa_2\Delta^2)\varphi\rangle_{\ell^2(\mathbb{Z}^d)}. \tag{4.2.1}$$

Let $d \geq 1$. Let D be a bounded domain in $\mathbb{R}^d$. For $N \in \mathbb{N}$, let $D_N = N\overline{D} \cap \mathbb{Z}^d$. Let us denote by Λ_N the set of points x in D_N such that, for every direction i, j, also the points $x \pm e_i$, $x \pm (e_i \pm e_j)$ are all in D_N. In other words, $\Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$ is the largest set satisfying $\partial_2\Lambda_N \subset N\overline{D} \cap \mathbb{Z}^d$ where $\partial_2\Lambda_N := \{y \in \mathbb{Z}^d \setminus \Lambda_N : \mathrm{dist}(y, \Lambda_N) \leq 2\}$ is the double (outer) boundary of Λ_N of points at ℓ^1 distance at most 2 from it. We consider the model with $\Lambda = \Lambda_N$, $\kappa_1 = 1/4d$, $\kappa_2 = \kappa(N)/2$ and want to study what happens when we tune suitably the parameter $\kappa(N)$ as N tends to infinity. We assume κ_1 to be constant as it is easy to state the results in this format. Also for simplicity we write κ for $\kappa(N)$. We just note here that if we write $G_{\Lambda_N}(x, y) := \mathbf{E}_{\Lambda_N}(\varphi_x\varphi_y)$, it follows from Lemma 1.2.1 that G_{Λ_N} solves the following discrete boundary value problem: for $x \in \Lambda_N$

$$\begin{cases} (-\Delta + \kappa\Delta^2)G_{\Lambda_N}(x,y) = \delta_x(y) & y \in \Lambda_N \\ G_{\Lambda_N}(x,y) = 0 & y \notin \Lambda_N \end{cases}. \tag{4.2.2}$$

To describe the main results we need some elliptic operators. We first introduce them and the corresponding Dirichlet problem. Let L denote one of the following three elliptic operators:

$$L = \begin{cases} -\Delta_c, \\ \Delta_c^2, \\ -\Delta_c + \Delta_c^2, \end{cases}$$

We consider the following continuum Dirichlet problem:

$$\begin{cases} Lu(x) = f(x) & x \in D \\ D^\alpha u(x) = 0 & |\alpha| \leq m-1,\, x \in \partial D. \end{cases} \tag{4.2.3}$$

where $m = 1$ if $L = -\Delta_c$ and $m = 2$ in the other cases.

4.2.1 Lower dimensional results

We first present the results in lower dimensions where we show that convergence takes place in the space of continuous functions. In this case we consider $D = (0,1)^d$. Also here, according to the behaviour of κ as $N \to \infty$ we have three different limits. To verify the convergence in the space of continuous functions we shall need to continuously interpolate the discrete model. In $d = 1$ the linear interpolation gives a continuous process but for higher dimensions there might be many ways. We stick to the following natural way. We will need this interpolation in $d = 2$ and 3 when $\kappa \gg N^2$ or $\kappa \sim 2dN^2$. We define the continuous interpolation $\{\psi_N\}_{N\in\mathbb{N}}$ in the following fashion:

- For $d = 1$ and $t \in \overline{D}$

$$\psi_N(t) = \mathbf{c}_N(1)\left[\varphi_{\lfloor Nt\rfloor} + (Nt - \lfloor Nt\rfloor)(\varphi_{\lfloor Nt\rfloor+1} - \varphi_{\lfloor Nt\rfloor})\right]. \tag{4.2.4}$$

- For $d = 2$ and $t = (t_1, t_2) \in \overline{D}$

$$\begin{aligned}\psi_N(t) = \mathbf{c}_N(2)\big[&\varphi_{\lfloor Nt\rfloor} + \{Nt_i\}\left(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}\right)\\ &+ \{Nt_j\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i}\right)\big], \quad \text{if}\,\{Nt_i\} \geq \{Nt_j\}\end{aligned} \tag{4.2.5}$$

  where $i,\, j \in \{1,\, 2\}$, $i \neq j$.

- For $d = 3$ and $t = (t_1, t_2, t_3) \in \overline{D}$

$$\begin{aligned}\psi_N(t) = \mathbf{c}_N(3)\big[&\varphi_{\lfloor Nt\rfloor} + \{Nt_i\}\left(\varphi_{\lfloor Nt\rfloor+e_i} - \varphi_{\lfloor Nt\rfloor}\right)\\ &+ \{Nt_j\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j} - \varphi_{\lfloor Nt\rfloor+e_i}\right)\\ &+ \{Nt_k\}\left(\varphi_{\lfloor Nt\rfloor+e_i+e_j+e_k} - \varphi_{\lfloor Nt\rfloor+e_i+e_j}\right)\big], \quad \text{if}\ \{Nt_i\} \geq \{Nt_j\} \geq \{Nt_k\}\end{aligned} \tag{4.2.6}$$

  where $i,\, j,\, k \in \{1,\, 2,\, 3\}$ and pairwise different. Here $\mathbf{c}_N(d)$, $d = 1,\, 2,\, 3$, are scaling factors which are specified in the following result.

Theorem 4.2.1. *We have the following convergence results.*

(i) $\kappa \gg N^2$. *Let* $1 \leq d \leq 3$. *Define a continuously interpolated field* ψ_N *as in* (4.2.4), (4.2.5) *and* (4.2.6) *with*

$$\mathbf{c}_N(d) = (2d)^{-1}\sqrt{\kappa}N^{\frac{d-4}{2}}.$$

Then we have, as $N \to \infty$, that the field ψ_N converges in distribution to $\psi_D^{\Delta^2}$ in the space of continuous functions on $\overline{D}$, where $\psi_D^{\Delta^2}$ is defined to be the centered continuous Gaussian process on $\overline{D}$ with covariance $G_D(\cdot,\cdot)$, the Green's function for the following biharmonic Dirichlet problem:

$$\begin{cases} \Delta_c^2 u(x) = f(x), & x \in D \\ D^\alpha u(x) = 0, & \forall\, |\alpha| \le 1,\, x \in \partial D. \end{cases} \tag{4.2.7}$$

(ii) $\kappa \sim 2dN^2$. *Let $1 \le d \le 3$. Define a continuously interpolated field ψ_N as in* (4.2.4)*,* (4.2.5) *and* (4.2.6) *with*

$$\mathbf{c}_N(d) = (2d)^{-1}\sqrt{\kappa}N^{\frac{d-4}{2}}.$$

Define $\psi_D^{-\Delta+\Delta^2}$ to be the continuous Gaussian process in $\overline{D}$ with covariance $G_D(\cdot,\cdot)$, where G_D is the Green's function for the problem

$$\begin{cases} (-\Delta_c + \Delta_c^2)u(x) = f(x), & x \in D \\ D^\alpha u(x) = 0, & \forall\, |\alpha| \le 1,\, x \in \partial D. \end{cases}$$

Then ψ_N converges in distribution to the field $\psi_D^{-\Delta+\Delta^2}$ in the space of continuous functions on $\overline{D}$.

(iii) $\kappa \ll N^2$. *Let $d = 1$. Define the continuously interpolated field ψ_N as in* (4.2.4) *with*

$$\mathbf{c}_N(1) = (2)^{-\frac{1}{2}}N^{-\frac{1}{2}}.$$

Then as $N \to \infty$, ψ_N converges in distribution to the Brownian bridge, $\psi_D^{-\Delta}$, in the space of continuous functions on $\overline{D}$.

Remark 4.2.2. *When $\kappa_1 = 0$ and $\kappa_2 = 1$ in* (4.1.1) *the $d = 1$ case was first studied in [21], where they showed that the limiting distribution is given by an integrated Brownian bridge (for a more precise definition see [21, Theorem 1.2]). The higher dimensional case has been studied in Chapter 2. It is shown that for $d = 2, 3$ the discrete membrane model converges to a Gaussian process with continuous paths and the methods in Chapter 2 can be seen to be valid in $d = 1$ also. By uniqueness of the limit in $C[0,1]$ it follows that the limiting Gaussian process in $d = 1$ for the case $\kappa \gg N^2$ (Theorem 4.2.1 (1)) can* *be described using the integrated Brownian bridge, the limit matching that of Caravenna and Deuschel [21].*

4.2.2 Higher dimensional results

We present now the results in higher dimensions where we show convergence in the space of distributions. We assume D to be any bounded domain with smooth boundary. In order to make our statements precise, we need to introduce three (negative ordered) Sobolev spaces denoted respectively as $\mathcal{H}^{-s}_{\Delta^2}(D)$, $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ and $\mathcal{H}^{-s}_{-\Delta}(D)$. We are going to recall some basic notations on Sobolev spaces and also some facts about the eigenvalues of the elliptic operators involved in our problem.

Continuum membrane model. We recall the definition of the Sobolev space $\mathcal{H}^{-s}_{\Delta^2}(D)$ and the continuum membrane model from Chapter 2. By the spectral theorem for compact self-adjoint operators and elliptic regularity one can show that there exist smooth eigenfunctions $\{u_j\}_{j\in\mathbb{N}}$ of Δ_c^2 corresponding to the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ such that $\{u_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $L^2(D)$. Now for any $s > 0$ we define the following inner product on $C_c^\infty(D)$:

$$\langle f\,,\,g\rangle_{s,\,\Delta^2} := \sum_{j\in\mathbb{N}} \lambda_j^{s/2}\,\langle f\,,\,u_j\rangle_{L^2}\,\langle u_j\,,\,g\rangle_{L^2}\,.$$

Then $\mathcal{H}^s_{\Delta^2,0}(D)$ is defined to be the Hilbert space completion of $C_c^\infty(D)$ with respect to this inner product. We define $\mathcal{H}^{-s}_{\Delta^2}(D)$ to be its dual and the dual norm is denoted by $\|\cdot\|_{-s,\,\Delta^2}$. Recall Proposition 2.3.10 which provides a description of the continuum membrane model $\Psi_D^{\Delta^2}$.

Definition 4.2.3. *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{\Delta^2} := \sum_{j\in\mathbb{N}} \lambda_j^{-1/2}\xi_j u_j.$$

Then $\Psi_D^{\Delta^2} \in \mathcal{H}^{-s}_{\Delta^2}(D)$ a.s. for all $s > (d-4)/2$ and is called the continuum membrane model.

Continuum mixed model. We define the space $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ analogously to $\mathcal{H}^{-s}_{\Delta^2}(D)$. One can find smooth eigenfunctions $\{v_j\}_{j\in\mathbb{N}}$ of $-\Delta_c+\Delta_c^2$ corresponding to eigenvalues $0<\mu_1\le\mu_2\le\cdots\to\infty$ such that $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(D)$. One can define, for $s>0$, the following inner product for functions from $C_c^\infty(D)$:

$$\langle f,g\rangle_{s,-\Delta+\Delta^2} := \sum_{j\in\mathbb{N}} \mu_j^{s/2}\left\langle f,v_j\right\rangle_{L^2}\left\langle v_j,g\right\rangle_{L^2}.$$

Let $\mathcal{H}^{s}_{-\Delta+\Delta^2,0}(D)$ be the completion of $C_c^\infty(D)$ with the above inner product and $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ be its dual. The dual norm is denoted by $\|\cdot\|_{-s,-\Delta+\Delta^2}$. We describe the details on this space in Section 4.6. The following definition is proved as Proposition 4.6.3 in Section 4.6.

Definition 4.2.4. *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{-\Delta+\Delta^2} := \sum_{j\in\mathbb{N}} \mu_j^{-1/2}\xi_j v_j.$$

Then $\Psi_D^{-\Delta+\Delta^2}\in\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ a.s. for all $s>(d-4)/2$ and is called the continuum mixed model.

Gaussian free field. Here also we recall the definition of the Sobolev space $\mathcal{H}^{-s}_{-\Delta}(D)$ and the Gaussian free field from Chapter 3. By the spectral theorem for compact self-adjoint operators and elliptic regularity we know that there exist smooth eigenfunctions $(w_j)_{j\in\mathbb{N}}$ of $-\Delta_c$ corresponding to the eigenvalues $0<\nu_1\le\nu_2\le\cdots\to\infty$ such that $(w_j)_{j\ge1}$ is an orthonormal basis of $L^2(D)$. Now for any $s>0$ we define the following inner product on $C_c^\infty(D)$:

$$\langle f\,,\,g\rangle_{s,\,-\Delta} := \sum_{j\in\mathbb{N}} \nu_j^s\langle f\,,\,w_j\rangle_{L^2}\langle w_j\,,\,g\rangle_{L^2}.$$

Then $\mathcal{H}^{s}_{-\Delta,0}(D)$ can be defined to be the completion of $C_c^\infty(D)$ with respect to this inner product. We define $\mathcal{H}^{-s}_{-\Delta}(D)$ to be its dual and the dual norm is denoted by $\|\cdot\|_{-s,\,-\Delta}$. We recall the definition of the Gaussian free field from Proposition 3.4.4.

Definition 4.2.5. *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{-\Delta} := \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\xi_j w_j.$$

Then $\Psi_D^{-\Delta} \in \mathcal{H}^{-s}_{-\Delta}(D)$ a.s. for all $s > d/2 - 1$ and is called the Gaussian free field.

Remark 4.2.6. *We define different spaces with respect to different eigenfunctions of the operators. It is not clear to us if these spaces coincide for a general domain. We are not aware of a result which gives the norm equivalence between the spaces $\mathcal{H}^s_{\Delta^2,0}(D)$, $\mathcal{H}^s_{-\Delta+\Delta^2,0}(D)$ and $\mathcal{H}^s_{-\Delta,0}(D)$. In this thesis we are not pursuing this line of research; what is important for us are the specific norms that determine the limiting variance of the discrete fields.*

We are now ready to state our main results in the higher dimensional case.

Theorem 4.2.7. *Assume that D has smooth boundary. Depending on the behaviour of κ as $N \to \infty$ we have the following three convergence results.*

(i) $\kappa \gg N^2$. *Let $d \geq 4$. Define Ψ_N by*

$$(\Psi_N, f) := (2d)^{-1}\sqrt{\kappa}N^{-\frac{d+4}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}^s_{\Delta^2,0}(D).$$

Then we have, as $N \to \infty$, that the field Ψ_N converges in distribution to the continuum membrane model $\Psi_D^{\Delta^2}$ in the topology of $\mathcal{H}^{-s}_{\Delta^2}(D)$ for $s > s_d$, where

$$s_d := \frac{d}{2} + 2\left(\left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 1\right)\right\rceil + \left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2} \right\rfloor + 6\right)\right\rceil - 1\right). \tag{4.2.8}$$

(ii) $\kappa \sim 2dN^2$. *Let $d \geq 4$. Define Ψ_N by*

$$(\Psi_N, f) := (2d)^{-1}\sqrt{\kappa}N^{-\frac{d+4}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}^s_{-\Delta+\Delta^2,0}(D).$$

Then, as $N \to \infty$, the field Ψ_N converges in distribution to $\Psi_D^{-\Delta+\Delta^2}$ in the topology of $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ for $s > s_d$ where s_d is as in (4.2.8).

(iii) $\kappa \ll N^2$. *Let $d \geq 2$. Define Ψ_N by*

$$(\Psi_N, f) := (2d)^{-\frac{1}{2}}N^{-\frac{d+2}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}^s_{-\Delta,0}(D).$$

Then, as $N \to \infty$, the field Ψ_N converges in distribution to the Gaussian free field $\Psi_D^{-\Delta}$ in the topology of $\mathcal{H}^{-s}_{-\Delta}(D)$ for $s > d/2 + \lfloor d/2 \rfloor + 2$.

