Mechanism Design In Sequencing Problems

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Contents

Chapter 1

Introduction

Collective decision making is an important social issue, since it depends on individual preferences that are not publicly observable. Therefore, the question is, whether it is possible to elicit the private information available to individuals and then how to extract the private information in various strategic environment; "Mechanism design" deals with these questions.

The difference between game theory and mechanism design is that, the former tries to predict the outcome of a strategic environment in some "equilibrium" but the latter tries to design or restrict the environment in such a way that the desired objective is attained, that is, the equilibrium outcome of that designed environment coincides with the objective of the designer. Note that, the message provided by the interacting agents may be quite abstract in nature but due to the famous"revelation principle" we restrict our attention to direct mechanism only.

In general, mechanism may or may not involve monetary payment to incentivize agents to reveal their private information. The voting environment is an example where monetary payment is not involved while designing a mechanism. A celebrated result in this environment is due to Gibbard (1973) and Satterthwaite (1975) where they show that the only unanimous and strategy-proof voting rule is dictatorial if there are at least three candidates or alternatives and the domain of preference is unrestricted. But if the domain of preference is quasi-linear then designing a mechanism involving money leads to positive outcome particularly in case of dominant strategy implementation. A few popular results in quasi-linear utility environment are due to Vickrey (1961), Clarke (1971) and Groves (1973) where the main course of discussion is the harmony of outcome efficient allocation along with dominant strategy implementation. Substantial part of this thesis deviates from outcome efficient allocation and resorts to other notions of allocation. While the essay in Chapter 2 finds the implication of Rawls' allocation, the essays in Chapter 3 and Chapter 4 deal with budget balanced affine cost minimizer rules and egalitarian allocation rules respectively. The notion of implementation used in all the three essays of this thesis is mainly strategy-proofness or dominant strategy incentive compatibility.

All the essays in this thesis are restricted to "sequencing" problem. There are vast amount of mechanism design literatures that address many important issues in this framework. Starting with Dolan (1978) this literature got enriched with the contribution of Suijs (1996), Mitra (2001), Mitra (2002), Hain and Mitra (2004) and many others. The general features of a sequencing problem are as follows: (1) There are n agents and a single server, (2) the server can provide services of non-identical processing length but can process only one particular service at a time. (3) Jobs may not be identical across agents, so their processing time may differ; we assume the processing time is common knowledge. (4) Waiting for the service is costly, monetary transfers are given to the agent to compensate them. (5) Agents have quasi-linear preferences over the positions in queue and monetary transfers. A real life example of a sequencing problem was given by Suijs (1996). He considered a large firm that has several divisions that need to have service facility provided by the maintenance and repairing unit of the firm. Since the maintenance and repairing unit can only serve one division at a time, when a number of divisions ask for service, each division has to incur a downtime cost. In order to minimize the total downtime cost, the firm has to use a true cost revelation mechanism since costs are private information to the corresponding units. Apart from the above example, we can have situations like a diagnostic center, installed with a machine (due to shortage of space) that can provide multiple services but can serve one agent at a time, where a certain number of enlisted patients visit for diagnosis or it can be software installation problem to PCs of a set of agents. All these examples capture the structure of sequencing problem.

Our structure of analysis is different from that of Dolan (1978) since the set of agents considered by Dolan is dynamic in nature; we follow the structure of Suijs (1996). We assume continuous time sequencing problem where the waiting cost perceived by the participating agent is linear in type (mostly) as well as in time. This assumption is, in fact, very crucial since linear time cost is necessary as well as sufficient for budget-balance [1] (see Suijs (1996), Mitra (2002)). With the quasi-linear utility function, participating agent's utility comprises of waiting cost in the queue and monetary transfer (may be received or paid by the agent). We will go into the details of the framework but at the very outset the results that we have in this thesis are the following.

> In the second chapter of the thesis, we identify the just sequencing rule that serves the agents in a non-increasing order of their waiting costs and prove that it is a Rawlsian rule. Further, with a particular kind of tie-breaking rule, we show that just sequencing rule weakly lexi-max cost dominates the outcome efficient sequencing rule. We also characterize the mechanism (ICJ mechanism) that implements the just sequencing rule. The other properties that we identify are the following: (1) just rule can be implemented with budget balanced transfers. (2) the generalized ICJ mechanism that ex-post implement the just sequencing rule and the budget balanced generalized ICJ mechanism.

> In third chapter, we have shown that the rules, for which any agent's job completion time is non-increasing in his/her own waiting cost, are implementable. We call such rules NI sequencing rules. We prove that any affine cost minimizer sequencing rule is an NI sequencing rule but the converse is not true. For two agent sequencing problems, we identify the complete class of NI sequencing rules that are implementable with balanced transfers. For sequencing problems with more than two agents, we identify a sufficient class of NI sequencing rules that are

[1] This result is in contrast with impossibility of budget-balance in case of pure public good problem (see Green and Laffont (1979)) where the nature of externality is severe compared to sequencing problem.

implementable with balanced transfers.

In the last chapter, we study strategy-proofness with each of the two fairness notions: egalitarian equivalence and identical preference lower bound. With a natural restriction on the reference bundle (specially the reference position in the bundle), we identify the complete class of mechanisms satisfying strategy-proofness, egalitarian equivalence and outcome efficiency. If we add either feasibility(or budget-balance) or weak group strategy-proofness then we get impossibility. Finally, for two agents, we characterize the entire class of mechanisms satisfying strategy-proofness, outcome efficiency and identical preference lower bound. For more than two agents, we provide an interesting sufficient class of mechanisms.

We briefly describe all the three essays in the thesis.

1.1 Incentives and justice for sequencing problems

Outcome efficiency in sequencing rule minimizes the aggregate job completion cost of agents. The algorithm to enforce outcome efficiency is to order the agents according to the decreasing values of their urgency index, that is, the ratio of agents waiting cost and processing time (see Smith (1956)), and allocate agents the queue position according to that order. Also, if the domain of private information (the type domain) is convex then outcome efficiency can be implemented in dominant strategies if and only if it is a VCG mechanism (see Vickrey (1961), Clarke (1971), Groves (1973)).

We deviate from outcome efficiency and focus on Rawlsian allocation based on John Rawls' principle (see Rawls (2009)) of distributive justice. It identifies the maximum agent specific job completion cost for each order of serving and picks up the order that minimizes the maximum cost. We Provide an algorithm to identify a rawlsian allocation and name this algorithm as just sequencing rule.

We then compare the outcome efficient rule and just sequencing rule in terms of completion cost vector and find that the just sequencing rule weakly lexi-max cost dominates outcome efficient sequencing rule. Clearly, just and outcome efficient sequencing rule has different objectives, hence we also provide the bound of relative efficiency loss.

Thereafter, we focus on the implementation issues of just sequencing rule by identifying the mechanism that implements just sequencing rule and call it ICJ mechanism. We further identify that subclass of ICJ mechanism that is budget balanced and compare ICJ and VCG mechanism in terms of utility vector but get ambiguous result even in the case of budget-balance.

We conclude this chapter with a nice property of just sequencing rule, that is, in case of multidimensional private information just sequencing rule can be ex-post implemented. But outcome efficiency lacks this property.

1.2 Balanced implementability of sequencing rules

What is the most general class of sequencing rule that can be implemented in dominant strategy? We begin our second chapter with this question. Although the answer lies in the existing literature on implementation (see Milgrom (2004), Myerson (1985), Bikhchandani et al. (2006)), we clarify the result in context of sequencing problem. We identify such rules (NI sequencing rules) and find the explicit transfer form that implements NI sequencing rule.

In the quasi-linear framework, Roberts'(Roberts (1979)) affine maximizer theorem holds in a multidimensional type space (unrestricted) for a finite set of agents with at least three alternatives. The appropriate transformation of affine maximizer allocation is the affine cost minimizer sequencing rule. We prove that any affine cost minimizer sequencing rule is an NI sequencing rule but the converse is not true.

In this chapter, our main focus is on implementability of NI sequencing rule with budget-balance. We completely identify the class of non-constant NI sequencing rule that are implementable with budget-balance when there are two agents. For more than two agents we identify a sufficient class of NI sequencing rule that are implementable with budget-balance. This class of NI sequencing rule is composed of a subset of non-affine cost minimizer and subset of affine cost minimizer allocation rule. We call that family of non affine cost minimizer allocation rule as group priority based cost minimizer (GPCM) sequencing rule.

1.3 Incentive and normative analysis on sequencing problems

Egalitarian equivalence, introduced by Pazner and Schmeidler (1978), is a normative concept that deals with equity, . The idea behind it, is the existence of a reference bundle containing reference position (in other words reference waiting time) and reference transfer such that such that every individuals are indifferent between his original bundle and the reference bundle.

We use this concept and focus on the compatibility issue of egalitarian equivalence with VCG mechanism. Our findings are more or less similar to that of Chun et al. (2014). The egalitarian equivalent allocation or the reference bundle is composed of reference position and reference transfer. While Chun et al. (2014) assumed the reference position can only take a few specific values, we generalize this idea; that is , the reference position is reference waiting time in our case and can take any positive value. With this change, we provide sufficient condition to achieve VCG mechanisms with egalitarian equivalence.

Apparently, we can achieve egalitarian equivalent VCG with various reference waiting cost functions that are non-constant. But they does not carry much sense in context of sequencing problems. For sequencing problems, it turns out to be more meaningful to consider only the last position as reference position, hence our reference time is now a specific constant value. We identify the class of egalitarian equivalent VCG mechanism with the above mentioned restriction. Also under this particular assumption we find that, feasibility is not possible along with VCG and egalitarian equivalent mechanism; as a result budget-balance is also impossible in this setup.

Next we focus on another normative criterion namely identical preference lower bound (IPLB), introduced by Moulin (1990), based on the idea that an agent's welfare is at least as that of consuming his/her equal share of resources. With the reference position fixed in the same way as described earlier, in case of two agents, we completely characterize the class of VCG mechanisms that satisfy egalitarian equivalence along with IPLB. For more than two agents, we identify the sufficient condition for implementing VCG mechanism with egalitarian equivalence and IPLB.

Chapter 2

Incentives and justice for sequencing problems

2.1 Introduction

In this chapter[1] we address the mechanism design issue for the sequencing problem in which agents have quasi-linear preferences. The setting comprises of a finite set of agents each of whom has one job to process using one facility. The facility can only handle one job at a time. No job can be interrupted once it starts processing. Each job is characterized by processing time and waiting cost. The latter represents the agent's disutility for waiting one unit of time. There is a well established literature in this direction (seeVan Den Brink et al. (2007), Dolan (1978), Duives et al. (2015), Hain and Mitra (2004), Mitra (2002), Moulin (2007) and Suijs (1996)).

A well-known and well studied concept is the outcome efficient sequencing rule that minimizes the aggregate job completion cost of the agents. Outcome efficiency, as pointed out by Smith (1956), requires that the jobs of the agents are processed in the non-increasing order of their urgency index. The urgency index of any agent is the ratio of his waiting cost and his processing time. It is well-known that, as long as preferences are "smoothly connected" (see Holmström (1979)), outcome efficient

[1]The similar version of this chapter is published in *Economic Theory*.

rules can be implemented in dominant strategies if and only if the mechanism is a Vickrey-Clarke-Groves (VCG) mechanism (see Clarke (1971), Groves (1973) and Vickrey (1961)). For the sequencing problem, outcome efficiency was analyzed by Dolan (1978), Mitra (2002) and Suijs (1996).

The main contribution of this chapter is to address the implementability issue of the Rawlsian sequencing rule. The Rawlsian sequencing rule is based on John Rawls' principle of distributive justice (see Rawls (2009)). From a planner's mechanism design perspective it is reasonable to think that the planner wants to devise a sequencing rule which is just by following Rawlsian difference principle (or maxi-min criterion) that requires that inequality across the agents is justified if it is beneficial for the least well off agent. Using this difference principle we define the Rawlsian sequencing rule. The Rawlsian sequencing rule first identifies the maximum agent specific job completion cost for each order of serving and then picks that order which minimizes this maximum agent specific job completion cost. We show that a sequencing rule for which agents are served in the non-increasing order of their waiting costs is a Rawlsian sequencing rule. We refer to this rule as the just sequencing rule.

There is a large literature on social welfare rankings of society's income that applies this Rawlsian difference principle (see Barbarà and Jackson (1988), d'Aspremont and Gevers (1977), Hammond (1976) and Moulin (1991) and Sen (2014)). The lexi-min and the family of k^{th}-rank dictator social orderings are all based on the Rawlsian difference principle and its extensions. Specifically, the k^{th}-rank dictator social orderings for $k = 1$ is the Rawlsian difference principle and it requires that, between two income vectors x and y for a society, income vector x dominates y if the minimum element in x is greater than the minimum element in y. The lexi-min social ordering is the lexicographic extension of the 1^{st}-rank dictator social ordering. Between two income vectors x and y for a society, x lexi-min dominates y if either the minimum element in x is greater than the minimum element in y or the minimum elements in x and y are equal but the second minimum element of x is greater than the second minimum element in y and so on. By taking appropriate tie-breaking rule and by taking completion cost vector of the agents, we show that the just sequencing rule weakly lexi-max cost dominates the outcome efficient sequencing rule, that is, for any profile of waiting cost vectors, either the maximum cost under the just sequencing rule is less than the maximum cost under the outcome efficient sequencing rule or the maximum cost under both rules are identical but the second highest cost under the just sequencing rule is less than the second highest cost under the outcome efficient sequencing rule and so on. Clearly, when we select the just sequencing rule there is an efficiency loss since for any profile, the aggregate cost under the just sequencing rule is no less than the aggregate cost under the outcome efficient rule. By looking at a notion of relative efficiency loss we show that the efficiency loss is bounded by $(n-1)$ where n is the total number of agents.

The just sequencing rule is an algorithm to achieve the Rawlsian objective of minimizing the maximum job completion cost. Typically, it is hard to find algorithms that achieve such min-max objectives. For example, consider the task allocation problem studied by Nisan and Ronen (2001) in the algorithmic mechanism design literature. The objective in Nisan and Ronen (2001) is to minimize the make-span of independent tasks on unrelated parallel machine (that is, to minimize the maximum completion time of all jobs assigned on all machines). However, this objective is NP-hard and hence the focus of Nisan and Ronen (2001) is to analyze schemes that are closely related to the make-span objective and can be achieved in polynomial time.[2]

We show that if agents have quasi-linear preferences and if the agents have private information about their respective waiting costs, then the just rule is implementable in dominant strategies. We also identify all mechanisms that implement the just sequencing rule. We call such mechanisms the incentive compatible just mechanisms or the ICJ mechanisms for short. Our result on implementation of the just sequencing rule shows compatibility of incentives and justice. In the mechanism design literature without transfers where preferences of the agents are defined using distance from the bliss points, Chichilnisky and Heal (1997) argued that Rawlsian rules are

[2]See Theorem 5.9. in Nisan and Ronen (2001)

"locally dictatorial" and hence implementable. However, in the mechanism design literature with transfers, this compatibility of incentives and justice is indeed rare. Conclusions of Deb et al. (2014) and Lavi et al. (2003) show that the Rawlsian allocation is incompatible with implementability in dominant strategies. A paper that shows this compatibility between incentives and justice is by Velez (2011) for the (house) allocation problems. Velez (2011) showed that the Generalized Money Rawlsian Fair solutions implements the no envy solution in Nash and Strong Nash equilibria. Thus, in Velez (2011), incentive compatibility is achieved in the Nash sense and not in dominant strategies sense like ours. In the algorithmic mechanism design literature, the task allocation problem in Nisan and Ronen (2001) also addresses the issue of truthful implementation in dominant strategies for schemes that are closely related to the make-span objective. However, the task allocation problem in Nisan and Ronen (2001) is significantly different from ours since in their problem, the machines have private information while in our problem agents (jobs) have private information.[3]

Consider the sequencing problem where the processing times of the agents is identical. Such situations are referred to as the queueing problem. Queueing problems have been analyzed extensively from both normative and strategic viewpoints (see Chun (2006), Chun et al. (2014), Hashimoto and Saitoh (2012), Kayı and Ramaekers (2010), Maniquet (2003), Mitra (2001), Mitra and Mutuswami (2011) and Mukherjee (2013)). For the queueing problem, the outcome efficient sequencing rule implies that the rule is also the just sequencing rule. However, for sequencing problems with non-identical processing time across agents, outcome efficient sequencing rule is different from the just sequencing rule and hence such sequencing problems bring out the trade-off between outcome efficient sequencing rule and the just sequencing rule. For sequencing problems that are not queueing problem, we have established that under complete information, the just sequencing rule lexi-max cost dominates the outcome efficient sequencing rule for an appropriate choice of tie breaking rules. However, we demonstrate that such unambiguous conclusion is not true in terms of lexi-min utility

[3]In our problem we have a single machine, implying that the make-span time is fixed throughout.

domination under asymmetric information when we compare the family of ICJ mechanisms with the family of VCG mechanisms.

The importance of finding balanced VCG mechanisms to implement outcome efficiency for canonical allocation problems was highlighted by Zhou (2007). However, for many economic environments implementing outcome efficiency with balanced transfers is difficult to achieve (see Hurwicz (1975), Hurwicz and Walker (1990) and Walker (1980)). For sequencing problem, it is known that we can have budget balanced (or first best) implementation with the outcome efficient sequencing rule (see Mitra (2002) and Suijs (1996)). We show that we can also find ICJ mechanisms that implement the just sequencing rule with balanced transfers and identify the set of all such balanced ICJ mechanisms. Again, for the queueing problem, the set of all balanced ICJ mechanisms coincide with the set of all balanced VCG mechanisms. The literature on balanced VCG mechanisms for the queueing problem includes the contributions of Chun et al. (2015), Kayı and Ramaekers (2010) and Mitra (2001). For the task allocation, problem Nisan and Ronen (2001) studies dominant strategy mechanisms with transfers satisfying a limited-budget restriction.[4] So Nisan and Ronen (2001) do not achieve budget balancedness for their task allocation problem.

The just sequencing rule is independent of the processing time of the agents. As a result, if we have a two-dimensional incentive problem, where waiting cost and processing time are private information, ex-post implementability of the just sequencing rule is possible. If processing times are private information, we have mechanism design problem under interdependent valuation, as the processing time generates interdependence across agents. Hence, the correct notion of implementation is ex-post implementation. Specifically, we show that the just sequencing rule is ex-post implementable by making some minor 'modification' in the ICJ mechanisms. Moreover, given the earlier results on implementability of the just sequencing rule with balanced ICJ mechanisms, it follows that ex-post implementability with balanced transfers is also a possibility. Jehiel et al. (2006) proved that the only deterministic social

[4]See Theorem 5.5 in Nisan and Ronen (2001) .

choice functions that are ex-post implementable in generic mechanism design frameworks with multidimensional signals and interdependent valuations are those rules for which the same alternative is chosen irrespective of agents' signals, that is, the outcome should be independent of the interdependent signals.

In sequencing with two-dimensional incentive problem, one dimension is waiting cost which is the private value and the other dimension is processing time that generates interdependence in terms of cost of completion time. The just sequencing rule is non-trivial in terms of the waiting cost or private value dimension and is independent of the interdependence inducing processing time (like the independence of the interdependent signal required by Jehiel et al. (2006) for ex-post implementability) and hence, the just sequencing rule is a non-trivial rule which is ex-post implementable. Moreover, for the outcome efficient sequencing rule, the profile contingent order is dependent on the processing time and hence it is not ex-post implementable under this two-dimensional incentive problem.[5]

This chapter is organized as follows. In Section 2, we provide the framework of the sequencing problems. In Section 3, we introduce and analyze the just sequencing rule and compare it with the outcome efficient sequencing rule. In Section 4, we address the implementability of the just sequencing rule. Section 5 deals with properties of the just sequencing rule. This is followed by the concluding section.

2.2 The framework

Consider a finite set of agents $N = \{1, 2, \dots, n\}$ in need of a facility that can be used sequentially. Using this facility, the agents want to process their jobs. The job processing time can be different for different agents. Specifically, for each agent $i \in N$, the job processing time is given by $s_i > 0$. Let $\mathbb{R}_{++}$ be the positive orthant of the real line $\mathbb{R}$ and let $\theta_i S_i$ measure the cost of job completion for agent $i \in N$ where $S_i \in \mathbb{R}_{++}$ is the

[5] Ex-post implementability literature includes contributions of Bergemann and Morris (2008), Bikhchandani (2006), Chung and Ely (2002), Jehiel et al. (2006), Jehiel and Moldovanu (2001), and, Fieseler et al. (2003). For the sequencing problem with private information only in processing time, incentive issues were addressed by Hain and Mitra (2004) and Moulin (2007).

job completion time for this agent and $\theta_i \in \Theta := \mathbb{R}_{++}$ denotes his constant per-period waiting cost. Due to the sequential nature of providing the service, the job completion time S_i for agent i depends not only on his own processing time s_i but also on the processing time of the agents who precede him in the order of service. By means of an order $\sigma = (\sigma_1, \ldots, \sigma_n)$ on N, one can describe the positions of each agent in the order. Specifically, $\sigma_i = k$ indicates that agent i has the k-th position in the order. Let $\Sigma(N)$ be the set of $n!$ possible orders on N. We define $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ to be the predecessor set of i in the order σ, that is, set of agents served before agent i in the order σ. Similarly, $P'_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ denotes the successor set of i in the order σ, that is, set of agents served after agent i in the order σ. Let $s = (s_1, \ldots, s_n) \in \mathcal{S} := \mathbb{R}^n_{++}$ denote the vector of processing time of the agents. Given a vector $s = (s_1, \ldots, s_n) \in \mathcal{S}$ and an order $\sigma \in \Sigma(N)$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the job completion time is $S_i(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. In the sum $S_i(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$, if $P_i(\theta) = \emptyset$, then we are assuming that $\sum_{j \in P_i(\sigma)} s_j = 0$. In general, we use the following convention on the summation operator: for any set $Y = \{X_1, \ldots, X_K\}$ and any $M \subseteq Y$, $\sum_{j \in M} X_j = 0$ if $M = \emptyset$. The agents have quasi-linear utility of the form $v_i(\sigma, \tau_i; m_i = (\theta_i, s_i); s_{-i}) = -\theta_i S_i(\sigma) + \tau_i$ where σ is the order, $\tau_i \in \mathbb{R}$ is the transfer that he receives and the parameters of the model are agents own parameter $m_i = (\theta_i, s_i)$ that consists of the waiting cost θ_i and the processing time s_i, and, more importantly, the processing time of the other agents that determines agent i's job completion time $S_i(\sigma)$. Specifically, given a commonly known job processing time vector $s = (s_1, \ldots, s_n) \in \mathcal{S}$ and an order $\sigma \in \Sigma(N)$, the utility of agent i, with just the waiting cost parameter θ_i, reduces to

$$v_i(\sigma, \tau_i; m_i = (\theta_i, s_i); s_{-i}) := U_i(\sigma, \tau_i; \theta_i) = -\theta_i S_i(\sigma) + \tau_i = -\theta_i(s_i + \sum_{j \in P_i(\sigma)} s_j) + \tau_i.$$

If we assume that both waiting cost and processing time are private information, then we have a *general sequencing problem* $\Omega = (N, \Theta^n, \mathcal{S})$ where N is the set of agents Θ^n is the domain of waiting cost of all agents assumed to be equal to $\mathbb{R}^n_{++}$ and $\mathcal{S}$ is the domain of processing time of all agents assumed to be equal to $\mathbb{R}^n_{++}$. In this context we associate the utility function $v_i(.)$ for each $i \in N$. If the processing time vector $s \in \mathcal{S}$ is given and waiting cost is private information, then we have a *sequencing problem* $\Omega(s) = (N, \Theta^n, s)$ and in that case the utility function reduces to $U_i(.)$ (from $v_i(.)$) for each $i \in N$. Except for Subsection 2.5.2, we will deal with $\Omega(s)$. Hence our first objective is to design direct revelation mechanisms for any given $\Omega(s)$.