4.2.3 Main ingredients in the proofs

We prove both Theorem 4.2.1 and Theorem 4.2.7 by first showing finite dimensional convergence and secondly tightness. As the measures are Gaussian with mean zero, the finite dimensional convergence follows from the convergence of the covariance. However the behaviour of the covariance of the model is not known explicitly. Therefore we use the expedient of boundary value problems to achieve both goals. The key fact which allows us to employ PDE techniques is that the covariance satisfies the discrete boundary value problem (4.2.2). For the proof of our main theorems we will compute in Theorem 4.2.8 the magnitude of the error one commits in approximating the solution of the Dirichlet problem (4.2.3) by its discrete counterpart. In the present section we only state the error estimate leaving the proof for Section 4.5. Let D be any bounded domain in $\mathbb{R}^d$ satisfying the uniform exterior ball condition (UEBC), which states that there exists $r > 0$ such that for any $z \in \partial D$ there is a ball $B_r(c)$ of radius r with center at some point c satisfying $\overline{B_r(c)} \cap \overline{D} = \{z\}$. We mention here that any domain with C^2 boundary satisfies the UEBC.

Let $h > 0$. We will call the points in $h\mathbb{Z}^d$ the grid points in $\mathbb{R}^d$. We consider L_h to be a discrete approximation of L given by

$$L_h u = \begin{cases} (-\Delta_h + \rho_1(h)\Delta_h^2)u & \text{if } L = -\Delta_c \\ (-\rho_2(h)\Delta_h + \Delta_h^2)u & \text{if } L = \Delta_c^2 \\ (-\Delta_h + \rho_3(h)\Delta_h^2)u & \text{if } L = -\Delta_c + \Delta_c^2 \end{cases} \tag{4.2.9}$$

where $\rho_i(h)$ are functions of h taking values in the positive real line such that

$$\lim_{h\to 0} \rho_i(h) = \begin{cases} 0 & i = 1, 2; \\ 1 & i = 3. \end{cases}$$

Let D_h be the set of grid points in $\overline{D}$ i.e. $D_h = \overline{D} \cap h\mathbb{Z}^d$. For any grid point x we define the points $x \pm he_i$, $x \pm h(e_i \pm e_j)$ with $1 \le i, j \le d$ to be its neighbours. We say that x is an interior grid point in D_h if all its neighbors are in D_h. Let R_h be the set of interior grid points in D_h and $B_h := D_h \setminus R_h$ be the set of grid points near the boundary. We divide R_h further into R_h^* and B_h^*, where R_h^* is the set of x in R_h such that all its neighbors are in R_h and B_h^* is the set of remaining points in R_h. Thus we have

$$D_h = B_h \cup R_h = B_h \cup B_h^* \cup R_h^*.$$

Denote by $\mathcal{D}_h$ the set of grid functions vanishing outside R_h. For a grid function f we define $R_h f \in \mathcal{D}_h$ by

$$R_h f(x) = \begin{cases} f(x) & x \in R_h \\ 0 & x \notin R_h. \end{cases} \tag{4.2.10}$$

Define for grid-functions vanishing outside a finite set

$$\begin{aligned} \langle u\,,\, v\rangle_{h,\,grid} &:= h^d \sum_{x\in h\,\mathbb{Z}^d} u(x)v(x), \\ \|u\|_{h,\,grid} &:= \langle u\,,\, u\rangle_{h,\,grid}^{1/2}. \end{aligned}$$

We now define the finite difference analogue of the Dirichlet problem (4.2.3). For given h, we look for a function $u_h(\cdot)$ defined on D_h such that

$$L_h u_h(x) = f(x), \quad x \in R_h \tag{4.2.11}$$

and

$$u_h(x) = 0, \quad x \in B_h. \tag{4.2.12}$$

The uniqueness of the solution of (4.2.11) and (4.2.12) is shown in Lemma 4.5.5. We are now ready to state the error estimate result which forms the core result of this chapter.

Theorem 4.2.8. *Depending on L we have the following error bounds.*

(i) $L = \Delta_c^2$. *Let $u \in C^5(\overline{D})$ be the solution of the Dirichlet problem (4.2.3). If $e_h := u - u_h$ then we have for all sufficiently small h*

$$\|R_h e_h\|_{h,\,grid}^2 \le C\left[M_5^2 h^2 + M_2^2(\rho_2(h))^2 + M_2^2 h\right].$$

(ii) $L = -\Delta_c + \Delta_c^2$. *Let* $u \in C^5(\overline{D})$ *be the solution of the Dirichlet problem* (4.2.3). *If* $e_h := u - u_h$ *then we have for all sufficiently small* h

$$\|R_h e_h\|^2_{h,\,grid} \le C\left[M_5^2 h^2 + M_4^2(\rho_3(h)-1)^2 + M_4^2 h^4 + M_2^2 h\right].$$

(iii) $L = -\Delta_c$. *Let* $u \in \mathcal{C}^4(\overline{D})$ *be a solution of the Dirichlet problem* (4.2.3). *If* $e_h := u - u_h$ *then for sufficiently small* h *we have*

$$\|R_h e_h\|^2_{h,\,grid} \le C\left[M_4^2\delta^4 + M_2^2\rho_1(h)\delta + M_1^2\delta\right],$$

where $\delta := \max\{h, \sqrt{\rho_1(h)}\}$.

In all the cases $M_k := \sum_{|\alpha|\le k}\sup_{x\in D}|D^\alpha u(x)|$.

4.3 Proof of Theorem 4.2.7

We now give the proof of each of the three parts of Theorem 4.2.7.

4.3.1 Proof of finite dimensional convergence

We first show that for $f \in C_c^\infty(D)$

$$(\Psi_N, f) \xrightarrow{d} \begin{cases} (\Psi_D^{\Delta^2}, f) & \kappa \gg N^2 \\ (\Psi_D^{-\Delta+\Delta^2}, f) & \kappa \sim 2dN^2 \\ (\Psi_D^{-\Delta}, f) & \kappa \ll N^2. \end{cases} \tag{4.3.1}$$

We begin by noting that (Ψ_N, f) is a centered Gaussian random variable. Hence to show the above convergence it is enough to show that $\mathbf{Var}(\Psi_N, f)$ converges to the variance of the Gaussian on the right hand side of (4.3.1). We denote $G_{\frac{1}{N}}(x,y) := \mathbf{E}_{\Lambda_N}[\varphi_{Nx}\varphi_{Ny}]$. Note that by (4.2.2), we have for all $x \in \frac{1}{N}\Lambda_N$,

$$\kappa \gg N^2: \qquad \begin{cases} \left(-\frac{2dN^2}{\kappa}\Delta_{\frac{1}{N}} + \Delta^2_{\frac{1}{N}}\right) G_{\frac{1}{N}}(x,y) = \frac{4d^2N^4}{\kappa}\delta_x(y), & y \in \frac{1}{N}\Lambda_N \\ G_{\frac{1}{N}}(x,y) = 0 & y \notin \frac{1}{N}\Lambda_N. \end{cases} \tag{4.3.2}$$

$$\kappa \sim 2dN^2: \qquad \begin{cases} \left(-\Delta_{\frac{1}{N}} + \frac{\kappa}{2dN^2}\Delta^2_{\frac{1}{N}}\right) G_{\frac{1}{N}}(x,y) = 2dN^2\delta_x(y), & y \in \frac{1}{N}\Lambda_N \\ G_{\frac{1}{N}}(x,y) = 0 & y \notin \frac{1}{N}\Lambda_N. \end{cases} \tag{4.3.3}$$

$$\kappa \ll N^2: \qquad \begin{cases} \left(-\Delta_{\frac{1}{N}} + \frac{\kappa}{2dN^2}\Delta^2_{\frac{1}{N}}\right) G_{\frac{1}{N}}(x,y) = 2dN^2\delta_x(y), & y \in \frac{1}{N}\Lambda_N \\ G_{\frac{1}{N}}(x,y) = 0 & y \notin \frac{1}{N}\Lambda_N. \end{cases} \tag{4.3.4}$$

Now considering all the three cases we can rewrite the variance as

$$\mathbf{Var}[(\Psi_N, f)] = N^{-d} \sum_{x \in \frac{1}{N}\Lambda_N} H_N(x) f(x)$$

where for $x \in \frac{1}{N}D_N$,

$$H_N(x) = \begin{cases} (2d)^{-2}\kappa N^{-4} \sum_{y \in \frac{1}{N}\Lambda_N} G_{\frac{1}{N}}(x,y) f(y) & \kappa \gg N^2 \\ (2d)^{-2}\kappa N^{-4} \sum_{y \in \frac{1}{N}\Lambda_N} G_{\frac{1}{N}}(x,y) f(y) & \kappa \sim 2dN^2 \\ (2d)^{-1} N^{-2} \sum_{y \in \frac{1}{N}\Lambda_N} G_{\frac{1}{N}}(x,y) f(y) & \kappa \ll N^2. \end{cases}$$

It is immediate from (4.3.2), (4.3.3), (4.3.4) that H_N is the solution of the following Dirichlet problem:

$$\kappa \gg N^2: \qquad \begin{cases} \left(-\frac{2dN^2}{\kappa}\Delta_{\frac{1}{N}} + \Delta^2_{\frac{1}{N}}\right) H_N(x) = f(x), & x \in \frac{1}{N}\Lambda_N \\ H_N(x) = 0, & x \notin \frac{1}{N}\Lambda_N \end{cases} \tag{4.3.5}$$

$$\kappa \sim 2dN^2: \qquad \begin{cases} \left(-\Delta_{\frac{1}{N}} + \frac{\kappa}{2dN^2}\Delta^2_{\frac{1}{N}}\right) H_N(x) = f(x) & x \in \frac{1}{N}\Lambda_N \\ H_N(x) = 0 & x \notin \frac{1}{N}\Lambda_N \end{cases} \tag{4.3.6}$$

$$\kappa \ll N^2: \qquad \begin{cases} \left(-\Delta_{\frac{1}{N}} + \frac{\kappa}{2dN^2}\Delta^2_{\frac{1}{N}}\right) H_N(x) = f(x), & x \in \frac{1}{N}\Lambda_N \\ H_N(x) = 0, & x \notin \frac{1}{N}\Lambda_N. \end{cases} \tag{4.3.7}$$

Observe that we get the discrete Dirichlet problem involving the operator L_h defined in (4.2.9) with $h = 1/N$ and

$$\rho_1(h) := \kappa h^2/2d, \quad \rho_2(h) := 2d/\kappa h^2, \quad \rho_3(h) := \kappa h^2/2d.$$

We now recall the continuum Dirichlet problem (4.2.3) with the elliptic operator L as in (4.2):

$$\begin{cases} Lu(x) = f(x) & x \in D \\ D^\alpha u(x) = 0 & |\alpha| \le m-1,\, x \in \partial D. \end{cases}$$

where $m = 1$ if $L = -\Delta_c$ and $m = 2$ in the other two cases. We set $L := \Delta_c^2$ when $\kappa \gg N^2$, $L := -\Delta_c$ when $\kappa \ll N^2$ and $L := -\Delta_c + \Delta_c^2$ when $\kappa \sim 2dN^2$. Define $e_N(x) = H_N(x) - u(x)$ for $x \in \frac{1}{N} D_N$. Then from Theorem 4.2.8 we have

$$N^{-d} \sum_{x \in \frac{1}{N}\Lambda_N} e_N(x)^2 \le \begin{cases} C\left(\frac{1}{N^2} + \frac{4d^2N^4}{\kappa^2} + \frac{1}{N}\right) & \kappa \gg N^2 \\ C\left(\frac{1}{N} + \left(\frac{\kappa}{2dN^2} - 1\right)^2\right) & \kappa \sim 2dN^2 \\ C \max\{\frac{1}{N}, \frac{\sqrt{\kappa}}{\sqrt{2d}N}\} & \kappa \ll N^2. \end{cases} \tag{4.3.8}$$

Hence we get that

$$\mathbf{Var}[(\Psi_N, f)] = \sum_{x \in \frac{1}{N}\Lambda_N} e_N(x) f(x) N^{-d} + \sum_{x \in \frac{1}{N}\Lambda_N} u(x) f(x) N^{-d}. \tag{4.3.9}$$

Note that by Cauchy-Schwarz inequality and (4.3.8) the first term goes to zero as $N \to \infty$. The second term converges to

$$\sum_{x \in \frac{1}{N}\Lambda_N} u(x) f(x) N^{-d} \to_{N\to\infty} \int_D u(x) f(x)\, \mathrm{d}\, x. \tag{4.3.10}$$

Notice that by integration by parts we have

$$\int_D u(x) f(x)\, \mathrm{d}\, x = \begin{cases} \|u\|^2_{2,\,\Delta^2} = \|f\|^2_{-2,\,\Delta^2} & L = \Delta_c^2 \\ \|u\|^2_{2,\,-\Delta+\Delta^2} = \|f\|^2_{-2,\,-\Delta+\Delta^2} & L = -\Delta_c + \Delta_c^2 \\ \|u\|^2_{1,-\Delta} = \|f\|^2_{-1,-\Delta} & L = -\Delta_c. \end{cases}$$

On the other hand from the definition it follows that

$$\begin{aligned}
\mathbf{Var}[(\Psi_D^{\Delta^2}, f)] &= \sum_{j\in\mathbb{N}} \lambda_j^{-1} \langle u_j, f\rangle_{L^2}^2 = \|f\|_{-2,\Delta^2}^2 \\
\mathbf{Var}[(\Psi_D^{-\Delta+\Delta^2}, f)] &= \sum_{j\in\mathbb{N}} \mu_j^{-1} \langle v_j, f\rangle_{L^2}^2 = \|f\|_{-2,-\Delta+\Delta^2}^2 \\
\mathbf{Var}[(\Psi^{\Delta}, f)] &= \sum_{j\in\mathbb{N}} \nu_j^{-1} \langle w_j, f\rangle_{L^2}^2 = \|f\|_{-1,-\Delta}^2.
\end{aligned}$$

Consequently we obtain (4.3.1).

4.3.2 Tightness

To show tightness we shall need the following bounds on the eigenfunctions $(u_j)_{j\in\mathbb{N}}$, $(v_j)_{j\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ of Δ_c^2, $-\Delta_c+\Delta_c^2$ and $-\Delta_c$ respectively. They can be obtained from the general Sobolev inequality ([35, Chapter 5, Theorem 6 (ii)]) and a repeated application of [39, Corollary 2.21].

Lemma 4.3.1. *Let*

$$l_k := \left\lceil \frac{1}{4}\left(\left\lfloor \frac{d}{2}\right\rfloor + k + 1\right)\right\rceil, \quad k \geq 0.$$

(i) *For the eigenfunctions $(u_j)_{j\in\mathbb{N}}$ of Δ_c^2 corresponding to eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ in Problem (4.2.3) there exists a constant $C > 0$ such that for $k \geq 0$*

$$\sum_{|\alpha|\leq k} \sup_{x\in D} |D^\alpha u_j(x)| \leq C\lambda_j^{l_k}. \tag{4.3.11}$$

(ii) *For the eigenfunctions $(v_j)_{j\in\mathbb{N}}$ of $-\Delta_c+\Delta_c^2$ corresponding to eigenvalues $(\mu_j)_{j\in\mathbb{N}}$ in Problem (4.2.3) there exists a constant $C > 0$ such that for $k \geq 0$*

$$\sum_{|\alpha|\leq k} \sup_{x\in D} |D^\alpha v_j(x)| \leq C\mu_j^{l_k}. \tag{4.3.12}$$

(iii) *For the eigenfunctions $(w_j)_{j\in\mathbb{N}}$ of $-\Delta_c$ corresponding to eigenvalues $(\nu_j)_{j\in\mathbb{N}}$ in Problem (4.2.3) there exists a constant $C > 0$ such that for $k \geq 0$*

$$\sum_{|\alpha|\leq k} \sup_{x\in D} |D^\alpha w_j(x)| \leq C\nu_j^{\frac{\lfloor \frac{d}{2}\rfloor + k + 1}{2}}. \tag{4.3.13}$$

In each instance, the constant C may depend on k.

We can now begin to show tightness.

Case 1: $\kappa \gg N^2$. Our target is to show that the sequence $(\Psi_N)_{N\in\mathbb{N}}$ is tight in $\mathcal{H}^{-s}_{\Delta^2}(D)$ for all $s > s_d$, where s_d is as in (4.2.8). It is enough to show that

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,\,\Delta^2}] < \infty \quad \forall\, s > s_d. \tag{4.3.14}$$

The tightness of $(\Psi_N)_{N\in\mathbb{N}}$ would then follow immediately from (4.3.14) and the fact that, for $0 \le s_1 < s_2$, $\mathcal{H}^{-s_1}_{\Delta^2}(D)$ is compactly embedded in $\mathcal{H}^{-s_2}_{\Delta^2}(D)$.

From the definition of dual norm it is immediate that we have

$$\mathbf{E}_{\Lambda_N}\left[\|\Psi_N\|^2_{-s,\,\Delta^2}\right] \le \sum_{j\in\mathbb{N}} \lambda_j^{-s/2}\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,u_j)^2].$$

Note that $u = \lambda_j^{-1}u_j$ is the unique solution of (4.2.3) with $L = \Delta_c^2$ for $f := u_j$. Define $e_{N,j}$ to be the error between the solution of the discrete Dirichlet problem (4.3.5) and the continuum one (4.2.3) with input datum $f := u_j$. Now as in (4.3.9) we have

$$\begin{aligned}\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,u_j)^2] &= \sum_{x\in\frac{1}{N}\Lambda_N} e_{N,j}(x)u_j(x)N^{-d} + \sum_{x\in\frac{1}{N}\Lambda_N} \lambda_j^{-1}u_j(x)u_j(x)N^{-d}\\ &\le C\sup_{x\in D}|u_j(x)|\left(N^{-d}\sum_{x\in\frac{1}{N}\Lambda_N} e_{N,j}(x)^2\right)^{1/2} + C\lambda_j^{-1}\left(\sup_{x\in D}|u_j(x)|\right)^2.\end{aligned} \tag{4.3.15}$$

Using Theorem 4.2.8 (1) along with the bounds (4.3.11) we obtain

$$\begin{aligned}\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,u_j)^2] &\le C\lambda_j^{l_0}\left[\lambda_j^{2l_5-2}N^{-2} + \lambda_j^{2l_2-2}4d^2N^4\kappa^{-2} + \lambda_j^{2l_2-2}N^{-1}\right]^{\frac{1}{2}} + C\lambda_j^{2l_0-1}\\ &\le C\lambda_j^{l_0+l_5-1}.\end{aligned}$$

Therefore we have

$$\mathbf{E}_{\Lambda_N}\left[\|\Psi_N\|^2_{-s,\,\Delta^2}\right] \le C\sum_{j\in\mathbb{N}} \lambda_j^{-\frac{s}{2}}\lambda_j^{l_0+l_5-1}.$$

Thus

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,\,\Delta^2}] < \infty \qquad \text{if} \qquad \sum_{j\in\mathbb{N}} \lambda_j^{-\frac{s}{2}+l_0+l_5-1} < \infty.$$

Now using $\lambda_j \sim c(d)j^{4/d}$ we obtain that $\sum_{j\in\mathbb{N}} \lambda_j^{-\frac{s}{2}+l_0+l_5-1}$ is finite whenever $s > s_d$. Thus we have proved (4.3.14).

Case 2: $\kappa \sim 2dN^2$**.** Due to the compact embedding of the spaces $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$, to show that the sequence $(\Psi_N)_{N\in\mathbb{N}}$ is tight in $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ for all $s > s_d$, it is enough to show that

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,-\Delta+\Delta^2}] < \infty \quad \forall\, s > s_d. \tag{4.3.16}$$

As in the previous case, by definition of dual norm we have

$$\mathbf{E}_{\Lambda_N}\left[\|\Psi_N\|^2_{-s,-\Delta+\Delta^2}\right] \le \sum_{j\in\mathbb{N}} \mu_j^{-s/2}\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,v_j)^2].$$

Note that $u = \mu_j^{-1}v_j$ is the unique solution of (4.2.3) with $L = -\Delta_c + \Delta_c^2$ for $f := u_j$. Define $e_{N,j}$ to be the error between the solution of the discrete Dirichlet problem (4.3.6) and the continuum one (4.2.3) with $f := v_j$. Now as in (4.3.15) we have

$$\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,v_j)^2] \le C\sup_{x\in D}|v_j(x)|\left(N^{-d}\sum_{x\in\frac{1}{N}\Lambda_N} e_{N,j}(x)^2\right)^{1/2} + C\mu_j^{-1}\left(\sup_{x\in D}|v_j(x)|\right)^2.$$

Using Theorem 4.2.8 (2) along with the bounds (4.3.12) we obtain

$$\begin{aligned}\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,v_j)^2] &\le C\mu_j^{l_0}\left[\mu_j^{2l_5-2}N^{-2} + \mu_j^{2l_5-2}\left(\frac{\kappa}{2dN^2}-1\right)^2 + \mu_j^{2l_2-2}N^{-1}\right]^{\frac{1}{2}} + C\mu_j^{2l_0-1}\\ &\le C\mu_j^{l_0+l_5-1}.\end{aligned}$$

Therefore we have

$$\mathbf{E}_{\Lambda_N}\left[\|\Psi_N\|^2_{-s,-\Delta+\Delta^2}\right] \le C\sum_{j\in\mathbb{N}} \mu_j^{-\frac{s}{2}}\mu_j^{l_0+l_5-1}.$$

Thus

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,-\Delta+\Delta^2}] < \infty \qquad \text{if} \qquad \sum_{j\in\mathbb{N}} \mu_j^{-\frac{s}{2}+l_0+l_5-1} < \infty.$$

From Proposition 4.6.2 we obtain that $\sum_{j\in\mathbb{N}} \mu_j^{-\frac{s}{2}+l_0+l_5-1} < \infty$ whenever $s > s_d$. Thus we have proved (4.3.16).

Case 3: $\kappa \ll N^2$. The arguments are similar to the previous two cases and hence we just indicate the required bounds. To show tightness in $\mathcal{H}^{-s}_{-\Delta}(D)$ it is enough to show

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,\,-\Delta}] \le \sum_{j\in\mathbb{N}} \nu_j^{-s}\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,w_j)^2] < \infty \quad \forall\, s > d/2 + \lfloor d/2\rfloor + 2. \tag{4.3.17}$$

Setting $e_{N,j}$ to be the error between the solution of the discrete Dirichlet problem (4.3.7) and the continuum one (4.2.3) with $f := w_j$ we obtain

$$\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,w_j)^2] \le C\sup_{x\in D}|w_j(x)|\left(N^{-d}\sum_{x\in\frac{1}{N}\Lambda_N} e_{N,j}(x)^2\right)^{1/2} + C\nu_j^{-1}\left(\sup_{x\in D}|w_j(x)|\right)^2.$$

Using Theorem 4.2.8 (3) along with the bounds (4.3.13) we can conclude the following upper bound for $\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,w_j)^2]$:

$$C\sup_{x\in D}|w_j(x)|\left[\left(\nu_j^{-1}\nu_j^{\frac{\lfloor\frac{d}{2}\rfloor+5}{2}}\right)^2 + \left(\nu_j^{-1}\nu_j^{\frac{\lfloor\frac{d}{2}\rfloor+3}{2}}\right)^2 + \left(\nu_j^{-1}\nu_j^{\frac{\lfloor\frac{d}{2}\rfloor+2}{2}}\right)^2\right]^{\frac{1}{2}}$$
$$+\, C\nu_j^{-1}\left(\sup_{x\in D}|w_j(x)|\right)^2.$$

Now a consequence of the above and (4.3.13) is that

$$\mathbf{E}_{\Lambda_N}[(\Psi_N\,,\,w_j)^2] \le C\nu_j^{\lfloor\frac{d}{2}\rfloor+2}. \tag{4.3.18}$$

Therefore we have

$$\mathbf{E}_{\Lambda_N}\left[\|\Psi_N\|^2_{-s,\,-\Delta}\right] \le C\sum_{j\in\mathbb{N}} \nu_j^{-s}\nu_j^{\lfloor\frac{d}{2}\rfloor+2}.$$

Thus

$$\limsup_{N\to\infty} \mathbf{E}_{\Lambda_N}[\|\Psi_N\|^2_{-s,\,-\Delta}] < \infty \qquad \text{if} \qquad \sum_{j\in\mathbb{N}} \nu_j^{-s+\lfloor\frac{d}{2}\rfloor+2} < \infty.$$

We now use the Weyl [72]'s asymptotic $\nu_j \sim Cj^{\frac{2}{d}}$ (see [73, Lemma 4.2]) and the fact that $\sum_{j\in\mathbb{N}} j^{\frac{2}{d}(-s+\lfloor\frac{d}{2}\rfloor+2)} < \infty$ whenever $s > d/2 + \lfloor d/2\rfloor + 2$ to conclude (4.3.17).

For all the cases we now have the tightness and the convergence of (Ψ_N, f) for all $f \in C_c^\infty(D)$. A standard uniqueness argument completes the proof of Theorem 4.2.7, using the fact that $C_c^\infty(D)$ is dense in $\mathcal{H}^s_{\Delta^2,0}(D)$, $\mathcal{H}^s_{-\Delta+\Delta^2,0}(D)$ and $\mathcal{H}^s_{-\Delta,0}(D)$ respectively.

4.4 Proof of Theorem 4.2.1

In this section we prove Theorem 4.2.1 by showing finite dimensional convergence and tightness. The proof is similar to the proofs of the lower dimensional results in the previous two chapters. First we will show tightness and then we will prove the finite dimensional convergence which is similar to the proof of Theorem 4.2.7. To show tightness we use Theorem 2.2.5.

We shall elaborate on the case when $\kappa \gg N^2$. The argument for other case $\kappa \sim 2dN^2$ is exactly the same. For the case $\kappa \ll N^2$, the proof is similar and hence we shall indicate only the crucial bounds which are required in the proof.

Case 1: $\kappa \gg N^2$

First we want to show that the sequence $\{\psi_N\}_{N\in\mathbb{N}}$ is tight in $C(\overline{D})$. We need the following bounds.

Lemma 4.4.1.

(1) For any $x, y \in \mathbb{Z}^d$

$$|G_{\Lambda_N}(x,y)| \leq C\kappa^{-1}N^{4-d}.$$

(2) For $x \in \mathbb{Z}^d$

$$\mathbf{E}_{\Lambda_N}\left[(\varphi_{x+e_i} - \varphi_x)^2\right] \leq \begin{cases} C\kappa^{-1}N & d=1 \\ C\kappa^{-1}\log N & d=2 \\ C\kappa^{-1} & d=3 \end{cases}.$$

Proof. To show the first inequality we bound $G_{\Lambda_N}(x,\, x)$. One can show using Theorem 5.1 in [18] that

$$G_{\Lambda_N}(x,\, x) \leq \kappa^{-1}\mathbf{E}^{MM}_{\Lambda_N}(\varphi_x^2)$$

where $\mathbf{P}^{MM}_{\Lambda_N}$ denotes the law of the membrane model on Λ_N with zero boundary conditions outside Λ_N. The bound for the $d=1$ case can now be obtained using the random walk representation of the model used in Lemma 4.6.1. For $d=2, 3$ we obtain the bound from [54, Theorem 1.1].

For the second part the Brascamp-Lieb inequality yields

$$\mathbf{E}_{\Lambda_N}[(\varphi_{x+e_i}-\varphi_x)^2] \le \kappa^{-1}\mathbf{E}^{MM}_{\Lambda_N}[(\varphi_{x+e_i}-\varphi_x)^2].$$

The bound now follows from Lemma 4.6.1 (for $d=1$) and [54, Theorem 1.1] (for $d=2,3$). □

Observe that the process $(\psi_N(t))_{t\in\overline{D}}$ is Gaussian. Using Lemma 4.4.1 (1) it is easy to see that $(\psi_N(0))$ is tight. Again, using the properties of Gaussian laws, to show (2.2.2) it is enough to prove the following lemma.

Lemma 4.4.2. *There exists $C>0$ such that*

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(s)|^2\right] \le C\|t-s\|^{1+b} \tag{4.4.1}$$

for all $t,s\in\overline{D}$, uniformly in N, where $b=1$ in $d=1$, $b\in(0,1)$ in $d=2$ and $b=0$ in $d=3$.

Proof of Lemma 4.4.2. First we consider $d=1$. To show (4.4.1) we consider the following two cases.

(I) Suppose $t,s\in[x,x+1/N]$ for some $x\in N^{-1}D_N$. Then we have

$$\psi_N(t)-\psi_N(s) = (2d)^{-1}\sqrt{\kappa}N^{-\frac{3}{2}}\left[(Nt-Ns)(\varphi_{Nx+1}-\varphi_{Nx})\right].$$

Now using Lemma 4.4.1 (2) we get (4.4.1).

(II) Next suppose $s\in[x,x+1/N)$ and $t\in[y,y+1/N)$ for some $x,\,y\in N^{-1}D_N$ and $t>x+1/N$. In this case if $|t-s|\le 1/N$ then one can obtain (4.4.1) using (I) and a suitable point in between. So we assume $|t-s|>1/N$. We first note that

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(y)-\psi_N(x)|^2\right] = (2d)^{-2}\kappa N^{-3}\mathbf{E}_{\Lambda_N}[(\varphi_{Ny}-\varphi_{Nx})^2]$$

$$\le C\kappa N^{-3}\kappa^{-1}\mathbf{E}^{MM}_{\Lambda_N}[(\varphi_{Ny}-\varphi_{Nx})^2]$$
$$\le C(y-x)^2,$$

where we have used Lemma 4.6.1 to get the last inequality. Now using (I) we obtain

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(s)|^2\right] \le C\left(\mathbf{E}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(y)|^2\right]+\mathbf{E}_{\Lambda_N}\left[|\psi_N(y)-\psi_N(x)|^2\right]\right.$$
$$\left.+\mathbf{E}_{\Lambda_N}\left[|\psi_N(x)-\psi_N(s)|^2\right]\right) \le C|t-s|^2.$$

Next we consider $d=2$. We fix $b\in(0,\,1)$ and let $t,\,s\in\overline{D}$. We split the proof into a few cases.