For any set X, let $|X|$ denote the cardinality of X. A typical profile of waiting costs is denoted by $\theta = (\theta_1, \dots, \theta_n) \in \Theta^n$, and, for any $i \in N$, $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$ denotes the profile $(\theta_1 \dots \theta_{i-1}, \theta_{i+1}, \dots \theta_n)$ which is obtained from the profile θ by eliminating i's waiting cost. For a given sequencing problem $\Omega(s)$, a (direct revelation) mechanism is (σ, τ) that consists of a sequencing rule σ and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \to \Sigma(N)$ that specifies for each profile $\theta \in \Theta^n$ a unique order $\sigma(\theta) = (\sigma_1(\theta), \dots, \sigma_n(\theta)) \in \Sigma(N)$.[6] A *transfer rule* is a function $\tau : \Theta^n \to \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_i(\theta), \dots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, given any sequencing problem $\Omega(s)$ and given any mechanism (σ, τ), if (θ'_i, θ_{-i}) is the announced profile when the true waiting cost of i is θ_i, then utility of i is $U_i(\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}); \theta_i) = -\theta_i S_i(\sigma(\theta'_i, \theta_{-i}) + \tau_i(\theta'_i, \theta_{-i})$.

DEFINITION **2.1** A mechanism (σ, τ) *implements* the sequencing rule σ in dominant strategies if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i, \theta'_i \in \Theta$ and any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$,

$$U_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq U_i(\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}); \theta_i). \tag{2.1}$$

Implementation of a rule σ via a mechanism (σ, τ) requires that the transfer rule τ is such that truthful reporting for any agent weakly dominates false reporting irrespective of other agents' report.

[6]The sequencing rule is a function and not a correspondence. Hence, we will require tie-breaking rule to reduce a correspondence to a function. For reasons to be clarified later, we will use different tie breaking rules for different sequencing rules.

2.2.1 The outcome efficient sequencing rule

DEFINITION **2.2** A sequencing rule σ^* is *outcome efficient* if for any profile $\theta \in \Theta^n$, $\sigma^*(\theta) \in \text{argmin}_{\sigma\in\Sigma(N)} \sum_{i\in N} \theta_i S_i(\sigma)$.

For each profile the outcome efficient sequencing rule selects an order that minimizes the aggregate cost of completion time. Define $\mu_i := \theta_i/s_i$ as the urgency index of agent i which is the ratio of his waiting cost and his processing time. From Smith (1956) we know that for any sequencing problem $\Omega(s)$ a sequencing rule σ^* is outcome efficient if and only if for any profile θ, the selected order $\sigma^*(\theta)$ satisfies condition (OE): For any $i, j \in N$, $\theta_i/s_i \geq \theta_j/s_j \Leftrightarrow \sigma_i^*(\theta) \leq \sigma_j^*(\theta)$. Therefore, outcome efficient sequencing rule requires that the agents are ordered in the non-increasing order of their urgency index. From (OE) it is clear that if $\theta_i/s_i \geq \theta_j/s_j$, then $S_i(\sigma^*(\theta)) \leq S_j(\sigma^*(\theta))$. Consider the outcome efficient sequencing rule σ^*. For outcome efficiency we will use the following tie-breaking rule.

TB(OE): There is a linear order $\leq_{oe}$ on N and if $\theta_i/s_i = \theta_j/s_j$ and $i \leq_{oe} j$, then $i \in P_j(\sigma)$.

It is well-known that VCG mechanisms are the only mechanisms that implement σ^* (see Holmström (1979)).

DEFINITION **2.3** For the outcome efficient sequencing rule σ^*, a mechanism (σ^*, τ) is a *VCG mechanism* if the transfer rule is such that for all $\theta \in \Theta^n$ and all $i \in N$,

$$\tau_i^*(\theta) = h_i(\theta_{-i}) - s_i \sum_{j\in P_i'(\sigma^*(\theta))} \theta_j, \tag{2.2}$$

where the function $h_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ is arbitrary.

Given a sequencing problem $\Omega(s)$, for any profile $\theta \in \Theta^n$ and any $i \in N$ and any $j \in N \setminus \{i\}$, let $\theta_j s_i$ be the *pivotal cost* of agent i on agent j. We call this the pivotal cost because $\theta_j s_i$ is the incremental cost that agent j has to incur if agent i precedes agent j in any order. The VCG transfer in condition (2.2) specifies that for any $i \in N$ and any $\theta_{-i} \in \Theta^{N\setminus\{i\}}$, if θ_i is such that agent i is served last in the outcome efficient order $\sigma^*(\theta_i, \theta_{-i})$, that is, if $P_i'(\sigma^*(\theta_i, \theta_{-i})) = \emptyset$, then $\tau_i^*(\theta_i, \theta_{-i}) = h_i(\theta_{-i})$. If, however, θ_i' is such that agent i is not served last in the order $\sigma^*(\theta'_i, \theta_{-i})$, that is, if $P'_i(\sigma^*(\theta'_i, \theta_{-i}) \neq \emptyset$, then agent i's transfer $\tau^*_i(\theta'_i, \theta_{-i})$ not only has $h_i(\theta_{-i})$ but he also has to pay the sum of the pivotal cost that agent i incurs on his followers in the order $\sigma^*(\theta'_i, \theta_{-i})$ (that is, the waiting cost of all the agent(s) served after him times his own processing time).

The VCG transfer $\tau^*_i(\theta)$ (in condition (2.2)) for which the agent specific constant functions $h_i(.)$ are always zero for all agents gives us the pivotal mechanism for implementing the outcome efficient order σ^*. The first work that identified the pivotal mechanism for any sequencing problem is by Dolan (1978). It must be pointed out that the specification (2.2) of the VCG transfers is the pivotal based representation of the VCG transfers and is not its standard representation. We show that an appropriate transformation of the standard VCG transfers gives us (2.2). The standard way of specifying the VCG transfers is that for all θ and for all $i \in N$,

$$\tau^*_i(\theta) = -\sum_{j \in N \setminus \{i\}} S_j(\sigma^*(\theta))\theta_j + g_i(\theta_{-i}).^7 \tag{2.3}$$

Consider the outcome efficient order $\sigma^*(\theta)$ for the profile $\theta \in \Theta^n$ and suppose that agent i leaves. We define the "induced" order $\sigma^*(\theta_{-i})$ (of length $|N \setminus \{i\}|$) for the agents in $N \setminus \{i\}$ as follows:

$$\sigma^*_j(\theta_{-i}) = \begin{cases} \sigma^*_j(\theta) - 1 & \text{if } j \in P'_i(\sigma^*(\theta)), \\ \sigma^*_j(\theta) & \text{if } j \in P_i(\sigma^*(\theta)). \end{cases} \tag{2.4}$$

In words, $\sigma^*(\theta_{-i})$ is the order formed by removing agent i and moving all agents behind him up by one position. Given the same tie-breaking rule for the economy with $N \setminus \{i\}$ agents, it is easy to see that if $\sigma^*(\theta)$ is outcome efficient for the profile θ, then $\sigma^*(\theta_{-i})$ is also outcome efficient in $N \setminus \{i\}$ for the profile θ_{-i}. Without loss of generality, we can write for all θ and all $i \in N$, (A) $g_i(\theta_{-i}) = \sum_{j \in N \setminus \{i\}} S_j(\sigma^*(\theta_{-i}))\theta_j + h_i(\theta_{-i})$. By substituting (A) in (2.3) we get

[7] See Mitra (2002) and Suijs (1996).

$$\tau_i^*(\theta) = -\sum_{j \in N \setminus \{i\}} [S_j(\sigma^*(\theta)) - S_j(\sigma^*(\theta_{-i}))]\theta_j + h_i(\theta_{-i}). \tag{2.5}$$

If for a profile $\theta \in \Theta^n$, the outcome efficient order is $\sigma^*(\theta)$ and agent i leaves, then the order $\sigma^*(\theta_{-i})$ is such that if $j \in P_i(\sigma^*(\theta))$, then j's completion time remains unaltered and if $k \in P_i'(\sigma^*(\theta))$, then k's completion time reduces by s_i. Hence

$$S_j(\sigma^*(\theta)) - S_j(\sigma^*(\theta_{-i})) = \begin{cases} s_i & \text{if } j \in P_i'(\sigma^*(\theta)), \\ 0 & \text{if } j \in P_i(\sigma^*(\theta)). \end{cases} \tag{2.6}$$

By substituting condition (2.6) in the transformed VCG transfer (2.5) and then simplifying it we get the VCG transfer (2.2).[8]

2.3 The just sequencing rule

DEFINITION **2.4** A sequencing rule σ' is *Rawlsian* if for each $\theta \in \Theta^n$, $\sigma'(\theta) \in \min\limits_{\sigma \in \Sigma(N)} \max\limits_{j \in N} S_j(\sigma)\theta_j$.

Given any profile $\theta \in \Theta^n$, for each order $\sigma \in \Sigma(N)$, let $M(\sigma) = \theta_j S_j(\sigma) \geq \theta_k S_k(\sigma)$ for all $k \in N$, that is, for the given profile θ and given the order σ, $M(\sigma)$ is the maximum value of the cost of completion time among all agents in N. The Rawlsian sequencing rule picks that order $\sigma' \in \Sigma(N)$ for which $M(\sigma')$ is minimum, that is $M(\sigma') \leq M(\sigma)$ for all $\sigma \in \Sigma(N)$.

EXAMPLE **2.1** Consider the sequencing problem $\Omega(s)$ such that $N = \{1, 2, 3\}$ and $s = (s_1 = 1, s_2 = 2, s_3 = 3)$. Let the waiting cost vector be $\theta = (\theta_1 = 100, \theta_2 = 5, \theta_3 = 3)$. For θ, outcome efficiency uniquely picks the order $\sigma^1 = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ since $\mu_1 > \mu_2 > \mu_3$. However, the Rawlsian selection is not unique. For profile θ we have the following: $M(\sigma^1 = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)) = \theta_1 s_1 = 100$, $M(\sigma^2 = (\sigma_1 = 1, \sigma_2 = 3, \sigma_3 = 2)) = \theta_1 s_1 = 100$, $M(\sigma^3 = (\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)) = \theta_1(s_2 + s_1) = 300$,

[8] A similar argument for the pivotal based representation of VCG transfers (like condition (2.2)) for the queueing problem can be found in Chun et al. (2014).

$M(\sigma^4 = (\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 1)) = \theta_1(s_3 + s_1) = 400$, $M(\sigma^5 = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)) = \theta_1(s_2 + s_3 + s_1) = 600$, and $M(\sigma^6 = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)) = \theta_1(s_3 + s_2 + s_1) = 600$. Therefore, for the profile θ, the Rawlsian rule can either pick σ^1 or σ^2 implying that the Rawlsian rule does not guarantee state contingent unique order selection. Note that the order σ^1 also has the property that it serves the agents in the decreasing (hence non-increasing) order of their waiting cost, that is given $\theta_1 > \theta_2 > \theta_3$, agent 1 is served first followed by agent 2 and then by agent 3.

Let the waiting cost vector be $\theta' = (\theta'_1 = 10, \theta_2 = 5, \theta_3 = 3)$. For profile θ' we have the following: $M(\sigma^1 = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)) = \theta_3(s_1 + s_2 + s_3) = 18$, $M(\sigma^2 = (\sigma_1 = 1, \sigma_2 = 3, \sigma_3 = 2)) = \theta_2(s_1 + s_3 + s_2) = 30$, $M(\sigma^3 = (\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)) = \theta'_1(s_2 + s_1) = 30$, $M(\sigma^4 = (\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 1)) = \theta'_1(s_3 + s_1) = 40$, $M(\sigma^5 = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)) = \theta'_1(s_2 + s_3 + s_1) = 60$, and $M(\sigma^6 = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)) = \theta'_1(s_3 + s_2 + s_1) = 60$. For the profile θ', $M(\sigma^1) < M(\sigma^k)$ for all $k \in \{2, \ldots, 6\}$ and hence any Rawlsian rule uniquely picks σ^1.

The above example suggests that we can have profiles for which more than one order is Rawlsian. It also seems likely that serving the agents in the non-increasing order of their waiting costs is always Rawlsian. In the next theorem we show that this is indeed the case.

DEFINITION **2.5** A sequencing rule $\tilde{\sigma}$ is called the just sequencing rule if for each profile $\theta \in \Theta^n$, the chosen order $\tilde{\sigma}(\theta)$ satisfies the following property: for any $i, j \in N$ such that $\theta_i \geq \theta_j$, $\tilde{\sigma}_i(\theta) \leq \tilde{\sigma}_j(\theta)$.

We use the following tie-breaking rule for the just sequencing rule $\tilde{\sigma}$.
TB(J): There is a linear order $\leq_r$ on N such that if $\theta_i = \theta_j$ and either $\theta_i/s_i > \theta_j/s_j$ or $\theta_i/s_i = \theta_j/s_j$ and $i \leq_r j$, then $i \in P_j(\sigma)$.

THEOREM **2.1** For any $\Omega(s)$, the just sequencing rule $\tilde{\sigma}$ is Rawlsian.

Proof: To prove that the just sequencing rule $\tilde{\sigma}$ is Rawlsian, consider any profile $\theta \in \Theta^n$ and the just order $\tilde{\sigma}(\theta)$. Consider that agent $i \in N$ for whom the cost of completion time $\theta_i S_i(\tilde{\sigma}(\theta)) = \theta_i(\sum_{j \in P_i(\tilde{\sigma}(\theta))} s_j + s_i)$ is maximum under $\tilde{\sigma}(\theta)$, that is $M(\tilde{\sigma}(\theta)) = \theta_i S_i(\tilde{\sigma}(\theta))$.[9] Define $O := P_i(\tilde{\sigma}(\theta)) \cup \{i\}$ as the set that includes the set of predecessors of i under the order $\tilde{\sigma}(\theta)$ and that also includes agent i. From the definition of just sequencing rule, $\theta_j \geq \theta_i$ for all $j \in O$. Consider any other order $\sigma \in \Sigma(N) \setminus \{\tilde{\sigma}(\theta)\}$. For this order σ, there is one agent in O who will be served last under σ relative to the other members of O, that is, there exists an agent $j \in O$ such that $\sigma_j > \sigma_k$ for all $k \in O \setminus \{j\}$. This means that $O \subseteq P_j(\sigma) \cup \{j\}$, and hence, $S_j(\sigma) \geq S_i(\tilde{\sigma}(\theta))$, that is, the completion time of agent j under the order σ is not less than the completion time of agent i under the order $\tilde{\sigma}(\theta)$. Therefore, the maximum cost of completion time $M(\sigma)$ under σ is at least as large as the maximum cost of completion time $M(\tilde{\sigma}(\theta)) = \theta_i S_i(\tilde{\sigma}(\theta))$ under $\tilde{\sigma}(\theta)$ since $M(\sigma) \geq \theta_j S_j(\sigma) \geq \theta_j S_i(\tilde{\sigma}(\theta)) \geq \theta_i S_i(\tilde{\sigma}(\theta)) = M(\tilde{\sigma}(\theta))$. Since the selection of σ was arbitrary, the result follows. ■

REMARK **2.1** Consider any sequencing problem $\Omega(s^*)$ with $s^* = (s_1^*, \ldots, s_n^*) \in \mathcal{S}$ and $s_1^* = \ldots = s_n^*$ so that we have the queueing problem. Then the outcome efficient sequencing rule implies the just sequencing rule and hence a Rawlsian sequencing rule. For the queueing problem $\Omega(s^*)$, for any profile $\theta \in \Theta^n$, the order of the urgency indexes and that of the waiting costs are identical and hence this implication.

2.3.1 Lexi-max domination and a bound on the efficiency loss

Consider any sequencing problem $\Omega(s)$. For any profile $\theta \in \Theta^n$ and any order $\sigma \in \Sigma(N)$, let $C(\sigma) = (C_1(\sigma), \ldots, C_n(\sigma)) \in \mathbb{R}^n_{++}$ be the agent specific completion cost vector so that $C_i(\sigma) = \theta_i S_i(\sigma)$ for all $i \in N$. For any $\sigma \in \Sigma(N)$ and $C(\sigma)$, define $C^*(\sigma) = (C_1^*(\sigma), \ldots, C_n^*(\sigma))$ as the reordering of $C(\sigma)$ such that $C_1^*(\sigma) \geq \ldots \geq C_n^*(\sigma)$.

DEFINITION **2.6** Consider any sequencing problem $\Omega(s)$ and let σ' and σ'' be two sequencing rules. We say that the sequencing rule σ' *weakly lexi-max dominates* the sequencing rule σ'' if for each $\theta \in \Theta^n$, either $C_1^*(\sigma'(\theta)) < C_1^*(\sigma''(\theta))$, or there exists a $k \in \{2, \ldots, n\}$ such that $C_r^*(\sigma'(\theta)) = C_r^*(\sigma''(\theta))$ for all $r = 1, \ldots, k-1$, and, $C_k^*(\sigma'(\theta)) < C_k^*(\sigma''(\theta))$, or the vector $C^*(\sigma'(\theta)) = C^*(\sigma''(\theta))$.

[9] If there are more than one agent for whom the cost is the maximum, pick any one of them arbitrarily.

From the definitions of outcome efficient and just sequencing rules, it is clear that for any $\theta \in \Theta^n$, $C_1^*(\tilde{\sigma}(\theta)) \leq C_1^*(\sigma^*(\theta))$ and $\sum_{r=1}^n C_r^*(\sigma^*(\theta)) \leq \sum_{r=1}^n C_r^*(\tilde{\sigma}(\theta))$.

THEOREM **2.2** If the linear orders on N for the tie-breaking rules TB(OE) and TB(J) are identical, then the just sequencing rule $\tilde{\sigma}$ weakly lexi-max dominates the outcome efficient sequencing rule σ^*.

Proof: From Theorem 2.1, it follows that for any $\theta \in \Theta^N$, $C_1^*(\tilde{\sigma}(\theta)) \leq C_1^*(\sigma)$ for any $\sigma \in \Sigma(N)$. Hence, for any $\theta \in \Theta^N$, $C_1^*(\tilde{\sigma}(\theta)) \leq C_1^*(\sigma^*(\theta))$. Consider any $\theta \in \Theta^n$ and let $i(1) < i(2) < \ldots < i(n)$ be the ordering of the agent set N induced by $C^*(\tilde{\sigma}(\theta))$, that is, $\theta_{i(r)} S_{i(r)}(\tilde{\sigma}(\theta)) \geq \theta_{i(r+1)} S_{i(r+1)}(\tilde{\sigma}(\theta))$ for all $r = 1, \ldots, n-1$ so that $C_r^*(\tilde{\sigma}(\theta)) = \theta_{i(r)} S_{i(r)}(\tilde{\sigma}(\theta))$ for all $r = 1, \ldots, n$.

Step 1: If for some $\theta \in \Theta^n$ and some $q = 1, \ldots, n$, $C_r^*(\tilde{\sigma}(\theta)) = C_r^*(\sigma^*(\theta))$ for all $r = 1, \ldots, q$, then $P_{i(r)}(\tilde{\sigma}(\theta)) = P_{i(r)}(\sigma^*(\theta))$ for all $r = 1, \ldots, q$.

Proof of Step 1: We prove this step by applying induction on q. First we prove that Step 1 is true for $q = 1$, that is, we show that if for some $\theta \in \Theta^n$, $C_1^*(\tilde{\sigma}(\theta)) = C_1^*(\sigma^*(\theta))$, then $P_{i(1)}(\tilde{\sigma}(\theta)) = P_{i(1)}(\sigma^*(\theta))$.

We first show that if for some $\theta \in \Theta^n$, $C_1^*(\tilde{\sigma}(\theta)) = \theta_{i(1)} S_{i(1)}(\tilde{\sigma}(\theta))$, $C_1^*(\tilde{\sigma}(\theta)) = C_1^*(\sigma^*(\theta))$ and $k \in P'_{i(1)}(\tilde{\sigma}(\theta))$, then $k \in P'_{i(1)}(\sigma^*(\theta))$. Assume to the contrary that there exists $k \in P'_{i(1)}(\tilde{\sigma}(\theta))$ such that $k \in P_{i(1)}(\sigma^*(\theta))$. Define $T := P_{i(1)}(\tilde{\sigma}(\theta)) \cup \{i(1)\}$ and consider $j \in T$ such that $S_j(\sigma^*(\theta)) = \max_{r \in T}\{S_r(\sigma^*(\theta))\}$. Clearly, $S_j(\sigma^*(\theta)) > S_{i(1)}(\tilde{\sigma}(\theta))$ since $k \in P_{i(1)}(\sigma^*(\theta))$ implies that $T \cup \{k\} \subseteq P_j(\sigma^*(\theta)) \cup \{j\}$. Moreover, either $j = i(1)$, or $j \neq i(1)$ and $j \in P_{i(1)}(\tilde{\sigma}(\theta))$, and in either case, $\theta_j \geq \theta_{i(1)}$. Therefore, $C_1^*(\tilde{\sigma}(\theta)) = \theta_{i(1)} S_{i(1)}(\tilde{\sigma}(\theta)) < \theta_j S_j(\sigma^*(\theta))$ which contradicts $C_1^*(\tilde{\sigma}(\theta)) = C_1^*(\sigma^*(\theta))$. Hence, if $k \in P'_{i(1)}(\tilde{\sigma}(\theta))$, then $k \in P'_{i(1)}(\sigma^*(\theta))$. To complete this proof, we show that if $k \in P_{i(1)}(\tilde{\sigma}(\theta))$, then $k \in P_{i(1)}(\sigma^*(\theta))$. If there exists $k \in P_{i(1)}(\tilde{\sigma}(\theta))$ such that $k \in P'_{i(1)}(\sigma^*(\theta))$, then consider $j \in T = P_{i(1)}(\tilde{\sigma}(\theta)) \cup \{i(1)\}$ such that $S_j(\sigma^*(\theta)) = \max_{r \in T}\{S_r(\sigma^*(\theta))\}$ so that $j \in P'_{i(1)}(\sigma^*(\theta))$, $S_j(\sigma^*(\theta)) \geq S_k(\sigma^*(\theta))$ and the equality holds only if $j = k$. Clearly, $j \neq i(1)$ (since $j \in P_{i(1)}(\tilde{\sigma}(\theta)) \cap P'_{i(1)}(\sigma^*(\theta))$), $\theta_j \geq \theta_{i(1)}$ (since $j \in P_{i(1)}(\tilde{\sigma}(\theta))$) and $S_j(\sigma^*(\theta)) \geq S_{i(1)}(\tilde{\sigma}(\theta))$. Hence, $\theta_j S_j(\sigma^*(\theta)) \geq$

$\theta_{i(1)} S_{i(1)}(\tilde{\sigma}(\theta))$. If $\theta_j > \theta_{i(1)}$, then $C_1^*(\tilde{\sigma}(\theta)) = \theta_{i(1)} S_{i(1)}(\tilde{\sigma}(\theta)) < \theta_j S_j(\sigma^*(\theta))$, which contradicts $C_1^*(\tilde{\sigma}(\theta)) = C_1^*(\sigma^*(\theta))$. If (A) $\theta_j = \theta_{i(1)}$, then, given TB(J) and $j \in P_{i(1)}(\tilde{\sigma}(\theta))$, either (B) $\theta_j / s_j > \theta_{i(1)}/s_{i(1)}$, or (C) $\theta_j/s_j = \theta_{i(1)}/s_{i(1)}$ and $j <_r i(1)$. If (B) holds, that is, if $\theta_j/s_j > \theta_{i(1)}/s_{i(1)}$, then we have a contradiction to $j \in P'_{i(1)}(\sigma^*(\theta))$. If (C) holds, that is, if $\theta_j/s_j = \theta_{i(1)}/s_{i(1)}$ and $j <_r i(1)$, then, given that the linear orders on N for the tie-breaking rules TB(OE) and TB(J) are identical, we have a contradiction to $j \in P'_{i(1)}(\sigma^*(\theta))$ since, given $j <_r i(1)$, we must also have $j <_{oe} i(1)$. Hence, we have established that if $k \in P_{i(1)}(\tilde{\sigma}(\theta))$, then $k \in P_{i(1)}(\sigma^*(\theta))$.