Case 1: Suppose t,s belong to the same smallest square box in the lattice $\frac{1}{N}\mathbb{Z}^2$. First assume $\lfloor Nt\rfloor=\lfloor Ns\rfloor$, that is, the points are in the interior and not touching the top and right boundaries. In this case if we have $\{Nt_1\}\ge\{Nt_2\}$ and $\{Ns_1\}\ge\{Ns_2\}$. Then by definition of the interpolation we have

$$\psi_N(t)-\psi_N(s)=(2d)^{-1}\sqrt{\kappa}[(t_1-s_1)\left(\varphi_{\lfloor Nt\rfloor+e_1}-\varphi_{\lfloor Nt\rfloor}\right)$$
$$+(t_2-s_2)\left(\varphi_{\lfloor Nt\rfloor+e_1+e_2}-\varphi_{\lfloor Nt\rfloor+e_1}\right)].$$

So from the above expression we have

$$\mathbf{E}_{\Lambda_N}\left[(\psi_N(t)-\psi_N(s))^2\right]\le 2(2d)^{-2}\kappa[(t_1-s_1)^2\mathbf{E}_{\Lambda_N}[\left(\varphi_{\lfloor Nt\rfloor+e_1}-\varphi_{\lfloor Nt\rfloor}\right)^2]$$
$$+(t_2-s_2)^2\mathbf{E}_{\Lambda_N}[\left(\varphi_{\lfloor Nt\rfloor+e_1+e_2}-\varphi_{\lfloor Nt\rfloor+e_1}\right)^2]].$$

Now from Lemma 4.4.1 (2) and $|t_1-s_1|\,,|t_2-s_2|<N^{-1}$ we obtain (4.4.1). The argument is similar if one has $\{Nt_1\}\le\{Nt_2\}$ and $\{Ns_1\}\le\{Ns_2\}$.

Again if $\{Nt_1\}\ge\{Nt_2\}$ and $\{Ns_1\}<\{Ns_2\}$, or if $\{Nt_1\}<\{Nt_2\}$ and $\{Ns_1\}\ge\{Ns_2\}$ then we consider the point u on the line segment joining t and s such that Nu is the point of intersection of the line segment joining Nt,Ns and the diagonal joining $\lfloor Nt\rfloor,\lfloor Nt\rfloor+e_1+e_2$. Then we have using the above computations

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(s)|^2\right]\le 2\mathbf{E}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(u)|^2\right]+2\mathbf{E}_{\Lambda_N}\left[|\psi_N(u)-\psi_N(s)|^2\right]$$

$$\leq C\left[\|t-u\|^{1+b}+\|u-s\|^{1+b}\right] \leq C\|t-s\|^{1+b}.$$

Now the other case, that is, when $\lfloor Nt\rfloor \neq \lfloor Ns\rfloor$, follows from above by continuity.

Case 2: Suppose t, s do not belong to the same smallest square box in the lattice $\frac{1}{N}\mathbb{Z}^2$. In this case if $\|t-s\| \leq 1/N$ then one can obtain (4.4.1) by Case 1 and a suitable point in between. So we assume $\|t-s\| > 1/N$. Depending on whether Nt and Ns belong to the discrete lattice we split the proof in two broad cases.

Sub-case 2 (a) Suppose $t, s \in \frac{1}{N}\mathbb{Z}^2$. Then using Brascamp-Lieb inequality we obtain

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(s)|^2\right] \leq \kappa^{-1}\mathbf{E}^{MM}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(s)|^2\right].$$

Now from Sub-case 2(a) of the proof of Lemma 2.2.6 we obtain

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t)-\psi_N(s)|^2\right] \leq C\|t-s\|^{1+b}.$$

Sub-case 2 (b) Suppose at least one between t, s does not belong to $\frac{1}{N}\mathbb{Z}^2$. Then

$$\begin{aligned}
\mathbf{E}_{\Lambda_N}[\| \ \psi_N(t)-\psi_N(s)|^2\Big] &\leq 3\mathbf{E}_{\Lambda_N}\left[\left|\psi_N(t)-\psi_N\left(\frac{\lfloor Nt\rfloor}{N}\right)\right|^2\right] \\
&+ 3\mathbf{E}_{\Lambda_N}\left[\left|\psi_N\left(\frac{\lfloor Nt\rfloor}{N}\right)-\psi_N\left(\frac{\lfloor Ns\rfloor}{N}\right)\right|^2\right] + 3\mathbf{E}_{\Lambda_N}\left[\left|\psi_N\left(\frac{\lfloor Ns\rfloor}{N}\right)-\psi_N(s)\right|^2\right] \\
&\leq C\left[\left\|t-\frac{\lfloor Nt\rfloor}{N}\right\|^{1+b}+\left\|\frac{\lfloor Nt\rfloor}{N}-\frac{\lfloor Ns\rfloor}{N}\right\|^{1+b}+\left\|\frac{\lfloor Ns\rfloor}{N}-s\right\|^{1+b}\right] \leq C\|t-s\|^{1+b}.
\end{aligned}$$

Note that for the last inequality we have used our assumption $\|t-s\| > 1/N$.

Finally we consider $d = 3$. Let $t, s \in \overline{D}$. We split the proof into cases similar to those of $d = 2$. We give a brief description. For Case 1, suppose t, s belong to the same smallest cube in the lattice $\frac{1}{N}\mathbb{Z}^3$. First assume $\lfloor Nt\rfloor = \lfloor Ns\rfloor$. In this case if $\{Nt_1\} \geq \{Nt_2\} \geq \{Nt_3\}$ and $\{Ns_1\} \geq \{Ns_2\} \geq \{Ns_3\}$ then it follows from the definition of interpolation

$$\begin{aligned}
\mathbf{E}_{\Lambda_N}\left[(\psi_N(t)-\psi_N(s))^2\right] \leq 3(2d)^{-2}\kappa N[(t_1-s_1)^2&\mathbf{E}_{\Lambda_N}[(\varphi_{\lfloor Nt\rfloor+e_1}-\varphi_{\lfloor Nt\rfloor})^2] \\
&+(t_2-s_2)^2\mathbf{E}_{\Lambda_N}[(\varphi_{\lfloor Nt\rfloor+e_1+e_2}-\varphi_{\lfloor Nt\rfloor+e_1})^2] \\
&+(t_3-s_3)^2\mathbf{E}_{\Lambda_N}[(\varphi_{\lfloor Nt\rfloor+e_1+e_2+e_3}-\varphi_{\lfloor Nt\rfloor+e_1+e_2})^2]].
\end{aligned}$$

Now from Lemma 4.4.1 (2) and the fact that $|t_1 - s_1|, |t_2 - s_2|, |t_3 - s_3| < 1/N$ we have (4.4.1). Note that this is a particular case of t, s lying in the same tetrahedral portion of the cube. Hence if t, s lie in the same tetrahedral portion of the cube then by similar arguments (4.4.1) holds. If t, s do not lie in the same tetrahedral part then we consider points (at most 3) on the line segment joining them such that two consecutive between t, the selected points and s lie in the same tetrahedral part. Then applying the previous argument we can obtain (4.4.1). The case when $\lfloor Nt \rfloor \neq \lfloor Ns \rfloor$ follows by continuity. For Case 2, we describe Sub-case 2(a) which turns out to be simpler in $d = 3$. The rest of the argument is similar to that in $d = 2$. Suppose $t, s \in \frac{1}{N}\mathbb{Z}^3$ with $\|t - s\| > 1/N$. Then using Brascamp-Lieb inequality we obtain

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t) - \psi_N(s)|^2\right] \leq \kappa^{-1}\mathbf{E}^{MM}_{\Lambda_N}\left[|\psi_N(t) - \psi_N(s)|^2\right].$$

Now similarly as in the proof of Lemma 2.2.6 we obtain

$$\mathbf{E}_{\Lambda_N}\left[|\psi_N(t) - \psi_N(s)|^2\right] \leq C\|t - s\|.$$

□

To conclude the finite dimensional convergence we first show the convergence of the covariance matrix. For $x, y \in \overline{D} \cap N^{-1}\mathbb{Z}^d$ we define

$$G_{\frac{1}{N}}(x, y) := (2d)^{-2}\kappa N^{d-4} G_{\Lambda_N}(Nx, Ny).$$

We now interpolate $G_{\frac{1}{N}}$ in a piece-wise constant fashion on small squares of $\overline{D} \times \overline{D}$ to get a new function $G^I_{\frac{1}{N}}$. We show that $G^I_{\frac{1}{N}}$ converges uniformly to G_D on $\overline{D} \times \overline{D}$. Indeed, let $F_N := G^I_{\frac{1}{N}} - G_D$. Similarly as in the proof of the finite dimensional convergence in Theorem 4.2.7 (1) it follows that, for any f, $g \in C^\infty_c(D)$,

$$\lim_{N\to\infty} \sum_{x,y \in \frac{1}{N}D_N} N^{-2d} G^I_{\frac{1}{N}}(x, y) f(x) g(y) = \iint_{D\times D} G_D(x, y) f(x) g(y) \,\mathrm{d}\,x \,\mathrm{d}\,y.$$

Again from Riemann sum convergence we have

$$\lim_{N\to\infty} \sum_{x,y \in \frac{1}{N}D_N} N^{-2d} G_D(x, y) f(x) g(y) = \iint_{D\times D} G_D(x, y) f(x) g(y) \,\mathrm{d}\,x \,\mathrm{d}\,y.$$

Thus we get

$$\lim_{N\to\infty}\sum_{x,y\in\frac{1}{N}D_N} N^{-2d}F_N(x,y)f(x)g(y)=0. \tag{4.4.2}$$

Note that G_D is bounded and

$$\sup_{x,y\in\frac{1}{N}D_N}|G_{\Lambda_N}(Nx,Ny)|\le C\kappa^{-1}N^{4-d}.$$

These imply that

$$\sup_{x,y\in\overline{D}}|F_N(x,y)|\le C.$$

Now one can show that F_N has a subsequence converging uniformly to some function F which is bounded by C. With abuse of notation we denote this subsequence by F_N. We then have

$$\lim_{N\to\infty}\sum_{x,y\in\frac{1}{N}D_N} N^{-2d}F_N(x,y)f(x)g(y)=\iint_{D\times D}F(x,y)f(x)g(y)\,\mathrm{d}\,x\,\mathrm{d}\,y.$$

Uniqueness of the limit gives

$$\iint_{D\times D}F(x,y)f(x)g(y)\,\mathrm{d}\,x\,\mathrm{d}\,y=0$$

by (4.4.2). From this we obtain that $F(x,y)=0$ for almost every x and almost every y. The definition by interpolation of $G^I_{\frac{1}{N}}$ ensures that F is pointwise equal to zero. Finally, the fact that the original sequence F_N converges uniformly to zero follows using the subsequence argument.

We now show the finite dimensional convergence. First let $t\in\overline{D}$. We write

$$\psi_N(t)=\psi_{N,1}(t)+\psi_{N,2}(t)$$

where $\psi_{N,1}(t):=(2d)^{-1}\sqrt{\kappa}N^{\frac{d-4}{2}}\varphi_{\lfloor Nt\rfloor}$ and $\psi_{N,2}(t):=\psi_N(t)-\psi_{N,1}(t)$. From Lemma 4.4.1 (2) it follows that $\mathbf{E}_{\Lambda_N}[\psi_{N,2}(t)^2]$ goes to zero as N tends to infinity. Therefore to show that $\psi_N(t)\xrightarrow{d}\psi^{\Delta^2}_D(t)$ it is enough to show that $\mathbf{Var}[\Psi_{N,1}(t)]\to G_D(t,t)$. But we have

$$\mathbf{Var}[\psi_{N,1}(t)]=(2d)^{-2}\kappa N^{d-4}G_{\Lambda_N}\left(\lfloor Nt\rfloor,\,\lfloor Nt\rfloor\right)=G^I_{\frac{1}{N}}(t,t)\to G_D(t,t)$$

since the sequence F_N converges to zero uniformly. Since the variables under consideration are Gaussian, one can show the finite dimensional convergence using the convergence of the Green's functions. $\square$

Case 3: $\kappa \ll N^2$

In this case also we use Theorem 2.2.5 to show tightness. Using the Brascamp-Lieb inequality and an argument similar to the proof of Lemma 3.4.5 we obtain the following bounds in both cases.

Lemma 4.4.3. *We have*

$$G_{\Lambda_N}(x,\,x) \leq \mathbf{E}^{GFF}_{\Lambda_N}(\varphi_x^2) \leq CN \quad \text{for all } x \in \mathbb{Z}\,. \tag{4.4.3}$$

Also there exists $C > 0$ such that for all $x,\, y \in \mathbb{Z}$

$$\mathbf{E}_{\Lambda_N}[(\varphi_x - \varphi_y)^2] \leq \mathbf{E}^{GFF}_{\Lambda_N}[(\varphi_x - \varphi_y)^2] \leq C|y-x|. \tag{4.4.4}$$

where $\mathbf{P}^{GFF}_{\Lambda_N}$ denote the law of the discrete Gaussian free field on Λ_N with zero boundary conditions outside Λ_N.

Once we have these bounds, the rest of the proof is similar to that of the one-dimensional result in the $\kappa \gg N^2$ case. In the case of $\kappa \ll N^2$ we need the following additional information for the identification of the limit. The Green's function G_D for the problem

$$\begin{cases} -\frac{\mathrm{d}^2 u}{\mathrm{d}\, x^2}(x) = f(x) & x \in D \\ u(x) = 0 & x \in \partial D \end{cases}$$

is given by

$$G_D(x,y) = \min\{x,y\} - xy, \quad x,y \in \overline{D}$$

which also turns out to be the covariance function of the Brownian bridge. To avoid repetitions of the arguments, we skip the details of these cases.

4.5 Proof of Theorem 4.2.8

This section is devoted to proof of the error estimation result in Theorem 4.2.8. To estimate the error we need to develop some Sobolev inequalities in the general setting which involves consistency between discrete and continuous operator. The content of this section can be of independent interest and can possibly be applied to general interface models. We would like to stress that although we follow the ideas involved in [69], we cannot quote the results from there *verbatim* as the coefficients of the discrete operators do not depend on the scaling of the lattice. Also another important remark is that the discrete Dirichlet problem involving the operators L_h introduced in (4.2.9) requires boundary conditions on points outside R_h which are within distance 2 from R_h but the definition of the limiting operator $-\Delta_c$ involves only one boundary condition. The ideas from [69] work well when $L = \Delta_c^2$ or $L = -\Delta_c + \Delta_c^2$. In the case when $L = -\Delta_c$, we assign a cut-off which helps in controlling the error around the boundary. The proof of Theorem 4.2.8 (3) should be applicable to many other models.

4.5.1 Sobolev-type norm inequalities

The main aim of this subsection is to have an estimate on the ℓ^2 norm of a function on the grid in terms of the operator L_h (and its truncated version). Later this turns out to be useful as we use the convergence of L_h to L. We continue with all the definitions and notations from Section 4.2.3.

The notion of discrete forward and backward derivatives will be essential in the following arguments.

$$\begin{aligned}
\partial_j u(x) &:= \frac{1}{h}(u(x+he_j) - u(x)),\\
\bar{\partial}_j u(x) &:= \frac{1}{h}(u(x) - u(x-he_j)),\\
\partial^\alpha &:= \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d},\\
\bar{\partial}^\alpha &:= \bar{\partial}_1^{\alpha_1}\cdots\bar{\partial}_d^{\alpha_d},
\end{aligned}$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index. It is easy to see that

$$\langle \partial_j u\,,\, v\rangle_{h,\,grid} = \langle u\,,\, \bar{\partial}_j v\rangle_{h,\,grid}$$

for grid-functions vanishing outside a finite set. We now define

$$\|u\|_{h,m} := \left(\sum_{|\alpha| \le m} \|\partial^\alpha u\|^2_{h,\,grid} \right)^{\frac{1}{2}}$$

and obtain the following lemma.

Lemma 4.5.1. *There are constants C independent of u and h such that*

$$\|u\|_{h,\,grid} \le C\|\partial_j u\|_{h,\,grid}, \quad u \in \mathcal{D}_h, \; j = 1, \dots, d, \tag{4.5.1}$$

and for fixed $m \ge 1$,

$$\|u\|_{h,\,grid} \le C\|u\|_{h,m}, \quad u \in \mathcal{D}_h. \tag{4.5.2}$$

Proof. Since $u \equiv 0$ outside R_h, we have for $x \in R_h$

$$u(x) = -h \sum_{l=0}^{\infty} \partial_j u(x + lhe_j).$$

As D is bounded, the number of non-zero terms is $O(h^{-1})$. Hence by Cauchy-Schwarz inequality we have

$$u(x)^2 \le Ch \sum_{z \in \{x + lhe_j : -\infty < l < \infty\}} (\partial_j u(z))^2 .$$

Summing over $\{x + lhe_j : -\infty < l < \infty\}$ and using the fact that the number of non-zero terms is $O(h^{-1})$ we obtain

$$\sum_{z \in \{x + lhe_j : -\infty < l < \infty\}} u(z)^2 \le C \sum_{z \in \{x + lhe_j : -\infty < l < \infty\}} (\partial_j u(z))^2 .$$

Now we obtain (4.5.1) by summing over the remaining components of x and multiplying by h^d. Then (4.5.2) follows from (4.5.1). □

Our aim is to estimate the error while approximating the solution of the boundary value problem involving the continuum operator L by its discrete counter part. In this error estimation we face some obstacle near the boundary due to boundary condition issues. To overcome this obstacle we define a new operator $L_{h,m}$, where we suitably truncate and modify the operator L_h near the boundary. To use this operator we need to prove that $\|u\|_{h,\,m} \le C\|L_{h,m}u\|_{h,\,grid}$ for any function u vanishing outside R_h. In order to prove this inequality we need the following norm which rescales the function near the boundary:

$$|||u|||_{h,m} := \left(h^d \left(\sum_{x\in R_h^*} u(x)^2 + \sum_{x\in B_h^*} (h^{-m}u(x))^2 \right) \right)^{\frac{1}{2}}, \quad u \in \mathcal{D}_h.$$

We can relate the weighted Sobolev norm $|||\cdot|||_{h,m}$ to $||\cdot||_{h,m}$ with this bound:

Lemma 4.5.2. *Let $m = 1$ or 2. There is a constant C independent of u and h such that*

$$|||u|||_{h,m} \leq C\|u\|_{h,m}, \quad u \in \mathcal{D}_h.$$

Proof. For this lemma we use the following fact. There is a natural number K such that for all sufficiently small h, the following is valid: consider for any $x \in B_h^*$ all half-rays through x. At least one of them contains m consecutive grid-points outside R_h within distance Kh from x. This fact is easy to observe when $m = 1$. For $m = 2$ it is proved in Section 2.6. Let $x \in B_h^*$. We first consider the case when the half-ray in the x_1-direction contains, within distance Kh from x, m consecutive grid-points outside R_h. Let $x - (K_0 + 1)he_1$, where $K_0 + m \leq K$, be the first of the m consecutive points. It is then easy to see that

$$h^{-m}u(x) = \sum_{j=0}^{K_0} \binom{m+j-1}{j} \bar{\partial}_1^m u(x - jhe_1).$$

So

$$\left(h^{-m}u(x)\right)^2 \leq C(K_0+1) \sum_{j=0}^{K_0} \left(\bar{\partial}_1^m u(x - jhe_1)\right)^2 = C \sum_{j=0}^{K_0} \left(\partial_1^m u(x - (j+2)he_1)\right)^2.$$

Similar inequalities hold in the cases of the other half-rays where in the above $\bar{\partial}_1$ has to be replaced by the derivative in the direction of the corresponding half-ray. With this observation we obtain

$$h^d \sum_{x\in B_h^*} \left(h^{-m}u(x)\right)^2 \leq C\|u\|_{h,m}^2.$$

And by definition

$$h^d \sum_{x \in R_h^*} u(x)^2 \leq \|u\|_{h,m}^2.$$

This completes the proof. $\square$

We rewrite L_h in (4.2.9) as

$$L_h u(x) = h^{-2m} \sum_{\eta} c_\eta u(x + \eta h), \quad x \in h\,\mathbb{Z}^d, \tag{4.5.3}$$

where $\eta = (\eta_1, \ldots, \eta_d)$ with the η_j's being integers, c_η's are real numbers which may depend on h. In (4.5.3) $m = 1$ when $L = -\Delta_c$ and $m = 2$ when $L = \Delta_c^2$ or $-\Delta_c + \Delta_c^2$. We now define the characteristic polynomial of L_h by

$$p(\theta) := \sum_{\eta} c_\eta \mathrm{e}^{\iota \langle \eta\,,\,\theta \rangle}, \tag{4.5.4}$$

where $\theta = (\theta_1, \ldots, \theta_d)$ and $\langle \eta\,,\,\theta \rangle = \sum_{j=1}^d \eta_j \theta_j$. We have the following lemma:

Lemma 4.5.3.

$$\langle L_h u\,,\,u \rangle_{h,\,grid} = h^{d-2m} (2\pi)^{-d} \int_S p(\theta) |\hat{u}(\theta)|^2 \mathrm{d}\theta, \quad u \in \mathcal{D}_h.$$

where

$$\hat{u}(\theta) = \sum_{\xi \in \mathbb{Z}^d} u(\xi h) \mathrm{e}^{-\iota \langle \xi\,,\,\theta \rangle}$$

and $S = \{\theta : |\theta_j| \leq \pi,\ j = 1, \ldots, d\}$.

Proof. We expand

$$\begin{aligned}
\langle L_h u\,,\,u \rangle_{h,\,grid} &= h^d \sum_{x \in h\,\mathbb{Z}^d} L_h u(x) u(x) \\
&\overset{(4.5.3)}{=} h^{d-2m} \sum_{x \in h\,\mathbb{Z}^d} \sum_{\eta \in \mathbb{Z}^d} c_\eta u(x + \eta h) u(x) \\
&= h^{d-2m} \sum_{x,\,\xi \in h\,\mathbb{Z}^d} c_{\frac{\xi - x}{h}} u(\xi) u(x).
\end{aligned}$$

By inverting (4.5.4) we have

$$c_\eta = (2\pi)^{-d} \int_S p(\theta) \mathrm{e}^{-\iota\langle \eta, \theta\rangle} \mathrm{d}\theta.$$

Thus

$$\begin{aligned} \langle L_h u\,,\, u\rangle_{h,\,grid} &= h^{d-2m} \sum_{x,\,\xi\in h\,\mathbb{Z}^d} (2\pi)^{-d} \int_S p(\theta) \mathrm{e}^{-\iota\left\langle \frac{\xi-x}{h},\, \theta\right\rangle} \mathrm{d}\theta u(\xi) u(x) \\ &= h^{d-2m} (2\pi)^{-d} \int_S p(\theta) |\hat{u}(\theta)|^2 \mathrm{d}\theta. \end{aligned}$$

□

We will also need

Lemma 4.5.4. *There is a constant C independent of u and h such that*

$$\|u\|^2_{h,m} \le C \sum_{j=1}^d \|\partial_j^m u\|^2_{h,\,grid}, \quad u \in \mathcal{D}_h.$$

Proof. We first prove that if α is a multi-index with $|\alpha| = m$ then

$$\left\langle \bar{\partial}^\alpha \partial^\alpha u,\, u\right\rangle_{h,\,grid} \le \langle Q_h u,\, u\rangle_{h,\,grid}\,, \quad u \in \mathcal{D}_h, \tag{4.5.5}$$

where Q_h is the difference operator

$$Q_h u := \sum_{j=1}^d \bar{\partial}_j^m \partial_j^m u. \tag{4.5.6}$$

Similar to (4.5.4) we can show the characteristic polynomial of $\bar{\partial}^\alpha \partial^\alpha$ and Q_h are respectively

$$q_1(\theta) = 2^m \prod_{j=1}^d (1 - \cos\theta_j)^{\alpha_j}$$

and

$$q_2(\theta) = 2^m \sum_{j=1}^d (1 - \cos\theta_j)^m.$$

Now by the inequality between arithmetic and geometric mean we have

$$q_1(\theta) \le 2^m \sum_{j=1}^d m^{-1} \alpha_j (1 - \cos\theta_j)^m \le q_2(\theta).$$

Using Lemma 4.5.3 we obtain (4.5.5), which implies

$$\|\partial^{\alpha} u\|^2_{h,\,grid} \leq \sum_{j=1}^{d} \|\partial_j^m u\|^2_{h,\,grid}, \quad u \in \mathcal{D}_h.$$

For $|\alpha| < m$, one can show using Lemma 4.5.1

$$\|\partial^{\alpha} u\|^2_{h,\,grid} \leq C \sum_{j=1}^{d} \|\partial_j^m u\|^2_{h,\,grid}, \quad u \in \mathcal{D}_h.$$

Hence the proof is complete. □

4.5.2 Errors in the Dirichlet problem

We have shown some discrete Sobolev inequalities till now. We now relate these directly to our discrete operators. We start dealing with each of the operators separately. Before we do so let us show here the existence and uniqueness of the solution of the discrete boundary value problem (4.2.11)-(4.2.12).

Lemma 4.5.5. *The finite difference Dirichlet problem* (4.2.11)-(4.2.12) *has exactly one solution for arbitrary f.*

Proof. We first show the following. There exists a constant $C > 0$ independent of u and h such that

$$\|u\|_{h,\,grid} \leq C \|L_h u\|_{h,\,grid}, \quad u \in \mathcal{D}_h. \tag{4.5.7}$$

In case $L = \Delta_c^2$ or $-\Delta_c + \Delta_c^2$, (4.5.7) follows Lemma 4.5.1 and from the proof of Lemmas 4.5.6, 4.5.7 respectively. For $L = -\Delta_c$ the argument is similar once we observe that

$$\begin{aligned}
p(\theta) &= -\sum_{i=1}^{d} (2\cos\theta_i - 2) \\
&\quad + \frac{\rho_1(h)}{h^2} \sum_{i,j=1}^{d} [2\cos(\theta_i + \theta_j) + 2\cos(\theta_i - \theta_j) - 4(\cos\theta_i + \cos\theta_j) + 4] \\
&= \sum_{i=1}^{d} (2 - 2\cos\theta_i) + \frac{\rho_1(h)}{h^2} \sum_{i,j=1}^{d} [4(1 - \cos\theta_i)(1 - \cos\theta_j)] \\
&\geq 2 \sum_{i=1}^{d} (1 - \cos\theta_i).
\end{aligned}$$

Now since $u \equiv 0$ in B_h, Equation (4.2.11) can be considered as a linear system of equations with the same number of equations as of unknowns (the number of points in R_h). Therefore it is sufficient to prove that the corresponding homogeneous system has only the trivial solution i.e. $u \equiv 0$ in R_h. This follows from (4.5.7). □

Bilaplacian case: proof of Theorem 4.2.8 (1)

In this subsection we consider $L := \Delta_c^2$. Recall $\rho_2(h) \to 0$ and we have for $x \in h\mathbb{Z}^d$,

$$\begin{aligned}
L_h u(x) = \frac{1}{h^4}\Bigg[&-h^2\rho_2(h)\sum_{i=1}^{d}(u(x+he_i)+u(x-he_i)-2u(x)) \\
&+\sum_{i,j=1}^{d}\{u(x+h(e_i+e_j))+u(x-h(e_i+e_j))+u(x+h(e_i-e_j))+u(x-h(e_i-e_j)) \\
&-2(u(x+he_i)-2u(x-he_i)-2(u(x+he_j)-2u(x-he_j)+4u(x))\}\Bigg].
\end{aligned}$$

We define the operator $L_{h,2}$ as follows:

$$L_{h,2}f(x) = \begin{cases} L_h f(x) & x \in R_h^* \\ h^2 L_h f(x) & x \in B_h^* \\ 0 & x \notin R_h. \end{cases} \tag{4.5.8}$$

Then we have the following lemma involving $L_{h,2}$.

Lemma 4.5.6. *There exists a constant $C > 0$ independent of u and h such that*

$$\|u\|_{h,2} \le C\|L_{h,2}u\|_{h,grid}, \quad u \in \mathcal{D}_h.$$

Proof. We consider the characteristic polynomial of L_h and observe that

$$\begin{aligned}
p(\theta) = &-h^2\rho_2(h)\sum_{i=1}^{d}(2\cos\theta_i - 2) \\
&+\sum_{i,j=1}^{d}[2\cos(\theta_i+\theta_j)+2\cos(\theta_i-\theta_j)-4\cos\theta_i-4\cos\theta_j+4] \\
&= h^2\rho_2(h)\sum_{i=1}^{d}(2-2\cos\theta_i)+\sum_{i,j=1}^{d}[4(1-\cos\theta_i)(1-\cos\theta_j)]
\end{aligned}$$

$$\geq 4\sum_{i=1}^{d}(1-\cos\theta_i)^2.$$

Hence by Lemmas 4.5.4 and 4.5.3 we obtain for $u \in \mathcal{D}_h$

$$\|u\|_{h,2}^2 \leq C\sum_{j=1}^{d}\|\partial_j^2 u\|_{h,\,grid}^2 = C\,\langle Q_h u,\, u\rangle_{h,\,grid} \leq C\,\langle L_h u,\, u\rangle_{h,\,grid}\,,$$

where Q_h is the difference operator defined in (4.5.6) with $m = 2$. Again we have

$$\langle L_h u,\, u\rangle_{h,\,grid} = h^d\left[\sum_{x\in B_h^*} L_{h,\,2}u(x)\left(h^{-2}u(x)\right) + \sum_{x\in R_h^*} L_{h,\,2}u(x)u(x)\right]$$

Therefore by Cauchy-Schwarz inequality we have

$$|\,\langle L_h u,\, u\rangle_{h,\,grid}\,| \leq C\|L_{h,\,2}u\|_{h,\,grid}|||u|||_{h,\,2}.$$

Thus from Lemma 4.5.2 we have

$$\|u\|_{h,\,2}^2 \leq C\|L_{h,\,2}u\|_{h,\,grid}\,|||u|||_{h,\,2} \leq C\|L_{h,\,2}u\|_{h,\,grid}\,\|u\|_{h,\,2}$$

This completes the proof. □

We have now all the ingredients to show Theorem 4.2.8 (1).