Assume that Step 1 is true for $q = 1, \ldots, m-1$. We then show that if Step 1 is true for $q = m$, that is, if $C_r^*(\tilde{\sigma}(\theta)) = C_r^*(\sigma^*(\theta))$ and $P_{i(r)}(\tilde{\sigma}(\theta)) = P_{i(r)}(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$, then $C_m^*(\tilde{\sigma}(\theta)) = C_m^*(\sigma^*(\theta))$ implies $P_{i(m)}(\tilde{\sigma}(\theta)) = P_{i(m)}(\sigma^*(\theta))$.

Let $C_m^*(\tilde{\sigma}(\theta)) = \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta)) = C_m^*(\sigma^*(\theta))$. Given $P_{i(r)}(\tilde{\sigma}(\theta)) = P_{i(r)}(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$, consider the set of agents $A(m-1) = \{i(1), \ldots, i(m-1)\}$ for whom the completion time is identical for both $\tilde{\sigma}(\theta)$ and $\sigma^*(\theta)$. Let $i(a) \in A(m-1)$ be such that $i(m) \notin P_{i(a)}(\tilde{\sigma}(\theta))$, and there does not exists $j \in A(m-1) \setminus \{i(a)\}$ such that $i(m) \notin P_j(\tilde{\sigma}(\theta))$ and $|P_j(\tilde{\sigma}(\theta))| > |P_{i(a)}(\tilde{\sigma}(\theta))|$. Therefore, $P_{i(a)}(\tilde{\sigma}(\theta))$ has the highest cardinality in comparison to the predecessor sets associated with the agents in $A(m-1)$ that do not include $i(m)$. Similarly, let $i(b) \in A(m-1)$ be such that $i(m) \notin P'_{i(b)}(\tilde{\sigma}(\theta))$ and there does not exists $j \in A(m-1) \setminus \{i(b)\}$ such that $i(m) \notin P'_j(\tilde{\sigma}(\theta))$ and $|P'_j(\tilde{\sigma}(\theta))| > |P'_{i(b)}(\tilde{\sigma}(\theta))|$. Thus, $P'_{i(b)}(\tilde{\sigma}(\theta))$ has the highest cardinality in comparison to the successor sets associated with the agents in $A(m-1)$ that do not include $i(m)$. Note that if $i(a)$ does not exist, then $i(b)$ exists, and, if $i(b)$ does not exist, then $i(a)$ exists. Define $S^* := T_1 \cap T_2 \subset N$, where

$$T_1 = \begin{cases} N \setminus [P_{i(a)}(\tilde{\sigma}(\theta)) \cup \{i(a)\} & \text{if } i(a) \text{ exists} \\ N & \text{if } i(a) \text{ does not exist,} \end{cases}$$

and

$$T_2 = \begin{cases} N \setminus [P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\} & \text{if } i(b) \text{ exists} \\ N & \text{if } i(b) \text{ does not exist.} \end{cases}$$

Observe that S^* is always nonempty since $i(m) \in S^*$. Moreover, $S^* \neq N$ since either $T_1 \neq N$ or $T_2 \neq N$. Given $P_{i(r)}(\tilde{\sigma}(\theta)) = P_{i(r)}(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$, if $|S^*| = 1$, then $P_{i(m)}(\tilde{\sigma}(\theta)) = P_{i(m)}(\sigma^*(\theta))$ and the proof is complete. Assume that $|S^*| > 1$. We first show that if $k \in P'_{i(m)}(\tilde{\sigma}(\theta))$, then $k \in P'_{i(m)}(\sigma^*(\theta))$. Suppose to the contrary that there exists $k \in P'_{i(m)}(\tilde{\sigma}(\theta))$ such that $k \in P_{i(m)}(\sigma^*(\theta))$. Observe that $k \in P'_{i(m)}(\tilde{\sigma}(\theta))$ implies $k \notin P_{i(a)}(\tilde{\sigma}(\theta)) \cup \{i(a)\} = P_{i(a)}(\sigma^*(\theta)) \cup \{i(a)\}$ and $k \in P_{i(m)}(\sigma^*(\theta))$ implies $k \notin P'_{i(b)}(\sigma^*(\theta)) \cup \{i(b)\} = P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\}$. In case either $i(a)$ or $i(b)$ does not exist, the above statements remain vacuously true. Hence, $k \in S^*$. Consider $j \in T := P_{i(m)}(\tilde{\sigma}(\theta)) \cup \{i(m)\}$ with $S_j(\sigma^*(\theta)) = \max_{w \in T}\{S_w(\sigma^*(\theta))\}$. Note that $j \notin N \setminus S^*$ since $S_j(\sigma^*(\theta)) > S_{j'}(\sigma^*(\theta))$ for all $j' \in P_{i(a)}(\sigma^*(\theta)) \cup \{i(a)\}$ and $j \notin P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\}$ since $\{P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\}\} \cap \{P_{i(m)}(\tilde{\sigma}(\theta)) \cup \{i(m)\}\} = \emptyset$. Therefore, $j \in S^*$. Note that $S_{i(m)}(\sigma^*(\theta)) > S_k(\sigma^*(\theta))$ since $k \in P_{i(m)}(\sigma^*(\theta))$. It is also clear that $S_j(\sigma^*(\theta)) \geq S_{i(m)}(\sigma^*(\theta)) > S_k(\sigma^*(\theta))$ implies $j \in P'_k(\sigma^*(\theta))$. Hence, $T \cup \{k\} \subseteq P_j(\sigma^*(\theta)) \cup \{j\}$. So, $S_j(\sigma^*(\theta)) \geq s_k + S_{i(m)}(\tilde{\sigma}(\theta)) > S_{i(m)}(\tilde{\sigma}(\theta))$. Moreover, $\theta_j \geq \theta_{i(m)}$. Consequently, $C^*_m(\tilde{\sigma}(\theta)) = \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta)) < \theta_j S_j(\sigma^*(\theta))$ and $j \in S^*$, which contradicts $C^*_m(\tilde{\sigma}(\theta)) = C^*_m(\sigma^*(\theta))$. Hence, if $k \in P'_{i(m)}(\tilde{\sigma}(\theta))$, then $k \in P'_{i(m)}(\sigma^*(\theta))$. To complete the proof, we show that if $k \in P_{i(m)}(\tilde{\sigma}(\theta))$, then $k \in P_{i(m)}(\sigma^*(\theta))$. Suppose to the contrary that there exists $k \in P_{i(m)}(\tilde{\sigma}(\theta))$ such that $k \in P'_{i(m)}(\sigma^*(\theta))$. Again, if either $i(a)$ or $i(b)$ does not exist, the statements remain vacuously true. Observe that $k \in P_{i(m)}(\tilde{\sigma}(\theta))$ implies $k \notin P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\} = P'_{i(b)}(\sigma^*(\theta)) \cup \{i(b)\}$ (if $i(b)$ exists) and $k \in P'_{i(m)}(\sigma^*(\theta))$ implies $k \notin P_{i(a)}(\sigma^*(\theta)) \cup \{i(a)\} = P_{i(a)}(\tilde{\sigma}(\theta)) \cup \{i(a)\}$ (if $i(a)$ exists). Hence, $k \in S^*$. Let $t \in T = P_{i(m)}(\tilde{\sigma}(\theta)) \cup \{i(m)\}$ be such that $S_t(\sigma^*(\theta)) = \max_{q \in T}\{S_q(\sigma^*(\theta))\}$. So, $t \notin P_{i(a)}(\tilde{\sigma}(\theta)) \cup \{i(a)\}$ (if $i(a)$ exists). Also $t \notin P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\}$ (if $i(b)$ exists) since $\{P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\}\} \cap \{P_{i(m)}(\tilde{\sigma}(\theta)) \cup \{i(m)\}\} = \emptyset$. Thus, $t \in S^*$. Observe that $S_t(\sigma^*(\theta)) \geq S_k(\sigma^*(\theta)) > S_{i(m)}(\sigma^*(\theta))$, which implies $t \in P'_{i(m)}(\sigma^*(\theta))$. Moreover, $\theta_t \geq \theta_{i(m)}$ since $t \in P_{i(m)}(\tilde{\sigma}(\theta))$. Also note that $S_t(\sigma^*(\theta)) \geq S_{i(m)}(\tilde{\sigma}(\theta))$. Hence, $\theta_t S_t(\sigma^*(\theta)) \geq \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta))$. If $\theta_t > \theta_{i(m)}$, then $C^*_m(\tilde{\sigma}(\theta)) = \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta)) < \theta_t S_t(\sigma^*(\theta))$ and $t \in S^*$ which contradicts $C^*_m(\tilde{\sigma}(\theta)) = C^*_m(\sigma^*(\theta))$. If (A) $\theta_t = \theta_{i(m)}$, then given TE(J) and $t \in P_{i(m)}(\tilde{\sigma}(\theta))$, either (B) $\theta_t/s_t > \theta_{i(m)}/s_{i(m)}$, or (B) $\theta_t/s_t = \theta_{i(m)}/s_{i(m)}$ and $t <_r i(m)$. If (B) holds, that is, $\theta_t/s_t > \theta_{i(m)}/s_{i(m)}$, then we have a contradiction to $t \in P'_{i(m)}(\sigma^*(\theta))$. If (C) holds, that is, $\theta_t/s_t = \theta_{i(m)}/s_{i(m)}$ and $t <_r i(m)$, then, given that the linear orders on N for the tie-breaking rules TB(OE) and TB(J) are identical, we have a contradiction to $t \in P'_{i(m)}(\sigma^*(\theta))$ since, given $t <_r i(m)$ we must also have $t <_{oe} i(m)$. Consequently, we have established that if $k \in P_{i(m)}(\tilde{\sigma}(\theta))$, then $k \in P_{i(m)}(\sigma^*(\theta))$.

Step 2: If for some $\theta \in \Theta^n$ and some $m = 1, \ldots, n$, $C_r^*(\tilde{\sigma}(\theta)) = C_r^*(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$, then $C_m^*(\tilde{\sigma}(\theta)) \leq C_m^*(\sigma^*(\theta))$.

Proof of Step 2: Suppose that for some $\theta \in \Theta^n$ and some $m = 1, \ldots, n$, $C_r^*(\tilde{\sigma}(\theta)) = C_r^*(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$. Then, from Step 1, $P_{i(r)}(\tilde{\sigma}(\theta)) = P_{i(r)}(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$. Let $C_m^*(\tilde{\sigma}(\theta)) = \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta))$. As in the proof of Step 1, consider the set of agents $A(m-1) = \{i(1), \ldots, i(m-1)\}$ such that $C_r^*(\tilde{\sigma}(\theta)) = \theta_{i(r)} S_{i(r)}(\tilde{\sigma}(\theta)) = \theta_{i(r)} S_{i(r)}(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$, and consider the set $S^* = T_1 \cap T_2$. If $|S^*| = 1$, then $P_{i(m)}(\tilde{\sigma}(\theta)) = P_{i(m)}(\sigma^*(\theta))$ and $C_m^*(\tilde{\sigma}(\theta)) = \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta)) = \theta_{i(m)} S_{i(m)}(\sigma^*(\theta)) = C_m^*(\sigma^*(\theta))$, and the proof is complete. Assume that $|S^*| > 1$. Define $T' = P_{i(m)}(\tilde{\sigma}(\theta)) \cup \{i(m)\}$. Consider $t \in T'$ such that $S_t(\sigma^*(\theta)) = \max_{p \in T'}\{S_p(\sigma^*(\theta))\}$. Note that either $t = i(m)$, or $t \neq i(m)$ and $t \in P_{i(m)}(\tilde{\sigma}(\theta)) \cap P'_{i(m)}(\sigma^*(\theta))$. Hence, in either case, $\theta_t \geq \theta_{i(m)}$, and $S_t(\sigma^*(\theta)) \geq S_{i(m)}(\tilde{\sigma}(\theta))$ so that $\theta_t S_t(\sigma^*(\theta)) \geq \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta))$. Moreover, $t \in S^*$ since $t \notin P_{i(a)}(\tilde{\sigma}(\theta)) \cup \{i(a)\} = P_{i(a)}(\sigma^*(\theta)) \cup \{i(a)\}$ (if $i(a)$ exists) and since $t \notin P'_{i(b)}(\tilde{\sigma}(\theta)) \cup \{i(b)\} = P'_{i(b)}(\sigma^*(\theta)) \cup \{i(b)\}$ (if $i(b)$ exists). Given $C_r^*(\tilde{\sigma}(\theta)) = C_r^*(\sigma^*(\theta))$ for all $r = 1, \ldots, m-1$, $C_m^*(\tilde{\sigma}(\theta)) = \theta_{i(m)} S_{i(m)}(\tilde{\sigma}(\theta)) \leq \theta_t S_t(\sigma^*(\theta))$ and $t \notin A(m-1)$, it follows that $C_m^*(\tilde{\sigma}(\theta)) \leq C_m^*(\sigma^*(\theta))$. This proves Step 2.

We get the result by applying Step 1 and Step 2. From Theorem 2.1, it follows that for any $\theta \in \Theta^N$, $C_1^*(\tilde{\sigma}(\theta)) \leq C_1^*(\sigma^*(\theta))$. If $C_1^*(\tilde{\sigma}(\theta)) < C_1^*(\sigma^*(\theta))$, then the just sequencing rule $\tilde{\sigma}$ lexi-max dominates the outcome efficient sequencing rule σ^*. If $C_1^*(\tilde{\sigma}(\theta)) = C_1^*(\sigma^*(\theta))$, then by Step 2, $C_2^*(\tilde{\sigma}(\theta)) \leq C_2^*(\sigma^*(\theta))$. Again, if $C_1^*(\tilde{\sigma}(\theta)) = C_1^*(\sigma^*(\theta))$ and $C_2^*(\tilde{\sigma}(\theta)) < C_2^*(\sigma^*(\theta))$, then the just sequencing rule $\tilde{\sigma}$ lexi-max dominates the outcome efficient sequencing rule σ^*. If $C_1^*(\tilde{\sigma}(\theta)) = C_1^*(\sigma^*(\theta))$

and $C_2^*(\tilde{\sigma}(\theta)) = C_2^*(\sigma^*(\theta))$, then, by Step 2, $C_3^*(\tilde{\sigma}(\theta)) \leq C_3^*(\sigma^*(\theta))$. Continuing this way the result follows. Note that if $C_r^*(\tilde{\sigma}(\theta)) = C_r^*(\sigma^*(\theta))$ for all $r = 1, \ldots, n-1$, then, by Step 2, $C_n^*(\tilde{\sigma}(\theta)) \leq C_n^*(\sigma^*(\theta))$. But, given $\sum_{r=1}^n C_r^*(\sigma^*(\theta)) \leq \sum_{r=1}^n C_r^*(\tilde{\sigma}(\theta))$, we get $C_n^*(\tilde{\sigma}(\theta)) = C_n^*(\sigma^*(\theta))$. ■

The importance of the tie-breaking rule TB(J) is transparent from the next example.

EXAMPLE **2.2** Consider $\Omega(s_1, s_2)$ such that $s_1 > s_2$. Assume that the tie-breaking rule TB(J') for $\tilde{\sigma}$ is simply $1 \leq_{r'} 2$, that is, if $\theta_1 = \theta_2$, then $S_1(\tilde{\sigma}(\theta)) = s_1$ and $S_2(\tilde{\sigma}(\theta)) = s_1 + s_2$. It is easy to see TB($J'$) is different from TB(J). Consider the profile $\theta' = (\theta_1', \theta_2')$ such that $\theta_1' = \theta_2'$ and, given $s_1 > s_2$, $\theta_2'/s_2 > \theta_1'/s_1$. Using this tie-breaking rule for $\tilde{\sigma}$ we get $S_1(\tilde{\sigma}(\theta')) = s_1$ and $S_2(\tilde{\sigma}(\theta')) = s_1 + s_2$ and, by outcome efficiency, $S_1(\sigma^*(\theta')) = s_1 + s_2$ and $S_2(\sigma^*(\theta')) = s_2$. Note that $C_1^*(\tilde{\sigma}(\theta')) = \theta_2'(s_1 + s_2) = \theta_1'(s_1 + s_2) = C_1^*(\sigma^*(\theta'))$ and $C_2^*(\tilde{\sigma}(\theta')) = \theta_1' s_1 > \theta_2' s_2 = C_2^*(\sigma^*(\theta'))$. Therefore, with TB($J'$), for θ', the outcome efficient rule lexi-max dominates the just sequencing rule. Moreover, Step 1 of Proposition 2.2 also fails to hold since $P_1(\tilde{\sigma}(\theta')) = \{2\} \neq P_1(\sigma^*(\theta')) = \emptyset$. Observe that if instead we use tie-breaking rule TB(J), then $\tilde{\sigma}(\theta') = \sigma^*(\theta')$ and then $C_1^*(\tilde{\sigma}(\theta')) = C_1^*(\sigma^*(\theta')) = \theta_1'(s_1 + s_2)$, $C_2^*(\tilde{\sigma}(\theta')) = C_2^*(\sigma^*(\theta')) = \theta_2' s_2$ and Step 1 also holds.

Is there a meaningful bound for the 'efficiency loss' that we may incur under the just sequencing rule? For any sequencing problem $\Omega(s)$, consider the just sequencing rule $\tilde{\sigma}$ and the outcome efficient sequencing rule σ^*. For any $\theta \in \Theta^n$, define

$$EL_n(\theta) = \left[\frac{\sum\limits_{j \in N} \theta_j S_j(\tilde{\sigma}(\theta)) - \sum\limits_{j \in N} \theta_j S_j(\sigma^*(\theta))}{\sum\limits_{j \in N} \theta_j S_j(\sigma^*(\theta))} \right].$$

$EL_n(\theta)$ is a measure of relative efficiency loss since, given any $\theta \in \Theta^n$, the numerator of $EL_n(\theta)$ is the aggregate cost difference between the just sequencing rule and the outcome efficient sequencing rule and the denominator of $EL_n(\theta)$ is the aggregate cost under the outcome efficient sequencing rule.

PROPOSITION **2.1** For any $\theta \in \Theta^n$, $EL_n(\theta) \in [0, n-1)$.

Proof: From the definition of the outcome efficient sequencing rule it follows that for any $\theta \in \Theta^n$, $\sum_{j \in N} \theta_j S_j(\tilde{\sigma}(\theta)) \geq \sum_{j \in N} \theta_j S_j(\sigma^*(\theta))$ implying $EL_n(\theta) \geq 0$. To prove $EL_n(\theta) < n-1$, we show that for any $\theta \in \Theta^n$, $n \left[\sum_{j \in N} \theta_j S_j(\sigma^*(\theta))\right] - \sum_{j \in N} \theta_j S_j(\tilde{\sigma}(\theta)) > 0$. Observe that for any $\theta \in \Theta^n$, $\sum_{j \in N} \theta_j S_j(\tilde{\sigma}(\theta)) = \sum_{j \in N} \theta_j s_j + \sum_{j \in N} s_j \left(\sum_{k \in P'_j(\tilde{\sigma}(\theta))} \theta_k\right)$ and $n \left[\sum_{j \in N} \theta_j S_j(\sigma^*(\theta))\right] = n \sum_{j \in N} \theta_j s_j + n \sum_{j \in N} s_j \left(\sum_{k \in P'_j(\sigma^*(\theta))} \theta_k\right)$. Therefore, $n \left[\sum_{j \in N} \theta_j S_j(\sigma^*(\theta))\right] - \sum_{j \in N} \theta_j S_j(\tilde{\sigma}(\theta)) = n \sum_{j \in N} s_j \left(\sum_{k \in P'_j(\sigma^*(\theta))} \theta_k\right) + \sum_{j \in N} s_j \left[(n-1)\theta_j - \sum_{k \in P'_j(\tilde{\sigma}(\theta))} \theta_k\right] > 0$. The strict inequality holds because the sum $\sum_{j \in N} s_j \left[(n-1)\theta_j - \sum_{k \in P'_j(\tilde{\sigma}(\theta))} \theta_k\right] \geq 0$ and the reason is the following. For any $j \in N$, $(n-1)\theta_j = |P_j(\tilde{\sigma}(\theta))|\theta_j + \sum_{k \in P'_j(\tilde{\sigma}(\theta))} \theta_j$ and $\theta_j \geq \theta_k$ for all $k \in P'_j(\tilde{\sigma}(\theta))$ implying that $\sum_{j \in N} s_j \left[(n-1)\theta_j - \sum_{k \in P'_j(\tilde{\sigma}(\theta))} \theta_k\right] = \sum_{j \in N} s_j \left[|P_j(\tilde{\sigma}(\theta)|\theta_j + \sum_{k \in P'_j(\tilde{\sigma}(\theta))} (\theta_j - \theta_k)\right] \geq 0$. ■

2.4 Implementability of the just sequencing rule

DEFINITION **2.7** For the just sequencing rule $\tilde{\sigma}$, a mechanism $(\tilde{\sigma}, \tilde{\tau})$ is an *ICJ mechanism* if the transfer rule is such that for all $\theta \in \Theta^n$ and all $i \in N$,

$$\tilde{\tau}_i(\theta) = \tilde{h}_i(\theta_{-i}) - \sum_{j \in P'_i(\tilde{\sigma}(\theta))} \theta_j s_j, \tag{2.7}$$

where the function $\tilde{h}_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary.

Given a sequencing problem $\Omega(s)$, for any profile $\theta \in \Theta^n$ and any $j \in N$, let $\theta_j s_j$ be the *minimum cost* of agent j, that is, the cost that agent j would have incurred if he was served first in any order. Like the VCG transfers (2.2), the ICJ transfers (2.7) specify that for any $i \in N$ and any $\theta_{-i} \in \Theta^{N \setminus \{i\}}$, if θ_i is such that agent i is served last in the just order $\tilde{\sigma}(\theta_i, \theta_{-i})$, then $\tilde{\tau}_i(\theta_i, \theta_{-i}) = \tilde{h}_i(\theta_{-i})$. If θ'_i is such that agent i is not served last in the order $\tilde{\sigma}(\theta'_i, \theta_{-i})$, then agent i's transfer $\tilde{\tau}_i(\theta'_i, \theta_{-i})$ not only has $\tilde{h}_i(\theta_{-i})$ but he also has to pay the sum of the minimum costs of his followers in the just order $\tilde{\sigma}(\theta'_i, \theta_{-i})$. As long as we are in a sequencing problem $\Omega(s)$ where the processing time of the agents are not identical, there will be some profile θ and some agent j for whom the minimum cost $\theta_j s_j$ will be different from the pivotal cost $\theta_j s_i$ of i on j for some $i \in N \setminus \{j\}$. Hence the payment amounts in the ICJ transfers are qualitatively different from the payment amounts under the VCG transfers (2.2) where, recall that, the payment amount is the sum of the pivotal cost of agent i on all his followers in the outcome efficient sequencing order. Therefore, it is easy to see that for the queueing problem $\Omega(s^*)$ we have the following: For any agent $j \in N$, any agent $i \in N \setminus \{i\}$ and any profile θ, the minimum cost $\theta_j s_j^*$ is identical to the pivotal cost $\theta_j s_i^*$ since $s_i^* = s_j^*$ and, given the same tie-breaking rule, $\sigma^*(\theta) = \tilde{\sigma}(\theta)$. Thus, the VCG-mechanisms and the ICJ mechanisms are identical for the queueing problem $\Omega(s^*)$.

THEOREM **2.3** The just sequencing rule $\tilde{\sigma}$ is implementable if and only if the mechanism $(\tilde{\sigma}, \tau)$ that implements it is an ICJ mechanism.