Proof of Theorem 4.2.8 (1). We denote all constants by C and they do not depend on $u,\, f$. Using Taylor expansion we have for all $x \in R_h$ and for small h

$$L_h u(x) = h^{-2}\rho_2(h)\mathcal{R}_2(x) + Lu(x) + h^{-4}\mathcal{R}_5(x)$$

where $|\mathcal{R}_2(x)| \leq CM_2h^2$ and $|\mathcal{R}_5(x)| \leq CM_5h^5$. We thus obtain, for $x \in R_h$,

$$\begin{aligned} L_h e_h(x) &= L_h u(x) - L_h u_h(x) \\ &= h^{-2}\rho_2(h)\mathcal{R}_2(x) + h^{-4}\mathcal{R}_5(x). \end{aligned} \tag{4.5.9}$$

For $x \in R_h^*$ we have

$$L_{h,2}R_h e_h(x) = L_h R_h e_h(x) = L_h e_h(x) = h^{-2}\rho_2(h)\mathcal{R}_2(x) + h^{-4}\mathcal{R}_5(x).$$

For $x \in B_h^*$ at least one among $x \pm h(e_i \pm e_j)$, $x \pm he_i$ is in $B_h \setminus \partial D$. For any $y \in B_h \setminus \partial D$ we consider a point $b(y)$ on ∂D of minimal distance to y. Note that this distance is at most $2h$. Now using Taylor expansion and the fact that the value of u and all its first order derivatives are zero at $b(y)$ one sees that

$$u(y) = u_h(y) + \mathcal{R}_2^{'}(y)$$

where $|\mathcal{R}_2^{'}(y)| \leq CM_2h^2$. For $x \in B_h^*$ denote by $S(x)$ the neighbors of x which are in $B_h \setminus \partial D$ i.e.

$$S(x) = \{y : y \in B_h \setminus \partial D \cap \{x \pm he_i, x \pm h(e_i \pm e_j) : 1 \leq i, j \leq d\}\}.$$

Therefore, for $x \in B_h^*$,

$$\begin{aligned}
L_{h,2}R_he_h(x) &= h^2 L_h R_h e_h(x) \\
&= h^2 \left\{ L_h e_h(x) - h^{-4} \sum_{y \in S(x)} \left(h^2 \rho_2(h) C(y) e_h(y) + C^{'}(y) e_h(y) \right) \right\} \\
&\overset{(4.5.9)}{=} h^2 \{h^{-2}\rho_2(h)\mathcal{R}_2(x) + h^{-4}\mathcal{R}_5(x)\} + (C\rho_2(h) + C^{'}h^{-2})\mathcal{R}_2^{''}(x)
\end{aligned}$$

where $|\mathcal{R}_2^{''}(x)| \leq CM_2h^2$. Hence

$$\begin{aligned}
\|L_{h,2}R_he_h\|_{h,\,grid}^2 &= h^d \sum_{x \in R_h} (L_{h,2}R_he_h(x))^2 \\
&= h^d \left[\sum_{x \in R_h^*} (L_{h,2}R_he_h(x))^2 + \sum_{x \in B_h^*} (L_{h,2}R_he_h(x))^2 \right] \\
&= h^d \left[\sum_{x \in R_h^*} \left(h^{-2}\rho_2(h)\mathcal{R}_2(x) + h^{-4}\mathcal{R}_5(x) \right)^2 \right. \\
&\left. + \sum_{x \in B_h^*} \left(\rho_2(h)\mathcal{R}_2(x) + h^{-2}\mathcal{R}_5(x) + (C\rho_2(h) + C^{'}h^{-2})\mathcal{R}_2^{''}(x) \right)^2 \right] \\
&\leq Ch^d \left[\sum_{x \in R_h^*} \left(M_2^2(\rho_2(h))^2 + M_5^2h^2 \right) + \sum_{x \in B_h^*} \left(M_2^2h^4(\rho_2(h))^2 + M_5^2h^6 + M_2^2 \right) \right] \\
&\leq C\left[\left(M_2^2(\rho_2(h))^2 + M_5^2h^2 \right) + h\left(M_2^2h^4(\rho_2(h))^2 + M_5^2h^6 + M_2^2 \right) \right]
\end{aligned}$$

where the last inequality holds as the number of points in B_h^* is $O(h^{-(d-1)})$. Finally to complete our proof we obtain

$$
\begin{aligned}
\|R_h e_h\|^2_{h,\,grid} &\le C\left[M_2^2(\rho_2(h))^2 + M_5^2 h^2 + M_2^2 h^5(\rho_2(h))^2 + M_5^2 h^7 + M_2^2 h\right] \\
&\le C\left[M_5^2 h^2 + M_2^2(\rho_2(h))^2 + M_2^2 h\right].
\end{aligned}
$$

using Lemmas 4.5.1 and 4.5.6. □

Laplacian + Bilaplacian case: proof of Theorem 4.2.8 (2)

In this subsection we consider $L = -\Delta_c + \Delta_c^2$. Recall $\rho_3(h) \to 1$ and we have for $x \in h\mathbb{Z}^d$,

$$
\begin{aligned}
L_h u(x) = \frac{1}{h^4}\Bigg[&-h^2\sum_{i=1}^{d}(u(x+he_i)+u(x-he_i)-2u(x)) \\
&+\rho_3(h)\sum_{i,\,j=1}^{d}\{u(x+h(e_i+e_j))+u(x-h(e_i+e_j))+u(x+h(e_i-e_j)) \\
&+u(x-h(e_i-e_j))-2(u(x+he_i)-2u(x-he_i)-2(u(x+he_j) \\
&-2u(x-he_j)+4u(x))\}\Bigg].
\end{aligned}
$$

We define the operator $L_{h,2}$ as in (4.5.8) and obtain

Lemma 4.5.7. *There exists a constant $C > 0$ independent of u and h such that*

$$
\|u\|_{h,\,2} \le C\|L_{h,\,2}u\|_{h,\,grid}, \quad u \in \mathcal{D}_h.
$$

Proof. We observe that

$$
\begin{aligned}
p(\theta) &= -h^2\sum_{i=1}^{d}(2\cos\theta_i - 2) \\
&+ \rho_3(h)\sum_{i,\,j=1}^{d}\left[2\cos(\theta_i+\theta_j)+2\cos(\theta_i-\theta_j)-4\cos\theta_i-4\cos\theta_j+4\right] \\
&= h^2\sum_{i=1}^{d}(2-2\cos\theta_i)+\rho_3(h)\sum_{i,\,j=1}^{d}\left[4(1-\cos\theta_i)(1-\cos\theta_j)\right] \\
&\ge 4\rho_3(h)\sum_{i=1}^{d}(1-\cos\theta_i)^2.
\end{aligned}
$$

Hence by Lemma 4.5.4 and 4.5.3 we obtain for $u \in \mathcal{D}_h$

$$\begin{aligned}\|u\|_{h,2}^2 &\le C\sum_{j=1}^{d}\|\partial_j^2 u\|_{h,\,grid}^2 = C\,\langle Q_h u,\,u\rangle_{h,\,grid} \le C(\rho_3(h))^{-1}\,\langle L_h u\,,\,u\rangle_{h,\,grid}\\ &\le C\,\langle L_h u\,,\,u\rangle_{h,\,grid}\,,\end{aligned}$$

where Q_h is the difference operator defined in (4.5.6) with $m = 2$. The rest of the proof is similar to Lemma 4.5.6 and hence omitted. □

We now prove the approximation result in this case.

Proof of Theorem 4.2.8 (2). As before the constant C does not depend on u and f. Using Taylor expansion we have for all $x \in R_h$ and for small h

$$L_h u(x) = Lu(x) + (\rho_3(h) - 1)\Delta_c^2 u(x) + h^{-2}\mathcal{R}_4(x) + \rho_3(h)h^{-4}\mathcal{R}_5(x)$$

where $|\mathcal{R}_4(x)| \le CM_4h^4$, $|\mathcal{R}_5(x)| \le CM_5h^5$. We obtain for $x \in R_h$

$$\begin{aligned}L_h e_h(x) &= L_h u(x) - L_h u_h(x)\\ &= Lu(x) + (\rho_3(h) - 1)\Delta_c^2 u(x) + h^{-2}\mathcal{R}_4(x) + \rho_3(h)h^{-4}\mathcal{R}_5(x) - L_h u_h(x)\\ &= (\rho_3(h) - 1)\Delta_c^2 u(x) + h^{-2}\mathcal{R}_4(x) + \rho_3(h)h^{-4}\mathcal{R}_5(x).\end{aligned}$$

For $x \in R_h^*$ we have

$$L_{h,2}R_h e_h(x) = L_h R_h e_h(x) = L_h e_h(x) = (\rho_3(h) - 1)\Delta_c^2 u(x) + h^{-2}\mathcal{R}_4(x) + \rho_3(h)h^{-4}\mathcal{R}_5(x). \tag{4.5.10}$$

As in the case of Δ_c^2 we have for any $y \in B_h \setminus \partial D$

$$u(y) = u_h(y) + \mathcal{R}_2(y)$$

where $|\mathcal{R}_2(y)| \le CM_2h^2$. Therefore, for $x \in B_h^*$,

$$\begin{aligned}L_{h,2}R_h e_h(x) &= h^2 L_h R_h e_h(x)\\ &= h^2\left\{L_h e_h(x) - h^{-4}\sum_{y\in S(x)}\Big(h^2C(y)e_h(y) + \rho_3(h)C^{'}(y)e_h(y)\Big)\right\}\end{aligned}$$

$$\overset{(4.5.10)}{=} h^2(\rho_3(h)-1)\Delta_c^2 u(x) + \mathcal{R}_4(x) + \rho_3(h)h^{-2}\mathcal{R}_5(x) + C\mathcal{R}_2'(x) + Ch^{-2}\rho_3(h)\mathcal{R}_2''(x) \tag{4.5.11}$$

where $S(x)$ is defined similarly as in Δ_c^2 case, $C(y)$, $C'(y)$ are constants depending on y and $|\mathcal{R}_2'(x)| \le CM_2h^2$, $|\mathcal{R}_2''(x)| \le CM_2h^2$. We have

$$\begin{aligned}\|L_{h,2}R_he_h\|^2_{h,grid} &= h^d \sum_{x\in R_h}(L_{h,2}R_he_h(x))^2 \\ &= h^d\left[\sum_{x\in R_h^*}(L_{h,2}R_he_h(x))^2 + \sum_{x\in B_h^*}(L_{h,2}R_he_h(x))^2\right]\end{aligned}$$

which, using the bounds (4.5.10)-(4.5.11), turns into

$$\begin{aligned}\|L_{h,2}R_he_h\|^2_{h,grid} &\le Ch^d\sum_{x\in R_h^*}\left((\rho_3(h)-1)^2M_4^2 + M_4^2h^4 + (\rho_3(h))^2M_5^2h^2\right) \\ &+ Ch^d\sum_{x\in B_h^*}\left(h^4(\rho_3(h)-1)^2M_4^2 + M_4^2h^8 + (\rho_3(h))^2M_5^2h^6 + M_2^2 + M_2^2(\rho_3(h))^2h^4\right) \\ &\le C[(\rho_3(h)-1)^2M_4^2 + M_4^2h^4 + (\rho_3(h))^2M_5^2h^2 + h^5(\rho_3(h)-1)^2M_4^2 \\ &+ M_4^2h^9 + (\rho_3(h))^2M_5^2h^7 + M_2^2h + M_2^2(\rho_3(h))^2h^5]\end{aligned}$$

where in the last inequality we have used that the number of points in B_h^* is $O(h^{-(d-1)})$. Finally to complete our proof we obtain using Lemma 4.5.1 and Lemma 4.5.7

$$\begin{aligned}\|R_he_h\|^2_{h,\,grid} &\le C[(\rho_3(h)-1)^2M_4^2 + M_4^2h^4 + (\rho_3(h))^2M_5^2h^2 + h^5(\rho_3(h)-1)^2M_4^2 \\ &+ M_4^2h^9 + (\rho_3(h))^2M_5^2h^7 + M_2^2h + M_2^2(\rho_3(h))^2h^5] \\ &\le C\left[M_5^2h^2 + M_4^2(\rho_3(h)-1)^2 + M_4^2h^4 + M_2^2h\right].\end{aligned}$$

□

Laplacian case: proof of Theorem 4.2.8 (3)

In this subsection we consider $L = -\Delta_c$. The continuum problem (4.2.3) is defined with one boundary condition, whereas in the discrete Dirichlet problem involving L_h two boundary conditions are needed. The contribution of Δ_h^2 is negligible in the limit but for finite h it is not. It is the effect of $\rho_1(h)$ which makes L_h vanish in the limit. However, if we simply apply the same proof of Theorem 4.2.8 (1)-(2) in this case the boundary condition effect and the discretisation effect are treated simultaneously. To take care of the different scales at which these effects are seen, we use a suitable cutoff function instead of truncating the discrete operator L_h near the boundary. Let us first define the cutoff function. Recall that $\delta := \max\{h, \sqrt{\rho_1(h)}\}$. We define

$$D^{l\delta} := \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial D) < l\delta\}, \quad l = 1, 2, \dots$$

where $\operatorname{dist}(x, \partial D) = \inf\{\|x - y\| : y \in \partial D\}$. Then we have the following proposition which follows from Theorem 1.4.1 and equation (1.4.2) of Hörmander [43].

Lemma 4.5.8. *One can find $\phi \in C_c^\infty\left(\overline{D^{7\delta}}\right)$ with $0 \le \phi \le 1$ so that $\phi = 1$ on $\overline{D^{5\delta}}$ and*

$$\sup_{x \in \mathbb{R}^d} |D^\alpha \phi(x)| \le C_\alpha \delta^{-|\alpha|}, \tag{4.5.12}$$

where C_α depends on α and d.

We now define a function $g : \overline{D} \to R$ so that $g = \tilde{\phi} u$ where $\tilde{\phi}$ is the restriction of ϕ to $\overline{D}$. We will use the following bounds of g and its derivatives.

Lemma 4.5.9. *We have*

(i)

$$\sup_{x \in D} |g(x)| \le C M_1 \delta,$$

(ii)

$$\sum_{|\alpha| \le 1} \sup_{x \in D} |D^\alpha g(x)| \le C M_1,$$

(iii)

$$\sum_{|\alpha| \le 2} \sup_{x \in D} |D^\alpha g(x)| \le C(M_1 \delta^{-1} + M_2).$$

Here we recall that $M_k = \sum_{|\alpha| \le k} \sup_{x \in D} |D^\alpha u(x)|$.

Proof. We first observe that $g = 0$ on $D \setminus \overline{D^{7\delta}}$. For any x in $D \cap \overline{D^{7\delta}}$ we use Taylor's theorem and the fact that $u = 0$ on ∂D to obtain $|u(x)| \le C M_1 \delta$. The bounds now follows from the definition of g and (4.5.12). $\square$

We are now ready to prove Theorem 4.2.8 (3).

Proof of Theorem 4.2.8 (3). For our convenience we denote by $\|\cdot\|_{\ell^2(A)}$ the $\|\cdot\|_{h,\,grid}$ norm of the projection of any grid-function onto the finite subset A of $h\,\mathbb{Z}^d$. More precisely, for any finite subset A of $h\,\mathbb{Z}^d$ and function $v : h\,\mathbb{Z}^d \to \mathbb{R}$ we define

$$\|v\|^2_{\ell^2(A)} := h^d \sum_{x\in A} v(x)^2. \tag{4.5.13}$$

We extend u and g on $\mathbb{R}^d$ by defining their values to be zero outside $\overline{D}$. Also let us extend u_h by defining it to be zero on $h\,\mathbb{Z}^d \setminus D_h$. Note that $B_h \subset \overline{D} \cap \overline{D^{5\delta}}$. Thus by definition we have $e_h = u = g$ on B_h. Therefore from Lemma 4.5.1 we have

$$\begin{aligned}\|R_h e_h\|^2_{h,\,grid} &\le 2\|e_h - g\|^2_{\ell^2(R_h)} + 2\|g\|^2_{\ell^2(R_h)} \\ &\le C\|\nabla_h(e_h - g)\|^2_{\ell^2(R_h\cup\partial R_h)} + 2\|g\|^2_{\ell^2(R_h)}\end{aligned} \tag{4.5.14}$$

where

$$\nabla_h v(x) := (\partial_j v(x))_{j=1}^d,$$

$$\|\nabla_h v\|^2_{\ell^2(A)} := \sum_{j=1}^d \|\partial_j v\|^2_{\ell^2(A)},$$

and $\partial R_h := \{x \in h\,\mathbb{Z}^d \setminus R_h : \mathrm{dist}_{h\,\mathbb{Z}^d}(x,\, R_h) = 1\}$ with $\mathrm{dist}_{h\,\mathbb{Z}^d}$ being the graph distance in the lattice $h\,\mathbb{Z}^d$. We have for $x \in R_h$

$$L_h(e_h - g)(x) = L_h u(x) - f(x) - L_h g(x).$$

Thus

$$\langle L_h(e_h - g), e_h - g\rangle_{h,\,grid} = \langle L_h u - f, e_h - g\rangle_{h,\,grid} + \langle -L_h g, e_h - g\rangle_{h,\,grid}\,. \tag{4.5.15}$$

Using summation by parts we obtain

$$\langle L_h(e_h - g), e_h - g\rangle_{h,\,grid} = \|\nabla_h(e_h - g)\|^2_{\ell^2(R_h\cup\partial R_h)} + \rho_1(h)\|\Delta_h(e_h - g)\|^2_{\ell^2(R_h\cup\partial R_h)}. \tag{4.5.16}$$

For the first term in equation (4.5.15) we have, using Lemma 4.5.1,

$$|\,\langle L_h u - f, e_h - g\rangle_{h,\,grid}\,| \le \|L_h u - f\|_{\ell^2(R_h)}\|e_h - g\|_{\ell^2(R_h)}$$

$$\begin{aligned}&\leq C\|L_h u-f\|_{\ell^2(R_h)}\|\nabla_h(e_h-g)\|_{\ell^2(R_h\cup\partial R_h)}\\&\leq C\|L_h u-f\|^2_{\ell^2(R_h)}+\frac{1}{4}\|\nabla_h(e_h-g)\|^2_{\ell^2(R_h\cup\partial R_h)}.\end{aligned}\tag{4.5.17}$$

For the second term of equation (4.5.15) we obtain using integration by parts

$$\begin{aligned}|\langle -L_h g, e_h-g\rangle_{h,\,grid}| &\leq |\langle -\Delta_h g, e_h-g\rangle_{h,\,grid}|+\rho_1(h)|\langle \Delta_h^2 g, e_h-g\rangle_{h,\,grid}|\\&\leq |\langle \nabla_h g, \nabla_h(e_h-g)\rangle_{h,\,grid}|+\rho_1(h)|\langle \Delta_h g, \Delta_h(e_h-g)\rangle_{h,\,grid}|\\&\leq \|\nabla_h g\|^2_{\ell^2(R_h\cup\partial R_h)}+\frac{1}{4}\|\nabla_h(e_h-g)\|^2_{\ell^2(R_h\cup\partial R_h)}\\&+\rho_1(h)\|\Delta_h g\|^2_{\ell^2(R_h\cup\partial R_h)}+\rho_1(h)\|\Delta_h(e_h-g)\|^2_{\ell^2(R_h\cup\partial R_h)}.\end{aligned}\tag{4.5.18}$$

Combining (4.5.15), (4.5.16), (4.5.17) and (4.5.18) we get

$$\begin{aligned}\|\nabla_h(e_h-g)\|^2_{\ell^2(R_h\cup\partial R_h)}\leq C\|L_h u-f\|^2_{\ell^2(R_h)}+C\|\nabla_h g\|^2_{\ell^2(R_h\cup\partial R_h)}\\+C\rho_1(h)\|\Delta_h g\|^2_{\ell^2(R_h\cup\partial R_h)}.\end{aligned}$$

Substituting this in (4.5.14) we obtain

$$\begin{aligned}\|R_h e_h\|^2_{h,\,grid}\leq C\|L_h u-f\|^2_{\ell^2(R_h)}+C\|\nabla_h g\|^2_{\ell^2(R_h\cup\partial R_h)}\\+C\rho_1(h)\|\Delta_h g\|^2_{\ell^2(R_h\cup\partial R_h)}+2\|g\|^2_{\ell^2(R_h)}.\end{aligned}\tag{4.5.19}$$

We now bound each of the term in the right hand side of the inequality (4.5.19). Using Taylor expansion we have for all $x\in R_h$

$$L_h u(x)=Lu(x)+h^{-2}\mathcal{R}_4(x)+h^{-4}\rho_1(h)\mathcal{R}_4'(x)$$

where $|\mathcal{R}_4(x)|\leq CM_4h^4$ and $|\mathcal{R}_4'(x)|\leq CM_4h^4$. Now

$$\begin{aligned}\|L_h u-f\|^2_{\ell^2(R_h)}&\leq h^d\sum_{x\in R_h}(M_4^2h^4+M_4^2\rho_1(h)^2)\\&\leq CM_4^2\delta^4.\end{aligned}$$

For the second term of (4.5.19) we have the bound

$$
\begin{aligned}
\|\nabla_h g\|^2_{\ell^2(R_h\cup\partial R_h)} &= h^d \sum_{x\in(R_h\cup\partial R_h)\cap\overline{D^{8\delta}}} h^{-2}\sum_{i=1}^{d}(g(x+he_i)-g(x))^2 \\
&\le Ch^d \sum_{x\in(R_h\cup\partial R_h)\cap\overline{D^{8\delta}}} M_1^2 \\
&\le CM_1^2\delta
\end{aligned}
$$

where in the first inequality we used Taylor expansion and Lemma 4.5.9 and in the last inequality we used the fact that number of points in $(R_h\cup\partial R_h)\cap\overline{D^{8\delta}}$ is $O(\delta h^{-d})$. Similarly, for the third term using Taylor expansion, Lemma 4.5.9 and the fact that number of points in $(R_h\cup\partial R_h)\cap\overline{D^{8\delta}}$ is $O(\delta h^{-d})$ we have

$$
\begin{aligned}
\rho_1(h)\|\Delta_h g\|^2_{\ell^2(R_h\cup\partial R_h)} &= \rho_1(h)h^d \sum_{x\in(R_h\cup\partial R_h)\cap\overline{D^{8\delta}}} (\Delta_h g(x))^2 \\
&\le C\rho_1(h)h^d\delta h^{-d}(M_1\delta^{-1}+M_2)^2 \\
&\le C\left(M_1^2\sqrt{\rho_1(h)}+M_2^2\rho_1(h)\delta\right).
\end{aligned}
$$

Finally we obtain

$$
\begin{aligned}
\|g\|^2_{\ell^2(R_h)} &= h^d \sum_{x\in R_h\cap\overline{D^{7\delta}}} g(x)^2 \\
&\le Ch^d \sum_{x\in R_h\cap\overline{D^{7\delta}}} M_1^2\delta^2 \\
&\le CM_1^2\delta^3.
\end{aligned}
$$

Here in the first inequality we used Lemma 4.5.9 and in the last inequality we used the fact that number of points in $R_h\cap\overline{D^{7\delta}}$ is $O(\delta h^{-d})$. Combining all these bounds we obtain from (4.5.19)

$$
\begin{aligned}
\|R_h e_h\|^2_{h,grid} &\le C\left(M_4^2\delta^4+M_1^2\delta+M_1^2\sqrt{\rho_1(h)}+M_2^2\rho_1(h)\delta+M_1^2\delta^3\right) \\
&\le C\left(M_4^2\delta^4+M_2^2\rho_1(h)\delta+M_1^2\delta\right).
\end{aligned}
$$

□

4.6 Some supplementary details

In this section we give some details which are supplementary to this Chapter.

4.6.1 Covariance bound for MM in $d = 1$

In this section we consider $d = 1$ and the membrane model $(\varphi_x)_{x \in V_N}$ on $V_N = \{1, \ldots, N-1\}$ with zero boundary conditions outside V_N. We want to show the following bound:

Lemma 4.6.1. *There exists a constant $C > 0$ such that*

$$\mathbf{E}_{V_N}[(\varphi_x - \varphi_{x+1})^2] \leq CN, \quad x \in \mathbb{Z}.$$

Proof. Let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of i.i.d. standard Gaussian random variables. We define $\{Y_i\}_{i \in \mathbb{Z}^+}$ to be the associated random walk starting at 0, that is,

$$Y_0 = 0, \quad Y_n = \sum_{i=1}^{n} X_i, \ n \in \mathbb{N},$$

and $\{Z_i\}_{i \in \mathbb{Z}^+}$ to be the integrated random walk starting at 0, that is, $Z_0 = 0$ and for $n \in \mathbb{N}$

$$Z_n = \sum_{i=1}^{n} Y_i.$$

Then one can show that $\mathbf{P}_{V_N}$ is the law of the vector $(Z_1, \ldots, Z_{N-1})$ conditionally on $Z_N = Z_{N+1} = 0$ [20, Proposition 2.2]. So we have that

$$\mathbf{E}_{V_N}\left[(\varphi_{i+1} - \varphi_i)^2\right] = \mathbf{E}\left[(Z_{i+1} - Z_i)^2 | Z_N = Z_{N+1} = 0\right] = \mathbf{E}\left[Y_{i+1}^2 | Z_N = Z_{N+1} = 0\right].$$

Hence it is enough to find a bound for $\mathbf{E}[Y_i^2 | Z_N = Z_{N+1} = 0]$ for $i = 1, \ldots, N-1$. The covariance matrix Σ for $(Y_1, \ldots, Y_{N-1}, Z_N, Z_{N+1})$ can be partitioned as

$$\Sigma = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

where A is a $(N-1) \times (N-1)$ matrix with entries

$$A(i,j) = \mathbf{Cov}(Y_i, Y_j) = \min\{i, j\}.$$

$B(i,j)$ and $C(i,j)$ are $(N-1)\times 2$ and $2\times(N-1)$ matrices respectively, with $C=B^T$ and

$$B(i,j)=\mathbf{Cov}(Y_i,Z_{j+N-1})=\sum_{l=1}^{j+N-1}\min\{i,\,l\}.$$

Finally, D is a 2×2 matrix with

$$D(i,j)=\mathbf{Cov}(Z_{i+N-1},Z_{j+N-1}).$$

It easily follows that

$$D=\frac{1}{6}\begin{bmatrix} N(N+1)(2N+1) & N(N+1)(2N+4) \\ N(N+1)(2N+4) & (N+1)(N+2)(2N+3)\end{bmatrix}. \tag{4.6.1}$$

It is well known that $(Y_1,\ldots,Y_{N-1}|Z_N=Z_{N+1}=0)$ is a Gaussian vector with mean zero and covariance matrix given by $A-BD^{-1}C$. The inverse of D is as follows. Observe

$$\gamma_N:=\det(D)=\frac{1}{36}N(N+1)^2(8N^2+3N+6)$$

and

$$D^{-1}=\frac{1}{\gamma_N}\begin{bmatrix} D(2,2) & -D(1,2) \\ -D(2,1) & D(1,1)\end{bmatrix}$$

Now the diagonal element of $BD^{-1}C$ can be determined:

$$\begin{aligned}(BD^{-1}C)(i,i)=\frac{1}{\gamma_N}\Bigg[&\left(\sum_{l=1}^{N}\min\{i,\,l\}\right)^2 D(2,2)-\left(\sum_{l=1}^{N}\min\{i,\,l\}\right)\left(\sum_{l=1}^{N+1}\min\{i,\,l\}\right)D(1,2)\\ &-\left(\sum_{l=1}^{N}\min\{i,\,l\}\right)\left(\sum_{l=1}^{N+1}\min\{i,\,l\}\right)D(1,2)+\left(\sum_{l=1}^{N+1}\min\{i,\,l\}\right)^2 D(1,1)\Bigg].\end{aligned}$$

Plugging in the entries $D(i,j)$ from (4.6.1) and simplifying we get

$$(BD^{-1}C)(i,i)=\frac{i^2(N+1)}{24\gamma_N}\left[6N^2-12Ni+6i^2+4N\right]>0$$

This shows that for $i=1,2,\ldots,N-1$,

$$\mathbf{E}[Y_i^2|Z_N=Z_{N+1}=0]=A(i,i)-(BD^{-1}C)(i,i)<i.$$

Similar bound can be obtained for $\mathbf{E}[Y_N^2|Z_N = Z_{N+1} = 0]$ and this completes the proof. $\square$

4.6.2 Details on the space $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$

In this section we briefly describe few of the details regarding the space $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ and also about the spectral theory of $-\Delta_c + \Delta_c^2$. This is an elliptic operator, and the spectral theory is similar to that of either $-\Delta_c$ or Δ_c^2. First recall the standard Sobolev inner products on $H_0^1(D)$ and $H_0^2(D)$. They are

$$\langle u, v\rangle_1 = \int_D \nabla_c u \cdot \nabla_c v \,\mathrm{d}\, x, \quad u,\, v \in H_0^1(D)$$

and

$$\langle u, v\rangle_2 = \int_D \Delta_c u \Delta_c v \,\mathrm{d}\, x, \quad u,\, v \in H_0^2(D)$$

and they induce norms on $H_0^1(D)$ and $H_0^2(D)$ respectively which are equivalent to the standard Sobolev norms [39, Corollary 2.29]. We now consider the following inner product on $H_0^2(D)$:

$$\langle u, v\rangle_{mixed} := \int_D \nabla_c u \cdot \nabla_c v \,\mathrm{d}\, x + \int_D \Delta_c u \Delta_c v \,\mathrm{d}\, x,\ u, v \in H_0^2(D).$$

Clearly the norm induced by this inner product is equivalent to the norm $\|\cdot\|_{H_0^2}$ (by integration by parts). We consider $H^{-2}(D)$ to be the dual of $(H_0^2(D), \|\cdot\|_{mixed})$.

We now give some results whose proofs are similar to Theorem 2.3.2 and 2.3.3.

(i) There exists a bounded linear isometry

$$T_0 : H^{-2}(D) \to (H_0^2(D), \|\cdot\|_{mixed})$$

such that, for all $f \in H^{-2}(D)$ and for all $v \in H_0^2(D)$,

$$(f\,,\, v) = \langle v,\, T_0 f\rangle_{mixed}\,.$$

Moreover, the restriction T on $L^2(D)$ of the operator $i \circ T_0 : H^{-2}(D) \to L^2(D)$ is a compact and self-adjoint operator, where $i : (H_0^2(D), \|\cdot\|_{mixed}) \hookrightarrow L^2(D)$ is the inclusion map.

(ii) There exist $v_1, v_2, \dots$ in $(H_0^2(D), \|\cdot\|_{mixed})$ and numbers $0 < \mu_1 \le \mu_2 \le \cdots \to \infty$ such that

- $\{v_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $L^2(D)$,
- $Tv_j = \mu_j^{-1} v_j$,
- $\langle v_j, v\rangle_{mixed} = \mu_j \langle v_j, v\rangle_{L^2}$ for all $v \in H_0^2(D)$,
- $\{\mu_j^{-1/2} v_j\}$ is an orthonormal basis for $(H_0^2(D), \|\cdot\|_{mixed})$.