Proof: Consider the just sequencing rule $\tilde{\sigma}$. We first prove that if a mechanism $(\tilde{\sigma}, \tau)$ implements $\tilde{\sigma}$, then it is necessarily the ICJ mechanism. Consider any agent $i \in N$ and fix any profile $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$. Take any $\theta_i, \theta_i' \in \Theta$ and apply the inequalities (1) $U_i(\tilde{\sigma}(\theta), \tau_i(\theta); \theta_i) \geq U_i(\tilde{\sigma}(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i)$ and (2) $U_i(\tilde{\sigma}(\theta), \tau_i(\theta); \theta_i') \leq U_i(\tilde{\sigma}(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i')$. From (1) and (2) we get

$$[S_i(\tilde{\sigma}(\theta_i', \theta_{_i})) - S_i(\tilde{\sigma}(\theta))]\theta_i \geq \tau_i(\theta_i', \theta_{-i}) - \tau_i(\theta) \geq [S_i(\tilde{\sigma}(\theta_i', \theta_{_i})) - S_i(\tilde{\sigma}(\theta))]\theta_i'. \quad (2.8)$$

If θ_i and θ_i' are such that $\tilde{\sigma}(\theta) = \tilde{\sigma}(\theta_i', \theta_{-i})$, then $S_i(\tilde{\sigma}(\theta_i', \theta_{_i})) = S_i(\tilde{\sigma}(\theta))$ and inequality (2.8) gives $\tau_i(\theta) = \tau_i(\theta_i', \theta_{-i})$.

Let $\theta_{(1)} \geq \theta_{(2)} \geq \ldots \geq \theta_{(n-1)}$ be the non-increasing order of the waiting cost for the fixed profile θ_{-i}. Consider any pair $(\theta_i^{t+1}, \theta_i^t) \in [\theta_{(t+1)}, \theta_{(t)}] \times [\theta_{(t)}, \theta_{(t-1)}]$. Using the just sequencing rule $\tilde{\sigma}$ and applying the implementability condition (2.8), if the actual profile is $(\theta_i^{t+1}, \theta_{-i})$ $((\theta_i^t, \theta_{-i}))$ and the misreport of agent i is θ_i^t (θ_i^{t+1}), then

$$\theta_i^{t+1} s_{(t)} \leq \tau_i(\theta_i^{t+1}, \theta_{-i}) - \tau_i(\theta_i^t, \theta_{-i}) \leq \theta_i^t s_{(t)}. \quad (2.9)$$

Since (2.9) must hold for all $(\theta_i^{t+1}, \theta_i^t) \in [\theta_i^{(t+1)}, \theta_i^{(t)}] \times [\theta_i^{(t)}, \theta_i^{(t-1)}]$, it follows that

$$\tau_i(\theta_i^{t+1}, \theta_{-i}) - \tau_i(\theta_i^t, \theta_{-i}) = \theta_{(t)} s_{(t)}. \tag{2.10}$$

Condition (2.10) must hold for all $t \in \{1, \ldots, n-1\}$. By setting $\tau_i(\theta_i^n, \theta_{-i}) = h_i(\theta_{-i})$ for any $\theta_i^n \in (0, \theta_{(n-1)})$ and then solving condition (2.10) recursively we get the ICJ transfers (2.7).

For the converse consider any agent $i \in N$ and any profile θ_{-i}. Let θ_i be the true waiting cost of i and $B_i(\theta_i'; \theta_i) := U_i(\tilde{\sigma}(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i) - U_i(\tilde{\sigma}(\theta_i, \theta_{-i}), \tau_i(\theta_i, \theta_{-i}); \theta_i)$ be the benefit of agent i from a misreport θ_i'.

(D1) If $\theta_i' > \theta_i$ and $P_i(\tilde{\sigma}(\theta_i, \theta_{-i})) \setminus P_i(\tilde{\sigma}(\theta_i', \theta_{-i})) \neq \emptyset$, then from the just sequencing rule we get $\theta_j \geq \theta_i$ for all $j \in P_i(\tilde{\sigma}(\theta_i, \theta_{-i})) \setminus P_i(\tilde{\sigma}(\theta_i', \theta_{-i}))$. Therefore, using any ICJ transfer we get $B_i(\theta_i'; \theta_i) = \sum_{j \in P_i(\tilde{\sigma}(\theta_i, \theta_{-i})) \setminus P_i(\tilde{\sigma}(\theta_i', \theta_{-i}))} (\theta_i - \theta_j) s_j \leq 0$.

(D2) If $\theta_i' < \theta_i$ and $P_i(\tilde{\sigma}(\theta_i', \theta_{-i})) \setminus P_i(\tilde{\sigma}(\theta_i, \theta_{-i})) \neq \emptyset$, then from the just sequencing rule we get $\theta_j \leq \theta_i$ for all $j \in P_i(\tilde{\sigma}(\theta_i', \theta_{-i})) \setminus P_i(\tilde{\sigma}(\theta_i, \theta_{-i}))$. Hence using any ICJ transfer we get $B_i(\theta_i'; \theta_i) = \sum_{j \in P_i(\tilde{\sigma}(\theta_i', \theta_{-i})) \setminus P_i(\tilde{\sigma}(\theta_i, \theta_{-i}))} (\theta_j - \theta_i) s_j \leq 0$.

(D3) Finally, if $\theta_i' \neq \theta_i$ and $P_i(\tilde{\sigma}(\theta_i', \theta_{-i})) = P_i(\tilde{\sigma}(\theta_i, \theta_{-i}))$, then $B_i(\theta_i'; \theta_i) = 0$.

Therefore, cases (D1)-(D3) prove that agent i cannot benefit from any deviation. Since the selection of agent i was arbitrary, the result follows. ■

REMARK **2.2** In this chapter we are interested in implementing the just sequencing rule. So we apply Rawlsian difference principle on the job completion cost $\theta_i S_i$ of the agents and not on utility/welfare $-\theta_i S_i + \tau_i$ of the agents. How to address the Rawlsian difference principle on welfare of the agents subject to the implementability constraint in a mechanism design framework like ours is a difficult but very pertinent open question.

2.4.1 Lexi-min domination

Consider any sequencing problem $\Omega(s)$. For any incentive compatible mechanism (σ, τ) with sequencing rule σ and transfer rule τ, consider any $\theta \in \Theta^n$ and define the vector of utilities as $b(\sigma(\theta), \tau(\theta)) = (U_1(\sigma(\theta), \tau(\theta); \theta_1), \ldots, U_n(\sigma(\theta), \tau(\theta); \theta_n))$. Define $b^*(\sigma(\theta), \tau(\theta)) = (b_1^*(\sigma(\theta), \tau(\theta)), \ldots, b_n^*(\sigma(\theta), \tau(\theta)))$ as the reordering of the utility vector $b(\sigma(\theta), \tau(\theta))$ such that $b_1^*(\sigma(\theta), \tau(\theta)) \leq \ldots \leq b_n^*(\sigma(\theta), \tau(\theta))$.

DEFINITION **2.8** Consider any sequencing problem $\Omega(s)$ and consider two incentive compatible mechanisms (σ', τ') and (σ'', τ''). The incentive compatible mechanism (σ', τ') *weakly lexi-min dominates* the incentive compatible mechanism (σ'', τ'') if for each $\theta \in \Theta^n$, either $b_1^*(\sigma'(\theta), \tau'(\theta)) > b_1^*(\sigma''(\theta), \tau''(\theta))$, or there exists $k \in \{2, \ldots, n\}$ such that $b_r^*(\sigma'(\theta), \tau'(\theta)) = b_r^*(\sigma''(\theta), \tau''(\theta))$ for all $r = 1, \ldots, k-1$ and $b_k^*(\sigma'(\theta), \tau'(\theta)) > b_k^*(\sigma''(\theta), \tau''(\theta))$, or the vector $b^*(\sigma'(\theta), \tau'(\theta))$ is identical to the vector $b^*(\sigma''(\theta), \tau''(\theta))$.

Consider any sequencing problem $\Omega(s)$ and, without loss of generality, assume that $s_1 \geq \ldots \geq s_n$ and $s_1 \neq s_n$.[10] The following observations are important in this context.

1. There exists a VCG mechanism and an ICJ mechanism such that *the ICJ mechanism weakly lexi-min dominates the VCG mechanism*. Consider the VCG mechanism (σ^*, τ^*) with the property that for all $\theta \in \Theta^n$ and all $i \in N$, $h_i^*(\theta_{-i}) = 0$ and consider the ICJ mechanism $(\tilde{\sigma}, \tilde{\tau})$ with the property that for all $\theta \in \Theta^n$ and all $i \in N$, $\tilde{h}_i(\theta_{-i}) = \sum_{j \in N \setminus \{i\}} \theta_j s_j$. It is easy to verify that for any $\theta \in \Theta^n$, $U_i(\tilde{\sigma}(\theta), \tilde{\tau}_i(\theta); \theta_i) = \sum_{j \in P_i(\tilde{\sigma}(\theta))} s_j(\theta_j - \theta_i) - \theta_i s_i$ and $U_i(\sigma^*(\theta), \tau_i^*(\theta); \theta_i) = -\theta_i \sum_{j \in P_i(\sigma^*(\theta))} s_j - s_i \sum_{j \in P_i'(\sigma^*(\theta))} \theta_j - \theta_i s_i$. Since $\theta_j \geq \theta_i$ for all $j \in P_i(\tilde{\sigma}(\theta))$, we have $\sum_{j \in P_i(\tilde{\sigma}(\theta))} s_j(\theta_j - \theta_i) \geq 0$ implying $U_i(\tilde{\sigma}(\theta), \tilde{\tau}_i(\theta); \theta_i) > U_i(\sigma^*(\theta), \tau_i^*(\theta); \theta_i)$ for all $i \in N$. Hence, the ICJ mechanism $(\tilde{\sigma}, \tilde{\tau})$ weakly lexi-min dominates the VCG mechanism (σ^*, τ^*).

[10]If $s_1 = s_n$, then we have the queueing problem.

2. There exists a VCG mechanism and an ICJ mechanism such that *the VCG mechanism weakly lexi-min dominates the ICJ mechanism*. Consider the VCG mechanism (σ^*, τ^*) with the property that for all $\theta \in \Theta^n$ and all $i \in N$, $h_i^*(\theta) = s_i \sum_{j \in N \setminus \{i\}} \theta_j$ and consider the ICJ mechanism $(\tilde{\sigma}, \tilde{\tau})$ with the property that for all $\theta \in \Theta^n$ and all $i \in N$, $\tilde{h}_i(\theta) = 0$. It is quite easy to verify that for any profile $\theta \in \Theta^n$, $U_i(\sigma^*(\theta), \tau_i^*(\theta); \theta_i) = \sum_{j \in P_i(\sigma^*(\theta))}(\theta_j s_i - \theta_i s_j) - \theta_i s_i$ and $U_i(\tilde{\sigma}(\theta), \tilde{\tau}_i(\theta); \theta_i) = -\theta_i \sum_{j \in P_i(\tilde{\sigma}(\theta))} s_j - \sum_{j \in P_i'(\tilde{\sigma}(\theta))} \theta_j s_j - \theta_i s_i$. Since for all $j \in P_i(\sigma^*(\theta))$, we have $\theta_j / s_j \geq \theta_i / s_i$ implying $\theta_j s_i \geq \theta_i s_j$, it follows that $\sum_{j \in P_i(\sigma^*(\theta))}(\theta_j s_i - \theta_i s_j) \geq 0$ and we have $U_i(\sigma^*(\theta), \tau_i^*(\theta); \theta_i) > U_i(\tilde{\sigma}(\theta), \tilde{\tau}_i(\theta); \theta_i)$ for all $i \in N$. Consequently, the VCG mechanism (σ^*, τ^*) weakly lexi-min dominates the ICJ mechanism $(\tilde{\sigma}, \tilde{\tau})$.

3. For any VCG mechanism and any ICJ mechanism with the property that for all $\theta \in \Theta^n$ and all $i \in N$, $h_i^*(\theta_{-i}) = \tilde{h}_i(\theta_{-i})$, *no such lexi-min domination relation exists.* Fix any VCG mechanism (σ^*, τ^*) and any ICJ mechanism $(\tilde{\sigma}, \tilde{\tau})$ with the property that for all $\theta \in \Theta^n$ and all $i \in N$, $h_i^*(\theta_{-i}) = \tilde{h}_i(\theta_{-i})$. Consider any profile $\theta = (\theta_1, \ldots, \theta_n)$ such that $\theta_1 / s_1 > \ldots > \theta_n / s_n$ and $\theta_1 > \ldots > \theta_n$ so that $\tilde{\sigma}(\theta) = \sigma^*(\theta)$ and $S_i(\tilde{\sigma}(\theta)) = S_i(\sigma^*(\theta)) = \sum_{j=1}^{i} s_j$ for all $i \in N$. Observe that for any $i \in N$, $s_i \geq s_j$ for all $j \in P_i'(\tilde{\sigma}(\theta)) = P_i'(\sigma^*(\theta))$ implying that $U_i(\tilde{\sigma}(\theta), \tilde{\tau}_i(\theta); \theta_i) - U_i(\sigma^*(\theta), \tau_i^*(\theta); \theta_i) = \sum_{j \in P_i'(\tilde{\sigma}(\theta))} \theta_j (s_i - s_j) \geq 0$. Given $s_1 \neq s_n$, $U_1(\tilde{\sigma}(\theta), \tilde{\tau}_1(\theta); \theta_1) - U_1(\sigma^*(\theta), \tau_1^*(\theta); \theta_1) = \sum_{j \in N \setminus \{1\}} \theta_j (s_1 - s_j) > 0$. Therefore, the VCG mechanism (σ^*, τ^*) cannot lexi-min dominate the ICJ mechanism $(\tilde{\sigma}, \tilde{\tau})$. Next, consider any profile $\theta' = (\theta_1', \ldots, \theta_n')$ such that $\theta_n' / s_n > \ldots > \theta_1' / s_1$ and $\theta_n' > \ldots > \theta_1'$ so that $\tilde{\sigma}(\theta') = \sigma^*(\theta')$ and $S_i(\tilde{\sigma}(\theta')) = S_i(\sigma^*(\theta')) = \sum_{j=i}^{n} s_j$ for all $i \in N$. Observe that for any $i \in N$, $s_i \leq s_j$ for all $j \in P_i'(\tilde{\sigma}(\theta')) = P_i'(\sigma^*(\theta'))$ implying that $U_i(\sigma^*(\theta'), \tau_i^*(\theta'); \theta_i') - U_i(\tilde{\sigma}(\theta'), \tilde{\tau}_i(\theta'); \theta_i') = \sum_{j \in P_i'(\tilde{\sigma}(\theta'))} \theta_j'(s_j - s_i) \geq 0$. Given $s_1 \neq s_n$, $U_n(\sigma^*(\theta'), \tau_n^*(\theta'); \theta_n') - U_n(\tilde{\sigma}(\theta'), \tilde{\tau}_n(\theta'); \theta_n') = \sum_{j \in N \setminus \{n\}} \theta_j'(s_j - s_n) > 0$. Thus, it is also true that the ICJ mechanism $(\tilde{\sigma}, \tilde{\tau})$ cannot lexi-min dominate the VCG mechanism (σ^*, τ^*).

2.5 Properties of the just sequencing rule

In this section we focus on two nice properties of the just sequencing rule.

2.5.1 Budget balanced implementability of the just sequencing rule

For implementation of outcome efficient allocation rules, Zhou (2007) provided examples of both public and private good allocation problems for which no budget balancing VCG mechanisms exists and for which all VCG mechanisms are inferior to other 'reasonable' non-VCG mechanisms. Zhou (2007) concluded that unless one can find a budget balancing VCG mechanism, one should limit its use. Implementation of outcome efficient rules with balanced transfers is difficult to get in many economic environments (see, for example, Hurwicz and Walker (1990) and Walker (1980)). That implementation of outcome efficiency is possible with balanced transfers for sequencing and queueing problems was established by Mitra (2001), Mitra (2002) and Suijs (1996). Mitra and Sen (2010) characterized domains in a heterogeneous objects model with private values where an outcome efficient rule can be implemented in dominant strategies with balanced transfers. They show that these domains are non-trivial and are 'closely' related to incentive problems in sequencing and queueing problems studied in Mitra Mitra (2001), Mitra (2002) and Suijs (1996). We prove that for sequencing problems, if we replace outcome efficient sequencing rule with the just sequencing rule, we get implementability with balanced transfers provided there are at least three agents.

DEFINITION **2.9** A sequencing rule σ is *implementable with balanced transfers* if the mechanism (σ, τ) that implements it has a budget balanced transfer, that is, for all $\theta \in \Theta^n$, $\sum_{j \in N} \tau_j(\theta) = 0$.

PROPOSITION **2.2** For any sequencing problem $\Omega(s_1, s_2)$ with two agents, implementation of the just sequencing rule $\tilde{\sigma}$ with balanced transfers is not possible.

Proof: Fix any processing time vector (s_1, s_2) and consider the sequencing problem $\Omega(s_1, s_2)$. To implement the refined sequencing rule it is necessary that the mechanism

(σ, τ) is an ICJ mechanism (Theorem 2.3). Pick any ICJ mechanism. Consider the waiting costs $\theta_1, \theta_1', \theta_2, \theta_2'$ such that $\theta_1 > \theta_2' > \theta_1' > \theta_2$ and $\theta_1' s_1 \neq \theta_2' s_2$. Budget balance for the profile (θ_1, θ_2) gives (B1) $\tilde{h}_1(\theta_2) + \tilde{h}_2(\theta_1) - \theta_2 s_2 = 0$. Budget balance for (θ_1, θ_2') gives (B2) $\tilde{h}_1(\theta_2') + \tilde{h}_2(\theta_1) - \theta_2' s_2 = 0$. Budget balance for (θ_1', θ_2) gives (B3) $\tilde{h}_1(\theta_2) + \tilde{h}_2(\theta_1') - \theta_2 s_2 = 0$. Finally, budget balance for the profile (θ_1', θ_2') gives (B4) $\tilde{h}_1(\theta_2') + \tilde{h}_2(\theta_1') - \theta_1' s_1 = 0$. By adding (B1) and (B4) and subtracting both (B2) and (B3) from it we get $\theta_2' s_2 - \theta_1' s_1 = 0$ which contradicts the restriction that $\theta_1' s_1 \neq \theta_2' s_2$. ■

DEFINITION **2.10** Consider any $\Omega(s)$ with at least three agents. A mechanism $(\tilde{\sigma}, \tilde{\tau}^*)$ is a *balanced ICJ mechanism* if the transfer rule is such that for all $\theta \in \Theta^n$ and all $i \in N$,

$$\tilde{\tau}_i^*(\theta) = \sum_{j \in P_i(\tilde{\sigma}(\theta))} \left(\frac{\tilde{\sigma}_j(\theta) - 1}{n-2} \right) \theta_j s_j - \sum_{j \in P_i'(\tilde{\sigma}(\theta))} \left(\frac{n - \tilde{\sigma}_j(\theta)}{n-2} \right) \theta_j s_j + g_i(\theta_{-i}), \tag{2.11}$$

where $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ for each $i \in N$ is such that for any $\theta \in \Theta^n$, $\sum_{i \in N} g_i(\theta_{-i}) = 0$.

The balanced ICJ transfer requires that for any profile $\theta \in \Theta^n$ and given any agent specific constant transfer $g_i(\theta_{-i})$ to agent i, each agent i, in addition, receives as reward a position specific weighted sum of the minimum cost of all agents served before him $(\sum_{j \in P_i(\tilde{\sigma}(\theta))} [(\tilde{\sigma}_j(\theta) - 1)/(n-2)] \theta_j s_j)$ under the just sequencing order $\tilde{\sigma}(\theta)$ (provided $P_i(\tilde{\sigma}(\theta)) \neq \emptyset$) and agent i also pays a position specific weighted sum of the minimum cost of all agents served after him $(\sum_{j \in P_i'(\tilde{\sigma}(\theta))} [(n - \tilde{\sigma}_j(\theta))/(n-2)] \theta_j s_j)$ under the just sequencing order $\tilde{\sigma}(\theta)$ (provided $P_i'(\tilde{\sigma}(\theta)) \neq \emptyset$). In addition, the balanced ICJ mechanism also requires that selection of the function $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ for each agent $i \in N$ must be such that for any $\theta \in \Theta^n$, $\sum_{j \in N} g_j(\theta_{-j}) = 0$.

THEOREM **2.4** Let $\Omega(s)$ be a sequencing problem with at least three agents. The just sequencing rule is implementable with balanced transfers if and only if the mechanism $(\tilde{\sigma}, \tau)$ that implements it is a balanced ICJ mechanism.

Proof: We know that the just sequencing rule is implementable if and only if the mechanism is an ICJ mechanism (Theorem 2.3). Therefore, identifying the complete class of mechanisms that implement the just sequencing rule with balanced transfers reduce to identifying the complete class of balanced ICJ mechanisms.

Let $\tilde{\sigma}(\theta)$ be the just order for the profile $\theta \in \Theta^n$ and agent i leaves. We define the "induced" order $\tilde{\sigma}(\theta_{-i})$ (of length $|N \setminus \{i\}|$) for the agents in $N \setminus \{i\}$ as follows:

$$\tilde{\sigma}_j(\theta_{-i}) = \begin{cases} \tilde{\sigma}_j(\theta) - 1 & \text{if } j \in P'_i(\tilde{\sigma}(\theta)), \\ \tilde{\sigma}_j(\theta) & \text{if } j \in P_i(\tilde{\sigma}(\theta)). \end{cases} \tag{2.12}$$

In words, $\tilde{\sigma}(\theta_{-i})$ is the order formed by removing agent i and moving all agents behind him up by one position. Given the same tie-breaking rule for the economy with $N \setminus \{i\}$ agents, it is easy to see that if $\tilde{\sigma}(\theta)$ is just order for the profile θ, then $\tilde{\sigma}(\theta_{-i})$ is also the just order in $N \setminus \{i\}$ for the profile θ_{-i}.

Given that the sequencing problem $\Omega(s)$ has at least three agents, without loss of generality, we redefine the ICJ mechanisms by setting for each agent $i \in N$ and each profile θ_{-i},

$$\tilde{h}_i(\theta_{-i}) = \sum_{j \in N \setminus \{i\}} \left(\frac{\tilde{\sigma}_j(\theta_{-i}) - 1}{n-2} \right) \theta_j s_j + g_i(\theta_{-i}), \tag{2.13}$$

where $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary. By substituting (2.13) in the ICJ transfer (2.7) we get the following for any $i \in N$ and any $\theta \in \Theta^n$.

(A) If $P'_i(\tilde{\sigma}(\theta)) = \emptyset$, then $\tilde{\sigma}_j(\theta_{-i}) = \tilde{\sigma}_j(\theta)$ for all $j \in N \setminus \{i\}$ and from (2.7) we get $\tilde{\tau}_i(\theta) = \tilde{h}_i(\theta_{-i}) = \sum_{j \in N \setminus \{i\}} \left(\frac{\tilde{\sigma}_j(\theta) - 1}{n-2} \right) \theta_j s_j + g_i(\theta_{-i})$.

(B) If $P_i(\tilde{\sigma}(\theta)) = \emptyset$, then $\tilde{\sigma}_j(\theta_{-i}) = \tilde{\sigma}_j(\theta) - 1$ for all $j \in N \setminus \{i\}$ and from (2.7) we get $\tilde{\tau}_i(\theta) = -\sum_{j \in N \setminus \{i\}} \theta_j s_j + \tilde{h}_i(\theta_{-i}) = -\sum_{j \in N \setminus \{i\}} \theta_j s_j + \sum_{j \in N \setminus \{i\}} \left(\frac{\tilde{\sigma}_j(\theta) - 2}{n-2} \right) \theta_j s_j + g_i(\theta_{-i}) = -\sum_{j \in N \setminus \{i\}} \left(\frac{n - \tilde{\sigma}_j(\theta)}{n-2} \right) \theta_j s_j + g_i(\theta_{-i})$.