For each $j \in \mathbb{N}$ one has $v_j \in C^\infty(D)$. Moreover v_j is an eigenfunction of $-\Delta_c + \Delta_c^2$ with eigenvalue μ_j. Indeed, we have for all $v \in H_0^2(D)$

$$\left\langle(-\Delta_c + \Delta_c^2)v_j,\, v\right\rangle_{L^2} = \langle(-\Delta_c)v_j,\, v\rangle_{L^2} + \left\langle(\Delta_c^2)v_j,\, v\right\rangle_{L^2} \overset{GI}{=} \langle v_j,\, v\rangle_{mixed} = \mu_j \langle v_j,\, v\rangle_{L^2}$$

where "GI" stands for Green's first identity

$$\int_D u\Delta_c v \,\mathrm{d}\,V = -\int_D \nabla_c u \cdot \nabla_c v \,\mathrm{d}\,V + \int_{\partial D} u \nabla_c v \cdot \mathbf{n} \,\mathrm{d}\,S.$$

Thus v_j is an eigenfunction of $-\Delta_c + \Delta_c^2$ with eigenvalue μ_j in the weak sense. The smoothness of v_j follows from the fact that $-\Delta_c + \Delta_c^2$ is an elliptic operator with smooth coefficients and the elliptic regularity theorem [37, Theorem 9.26]. Hence v_j is an eigenfunction of $-\Delta_c + \Delta_c^2$ with eigenvalue μ_j.

As a consequence of the above, one easily has that

$$\|f\|_{mixed}^2 = \sum_{j\ge1} \mu_j \langle f, v_j\rangle_{L^2}^2 \tag{4.6.2}$$

for any $f \in H_0^2(D)$.

For any $v \in C_c^\infty(D)$ and for any $s > 0$ we define

$$\|v\|_{s,-\Delta+\Delta^2}^2 := \sum_{j\in\mathbb{N}} \mu_j^{s/2} \langle v, v_j\rangle_{L^2}^2 .$$

We define $\mathcal{H}^s_{-\Delta+\Delta^2,0}(D)$ to be the Hilbert space completion of $C_c^\infty(D)$ with respect to the norm $\|\cdot\|_{s,-\Delta+\Delta^2}$. Then $\left(\mathcal{H}^s_{-\Delta+\Delta^2,0}(D)\,,\, \|\cdot\|_{s,-\Delta+\Delta^2}\right)$ is a Hilbert space for all $s > 0$. Moreover, we also notice the following.

- Note that for $s = 2$ we have $\mathcal{H}^2_{-\Delta+\Delta^2,0}(D) = (H_0^2(D), \|\cdot\|_{mixed})$ by 4.6.2.

- $i : \mathcal{H}^s_{-\Delta+\Delta^2,0}(D) \hookrightarrow L^2(D)$ is a continuous embedding.

For $s > 0$ we define $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D) = (\mathcal{H}^s_{-\Delta+\Delta^2,0}(D))^*$, the dual space of $\mathcal{H}^s_{-\Delta+\Delta^2,0}(D)$. Then we have

$$\mathcal{H}^s_{-\Delta+\Delta^2,0}(D) \subseteq L^2(D) \subseteq \mathcal{H}^{-s}_{-\Delta+\Delta^2}(D).$$

One can show using the Riesz representation theorem that for $s > 0$, and $v \in L^2(D)$ the norm of $\mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ is given by

$$\|v\|^2_{-s,-\Delta+\Delta^2} := \sum_{j\in\mathbb{N}} \mu_j^{-s/2} \langle v,\, v_j\rangle^2_{L^2}\,.$$

Before we show the definition of the continuum mixed model, we need an analog of Weyl's law for the eigenvalues of the operator $-\Delta_c + \Delta_c^2$.

Proposition 4.6.2 ([4, Theorem 5.1], [57])**.** *There exists an explicit constant c such that, as $j \uparrow +\infty$,*

$$\mu_j \sim c^{-d/4} j^{4/d}.$$

Proof. We want to apply [4, Theorem 5.1] for $A := -\Delta_c + \Delta_c^2$. First note that A is an elliptic operator of order $m = 4$ defined on D having smooth coefficients. Let us consider $A_1 := (-\Delta_c + \Delta_c^2)|_{H^4(D)\cap H_0^2(D)}$. Clearly, $A_1 : H^4(D) \cap H_0^2(D) \to L^2(D)$ and also $C_c^\infty(D) \subset D(A_1) \subset H^4(D)$, where $D(A_1)$ is the domain of A_1. By elliptic regularity we have $D(A_1^p) \subset H^{4p}$, $p = 1, 2, \ldots$ We first show that A_1 is self-adjoint. Note that as $C_c^\infty(D) \subset D(A_1)$ and $C_c^\infty(D)$ is dense in $L^2(D)$, A_1 is densely defined. Again, by Green's identity we have for all $u, v \in H^4(D) \cap H_0^2(D)$

$$\left\langle(-\Delta_c + \Delta_c^2)u,\, v\right\rangle_{L^2} = \langle\nabla_c u,\, \nabla_c v\rangle_{L^2} + \langle\Delta_c u,\, \Delta_c v\rangle_{L^2} = \left\langle u,\, (-\Delta_c + \Delta_c^2)v\right\rangle_{L^2}.$$

Thus A_1 is symmetric. Also by [39, Corollary 2.21] we observe that image of A_1 is $L^2(D)$. The self-adjointness of A_1 now follows from [58, Theorem 13.11]. Also we conclude from [58, Theorem 13.9] that A_1 is closed. Now applying [4, Theorem 5.1] we get the asymptotic. □

The result we will prove now shows the well-posedness of the series expansion for $\Psi_D^{-\Delta+\Delta^2}$.

Proposition 4.6.3. *Let $(\xi_j)_{j\in\mathbb{N}}$ be a collection of i.i.d. standard Gaussian random variables. Set*

$$\Psi_D^{-\Delta+\Delta^2} := \sum_{j\in\mathbb{N}} \mu_j^{-1/2}\xi_j v_j.$$

Then $\Psi_D^{-\Delta+\Delta^2} \in \mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$ a.s. for all $s > (d-4)/2$.

Proof. Fix $s > (d-4)/2$. Clearly $v_j \in L^2(D) \subseteq \mathcal{H}^{-s}_{-\Delta+\Delta^2}(D)$. We need to show that $\|\Psi_D^{-\Delta+\Delta^2}\|_{-s,-\Delta+\Delta^2} < +\infty$ almost surely. Now this boils down to showing the finiteness of the random series

$$\|\Psi_D^{-\Delta+\Delta^2}\|^2_{-s,-\Delta+\Delta^2} = \sum_{j\geq 1} \mu_j^{-s/2}\left(\sum_{k\geq 1}\mu_k^{-1/2}u_k\xi_k\,, v_j\right)^2 = \sum_{j\geq 1}\mu_j^{-\frac{s}{2}-1}\xi_j^2$$

where the last equality is true since $(v_j)_{j\geq 1}$ form an orthonormal basis of $L^2(D)$. Observe that the assumptions of Kolmogorov's two-series theorem are satisfied: indeed using Proposition 4.6.2 one has

$$\sum_{j\geq 1}\mathbf{E}\left(\mu_j^{-\frac{s}{2}-1}\xi_j^2\right) \asymp \sum_{j\geq 1} j^{-\frac{4}{d}\left(\frac{s}{2}+1\right)} < +\infty$$

for $s > (d-4)/2$ and

$$\sum_{j\geq 1}\mathbf{Var}\left(\mu_j^{-\frac{s}{2}-1}\xi_j^2\right) \asymp \sum_{j\geq 1} j^{-\frac{4}{d}(s+2)} < +\infty$$

for $s > (d-8)/4$. The result then follows. ∎

4.6.3 Random walk representation of the $(\nabla+\Delta)$-model in $d = 1$ and estimates

In this section we recall some of the notations about the $d = 1$ case which were used in the heuristic explanations in the introduction of this chapter. We take advantage of the representation of the mixed model given in [14, Subsection 3.3.1] in our setting. To do that we set $\beta_N := 16\kappa_N$.

Let

$$\gamma = \left(\frac{1+\beta_N-\sqrt{1+2\beta_N}}{1+\beta_N+\sqrt{1+2\beta_N}}\right)^{1/2} \tag{4.6.3}$$

and let $(\varepsilon_i)_{i\in\mathbb{Z}^+}$ be i.i.d. $\mathcal{N}(0,\sigma^2)$ with

$$\sigma^2 = 4/(1+\beta_N+\sqrt{1+2\beta_N}). \tag{4.6.4}$$

Define

$$Y_n = \gamma^{n-1}\varepsilon_1 + \ldots + \gamma^0\varepsilon_n = \sum_{i=1}^n \gamma^{n-i}\varepsilon_i.$$

Let the integrated walk be denoted by

$$W_n = \sum_{i=1}^n Y_i = r_{n-1}\varepsilon_1 + \ldots + r_0\varepsilon_n = \sum_{i=1}^n r_{n-i}\varepsilon_i$$

where $r_{n-i} = \sum_{i=0}^{n-i}\gamma^i$.

We consider the case when $\kappa_N \to \infty$ and note that then $\gamma = \gamma_N \to 1$ and $\sigma_N^2 = \sigma^2 \to 0$. The following representation will give an idea on how the phase transition occurs in the mixed model:

$$W_n = \frac{1}{1-\gamma}(\varepsilon_1+\cdots+\varepsilon_n) - \frac{1}{1-\gamma}(\gamma^n\varepsilon_1+\gamma^{n-1}\varepsilon_2+\cdots+\gamma\varepsilon_n).$$

We recall the following proposition from [14, Proposition 1.10].

Proposition 4.6.4. *Let $\mathbf{P}_N(\cdot)$ be the mixed model with 0 boundary conditions. Then*

$$\mathbf{P}_N(\cdot) = \mathbf{P}\left((W_1,\ldots,W_{N-1}) \in \cdot | W_N = W_{N+1} = 0\right)$$

Let $(\widetilde{\varepsilon}_i)_{i\in\mathbb{Z}^+}$ be i.i.d. $\mathcal{N}\left(0, \frac{\sigma^2}{(1-\gamma)^2}\right)$. Then W_n can be written as

$$W_n = S_n - U_n$$

where $S_n = \sum_{k=1}^n \widetilde{\varepsilon}_k$ and $U_n = \gamma^n\widetilde{\varepsilon}_1 + \gamma^{n-1}\widetilde{\varepsilon}_2 + \cdots + \gamma\widetilde{\varepsilon}_n$. The conditional integrated random walk process has a representation, stated in [14, Proposition 3.7]. Let

$$\mathbf{P}\left((\widehat{W}_1,\ldots,\widehat{W}_{N-1}) \in \cdot\right) = \mathbf{P}\left((W_1,\ldots,W_{N-1}) \in \cdot | W_N = W_{N+1} = 0\right)$$

Then

$$\widehat{W}_k = W_k - W_N r_1(k) - W_{N+1} r_2(k)$$

where $r_1(k) = s_1(k)/r(k)$ and $r_2(k) = s_2(k)/r(k)$. The definitions of $r(k)$ and $s_i(k)$ for $i = 1, 2$ are as follows:

$$r(k) = (-1+\gamma)(-1+\gamma^{N+1})\left(-N+\gamma(2+N+\gamma^N(-2+(-1+\gamma)N))\right),$$
$$s_1(k) = (-k+\gamma(1-\gamma^k+k)) + \gamma^{3+2N+k}(1+\gamma^k(-1+(-1+\gamma)k))$$
$$+\gamma^{N-k}(\gamma^k(-\gamma+\gamma^3)(1-k+N)+\gamma^{2+2k}(2+N-\gamma(1+N))+\gamma(1+N-\gamma(2+N))),$$

and

$$s_2(k) = \gamma(\gamma^{1+k}+k-\gamma(1+k)) + \gamma^{2+2N-k}(-1+\gamma^k(1+k-\gamma k))$$
$$+\gamma^{1+N-k}(\gamma+\gamma^k(-1+\gamma^2)(k-N)-N+\gamma N+\gamma^{1+2k}(-1+(-1+\gamma)N)).$$

Let us consider the unconditional process W_n. Note that

$$\mathbf{Var}(S_n) = \frac{n\sigma^2}{(1-\gamma)^2}, \qquad \mathbf{Var}(U_n) = \frac{\sigma^2\gamma^2(1-\gamma^{2n})}{(1-\gamma)^2(1-\gamma^2)}$$

and

$$\mathbf{Cov}(S_n, U_n) = \frac{\gamma\sigma^2(1-\gamma^n)}{(1-\gamma)^2(1-\gamma)}.$$

So from here we have

$$\mathbf{Var}(W_n) = \frac{n\sigma^2}{(1-\gamma)^2} - \frac{\sigma^2\gamma^2(1-\gamma^n)^2}{(1-\gamma)^3(1+\gamma)} - \frac{2\sigma^2\gamma(1-\gamma^N)}{(1-\gamma)^3(1+\gamma)}. \tag{4.6.5}$$

From the above expressions one can show that $\mathbf{Var}(W_{N-1}) \sim N$ when $\kappa = \kappa_N \ll N^2$. We now derive the variance estimate when $\kappa \gg N^2$. For ease of writing, denote

$$\zeta = \frac{1}{\beta_N} + \sqrt{\frac{1}{\beta_N}}\sqrt{\frac{1}{\beta_N}+2} \to 0.$$

Furthermore $\gamma = 1/(1+\zeta)$ and $\sigma^2 = 2/\beta_N(1+\zeta)$. Rewriting (4.6.5) in terms of ζ we have

$$\begin{aligned}\mathbf{Var}(W_{N-1}) &= \frac{2(N-1)(1+\zeta)^2}{\zeta^2\beta_N(1+\zeta)} - \frac{2(1+\zeta)(1-(1+\zeta)^{-(N-1)})^2}{\beta_N\zeta^3(2+\zeta)} \\ &\quad - 4\frac{(1+\zeta)^2(1-(1+\zeta)^{-(N-1)})}{\beta_N\zeta^3(2+\zeta)} \\ &= \frac{2(1+\zeta)}{\beta_N(2+\zeta)\zeta^3}\Big[(N-1)(2+\zeta)\zeta - (1-(1+\zeta)^{-(N-1)})^2\end{aligned}$$

$$-2(1+\zeta)(1-(1+\zeta)^{-(N-1)})\Big]. \tag{4.6.6}$$

Using a Taylor series expansion of the fourth order for the second and third summands in (4.6.6) (since coefficients up to ζ^2 get cancelled) we obtain that

$$\mathbf{Var}(W_{N-1}) \approx \frac{(1+\zeta)N(N-1)^2}{\beta_N(2+\zeta)} \approx \frac{N^3}{\beta_N} \approx \frac{N^3}{\kappa_N}.$$

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List of Publications

(i) A. Cipriani, B. Dan, and R. S. Hazra. The scaling limit of the $(\nabla + \Delta)$-model. *arXiv preprint arXiv:1808.02676*, 2018.

(ii) A. Cipriani, B. Dan, and R. S. Hazra. Scaling limit of semiflexible polymers: a phase transition. *Communications in Mathematical Physics*, 377(2), 1505-1544, 2020. URL http://link.springer.com/article/10.1007/s00220-020-03762-9

(iii) A. Cipriani, B. Dan, and R. S. Hazra. The scaling limit of the membrane model. *The Annals of Probability*, 47(6):3963-4001, 2019. URL https://doi.org/10.1214/19-AOP1351