(C) If $P'_i(\tilde{\sigma}(\theta)) \neq \emptyset$ and $P_i(\tilde{\sigma}(\theta)) \neq \emptyset$, then $\tilde{\sigma}_j(\theta_{-i}) = \tilde{\sigma}_j(\theta)$ for all $j \in P_i(\tilde{\sigma}(\theta)) \neq \emptyset$, $\tilde{\sigma}_j(\theta_{-i}) = \tilde{\sigma}_j(\theta) - 1$ for all $j \in P'_i(\tilde{\sigma}(\theta)) \neq \emptyset$ and from (2.7) we get

$$\tilde{\tau}_i(\theta) = -\sum_{j \in P'_i(\tilde{\sigma}(\theta))} \theta_j s_j + \tilde{h}_i(\theta_{-i})$$

$$= -\sum_{j \in P'_i(\tilde{\sigma}(\theta))} \theta_j s_j + \sum_{j \in P_i(\tilde{\sigma}(\theta))} \left(\frac{\tilde{\sigma}_j(\theta) - 1}{n-2} \right) \theta_j s_j + \sum_{j \in P'_i(\tilde{\sigma}(\theta))} \left(\frac{\tilde{\sigma}_j(\theta) - 2}{n-2} \right) \theta_j s_j + g_i(\theta_{-i})$$

$$= \sum_{j \in P_i(\tilde{\sigma}(\theta))} \left(\tfrac{\tilde{\sigma}_j(\theta)-1}{n-2}\right) \theta_j s_j - \sum_{j \in P_i'(\tilde{\sigma}(\theta))} \left(\tfrac{n-\tilde{\sigma}_j(\theta)}{n-2}\right) \theta_j s_j + g_i(\theta_{-i}).$$

Conditions (A)-(C) establish that if the sequencing problem $\Omega(s)$ has at least three agents, then an equivalent representation of the ICJ transfers is that for any $i \in N$ and any $\theta \in \Theta^n$,

$$\tilde{\tau}_i'(\theta) = \sum_{j \in P_i(\tilde{\sigma}(\theta))} \left(\frac{\tilde{\sigma}_j(\theta) - 1}{n-2}\right) \theta_j s_j - \sum_{j \in P_i'(\tilde{\sigma}(\theta))} \left(\frac{n - \tilde{\sigma}_j(\theta)}{n-2}\right) \theta_j s_j + g_i(\theta_{-i}), \tag{2.14}$$

where for each $i \in N$, the function $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary. By taking representation (2.14) of the ICJ transfers we get for any $\theta \in \Theta^n$,

$$\sum_{k \in N} \tilde{\tau}_k'(\theta) = \sum_{k \in N} \left\{ \sum_{j \in P_k(\tilde{\sigma}(\theta))} \left(\tfrac{\tilde{\sigma}_j(\theta)-1}{n-2}\right) \theta_j s_j - \sum_{j \in P_k'(\tilde{\sigma}(\theta))} \left(\tfrac{n-\tilde{\sigma}_j(\theta)}{n-2}\right) \theta_j s_j \right\} + \sum_{k \in N} g_k(\theta_{-k})$$
$$= \sum_{k \in N} \sum_{j \in P_k(\tilde{\sigma}(\theta))} \left(\tfrac{\tilde{\sigma}_j(\theta)-1}{n-2}\right) \theta_j s_j - \sum_{k \in N} \sum_{j \in P_k'(\tilde{\sigma}(\theta))} \left(\tfrac{n-\tilde{\sigma}_j(\theta)}{n-2}\right) \theta_j s_j + \sum_{k \in N} g_k(\theta_{-k})$$
$$= \sum_{j \in N} \left\{ |P_j'(\tilde{\sigma}_j(\theta))| \left(\tfrac{\tilde{\sigma}_j(\theta)-1}{n-2}\right) \right\} \theta_j s_j - \sum_{j \in N} \left\{ |P_j(\tilde{\sigma}_j(\theta))| \left(\tfrac{n-\tilde{\sigma}_j(\theta)}{n-2}\right) \right\} \theta_j s_j + \sum_{k \in N} g_k(\theta_{-k})$$
$$= \sum_{j \in N} \left\{ (n - \tilde{\sigma}_j(\theta)) \left(\tfrac{\tilde{\sigma}_j(\theta)-1}{n-2}\right) \right\} \theta_j s_j - \sum_{j \in N} \left\{ (\tilde{\sigma}_j(\theta) - 1) \left(\tfrac{n-\tilde{\sigma}_j(\theta)}{n-2}\right) \right\} \theta_j s_j + \sum_{k \in N} g_k(\theta_{-k})$$
$$= \sum_{k \in N} g_k(\theta_{-k})$$

Therefore, by taking representation (2.14) of the ICJ transfers, we have proved that for any $\theta \in \Theta^n$, $\sum_{k \in N} \tilde{\tau}_k'(\theta) = \sum_{k \in N} g_k(\theta_{-k})$. Thus, an ICJ mechanism represented by (2.14) is budget balanced if and only if (I) for all $\theta \in \Theta^n$, $\sum_{k \in N} g_k(\theta_{-k}) = 0$. From representation (2.14) of the ICJ transfers and from condition (I), the result follows. ■

REMARK **2.3** There are papers that show that for the family of sequencing problems $\Omega(s)$ with three or more agents one can implement the outcome efficient sequencing rule with balanced VCG mechanisms (see Mitra (2002) and Suijs (1996)). A balanced VCG mechanism has the following functional form of the transfer: For each $\theta \in \Theta^n$

the transfer vector $\tau^*(\theta) = (\tau_1^*(\theta), \tau_2^*(\theta), \ldots, \tau_n^*(\theta))$ is such that for each $i \in N$,

$$\tau_i^*(\theta) = \left[\sum_{j \in P_i(\sigma^*(\theta))} \theta_j \right] s_i - \frac{1}{(n-2)} \sum_{j \neq i} \left[\theta_j \left\{ \sum_{k \in P_j'(\sigma^*(\theta)), k \neq i\}} s_k \right\} \right].^{11} \tag{2.15}$$

From Remark 2.1 it follows that for the queueing problem $\Omega(s^*)$, the set of all balanced ICJ mechanisms coincide with the set of all balanced VCG mechanisms.

In the next example we select a balanced VCG mechanism and a balanced ICJ mechanism and show that neither lexi-min dominates the other.

EXAMPLE **2.3** Consider the sequencing problem $\Omega(s)$ where $N = \{1, 2, 3\}$, $s = (s_1, s_2, s_3)$ is such that $s_1 \geq s_2 \geq s_3 > 0$ and $s_1 \neq s_3$. Consider the balanced VCG mechanism (σ^*, τ^*) where $\tau^*(\theta)$ is given by equation (2.15) and consider balanced ICJ mechanism $(\tilde{\sigma}, \tilde{\tau}^*)$ with the transfer given by equation (2.11) of Section 2.5.1 by imposing an added restriction that for all $\theta \in \Theta^3$, $g_1(\theta_{-1}) = -s_3(s_1 - s_3)$, $g_2(\theta_{-2}) = 0$ and $g_3(\theta_{-3}) = s_3(s_1 - s_3)$. Consider any $m > s_1$ and let $\theta = (\theta_1, \theta_2, \theta_3)$ be such that $\theta_1 = m(s_1 - s_3)/s_1$, $\theta_2 = 2m(s_1 - s_3)/s_2$ and $\theta_3 = 3m(s_1 - s_3)/s_3$. This implies $\theta_3 > \theta_2 > \theta_1$, $\theta_3 s_3 > \theta_2 s_2 > \theta_1 s_1$ and $\theta_3/s_3 > \theta_2/s_2 > \theta_1/s_1$. Hence, by construction, for all $i \in N$, $P_i(\tilde{\sigma}(\theta)) = P_i(\sigma^*(\theta))$ and one can verify that $U_1(\tilde{\sigma}(\theta), \tilde{\tau}_1^*(\theta); \theta_1) > U_2(\tilde{\sigma}(\theta), \tilde{\tau}_2^*(\theta); \theta_2) > U_3(\tilde{\sigma}(\theta), \tilde{\tau}_3^*(\theta); \theta_3)$. Therefore, $b_1^*(\tilde{\sigma}(\theta), \tilde{\tau}^*(\theta)) = U_3(\tilde{\sigma}(\theta), \tilde{\tau}_3^*(\theta); \theta_3)$. Finally, we get $U_3(\tilde{\sigma}(\theta), \tilde{\tau}_3^*(\theta); \theta_3) - U_3(\sigma^*(\theta), \tau_3^*(\theta); \theta_3) = \theta_2(s_1 - s_2) + s_3(s_1 - s_3) > 0$. Hence $b_1^*(\tilde{\sigma}(\theta), \tilde{\tau}^*(\theta)) = U_3(\tilde{\sigma}(\theta), \tilde{\tau}_3^*(\theta); \theta_3) > U_3(\sigma^*(\theta), \tau_3^*(\theta); \theta_3) \geq b_1^*(\sigma^*(\theta), \tau^*(\theta))$ implying that the balanced VCG mechanism (σ^*, τ^*) cannot lexi-min dominate the balanced ICJ mechanism $(\tilde{\sigma}, \tilde{\tau}^*)$. Consider any $\theta' = (\theta_1', \theta_2', \theta_3')$ such $\theta_1' > \theta_2' > \theta_3'$, $\theta_1' s_1 > \theta_2' s_2 > \theta_3' s_3$ and $\theta_1'/s_1 > \theta_2'/s_2 > \theta_3'/s_3$. Hence, by construction, for all $i \in N$, $P_i(\tilde{\sigma}(\theta')) = P_i(\sigma^*(\theta'))$ and one can verify that $b_1^*(\sigma^*(\theta'), \tau^*(\theta')) = U_1(\sigma^*(\theta'), \tau_3^*(\theta'); \theta_3')$. One can also verify that $U_1(\tilde{\sigma}(\theta'), \tilde{\tau}_1^*(\theta'); \theta_1') - U_1(\sigma^*(\theta'), \tau_1^*(\theta'); \theta_1') = \theta_2'(s_3 - s_2) - s_3(s_1 - s_3) < 0$. Therefore, we have the following: $b_1^*(\sigma^*(\theta'), \tau^*(\theta')) = U_1(\sigma^*(\theta'), \tau_1^*(\theta'); \theta_1') >$

[11] See equation (3.1) in Mitra (2002).

$U_1(\tilde{\sigma}(\theta'), \tilde{\tau}_1^*(\theta'); \theta_1') \geq b_1^*(\tilde{\sigma}(\theta'), \tilde{\tau}^*(\theta'))$. Hence the balanced ICJ mechanism $(\tilde{\sigma}, \tilde{\tau}^*)$ cannot lexi-min dominate the balanced VCG mechanism (σ^*, τ^*).

2.5.2 Two-dimensional incentives and the just sequencing rule

If we assume that both waiting cost and the processing time are agent specific private information, then we have the mechanism design problem for the general sequencing problem $\Omega = (N, \Theta^n, \mathcal{S})$. Specifically, we have an interdependent value situation and therefore the correct notion is ex-post implementability. The type of any agent $i \in N$ is $m_i = (\theta_i, s_i) \in \Theta \times \mathbb{R}_{++}$ that constitutes of his waiting cost as well as his processing time. A profile is $m = (m_1, \ldots, m_n) \in \Theta^n \times \mathcal{S}$. For any $i \in N$, let m_{-i}, denote the profile $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n) \in \Theta^{|N \setminus \{i\}|} \times \mathbb{R}_{++}^{|N \setminus \{i\}|}$ which is obtained from the profile m by eliminating i's type. For the general sequencing problem Ω, a (direct revelation) mechanism is (σ^g, τ) that constitutes of a general sequencing rule σ^g and a transfer rule τ. A *general sequencing rule* is a function $\sigma^g : \Theta^n \times \mathcal{S} \to \Sigma(N)$ that specifies for each profile $m \in \Theta^n \times \mathcal{S}$, a unique order $\sigma^g(m) = (\sigma_1^g(m), \ldots, \sigma_n^g(m)) \in \Sigma(N)$. A *transfer rule* is a function $\tau : \Theta^n \times \mathcal{S} \to \mathbb{R}^n$ that specifies for each profile $m \in \Theta^n \times \mathcal{S}$ a transfer vector $\tau(m) = (\tau_1(m), \ldots, \tau_n(m)) \in \mathbb{R}^n$. Specifically, for Ω and given any mechanism (σ^g, τ), if m_i' is the announced type of agent i when his true type is m_i and m_{-i} is the true profile for agents $N \setminus \{i\}$, then the utility of i is given by $v_i(\sigma^g(m_i', m_{-i}), \tau_i(m_i', m_{-i}); m_i; s_{-i}(m_{-i})) = -\theta_i \left(s_i + \sum_{j \in P_i(\sigma^g(m_i', m_{-i}))} s_j\right) + \tau_i(m_i', m_{-i})$.

DEFINITION **2.11** A mechanism (σ^g, τ) *ex-post implements* the general sequencing rule σ^g if the transfer rule $\tau : \Theta^n \times \mathcal{S} \to \mathbb{R}^n$ is such that for any $i \in N$, any m_i, m_i' and any true profile m_{-i},

$$v_i(\sigma^g(m), \tau_i(m); m_i; s_{-i}(m_{-i})) \geq v_i(\sigma^g(m_i', m_{-i}), \tau_i(m_i', m_{-i}); m_i; s_{-i}(m_{-i})). \quad (2.16)$$

Ex-post implementability requires truth-telling is a Nash equilibrium for any agent and for every true type profile m.

DEFINITION **2.12** A mechanism (σ^g, τ) *ex-post implements with balanced transfers* the general sequencing rule σ^g if the transfer rule $\tau : \Theta^n \times \mathcal{S} \to \mathbb{R}^n$ satisfies ex-post implementability condition (2.16) and is also budget balanced, that is for all $m \in \Theta^n \times \mathcal{S} \to \mathbb{R}^n$, $\sum_{j \in N} t_j(m) = 0$.

DEFINITION **2.13** A general sequencing rule $\tilde{\sigma}^g$ is *just* if for each profile $m \in \Theta^n \times \mathcal{S}$, the chosen order $\tilde{\sigma}^g(m)$ satisfies the following property: for any $i, j \in N$ such that $\theta_i \geq \theta_j$, $\tilde{\sigma}_i^g(m) \leq \tilde{\sigma}_j^g(m)$.

For any true $m = (m_1 = (\theta_1, s_1), \ldots, m_n = (\theta_n, s_n))$ we say θ *is obtained from* m if it is a collection of the first element from $m_j = (\theta_j, s_j)$ for each $j \in N$ and we say s *is obtained from* m if it is a collection of the second element from $m_j = (\theta_j, s_j)$ for each $j \in N$. Similarly, for any $i \in N$ and any true $m_{-i} = (m_1 = (\theta_1, s_1), \ldots, m_{i-1} = (\theta_{i-1}, s_{i-1}), m_{i+1} = (\theta_{i+1}, s_{i+1}), \ldots, m_n = (\theta_n, s_n))$ we say θ_{-i} is obtained from m_{-i} if it is a collection of the first element from $m_j = (\theta_j, s_j)$ for each $j \in N \setminus \{i\}$ and we say s_{-i} is obtained from m_{-i} if it is a collection of the second element from $m_j = (\theta_j, s_j)$ for each $j \in N \setminus \{i\}$. For Ω, the just general sequencing rule satisfies the following: For any $m = (m_1, \ldots, m_n) \in \Theta^n \times \mathcal{S}$, $\tilde{\sigma}^g(m) = \tilde{\sigma}(\theta)$ where θ is obtained from m.

DEFINITION **2.14** For $\tilde{\sigma}^g$, a mechanism $(\tilde{\sigma}^g, \tilde{\tau})$ is a *generalized ICJ mechanism* if the transfer rule is such that for all $m \in \Theta^n \times \mathcal{S}$ and all $i \in N$,

$$\tilde{\tau}_i^g(\theta_i, m_{-i}) = h_i^g(m_{-i}) - \sum_{j \in P_i'(\tilde{\sigma}(\theta_i, \theta_{-i}))} \theta_j s_j, \tag{2.17}$$

where θ_{-i} is obtained from m_{-i} and $h_i^g : \Theta^{|N \setminus \{i\}|} \times \mathbb{R}_{++}^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary.

PROPOSITION **2.3** The just general sequencing rule $\tilde{\sigma}^g$ is ex-post implementable if and only if the mechanism $(\tilde{\sigma}^g, \tau)$ that implements it is a generalized ICJ mechanism.

Proof: We only prove that if the just general sequencing rule $\tilde{\sigma}^g$ is ex-post implementable via a mechanism $(\tilde{\sigma}^g, \tau)$, then the mechanism is a generalized ICJ mechanism. The other part is easy and hence omitted.

Consider any $i \in N$ and fix any true type m_{-i} for all agents other than i. For any true type $m_i = (\theta_i, s_i)$ of i and any misreport $m_i' = (\theta_i', s_i')$ by i, ex-post implementability of $\tilde{\sigma}^g$ requires $v_i(\tilde{\sigma}^g(m), \tau_i(m); m_i; s_{-i}(m_{-i})) \geq v_i(\tilde{\sigma}^g(m_i', m_{-i}), \tau_i(m_i', m_{-i}); m_i; s_{-i}(m_{-i}))$ which is the same as requiring

$$v_i(\tilde{\sigma}(\theta_i, \theta_{-i}), \tau_i(m_i, m_{-i}); m_i; s_{-i}(m_{-i})) \geq v_i(\tilde{\sigma}(\theta_i', \theta_{-i}), \tau_i(m_i', m_{-i}); m_i, s_{-i}(m_{-i})). \tag{2.18}$$

Moreover, given the true type m_{-i} of all agents and given any pair of types $(m_i' = (\theta_i', s_i'), m_i'' = (\theta_i', s_i''))$ for i with the property the waiting cost is identical in m_i' and m_i'', $\tilde{\sigma}^g(m_i', m_{-i}) = \tilde{\sigma}(\theta_i', \theta_i) = \tilde{\sigma}^g(m_i'', m_{-i})$ since just sequencing rule ignores the processing time. Hence, using ex-post implementability, we get the following inequalities:

$$\text{(I1)}\ -\theta_i'(s_i' + \sum_{j \in P_i(\tilde{\sigma}(\theta_i', \theta_{-i}))} s_j) + \tau_i(m_i', m_{-i}) \geq -\theta_i'(s_i' + \sum_{j \in P_i(\tilde{\sigma}(\theta_i', \theta_{-i}))} s_j) + \tau_i(m_i'', m_{-i}).$$

$$\text{(I2)}\ -\theta_i'(s_i'' + \sum_{j \in P_i(\tilde{\sigma}(\theta_i', \theta_{-i}))} s_j) + \tau_i(m_i', m_{-i}) \leq -\theta_i'(s_i'' + \sum_{j \in P_i(\tilde{\sigma}(\theta_i', \theta_{-i}))} s_j) + \tau_i(m_i'', m_{-i}).$$

Condition (I1) gives $\tau_i(m_i', m_{-i}) \geq \tau_i(m_i'', m_{-i})$ and (I2) gives $\tau_i(m_i', m_{-i}) \leq \tau_i(m_i'', m_{-i})$. Therefore, $\tau_i(m_i', m_{-i}) = \tau_i(m_i'', m_{-i})$. Hence, we have

$$\tau_i(m_i', m_{-i}) = \tau_i(m_i'', m_{-i}) := \tau_i(\theta_i', m_{-i}). \tag{2.19}$$

Using (2.19) in (2.18) we get

$$v_i(\tilde{\sigma}(\theta_i, \theta_{-i}), \tau_i(\theta_i, m_{-i}); m_i; s_{-i}(m_{-i})) \geq v_i(\tilde{\sigma}(\theta_i', \theta_{-i}), \tau_i(\theta_i', m_{-i}); m_i, s_{-i}(m_{-i})). \tag{2.20}$$

If the true type m_{-i} of agents in the set $N \setminus \{i\}$ is known, then the processing time of these agents s_{-i} obtained from m_{-i} is also known and θ_{-i} obtained from m_{-i} is also known. Therefore, the re-ordering $(\theta_{(1)}, \ldots, \theta_{(n-1)})$ of θ_{-i} such that $\theta_{(1)} \geq \ldots \geq \theta_{(n-1)}$, is also known. Moreover, the calculation of the transfer for agent i is independent of the processing time of agent i. Therefore, from inequality (2.20) we get that for any given $s_i > 0$, the following inequality must be true.

$$U_i(\tilde{\sigma}(\theta_i, \theta_{-i}), \tau_i(\theta_i, m_{-i}); \theta_i) \geq U_i(\tilde{\sigma}(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, m_{-i}); \theta_i). \tag{2.21}$$

Note that if inequality (2.21) holds for some given $s_i > 0$ and for all $\theta_i \in \Theta$, then for any other $s'_i > 0$ with the same transfer, inequality (2.21) holds for any $\theta_i \in \Theta$ since s_i (either actual or misreported) does not enter the calculation of the transfer $\tau_i(\theta_i, m_{-i})$. The true processing time s_i only enters in the calculation of the completion time $S_i(\tilde{\sigma})$.

Inequality (2.21) is similar to inequality (3.1) provided $\tau_i(\theta_i, \theta_{-i})$ of inequality (3.1) is replaced by $\tau_i(\theta_i, m_{-i})$ in (2.21) where θ_{-i} is the waiting cost vector obtained from m_{-i}. Moreover, the just sequencing rule is same for both m_{-i} and θ_{-i} when θ_{-i} is obtained from m_{-i}. Therefore, for any true m_{-i}, true θ_{-i} obtained from m_{-i} and any $\theta_i, \theta'_i \in \Theta$, we must have (B) $\tau_i(\theta_i, \theta_{-i}) - \tau_i(\theta'_i, \theta_{-i}) = \tau_i(\theta_i, m_{-i}) - \tau_i(\theta'_i, m_{-i})$. From (B) it follows that for any $\theta_i \in \Theta$, any true m_{-i} and θ_{-i} obtained from m_{-i}, (C) $\tau_i(\theta_i, m_{-i}) - \tau_i(\theta_i, \theta_{-i}) := g_i(m_{-i})$. Recall that implementability of the just sequencing rule $\tilde{\sigma}$ requires that the transfer must be an ICJ transfer, that is for any θ_{-i}, and any $\theta_i \in \Theta$, the transfer $\tau_i(\theta_i, \theta_{-i})$ in (C) must be equal to the ICJ transfer $\tilde{\tau}_i(\theta_i, \theta_{-i})$ given by (2.7). Hence, from (C) we get (D) $\tau_i(\theta_i, m_{-i}) = h_i(\theta_{-i}) + g_i(m_{-i}) - \sum_{j \in P'_i(\tilde{\sigma}(\theta_i, \theta_{-i}))} \theta_j s_j$, where the functions $h_i : \Theta^{|N \setminus \{i\}|} \rightarrow \mathbb{R}$ and $g_i : \Theta^{|N \setminus \{i\}|} \times \mathbb{R}_{++}^{|N \setminus \{i\}|} \rightarrow \mathbb{R}$ are arbitrary. Without loss of generality, if we set $h_i(\theta_{-i}) + g_i(m_{-i}) = h_i^g(m_{-i})$ in (D), then we get $\tau_i(\theta_i, m_{-i}) := \tilde{\tau}_i^g(\theta_i, m_{-i})$ and the result follows. ■

Ex-post implementability with balanced transfers is not possible with two agents is a natural extension of the arguments in Proposition 2.2. By applying arguments similar to that in Theorem 2.4, it is easy to see that the general just sequencing rule $\tilde{\sigma}^g$ is ex-post implementable with balanced transfers.

DEFINITION **2.15** Consider the general sequencing problem Ω with at least three agents. For $\tilde{\sigma}^g$, a mechanism $(\tilde{\sigma}^g, \tilde{\tau}^{g*})$ is a *balanced generalized ICJ mechanism* if the transfer rule is such that for all $m \in \Theta^n \times \mathcal{S}$ and all $i \in N$,

$$\tilde{\tau}_i^{g*}(\theta_i, m_{-i}) = \sum_{j \in P_i(\tilde{\sigma}(\theta_i, \theta_{-i}))} A_j(\theta)\theta_j s_j - \sum_{j \in P_i'(\tilde{\sigma}(\theta_i, \theta_{-i}))} B_j(\theta)\theta_j s_j + g_i^g(m_{-i}), \tag{2.22}$$

where θ_{-i} is obtained from the message m_{-i}, $A_j(\theta) := (\tilde{\sigma}_j(\theta_i, \theta_{-i}) - 1)/(n-2)$, $B_j(\theta) := (n - \tilde{\sigma}_j(\theta_i, \theta_{-i}))/(n-2)$ and for any $i \in N$, $g_i^g : \Theta^{|N \setminus \{i\}|} \times \mathbb{R}_{++}^{|N \setminus \{i\}|} \rightarrow \mathbb{R}$ is such that for any $m \in \Theta^n \times \mathcal{S}$, $\sum_{i \in N} g_i^g(m_{-i}) = 0$.

PROPOSITION **2.4** Let Ω be the general sequencing problem with at least three agents. The general just sequencing rule $\tilde{\sigma}^g$ is ex-post implementable with balanced transfers if and only if the mechanism $(\tilde{\sigma}^g, \tau)$ that implements it is a balanced generalized ICJ mechanism.

2.6 Conclusion

The just sequencing rule is a simple algorithm to obtain justice in terms of Rawlsian difference principle. An attractive feature of the just sequencing rule is that it weakly lexi-max dominates the outcome efficient sequencing rule in spite of the fact that it can lead to efficiency loss since the aggregate completion cost under the just sequencing rule is no less than that under the outcome efficient sequencing rule. When the processing time of the agents are non-identical, this efficiency loss is positive for profiles where the order under the just sequencing rule is different from that of the outcome efficient sequencing rule. However, the relative efficiency loss is bounded by $(n-1)$. One can summarize the properties of the just sequencing rule from a mechanism design perspective as follows.

(C1) Like the outcome efficient sequencing rule, the just sequencing rule is implementable with balanced transfers. However, the lexi-min comparison of ICJ mechanisms and VCG mechanisms are ambiguous. We can find ICJ mechanism and VCG mechanism such that the ICJ mechanism lexi-min dominates the VCG mechanism and can find ICJ mechanism and VCG mechanism such that the VCG mechanism lexi-min dominates the ICJ mechanism. We can also find ICJ and VCG (balanced ICJ and balanced VCG) mechanism pairs such that no such lexi-min domination exists.

(C2) Unlike the outcome efficient sequencing rule, even when both waiting cost and processing time are private information, the just sequencing rule is ex-post implementable with balanced transfers. Therefore, two-dimensional incentive problem does not matter for the just sequencing rule since the agents do not benefit by misreporting their processing time as this information is not required to specify the order of using the facility under the just sequencing rule.

Chapter 3

Balanced implementability of sequencing rules

3.1 Introduction

In this chapter we address the problem of balanced implementability in sequencing problems. In a sequencing problem we have a finite set of agents each of whom has one job to process using one facility. The facility can only handle one job at a time. Once the processing of a job starts, it cannot be interrupted. Each job is characterized by processing time and waiting cost. The waiting cost represents the agent's disutility for waiting one unit of time. There is a fair amount of literature on sequencing problems (see De and Mitra (2016), Dolan (1978), Duives et al. (2015)). Assuming that processing time of the agents are common knowledge and waiting costs are private information, we identify the complete class of sequencing rules that are implementable in dominant strategies. Any rule[1] for which any agent's job completion time is non-increasing in his own waiting cost is implementable in dominant strategies. We call such rules NI rules. These results follow from the existing literature on implementation (Bikhchandani et al. (2006), Rochet (1987) and Rockafellar (2015)). For any given NI rule, we also identify all direct mechanisms that implement it. We refer to such

[1]Since this is a context of sequencing problems, whenever we use the term 'rule' (as we have done quite often in this section) we explicitly mean 'sequencing rule'.

mechanisms as "Cut off" based mechanisms. Such "Cut off" based mechanisms are present in the existing literature (see Milgrom (2004) and Myerson (1985)). Similar "Cut off" based mechanisms were also derived for scheduling problems with multiple machines and varying speed by Mishra and Mitra (2010) and for multi-dimensional dichotomous domains by Mishra and Roy (2013).

A classic result in mechanism design in quasi-linear framework is the Roberts' affine maximizer theorem (see Roberts (1979)) for multidimensional type spaces with finite set of alternatives. Roberts (1979) showed that if there are at least three alternatives and the type space is unrestricted, then every onto implementable allocation rule is an affine maximizer. There are many papers that analyze the affine maximizer allocation rules for different allocation problems (see Carbajal et al. (2013), Dobzinski and Nisan (2009), Lavi et al. (2009), Marchant and Mishra (2015), Mishra and Quadir (2014), Mishra and Sen (2012) and Nath and Sen (2015)).

Sequencing problems deal with agents' cost and hence the appropriate transformed concept of affine maximizer allocation rule is the affine cost minimizer rule. In any affine cost minimizer rule, the objective is to select that order of servicing the agents (from the set of all possible orders of servicing or from some subset of it) so as to minimize the sum of an *order-specific number* and the *weighted* sum of completion time of the agents[2]. This order-specific numbers are captured by a function κ which maps from the set (or subset) of orders to the real line and we call them the *κ-functions*. All the agent-specific weights are non-negative real numbers. We prove that any affine cost minimizer sequencing rule (whether onto or not onto) is an NI rule but the converse is not true. Specifically, for any sequencing problem with a given number of agents, we provide an example of NI rule that is not an affine cost minimizer.[3] That under different domain restrictions we can have implementable rules that are different from affine maximizers was also pointed out by Carbajal et al. (2013), Marchant and Mishra (2015) and Mishra and Quadir (2014).

[2]The term 'completion time' denotes job completion time

[3]Roberts (1979) result uses affine maximizers that are onto. Hence our result shows that class of affine cost minimizer rules, which is a generalization of Roberts (1979)'s class of affine maximizer allocation rules, is a strict subset of the class of NI rules.

The main contribution in this chapter is to identify NI sequencing problems that are implementable with balanced transfers. For the sequencing problem, implementing any NI rule with balanced transfer simply ensures that the resulting utility allocation is Pareto indifferent to the utility under the rule in the absence of private information and with zero monetary transfers. For many economic environments, implementing outcome efficiency with balanced transfers is not possible (see Hurwicz (1975), Hurwicz and Walker (1990) and Walker (1980)). However, for sequencing problems with more than two agents, it is possible to implement the outcome efficient rule with balanced transfers (see Mitra (2002) and Suijs (1996)) and it is also possible to implement the just sequencing rule (in the Rawlsian sense) with balanced transfers (see De and Mitra (2016)).

Our work establishes that there are many NI sequencing rules having real life significance, other than the outcome efficient sequencing rule and the just sequencing rule, that are implementable with balanced transfers. One obvious type of NI rules that are implementable with balanced transfers are the constant rules like the shortest processing time sequencing rule (where the shortest jobs are handled first) and the longest processing time sequencing rule (where the longer jobs are often very important and are selected first). Specifically, for sequencing problems with two agents we identify the complete class of non-constant NI rules that are implementable with balanced transfers. Specifically we show that there are exactly two types of NI rules that are implementable with balanced transfers. The first type consists of onto affine cost minimizers (that includes neither the outcome efficient sequencing rule nor the just sequencing rule) and the second type are NI rules that are not affine cost minimizers. For sequencing problems with more than two agents we identify a sufficient family of NI rules that are implementable with balanced transfers. This sufficient family of NI rules include a subset of affine cost minimizer rules with constant κ-functions (normalized to zero) and also includes a subset of NI rules that are not affine cost minimizers. We refer to this family of rules as *group priority based cost minimizer* (GP-CM) sequencing rules. These rules are defined by imposing all types of priority based partition on the set of agents. This family includes the following types of rules.

1. The family of GP-CM sequencing rules includes priority based partition where all elements of the partition are singletons so that all constant rules are included.

2. The GP-CM sequencing rules also includes grand coalition as a partition and hence includes all affine cost minimizer rules for which the agent-specific weights are positive and the κ-functions are constant (and the value of the constant is normalized to zero). Hence the outcome efficient sequencing rule (see Mitra (2002) and Suijs (1996)) and the just sequencing rule (in the Rawlsian sense) (see De and Mitra (2016)) are also members of this family of GP-CM sequencing rules.

3. There are sequencing situations where we have a well-defined priority across the set of agents. In an academic institute, faculty members may be given priority over students in using computers (or printers or photocopiers). A similar situation rises for emergency related treatment of patients where priority in treatment needs to be given based on the degree of emergency of the patients' diseases. When the number of agents to be served is known, all such situations are captured under GP-CM sequencing rules.

4. The non-affine cost minimizers NI rules included in the family of GP-CM sequencing rules are a generalization of the affine cost minimizer rules included in this family of GP-CM sequencing rules. This generalization is done replacing any subset of agents' waiting cost with a non-linear function of the waiting cost which is increasing and onto.

The chapter is organized as follows. In Section 3.2, we introduce the framework. In Section 3.3, we address the implementability issue. In Section 3.4, we obtained results on implementability with balanced transfers. This is followed by an appendix where we provide the proofs of our results.

3.2 The framework

Consider a finite set of agents $N = \{1, 2, \ldots, n\}$ in need of a facility that can be used sequentially. Using this facility, the agents want to process their jobs. The job processing time can be different for different agents. Specifically, for each agent $i \in N$, the job processing time is given by $s_i > 0$. Let $\theta_i S_i$ measure the cost of job completion for agent $i \in N$ where $S_i \in \mathbb{R}_{++}$ is the completion time for this agent and $\theta_i \in \Theta := \mathbb{R}_{++}$ denotes his constant per-period waiting cost. Due to the sequential nature of providing the service, the completion time S_i for agent i depends not only on his own processing time s_i but also on the processing time of the agents who precede him in the order of service. By means of an order $\sigma = (\sigma_1, \ldots, \sigma_n)$ on N, one can describe the positions of each agent in the order. Specifically, $\sigma_i = k$ indicates that agent i has the k-th position in the order. Let $\Sigma(N)$ be the set of $n!$ possible orders on N. We define $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ to be the predecessor set of i in the order σ, that is, set of agents served before agent i in the order σ. Similarly, $P'_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ denotes the successor set of i in the order σ, that is, set of agents served after agent i in the order σ. Let $s = (s_1, \ldots, s_n) \in \mathcal{S} := \mathbb{R}^n_{++}$ denote the vector of processing time of the agents. Given a vector $s = (s_1, \ldots, s_n) \in \mathcal{S}$ and an order $\sigma \in \Sigma(N)$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the completion time is $S_j(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. The agents have quasi-linear utility of the form $U_i(\sigma, \tau_i; \theta_i); s_{-i}) = -\theta_i S_i(\sigma) + \tau_i$ where σ is the order, $\tau_i \in \mathbb{R}$ is the transfer that he receives and the parameter of the model θ_i that constitutes of the waiting cost θ_i. If the processing time vector $s \in \mathcal{S}$ is given and waiting cost is private information, then we have a sequencing problem $\Omega^s_N = (\Theta^n, s)$.

A typical profile of waiting costs is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$. For any $i \in N$, let θ_{-i}, denote the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \Theta^{|N \setminus \{i\}|}$ which is obtained from the profile θ by eliminating i's waiting cost where for any set X, $|X|$ denotes the cardinality of X. For a given sequencing problem Ω^s_N, a (direct revelation) mechanism is (σ, τ) that constitutes of a sequencing rule σ and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \to \Sigma(N)$ that specifies for each profile $\theta \in \Theta^n$ a unique order

$\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma(N)$. Because the sequencing rule is a function (and not a correspondence) we will require tie-breaking rule to reduce a correspondence to a function which, unless explicitly discussed, is assumed to be fixed. We use the following tie-breaking rule. We take the linear order $1 \succ 2 \succ \ldots \succ n$ on the set of agents N. For any sequencing rule σ and any profile $\theta \in \Theta^n$ with a tie between agents $i, j \in N$, we pick the order $\sigma(\theta)$ with $\sigma_i(\theta) < \sigma_j(\theta)$ if and only if $i \succ j$. A *transfer rule* is a function $\tau : \Theta^n \to \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_i(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, given any sequencing problem Ω_N^s and given any mechanism (σ, τ), if (θ_i', θ_{-i}) is the announced profile when the true waiting cost of i is θ_i, then utility of i is $U_i(\sigma(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i) = -\theta_i S_i(\sigma(\theta_i', \theta_{-i})) + \tau_i(\theta_i', \theta_{-i})$.

3.3 Implementability criterion for sequencing rules

DEFINITION **3.1** A mechanism (σ, τ) is *strategy-proof* if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i, \theta_i' \in \Theta$ and any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$,

$$U_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq U_i(\sigma(\theta_i', \theta_{-i}), \tau_i(\theta_i', \theta_{-i}); \theta_i). \tag{3.1}$$

When the mechanism is strategy-proof then it can implement the allocation rule in true sense. In other words, implementation of a rule σ via a mechanism (σ, τ) requires that the transfer rule τ is such that truthful reporting for any agent weakly dominates false report irrespective of other agents' report.

DEFINITION **3.2** A sequencing rule σ is *non-increasing* (or NI) if for any $i \in N$ and any $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the chosen order $\sigma(\theta_i, \theta_{-i})$ for each $\theta_i \in \Theta$ is such that the completion time $S_i(\sigma(\theta_i, \theta_{-i}))$ is non-increasing in θ_i.

PROPOSITION **3.1** A sequencing rule σ is implementable if and only if it is an NI sequencing rule.

The proof is obvious from the existing literature and hence omitted. In particular, non-increasingness is the weak monotonicity (or two-cycle monotonicity) for the sequencing problems. From Bikhchandani et al. (2006) we know that, for any deterministic rule (like the sequencing rules we have for Ω_N^s), weak monotonicity is necessary and sufficient for implementation in dominant strategies. Let $NI(\Omega_N^s)$ denote the set of all NI sequencing rules. In the next proposition we derive the complete class of mechanisms that implement any NI sequencing rule.

Given a processing time vector $s \in \mathcal{S}$ and the sequencing problem Ω_N^s, consider an agent $i \in N$. Depending on his waiting cost $\theta_i \in \Theta$, agent i can face a maximum of 2^{n-1} (specifically, $\sum_{j=0}^{n-1} \binom{n-1}{j}$) different completion times. But the number of different completion time that any agent i actually faces depends on the profile $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$ and, more importantly, on the underlying sequencing rule whose implementation is under consideration. So depending on the sequencing rule and the profile θ_{-i}, the agent can face a single completion time (like in the constant sequencing rule) or more than one completion time (like in any outcome efficient sequencing rule).

Consider any $\sigma \in NI(\Omega_N^s)$ and any $i \in N$. Fix a profile θ_{-i} and let us assume that the number of different completion time that agent i faces, as θ_i varies over Θ, is T. Given that σ is NI, this means that either $T = 1$ or $T \geq 2$ and there exists a waiting cost cut off vector $(\theta_i^{(0)}, \theta_i^{(1)}, \ldots, \theta_i^{(T-1)}, \theta_i^{(T)})$ where $0 := \theta_i^{(T)} < \theta_i^{(T-1)} < \ldots < \theta_i^{(2)} < \theta_i^{(1)} < \theta_i^{(0)} := \infty$ such that for any $t \in \{1, \ldots, T\}$, $S_i(\sigma(\theta_i^t, \theta_{-i})) := \bar{S}(t, \theta_{-i})$ for all $\theta_i^t \in (\theta_i^{(t)}, \theta_i^{(t-1)})$. Define $D_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - \bar{S}(t, \theta_{-i})$ and $\overline{D}_t(\theta_{-i}) := \bar{S}(t+1, \theta_{-i}) - S_i(\sigma(\theta_i^{(t)}, \theta_{-i}))$ for any $t \in \{1, \ldots, T-1\}$. Observe that the difference in the definitions of $D_t(\theta_{-i})$ and $\overline{D}_t(\theta_{-i})$ lies in the second term. While for the $D_t(\theta_{-i})$ case, $\bar{S}(t, \theta_{-i})$ is the completion time of agent i when his waiting cost is any number θ_i^t that lies in the open interval $(\theta_i^{(t)}, \theta_i^{(t-1)})$ and for the $\overline{D}_t(\theta_{-i})$ case, $S_i(\sigma(\theta_i^{(t)}, \theta_{-i}))$ is the completion time of agent i when his waiting cost is exactly $\theta_i^{(t)}$ which is a cut off point. Depending on the tie-breaking rule, the numbers $\bar{S}(t, \theta_{-i})$ and $S_i(\sigma(\theta_i^{(t)}, \theta_{-i}))$ may or may not be different and hence for completeness of the analysis, the distinction between $D_t(\theta_{-i})$ and $\overline{D}_t(\theta_{-i})$ is necessary.

DEFINITION **3.3** Consider any $\sigma \in NI(\Omega_N^s)$ and a mechanism (σ, τ) with transfer rule $\tau : \Theta^n \to \mathbb{R}^n$. The mechanism is *"Cut off" based* if the transfer rule τ

is obtained from the following procedure. For each $i \in N$, we first select any function $h_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ and then, given any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, we consider the waiting cost cut off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ where $0 := \theta_i^{(T)} < \theta_i^{(T-1)}(\theta_{-i}) < \ldots < \theta_i^{(2)}(\theta_{-i}) < \theta_i^{(1)}(\theta_{-i}) < \theta_i^{(0)} := \infty$. Given the selected function $h_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$, for any profile θ_{-i} of all but agent i and the associated cut off vector $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$, the transfer of agent i is the following:

(PI1) For any $\theta_i \in \Theta \setminus \{\theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i})\}$, $\tau_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i, \theta_{-i})$ where

$$I_i(\theta_i, \theta_{-i}) = \begin{cases} 0 & \text{if } \theta_i \in (\theta_i^{(T)}, \theta_i^{(T-1)}(\theta_{-i})), \\ \sum\limits_{r=t}^{T-1} \theta_i^{(r)}(\theta_{-i}) D_r(\theta_{-i}) & \text{if } \theta_i \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i})), t = \{1, \ldots, T-1\} \;\&\; T \geq 2. \end{cases} \tag{3.2}$$

(PI2) For $T \geq 2$, any $t \in \{1, \ldots, T-1\}$ and cut off point $\theta_i^{(t)}(\theta_{-i})$, $\tau_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = h_i(\theta_{-i}) - I_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i})$ where the incentive payment $I_i(\theta_i^{(t)}(\theta_{-i}), \theta_{-i}) = I_i(\theta_i^t, \theta_{-i}) - \theta_i^{(t)}(\theta_{-i})\overline{D}_t(\theta_{-i}) + \theta_i^{(t)}(\theta_{-i}) D_t(\theta_{-i})$ and $\theta_i^t \in (\theta_i^{(t)}(\theta_{-i}), \theta_i^{(t-1)}(\theta_{-i}))$.

Definition 3.3 specifies the following. For each agent i and each profile θ_{-i} of waiting costs of all but agent i, we get a set of cut off points $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$ for agent i that depends on the specific NI sequencing rule. The transfers associated with "Cut off" based mechanism requires that each agent i gets an agent-specific constant $h_i(\theta_{-i})$ that depends on the waiting cost of all other agents and, if agents i's waiting cost θ_i is greater than the smallest non-zero cut off value $\theta_i^{(T-1)}(\theta_{-i})$, agent i also has to make an *incentive payment* $I_i(\theta_i, \theta_{-i})$ that depends of the set of cut off values that are less than the waiting costs of agent i. For each such cut off value $\theta_i^{(r)}(\theta_{-i})$, agent i pays $\theta_i^{(r)}(\theta_{-i}) D_r(\theta_{-i})$ which is the cut off value times the absolute difference between the completion time of agent i below and above this cut off value. If agent i's waiting cost coincides with a cut off point then, ceteris paribus, his incentive payment needs to be adjusted by changing the difference in completion time term $D_r(\theta_{-i})$ to the difference in completion time below and at the cut off point $\overline{D}_r(\theta_{-i})$ only for the highest cut off value less than the waiting cost of agent i. Whenever the dependence of the cut off points of agent i for any given

θ_{-i} is clear, we will write the cut off vector as $(\theta_i^{(0)}, \theta_i^{(1)}, \ldots, \theta_i^{(T-1)}, \theta_i^{(T)})$ instead of $(\theta_i^{(0)}, \theta_i^{(1)}(\theta_{-i}), \ldots, \theta_i^{(T-1)}(\theta_{-i}), \theta_i^{(T)})$.

PROPOSITION **3.2** Any $\sigma \in NI(\Omega_N^s)$ is implementable via a mechanism (σ, τ) if and only if the mechanism is "Cut off" based.

The proof is again obvious from the existing literature and hence omitted. Specifically, such "Cut off" based mechanisms is just a manifestation of the usual integral form of incentive compatible transfers derived in Myerson (1985) and is referred to as Holmström's Lemma in Milgrom (2004). Such "Cut off" based mechanisms for multi-dimensional dichotomous preferences was derived by Mishra and Roy (2013). For scheduling problems, such "Cut off" based mechanisms were derived by Mishra and Mitra (2010).

There are many natural examples of NI sequencing rules.

DEFINITION **3.4** A sequencing rule $\bar{\sigma}$ is a *constant sequencing rule* if there is a fixed order $\bar{\sigma} \in \Sigma(N)$ such that the agents are always served in this fixed order $\bar{\sigma}$, that is, for any $\theta \in \Theta^n$, $\sigma(\theta) = \bar{\sigma}$.

There are many priority rules that are constant sequencing rules. For the constant sequencing rule with $\bar{\sigma}$ as the state independent order, for each $i \in N$ and for any given $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the completion time of agent i is fixed at $S_i(\bar{\sigma}) = s_i + \sum_{j \in P_i(\bar{\sigma})} s_j$ for all $\theta_i \in \Theta$ implying non-increasingness in θ_i. Hence it satisfies NI. Two other NI sequencing rules from the existing literature on sequencing problems are the following.

DEFINITION **3.5** A sequencing rule σ^* is *outcome efficient* if for any profile $\theta \in \Theta^n$, $\sigma^*(\theta) \in \operatorname{argmin}_{\sigma \in \Sigma(N)} \sum_{i \in N} \theta_i S_i(\sigma)$.

For each profile the outcome efficient sequencing rule selects an order to minimize the aggregate cost of completion time. Define $u_i := \theta_i / s_i$ as the urgency index of agent i which is the ratio of his waiting cost and his processing time. From Smith (1956) we know that for any sequencing problem Ω_N^s a sequencing rule σ^* is outcome efficient if and only if the following condition holds.

(OE) For any profile $\theta \in \Theta^n$, the selected order $\sigma^*(\theta)$ satisfies the following condition: for any $i, j \in N, \theta_i/s_i \geq \theta_j/s_j \Leftrightarrow \sigma^*_i(\theta) \leq \sigma^*_j(\theta)$.

Clearly, outcome efficient sequencing rule σ^* is NI. Outcome efficiency and incentives has been extensively analyzed in the sequencing literature (see Mitra (2002),Suijs (1996) and De and Mitra (2016)).

DEFINITION **3.6** A sequencing rule $\tilde{\sigma}$ is *just* if for each profile $\theta \in \Theta^n$, the chosen order $\tilde{\sigma}(\theta)$ satisfies the following property: for any $i, j \in N$ such that $\theta_i \geq \theta_j$, $\tilde{\sigma}_i(\theta) \leq \tilde{\sigma}_j(\theta)$.[4]

Just sequencing rule was analyzed in De and Mitra (2016). Clearly, the just sequencing rule $\tilde{\sigma}$ is NI. The constant sequencing rule, the outcome efficient sequencing rule and the just sequencing rule are all affine cost minimizer sequencing rules.

DEFINITION **3.7** A sequencing rule $\sigma^{w,\kappa} : \Theta^n \to \Sigma(N)$ is an *affine cost minimizer* (ACM) if for each $\theta \in \Theta^n$, $\sigma^{w,\kappa}(\theta) \in \arg\min_{\sigma \in \Sigma'(N)} \left\{\kappa(\sigma) + \sum_{j \in N} w_j \theta_j S_j(\sigma)\right\}$, where $\Sigma'(N) \subseteq \Sigma(N)$, $w_j \geq 0$ for all $j \in N$ and $\kappa : \Sigma'(N) \to \mathbb{R}$.

The next two examples are NI sequencing rules that are not ACM.

EXAMPLE **3.1** Consider any sequencing problem Ω^s_N with $|N| = 2$. Define the sequencing rule σ^V such that, given any two positive numbers a_1 and a_2, it satisfies the following: For any profile $\theta = (\theta_1, \theta_2)$ such that $\theta_1 < a_1$ and $\theta_2 > a_2$, $\sigma^V(\theta) = (\sigma^V_1(\theta) = 2, \sigma^V_2(\theta) = 1)$. For all other profiles $\theta' = (\theta'_1, \theta'_2)$ such that either $\theta'_1 \geq a_1$ or $\theta'_2 \leq a_2$, $\sigma^V(\theta') = (\sigma^V_1(\theta') = 1, \sigma^V_2(\theta') = 2)$.

One can easily verify that σ^V is NI. If $\theta''_2 \leq a_2$, then for any $\theta_1 \in \Theta$, $\sigma^V_1(\theta_1, \theta''_2) = 1$ and hence $S_1(\sigma^V(\theta_1, \theta_2)) = s_1$ is non-increasing in θ_1 for any given $\theta_2 \geq a_2$. If $\theta'_2 > a_2$, then for any $\theta_1 \in (0, a_1)$, $\sigma^V_1(\theta_1, \theta'_2) = 2$ and agent 1's completion time is $S_1(\sigma^V(\theta_1, \theta'_2)) = s_2 + s_1$ and for any $\theta'_1 \geq a_1$, $\sigma^V_1(\theta'_1, \theta'_2) = 1$ and agent 1's completion time is $S_1(\sigma^V(\theta'_1, \theta'_2)) = s_1$. Hence, we have non-increasingness of completion time $S_1(\sigma^V(\theta_1, \theta'_2))$ in θ_1 for any given $\theta'_2 > a_2$. Similarly, if we fix $\theta''_1 \geq a_1$, then for any

[4]Given the tie-breaking rule, for any profile $\theta \in \Theta^n$, both the selections $\tilde{\sigma}(\theta)$ for the just sequencing rule and $\sigma^*(\theta)$ for the outcome efficient sequencing rule satisfy profile contingent uniqueness.

$\theta_2 \in \Theta$, $\sigma_2^V(\theta_1'', \theta_2) = 2$ and hence $S_2(\sigma^V(\theta_1'', \theta_2)) = s_1 + s_2$ is non-increasing in θ_2 for any given $\theta_1'' \geq a_1$. If $\theta_1 < a_1$, then for any $\theta_2 \in (0, a_2]$, $\sigma_2^V(\theta_1, \theta_2) = 2$ and agent 2's completion time is $S_2(\sigma^V(\theta_1, \theta_2)) = s_2 + s_1$. For any $\theta_2' > a_1$, $\sigma_2^V(\theta_1, \theta_2') = 1$ and agent 2's completion time is $S_2(\sigma^V(\theta_1, \theta_2')) = s_2$. Hence $S_2(\sigma^V(\theta_1, \theta_2))$ is non-increasing in θ_2 for any given $\theta_1 < a_1$. That σ^V is not an ACM sequencing rule will follow from Proposition3.3.

"Cut off" based mechanisms: Consider the sequencing problem Ω_N^s with $|N| = 2$ and consider the sequencing rule σ^V. If we fix $\theta_2'' \leq a_2$, then for any $\theta_1 \in \Theta$, $\sigma^V(\theta_1, \theta_2'') = (\sigma_1^V(\theta_1, \theta_2'') = 1, \sigma_2^V(\theta_1, \theta_2'') = 2)$. In that case the "Cut off" based transfer gives $\tau_1^V(\theta_1, \theta_2'') = h_1(\theta_2'')$ for all $\theta_1 \in \Theta$ since, given θ_2'', the cut off point for agent 1 is $\theta_1^{(1)} = \theta_1^{(T)} = 0$. Therefore, given any $\theta_2'' \leq a_2$, the incentive payment of agent 1 is $I_1^V(\theta_1, \theta_2'') = 0$ for all $\theta_1 \in \Theta$. If we fix $\theta_2' > a_2$, then for any $\theta_1 \in (0, a_1)$, $\sigma^V(\theta_1, \theta_2') = (\sigma_1^V(\theta_1, \theta_2') = 2, \sigma_2^V(\theta_1, \theta_2') = 1)$ and for any $\theta_1' \geq a_1$, $\sigma^V(\theta_1', \theta_2') = (\sigma_1^V(\theta_1', \theta_2') = 1, \sigma_2^V(\theta_1', \theta_2') = 2)$. Hence, given θ_2', the cut off point for agent 1 is $\theta_1^{(1)} = \theta_1^{(T-1)} = a_1$. Therefore, given any $\theta_2' > a_2$, the incentive payment of agent 1 is

$$I_1^V(\theta_1, \theta_2') = \begin{cases} 0 & \text{if } \theta_1 \in (0, a_1), \\ a_1 s_2 & \text{if } \theta_1 \geq a_1. \end{cases}$$

The "Cut off" based transfer for agent 1 is $\tau_1^V(\theta_1, \theta_2') = h_1(\theta_2')$ for all $\theta_1 \in (0, a_1)$ and $\tau_1^V(\theta_1', \theta_2') = h_1(\theta_2') - a_1 s_2$ for all $\theta_1' \geq a_1$.

If we fix $\theta_1'' \geq a_1$, then for any $\theta_2 \in \Theta$, $\sigma^V(\theta_1'', \theta_2) = (\sigma_1^V(\theta_1'', \theta_2) = 1, \sigma_2^V(\theta_1'', \theta_2) = 2)$. In that case the "Cut off" based transfer gives $\tau_2^V(\theta_1'', \theta_2) = h_2(\theta_1'')$ for all $\theta_2 \in \Theta$ since, given θ_1'', the cut off point for agent 2 is $\theta_2^{(1)} = \theta_2^{(T)} = 0$. Therefore, given any $\theta_1'' \geq a_1$, the incentive payment of agent 2 is $I_2^V(\theta_1'', \theta_2'') = 0$ for all $\theta_2 \in \Theta$. If we fix $\theta_1 < a_1$, then for any $\theta_2 \in (0, a_2]$, $\sigma^V(\theta_1, \theta_2) = (\sigma_1^V(\theta_1, \theta_2) = 1, \sigma_2^V(\theta_1, \theta_2) = 2)$ and for any $\theta_2' > a_2$, $\sigma^V(\theta_1, \theta_2') = (\sigma_1^V(\theta_1, \theta_2') = 2, \sigma_2^V(\theta_1, \theta_2') = 1)$. Hence, given $\theta_1 < a_1$, the cut off point for agent 2 is $\theta_2^{(1)} = \theta_2^{(T-1)} = a_2$. Therefore, given any $\theta_1 < a_1$, the incentive payment of agent 2 is

$$I_2^V(\theta_1, \tilde{\theta}_2) = \begin{cases} 0 & \text{if } \tilde{\theta}_2 \in (0, a_2], \\ a_2 s_1 & \text{if } \tilde{\theta}_2 > a_2. \end{cases}$$

The “Cut off" based transfer for agent 2 is $\tau_2^V(\theta_1, \theta_2) = h_2(\theta_1)$ for all $\theta_2 \in (0, a_2]$ and $\tau_2^V(\theta_1, \theta_2') = h_2(\theta_1) - a_2 s_1$ for all $\theta_2' > a_2$. Therefore, from all these cases, the “Cut off" based transfer of the two agents is the following: For any profile $\theta \in \Theta^2$,

$$\tau_1^V(\theta) = \begin{cases} h_1(\theta_2) & \text{if } \theta_2 > a_2 \text{ and } \theta_1 \in (0, a_1), \\ h_1(\theta_2) - a_1 s_2 & \text{otherwise.} \end{cases} \tag{3.3}$$

$$\tau_2^V(\theta) = \begin{cases} h_2(\theta_1) - a_2 s_1 & \text{if } \theta_2 > a_2 \text{ and } \theta_1 \in (0, a_1), \\ h_2(\theta_1) & \text{otherwise.} \end{cases} \tag{3.4}$$

EXAMPLE **3.2** Consider any sequencing problem Ω_N^s with $|N| \geq 3$. Define the sequencing rule σ^{NA} that satisfies the following properties:

1. For any profile such that the urgency index of agent 1 is no smaller than the smallest urgency index of all other agents, agent 1 is served first and all other agents are served, after agent 1 completes his jobs, in the non-increasing order of their urgency indexes. Formally, let θ be a profile such that $\theta_1/s_1 \geq \min_{j \in N \setminus \{1\}}(\theta_j/s_j)$. Then $\sigma^{NA}(\theta)$ specifies that $1 = \sigma_1^{NA}(\theta) < \sigma_j^{NA}(\theta)$ for any $j \in N \setminus \{1\}$, and, for any $j, k \in N \setminus \{1\}$, $\sigma_j^{NA}(\theta) \leq \sigma_j^{NA}(\theta)$ if and only if $(\theta_j/s_j) \geq (\theta_k/s_k)$.

2. For any profile such that the urgency index of agent 1 is smaller than the smallest urgency index of all other agents, agent 1 is served last and all other agents are served, before agent 1, according to the non-increasing order of their urgency indexes. Formally, let θ' be a profile such that $\theta_1'/s_1 < \min_{j \in N \setminus \{1\}}(\theta_j'/s_j)$. Then $\sigma^{NA}(\theta')$ specifies that $n = \sigma_1^{NA}(\theta') > \sigma_j^{NA}(\theta')$ for any $j \in N \setminus \{1\}$, and, for any $j, k \in N \setminus \{1\}$, $\sigma_j^{NA}(\theta') \leq \sigma_k^{NA}(\theta')$ if and only if $(\theta_j'/s_j) \geq (\theta_k'/s_k)$.

It is quite easy to see that the sequencing rule σ^{NA} satisfies NI. That σ^{NA} is not an affine cost minimizer will follow from Proposition 3.3.

"Cut off" based mechanisms: Consider the sequencing problem Ω^s_N with $|N| \geq 3$ and consider the sequencing rule σ^{NA}. To determine the transfer of agent 1, consider any profile $\theta_{-i} \in \Theta^{|N\setminus\{1\}|}$ and find the minimum urgency index for the agents $N \setminus \{1\}$, that is, find $\min_{k\in N\setminus\{1\}}(\theta_k/s_k)$. If $\theta_1 < s_1 \min_{k\in N\setminus\{1\}}(\theta_k/s_k)$, then $\sigma^{NA}_1(\theta_1, \theta_{-1}) = n$ and agent 1's transfer is $h_1(\theta_{-1})$. If $\theta'_1 \geq s_1 \min_{k\in N\setminus\{1\}}(\theta_k/s_k)$, then $\sigma^{NA}_1(\theta'_1, \theta_{-1}) = 1$ and agent 1 gets $h_1(\theta_{-1})$ and his payment is $s_1 \min_{k\in N\setminus\{1\}}(\theta_k/s_k)[S_1(\sigma^{NA}(\theta_1, \theta_{-1})) - S_1(\sigma^{NA}(\theta'_1, \theta_{-1}))]$. Therefore, agent 1's cut off point given the profile θ_{-1} is $\theta^{T-1}_1 = s_1 \min_{k\in N\setminus\{1\}}(\theta_k/s_k)$ and his incentive payment is the following:

$$I^{NA}_1(\theta) = \begin{cases} 0 & \text{if } P'_1(\sigma^{NA}(\theta)) = \emptyset, \\ s_1 \left\{ \min\limits_{k\in N\setminus\{1\}} \left(\frac{\theta_k}{s_k}\right) \right\} \sum\limits_{j\in N\setminus\{1\}} s_j & \text{otherwise.} \end{cases}$$

Hence for agent 1, the "Cut off" based transfer for any profile $\theta \in \Theta^n$ is the following:

$$\tau^{NA}_1(\theta) = \begin{cases} h_1(\theta_{-1}) & \text{if } P'_1(\sigma^{NA}(\theta)) = \emptyset, \\ h_1(\theta_{-1}) - s_1 \left\{ \min\limits_{k\in N\setminus\{1\}} \left(\frac{\theta_k}{s_k}\right) \right\} \sum\limits_{j\in N\setminus\{1\}} s_j & \text{otherwise.} \end{cases} \tag{3.5}$$

For any agent $i \in N \setminus \{1\}$, consider any profile $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$. We can have two possibilities-(a) $(\theta_1/s_1) \geq \min_{k\in N\setminus\{1,i\}}(\theta_k/s_k)$ and (b) $(\theta_1/s_1) < \min_{k\in N\setminus\{1,i\}}(\theta_k/s_k)$. If possibility (a) holds, then $\sigma^{NA}_1(\theta_i, \theta_{-i}) = 1$ for all $\theta_i \in \Theta$. Assume that the order of the urgency indexes for the $N \setminus \{1, i\}$ agents is $u_{(1)} \geq \ldots \geq u_{(n-2)}$, then due to sequencing rule σ^{NA}, we have the following: As θ_i increases from any positive number $\theta^n_i \in (0, s_i u_{(n-2)})$ to any positive number $\theta^2_i > s_i u_{(1)}$, the completion time $S_i(\sigma^{NA}(\theta_i, \theta_{-i}))$ weakly decreases from $S_i(\sigma^{NA}(\theta^n_i, \theta_{-i})) = s_1 + s_i + \sum_{j\in N\setminus\{1,i\}} s_j$ to $S_i(\sigma^{NA}(\theta^2_i, \theta_{-i})) = s_1 + s_i$. The cut off points where agent i's S_i changes are the distinct numbers from the set $(u_{(1)}, \ldots, u_{(n-2)})$. Assume that there $T - 1$ distinct urgency indexes in $(u_{(1)}, \ldots, u_{(n-2)})$, that is, $u_{\mu(1)} > \ldots > u_{\mu(T-1)}$. The difference in transfer between any $\theta^{r+1}_i \in (s_i u_{\mu(r+1)}, s_i u_{\mu(r)})$ and $\theta^r_i \in (s_i u_{\mu(r)}, s_i u_{\mu(r-1)})$ is

$$\tau^{NA}_i(\theta^{r+1}_i, \theta_{-i}) - \tau^{NA}_i(\theta^r_i, \theta_{-i}) = s_i u_{(r)}[S_i(\sigma^{NA}(\theta^{r+1}_i, \theta_{-i})) - S_i(\sigma^{NA}(\theta^r_i, \theta_{-i}))].$$

If possibility (b) holds, then if $\theta_i' \leq s_i(\theta_1/s_1)$, then $\sigma_i^{NA}(\theta_i, \theta_{-i}) = n$ and agent i has a transfer of $h_i(\theta_{-i})$. However, $\sigma_1^{NA}(\theta_i, \theta_{-i}) = n$ for all $\theta_i > s_i(\theta_1/s_1)$. Assume that the order of the urgency indexes for the $N \setminus \{1, i\}$ agents is $u_{(1)} \geq \ldots \geq u_{(n-2)}$, then due to sequencing rule σ^{NA}, we have the following: As θ_i increases from any positive number $\theta_i^{n-1} \in (s_i(\theta_1/s_s), s_i u_{(n-2)})$ to any positive number $\theta_i^1 > s_i u_{(1)}$, the completion time $S_i(\sigma^{NA}(\theta_i, \theta_{-i}))$ weakly decreases from $S_i(\sigma^{NA}(\theta_i^{n-1}, \theta_{-i})) = s_i + \sum_{j \in N \setminus \{1,i\}} s_j$ to $S_i(\sigma^{NA}(\theta_i^1, \theta_{-i})) = s_i$. The cut off points where agent i's S_i changes are the distinct numbers from the set $(u_{(1)}, \ldots, u_{(n-2)})$. The remaining argument is similar to possibility (a).

From possibilities (a) and (b) we get that the incentive payment of any $i \in N \setminus \{1\}$ is the following:

$$I_i^{NA}(\theta) = \begin{cases} 0 & \text{if } P_i'(\sigma^{NA}(\theta)) = \emptyset, \\ s_i \sum_{j \in P_i'(\sigma^{NA}(\theta))} \theta_j & \text{otherwise.} \end{cases}$$

Therefore, the "Cut off" based transfer for any $i \in N \setminus \{1\}$ and any profile $\theta \in \Theta^n$ is the following:

$$\tau_i^{NA}(\theta) = \begin{cases} h_i(\theta_{-i}) & \text{if } P_i'(\sigma^{NA}(\theta)) = \emptyset, \\ h_i(\theta_{-i}) - s_i \sum_{j \in P_i'(\sigma^{NA}(\theta))} \theta_j & \text{otherwise.} \end{cases} \tag{3.6}$$

Let $ACM(\Omega_N^s)$ denote the set of all affine cost minimizer sequencing rules for any given sequencing problem Ω_N^s.

PROPOSITION **3.3** For any Ω_N^s, $ACM(\Omega_N^s) \subseteq NI(\Omega_N^s)$ and $ACM(\Omega_N^s) \neq NI(\Omega_N^s)$.

3.4 Balanced implementability

DEFINITION **3.8** A sequencing rule σ is *implementable with balanced transfers* if there exists a mechanism (σ, τ) that implements it with budget balancing transfers where budget balancing transfers require that for all $\theta \in \Theta^n$, $\sum_{j \in N} \tau_j(\theta) = 0$.

Implementing a sequencing rule with balanced transfers simply means information extraction is done costlessly. That is, if any sequencing rule is implementable with balanced transfer, then, for any given profile θ, the utility of the agents are such that neither it is Pareto dominated nor it Pareto dominates the utility allocation of the agents under the same profile θ when there is complete information with no monetary transfers.

Consider any $\sigma \in NI(\Omega_N^s)$ and any "Cut off" based mechanism (σ, τ). For any $\theta \in \Theta^n$, the "Cut off" based transfer for any $i \in N$ is $\tau_i(\theta) = h_i(\theta_{-i}) - I_i(\theta)$ where $h_i(\theta_{-i})$ is the agent-specific number that depends on the profile θ_{-i} of all but i and $I_i(\theta)$ is his incentive payment. Define $I(\theta) := \sum_{i \in N} I_i(\theta)$ as the *aggregate incentive payment* for the profile $\theta \in \Theta^n$. Any NI sequencing rule σ is implementable with balanced transfers if and only if there exists a "Cut off" based mechanism (σ, τ) with given functions $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ for all $i \in N$ such that for any profile $\theta \in \Theta^n$, $\sum_{i \in N} \tau_i(\theta) = I(\theta) - \sum_{i \in N} h_i(\theta_{-i}) = 0 \Leftrightarrow I(\theta) = \sum_{i \in N} h_i(\theta_{-i})$. Therefore any $\sigma \in NI(\Omega_N^s)$ is implementable with balanced transfer if and only if for any $\theta \in \Theta^n$,

$$I(\theta) = \sum_{i \in N} h_i(\theta_{-i}). \tag{3.7}$$

Thus for budget balance we require that the profile contingent aggregate incentive payment is $(n-1)$ type separable.

REMARK **3.1**

1. Any constant sequencing rule $\bar{\sigma}$ satisfies condition (3.7). In particular, for any profile $\theta \in \Theta^n$, the incentive payment of any $i \in N$ is $I_i(\theta) = 0$. Hence for any profile $\theta \in \Theta^n$, $I(\theta) = \sum_{i \in N} I_i(\theta) = 0$ and condition (3.7) holds. For any constant sequencing rule $\bar{\sigma}$, the "Cut off" based transfer specifies that for any $\theta \in \Theta^n$, $\tau_i(\theta) = h_i(\theta_{-i})$ for all $i \in N$. By setting the transfer ($h_i(\theta_{-i})$) of all agents at zero we can achieve implementability with balanced transfers.

2. Let $\sigma^{w,\kappa} \in ACM(\Omega_N^s)$ with the property that there exists $j \in N$ such that $w_j = 0$. For this $\sigma^{w,\kappa}$, the following property holds: For any $\theta_{-j} \in \Theta^{|N \setminus \{j\}}$ and any

$\theta_j, \theta'_j \in \Theta$, $\sigma^{w,\kappa}(\theta_j, \theta_{-j}) = \sigma^{w,\kappa}(\theta'_j, \theta_{-j})$. Hence the incentive payment of agent j is $I_j(\theta) = 0$ for all $\theta \in \Theta^n$ and for any $i \in N \setminus \{j\}$, the incentive payment is independent of θ_j so that $I_i(\theta) := K_i(\theta_{-j})$ for any $\theta \in \Theta^N$. Therefore, for any $\theta \in \Theta^n$, $I(\theta) = \sum_{i \in N} I_i(\theta) = \sum_{i \in N \setminus \{j\}} K_i(\theta_{-j}) := K(\theta_{-j})$ and condition (3.7) holds. If we take the 'cut off' based transfer such that for any $\theta \in \Theta^n$, $h_j(\theta_{-j}) = K(\theta_{-j})$ and $h_i(\theta_{-i}) = 0$ for all $i \in N \setminus \{j\}$, then we get budget balance.

An implication of budget balanced VCG mechanism for outcome efficient allocation rules was provided by Walker (1980) which is better known as the Cubical Array Lemma. For any NI sequencing rule σ with "Cut off" based mechanisms we get something similar in terms of aggregate incentive payment which is stated in the next proposition. Before stating the next proposition we introduce some more notations. For any pair of profiles $\theta = (\theta_1, \theta_2, \ldots, \theta_n), \theta' = (\theta'_1, \theta'_2, \ldots, \theta'_n) \in \Theta^n$ and any $S \subseteq N$, let $\theta(S) = (\theta_1(S), \theta_2(S) \ldots, \theta_n(S)) \in \Theta^n$ be a profile such that

$$\theta_j(S) = \begin{cases} \theta_j & \text{if } j \notin S, \\ \theta'_j & \text{if } j \in S. \end{cases} \tag{3.8}$$

Observe that $\theta(S = \emptyset) = \theta$, $\theta(S = \{i\}) = (\theta'_i, \theta_{-i})$, $\theta(S = \{i, j\}) = (\theta'_i, \theta'_j, \theta_{-i-j})$ and so on, $\theta(S = N \setminus \{i\}) = (\theta_i, \theta'_{-i})$ and $\theta(S = N) = \theta'$.

LEMMA **3.1** For any $\sigma \in NI(\Omega^s_N)$, we can find a "Cut off" based mechanism (σ, τ) that implements σ with balanced transfers *only if* for all pairs of profiles $\theta, \theta' \in \Theta^n$,

$$\sum_{S \subseteq N} (-1)^{|S|} I(\theta(S)) = 0. \tag{3.9}$$

Condition (3.9) in Lemma 3.1 states that the weighted aggregate incentive payment must add up to zero while moving from profile θ to any other profile θ' by allowing for all possible group deviations. The weights are all (-1) for groups with odd number of agents and are 1 for groups with even number of agents. The proof of Lemma 3.1 is similar to the proof of the Cubical Array Lemma due to Walker (1980) and hence a formal proof is not provided. For $N = \{1, 2\}$ implementation of any NI sequencing rule σ with balanced transfer requires that condition (3.7) holds, that is for all $\theta = (\theta_1, \theta_2) \in \Theta^2$, (a) $I(\theta_1, \theta_2) = h_1(\theta_2) + h_2(\theta_1)$. Using (a) it follows that for any $\theta, \theta' \in \Theta^2$, $\sum_{S \subseteq N} (-1)^{|S|} I(\theta(S)) = I(\theta_1, \theta_2) - I(\theta'_1, \theta_2) - I(\theta_1, \theta'_2) + I(\theta'_1, \theta'_2) = [h_1(\theta_2) + h_2(\theta_1)] - [h_1(\theta_2) + h_2(\theta'_1)] - [h_1(\theta'_2) + h_2(\theta_1)] + [h_1(\theta'_2) + h_2(\theta'_1)] = 0$. Lemma 3.1 is a generalization of this idea.

The next remark shows how Lemma 3.1 can help us identify necessary restrictions for implementability of any sequencing rule with balanced transfers.

REMARK **3.2**

1. For any NI sequencing rule σ^V of Example 3.1, Lemma 3.1 puts a restriction on the vector $a = (a_1, a_2)$. Specifically, for any pair of profiles $\theta = (\theta_1 = a_1 + \delta_1, \theta_2 = a_2 + \delta_2)$ and $\theta' = (\theta'_1 = a_1 - \delta_1, \theta'_2 = a_2 - \delta_2)$ with $\delta_1 \in (0, a_1)$, $\delta_2 \in (0, a_2)$, condition (3.9) requires that $\sum_{S \subseteq N} (-1)^{|S|} I^V(\theta(S)) = I^V(\theta_1, \theta_2) - I^V(\theta'_1, \theta_2) - I^V(\theta_1, \theta'_2) + I^V(\theta'_1, \theta'_2) = a_1 s_2 - a_2 s_1 - 0 + 0 = a_1 s_2 - a_2 s_1 = 0$ implying that for any σ^V to be implementable with balanced transfer it is necessary that $a_1 s_2 = a_2 s_1$. Therefore, from Lemma 3.1 it follows that for a sequencing problem $\Omega^{(s_1,s_2)}_{\{1,2\}}$ with two agents, any sequencing rule σ^V of Example 3.1, satisfying the added restriction that $a_1 s_2 \neq a_2 s_1$, is not implementable with balanced transfers.

2. For the NI sequencing rule σ^{NA} of Example 3.2, Lemma 3.1 fails to hold. For any two profiles $\theta = (\theta_1, \dots, \theta_n), \theta' = (\theta'_1, \dots, \theta'_n) \in \Theta^n$ such that $\theta_1/s_1 > \dots > \theta_n/s_n > \theta'_1/s_1 > \dots > \theta'_n/s_n$, one can verify that $\sum_{S \subseteq N} (-1)^{|S|} I^{NA}(\theta(S)) = s_1 \left(\sum_{j \in N \setminus \{i\}} s_j \right) [(\theta'_1/s_1) - (\theta_n/s_n)] < 0$ and we have a violation of condition (3.9) of Lemma 3.1. Therefore, the NI sequencing rule σ^{NA} is not implementable with balanced transfers.

3.4.1 Case 1: Two agents

Consider any two agent sequencing problem $\Omega^{(s_1,s_2)}_{\{1,2\}}$ and consider any sequencing rule σ. Recall that in Remark 3.1 we have already established that a constant sequencing rule is implementable with balanced transfers. In this sub-section we concentrate only on non-constant NI sequencing rules. For any $i \in \{1,2\}$, let $A_i(\sigma) = \{\theta \in \Theta^2 \mid \sigma_i(\theta) = 1\}$ be the set of profiles such that agent i is first in the order. Clearly, for any two agent sequencing rule σ, $A_1(\sigma) \cup A_2(\sigma) = \Theta^2$. If the sequencing rule $\bar{\sigma}$ is such that $A_1(\bar{\sigma}) = \Theta^2$, then it is the constant sequencing rule with the fixed order $\bar{\sigma} = (\bar{\sigma}_1 = 1, \bar{\sigma}_2 = 2)$. If σ^* is a sequencing rule such that $A_1(\sigma^*) = \{\theta \in \Theta^2 \mid \theta_1/s_1 \geq \theta_2/s_2\}$, then it is the outcome efficient sequencing rule. If $\tilde{\sigma}$ is a sequencing rule such that $A_1(\tilde{\sigma}) = \{\theta \in \Theta^2 \mid \theta_1 \geq \theta_2\}$, then it is the just sequencing rule. For any $i \in \{1,2\}$, an obvious consequence of any NI sequencing rule σ is the following:

(ni): If $\theta \in A_i(\sigma)$, then $Q_i(\theta) = \{\theta' = (\theta'_1, \theta'_2) \in \Theta^2 \mid \theta'_i \geq \theta_i \ \& \ \theta'_j \leq \theta_j\} \subseteq A_i(\sigma)$. Moreover, $Q'_i(\theta) = \{\theta' = (\theta'_1, \theta'_2) \in \Theta^2 \mid \theta'_i > \theta_i \ \& \ \theta'_j < \theta_j\} \subseteq A_i(\sigma)$ since $Q'_i(\theta) \subset Q_i(\theta)$.

Consider any NI sequencing rule σ and consider any pair $\theta, \theta' \in \Theta^2$ such that $\theta'_i > \theta_i$ for $i \in \{1,2\}$ and define $X_0 = \theta$, $X_1 = (\theta'_1, \theta_2)$, $X_2 = (\theta_1, \theta'_2)$ and $X_{12} = \theta'$. Given **(ni)**, for any $i, j \in \{1,2\}$ with $i \neq j$, it is *not possible* to have the following: (a) $X_0, X_{12} \in A_i(\sigma)$ and $X_i, X_j \in A_j(\sigma)$, (b) $X_0, X_i \in A_j(\sigma)$ and $X_j, X_{12} \in A_i(\sigma)$ and (c) $X_0, X_i, X_{12} \in A_j(\sigma)$ and $X_j \in A_i(\sigma)$.

LEMMA **3.2** If a non-constant $\sigma \in NI(\Omega^{(s_1,s_2)_{\{1,2\}}})$ is implementable with balanced transfers, then, for any $i, j \in \{1,2\}$ with $i \neq j$ and any pair $\theta, \theta' \in \Theta^2$ such that $\theta'_1 > \theta_1$, $\theta'_2 > \theta_1$, $X_0 = \theta$, $X_1 = (\theta'_1, \theta_2)$, $X_2 = (\theta_1, \theta'_2)$ and $X_{12} = \theta'$, the following conditions must hold.

(B1) If $X_0, X_i, X_{12} \in A_i(\sigma)$ and $X_j \in A_j(\sigma)$, then the cut off point of agent i for θ'_j and the cut off point of agent j for θ_i have the following relation: $\theta_i^{(1)}(\theta'_j)s_j = \theta_j^{(1)}(\theta_i)s_i$.

(B2) If $X_0, X_i \in A_i(\sigma)$ and $X_j, X_{12} \in A_j(\sigma)$, then the cut off points of agent j for θ_i and θ'_i are equal, that is, $\theta_j^{(1)}(\theta_i) = \theta_j^{(1)}(\theta'_i)$.

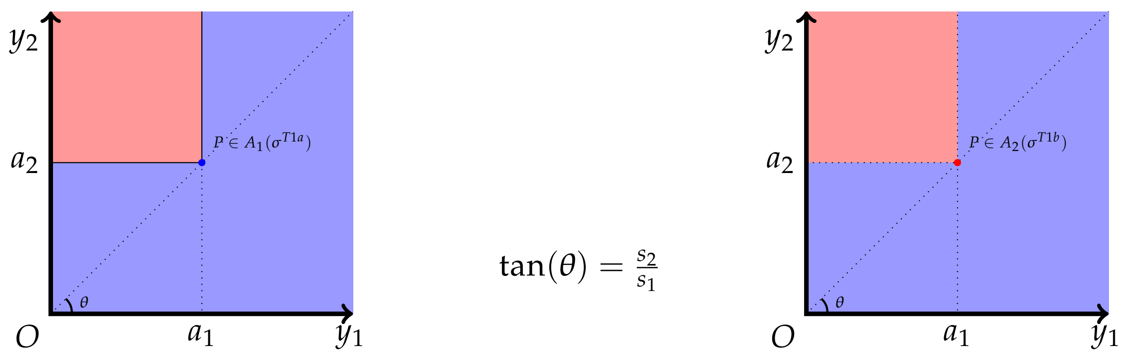
DEFINITION **3.9** Let $\Omega^{(s_1,s_2)}_{\{1,2\}}$ be a two-agent sequencing problem. A sequencing rule σ^{Tx} is a *two agent balancing* (TAB) sequencing rule if there exists an agent $k \in \{1,2\}$

such that any one of the following conditions hold.

(T1) There exists $a_k > 0$ such that $a_l = (a_k s_l)/s_k > 0$ and either $A_k(\sigma^{T1a}) = \{\theta \in \Theta^2 \mid$ *either* $\theta_k \geq a_k$ *or* $\theta_l \leq a_l\}$ or $A_k(\sigma^{T1b}) = \{\theta \in \Theta^2 \mid$ *either* $\theta_k > a_k$ *or* $\theta_l < a_l\}$ (see Figure 1 where we have (T1a) and (T1b) for $k = 1$).

(T2) There exists a real number $a_k > 0$ such that either $A_k(\sigma^{T2a}) = \{\theta \in \Theta^2 \mid \theta_k \geq a_k\}$ or $A_k(\sigma^{T2b}) = \{\theta \in \Theta^2 \mid \theta_k > a_k\}$ (See Figure 2 where we have (T2a) and (T2b) for $k = 1$).

(T3) There exists $a_k > 0$ such that $a_l = (a_k s_l)/s_k > 0$ and either $A_k(\sigma^{T3a}) = \{\theta \in \Theta^2 \mid$ *either* $\theta_k \geq a_k$ *or* $\theta_l < a_l\}$ or $A_k(\sigma^{T3b}) = \{\theta \in \Theta^2 \mid$ *either* $\theta_k > a_k$ *or* $\theta_l \leq a_l\}$ (see Figure 3 where we have (T3a) and (T3b) for $k = 1$).

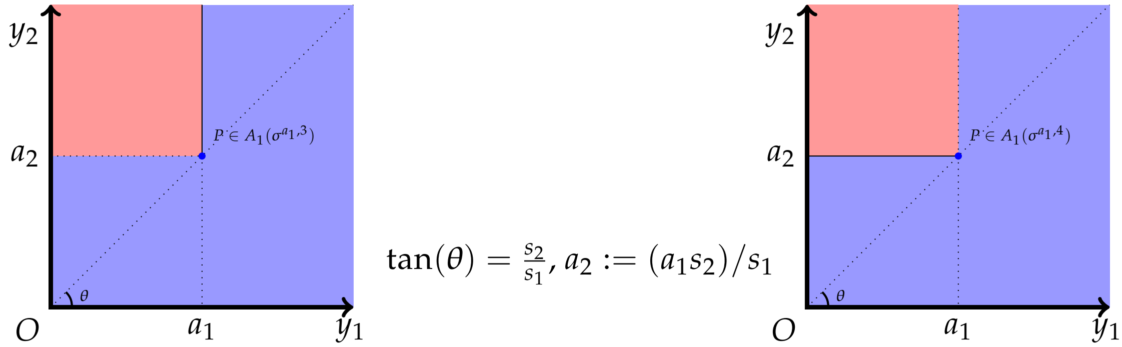


(T1a) *(T1b)*

Figure 1: (T1)

(T2a) *(T2b)*

Figure 2: (T2)



(T3a) Figure 3: (T3) *(T3b)*

Any ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_1 > 0$, $w_2 = 0$ and $\kappa(\sigma_1 = 1, \sigma_2 = 2) > \kappa(\sigma_1 = 2, \sigma_2 = 1)$ is a sequencing rule σ^{T2a} given in (T2) with $k = 1$. A sequencing rule σ^{T1a} given in (T1) is the special case of any NI sequencing rule σ^V of Example 3.1 with the added restriction that $a_2 s_1 = a_1 s_2$. Therefore, as established in the proof of Proposition 3.3, any sequencing rule defined in (T1) is not an ACM sequencing rule.

THEOREM **3.1** A non-constant $\sigma \in NI(\Omega^{(s_1,s_2)}_{\{1,2\}})$ is implementable with balanced transfers if and only if it is a TAB sequencing rule σ^{Tx}.

Therefore, a consequence of Theorem 3.1 is that any ACM sequencing rule $\sigma^{w,\kappa}$ such that the agent-specific weights w_1 and w_2 are both positive are not implementable with balanced transfers. Hence the outcome efficient sequencing rule σ^* and the just sequencing rule $\tilde{\sigma}$ are not implementable with balanced transfers.

3.4.2 Case 2: More than two agents

For any sequencing problem Ω^s_N with more than two agents it is difficult to identify the complete class of NI sequencing rules that are implementable with balanced transfers.

Consider any sequencing problem Ω^s_N with three or more agents. In Remark 3.1 we have argued that any ACM sequencing rule $\sigma^{w,\kappa}$ with the property that there exists $i \in N$ such that $w_i = 0$ is implementable with balanced transfers. What can we say about implementability with balanced transfers for any ACM sequencing rule $\sigma^{w,\kappa}$ with the property that for all $i \in N$, $w_i > 0$? We identify an ACM sequencing rule $\sigma^{w,\kappa}$

with the properties that for all $i \in N$, $w_i > 0$ and κ function is not a constant function, that cannot be implemented with balanced transfers.

EXAMPLE **3.3** Consider any sequencing problem $\Omega^{(s_1,s_2,s_3)}_{\{1,2,3\}}$ with three agents. Consider the ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_1 = w_2 = w_3 = 1 > 0$ and

$$\kappa(\sigma) = \begin{cases} \bar{\kappa} > 0 & \text{if } \sigma^1 = (\sigma^1_1 = 1, \sigma^1_2 = 2, \sigma^1_3 = 3), \\ 0 & \text{if } \sigma \in \Sigma(\{1,2,3\}) \setminus \{\sigma^1\}. \end{cases} \tag{3.10}$$

Given the selection of the $\kappa(\sigma)$ function it may so happen that agent 1 has the highest urgency index, agent 2 has the second highest urgency index and agent 3 has the lowest urgency index and yet the order $(\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$ is less costly compared to the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ simply because we have an added cost of $\bar{\kappa} > 0$ associated with selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$. This aspect can be used to demonstrate that it is impossible to implement this sequencing rule with balanced transfers.

Consider any two profiles $\theta = (\theta_1, \theta_2, \theta_3), \theta' = (\theta'_1, \theta'_2, \theta'_3) \in \Theta^3$ such that $\theta_3/s_3 > \theta_2/s_2 > \theta_1/s_1 > \theta'_2/s_2 > \theta'_3/s_3 > \theta'_1/s_1$, $\theta_1 s_2 - \theta'_2 s_1 = \bar{\kappa}/2$ and $\theta'_3 = \bar{\kappa}/s_2$. We provide the chosen order and the incentive payment of the three agents for the eight possible profiles.

(a) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$, $I_1(\theta_1, \theta_2, \theta_3) = 0$, $I_2(\theta_1, \theta_2, \theta_3) = s_2\theta_1$ and $I_3(\theta_1, \theta_2, \theta_3) = s_3(\theta_2 + \theta_1)$.

(b) $\sigma^{w,\kappa}(\theta'_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$, $I_1(\theta'_1, \theta_2, \theta_3) = 0$, $I_2(\theta'_1, \theta_2, \theta_3) = s_2\theta'_1$ and $I_3(\theta'_1, \theta_2, \theta_3) = s_3(\theta_2 + \theta'_1)$.

(c) $\sigma^{w,\kappa}(\theta_1, \theta'_2, \theta_3) = (\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 1)$, $I_1(\theta_1, \theta'_2, \theta_3) = s_1\theta'_2$, $I_2(\theta_1, \theta'_2, \theta_3) = 0$ and $I_3(\theta_1, \theta'_2, \theta_3) = s_3(\theta_1 + \theta'_2)$.

(d) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta'_3) = (\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$, $I_1(\theta_1, \theta_2, \theta'_3) = s_1\theta'_3$, $I_2(\theta_1, \theta_2, \theta'_3) = s_2(\theta_1 + \theta'_3)$ and $I_3(\theta_1, \theta_2, \theta'_3) = 0$.

(e) $\sigma^{w,\kappa}(\theta_1',\theta_2',\theta_3) = (\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$, $I_1(\theta_1',\theta_2',\theta_3) = 0$, $I_2(\theta_1',\theta_2',\theta_3) = s_2\theta_1'$ and $I_3(\theta_1',\theta_2',\theta_3) = s_3(\theta_2' + \theta_1')$.

(f) The profile $(\theta_1,\theta_2',\theta_3')$ shows how we cannot rely only on the urgency index when κ-function is not a constant function. In particular, we have $\theta_1/s_1 > \theta_2'/s_2$ and yet agent 1 is served after agent 2 simply because the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ less the cost of selecting the order $(\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$ equals $\bar{\kappa} + \theta_2's_1 - \theta_1s_2 = \bar{\kappa} - \bar{\kappa}/2 = \bar{\kappa}/2 > 0$. Hence, $\sigma^{w,\kappa}(\theta_1,\theta_2',\theta_3') = (\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$. Further, the relevant cut off point for agent 1 is $\theta_1^{(2)} = (s_1\theta_3')/s_3$ and hence his incentive payment is $I_1(\theta_1,\theta_2',\theta_3') = s_1\theta_3'$. The relevant cut off points for agent 2 are $\theta_2^{(1)} = (\theta_1s_2 - \bar{\kappa})/s_1$ and $\theta_2^{(2)} = (s_2\theta_3')/s_3$. Specifically, given (θ_1,θ_3'), $\theta_2^{(1)}$ is that waiting cost of agent 2 for which the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ less the cost of selecting the order $(\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 3)$ equals zero. Hence his incentive payment is $I_2(\theta_1,\theta_2',\theta_3') = s_2(\theta_1 + \theta_3') - \bar{\kappa}$. Finally, since agent 3 is served last, $I_3(\theta_1,\theta_2',\theta_3') = 0$.

(g) $\sigma^{w,\kappa}(\theta_1',\theta_2,\theta_3') = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$, $I_1(\theta_1',\theta_2,\theta_3') = 0$, $I_2(\theta_1',\theta_2,\theta_3') = s_2(\theta_3' + \theta_1')$ and $I_3(\theta_1',\theta_2,\theta_3') = s_3\theta_1'$.

(h) $\sigma^{w,\kappa}(\theta_1',\theta_2',\theta_3') = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$, $I_1(\theta_1',\theta_2',\theta_3') = 0$, $I_2(\theta_1',\theta_2',\theta_3') = s_2(\theta_3' + \theta_1')$ and $I_3(\theta_1',\theta_2',\theta_3') = s_3\theta_1'$.

Taking the left hand side of condition (3.9) of Lemma 3.1 and then simplifying it using (a)-(h) above we get

$$\sum_{S\subseteq N}(-1)^{|S|}I(\theta(S)) = \theta_3's_2 + (\theta_1s_2 - \theta_2's_1 - \bar{\kappa}) = \bar{\kappa} + \left(\frac{\bar{\kappa}}{2} - \bar{\kappa}\right) = \frac{\bar{\kappa}}{2} \neq 0. \tag{3.11}$$

Condition (3.11) is a violation of condition (3.9) in Lemma 3.1. Hence the ACM sequencing rule $\sigma^{w,\kappa}$ with $w_1 = w_2 = w_3 = 1 > 0$ and the $\kappa(\sigma)$ function given by condition (3.10) is not implementable with balanced transfers.

Given that the agent-specific weights are all positive, Example 3.3 shows that it is difficult to check the prospect of implementability with balanced transfers for any ACM sequencing rule with the property that the κ-function is not a constant. Keeping this difficulty in mind, we identify a sufficient family of NI sequencing rules that are implementable with balanced transfers.

Consider any sequencing problem Ω_N^s with more than two agents and let Π_N be the set of all possible *priority* partitions of the agents where the order of representing the partition is important in terms of priority. For example, if $\pi(N) = (\pi_1, \pi_2, \ldots, \pi_K)$ is any priority partition, then group π_1 is given priority over group π_2 and so on. Let $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi_N$ be any priority partition of the set of agents. The set of $\pi(N)$ induced orders is

$$\Sigma(\pi(N)) = \begin{cases} \{\sigma \in \Sigma(N) \mid \forall k \in \{1, \ldots, K-1\}, \sigma_i < \sigma_j, \forall i \in \pi_k, \forall j \in \pi_{k+1}\} & \text{if } K \geq 2, \\ \Sigma(N) & \text{if } K = 1. \end{cases} \tag{3.12}$$

Therefore, the set of priority partition $\pi(N)$ induced orders $\Sigma(\pi(N))$ are those orders where agents in π_1 are always served first, agents in π_2 are always served after agents in π_1 but before agents in π_3 (if any) and so on. If $K = 1$ so that $\pi(N) = (\pi_1 = \pi_K = \{N\})$, then $\Sigma(\pi(N)) = \Sigma(N)$ which is the set of all possible ordering on the set of agents N. For example, for $\Pi_{\{1,2,3\}}$, there are four types of priority partitions. These are $\pi^c = (\pi_1 = \{i\}, \pi_2 = \{j\}, \pi_3 = \{k\})$, $\pi^{21} = (\pi_1 = \{i, j\}, \pi_2 = \{k\})$, $\pi^{12} = (\pi_1 = \{i\}, \pi_2^{12} = \{j, k\})$ and $\bar{\pi} = (\pi_1 = \{1, 2, 3\})$ where $i \neq j \neq k \neq i$. For π^c, $\Sigma(\pi^c) = \{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\}$, for π^{21}, $\Sigma(\pi^{21}) = \{\{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\} \cup \{(\sigma_i = 2, \sigma_j = 1, \sigma_k = 3)\}\}$, for π^{12}, $\Sigma(\pi^{12}) = \{\{(\sigma_i = 1, \sigma_j = 2, \sigma_k = 3)\} \cup \{(\sigma_i = 1, \sigma_j = 3, \sigma_k = 2)\}\}$ and finally for $\bar{\pi}$, we have the set of all possible orders on the set of agents, that is, $\Sigma(\bar{\pi}) = \Sigma(\{1, 2, 3\})$.

DEFINITION **3.10** Consider any priority partition $\pi(N) \in \Pi_N$ and let $f = \{f_1, \ldots, f_n\}$ be a set of agent-specific increasing and one-to-one functions $f_j : \Theta \to \mathbb{R}_+$. The sequencing rule $\sigma^{\pi(N), f} : \Theta^n \to \Sigma(N)$ satisfies *group priority based cost minimization* (GP-CM) if for each $\theta \in \Theta^n$, $\sigma^{\pi(N), f}(\theta) \in \arg\min_{\sigma \in \Sigma(\pi(N))} \sum_{j \in N} f_j(\theta_j) S_j(\sigma)$.

By appropriately modifying the arguments used to prove that an ACM sequencing rule is NI in Proposition 3.3, one can easily show that any GP-CM sequencing rule $\sigma^{\pi(N),f}$ is NI. The following observations are important for our understanding of GP-CM sequencing rules.

1. For any $\pi(N) \in \Pi_N$, any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with the property that there exists an agent $j \in N$ such that $f_j(.)$ is non-linear is an NI sequencing rule which is not an ACM.

2. For any $\pi(N) \in \Pi_N$, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ where $f_j(.)$ is linear for all $j \in N$ is an ACM sequencing rule. Specifically, any ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_j > 0$ for all $j \in N$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma'(N)$ and there exists a priority partition $\pi(N) \in \Pi_N$ such that $\Sigma'(N) = \Sigma(\pi(N))$ is a GP-CM sequencing rule.

3. The GP-CM sequencing rule is not onto for any $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi_N$ such that $K \geq 2$ since, in that case, $\Sigma(N) \setminus \Sigma(\pi(N)) \neq \emptyset$ and any order $\sigma \in \Sigma(N) \setminus \Sigma(\pi(N))$ is never chosen.

4. For $\pi(N) \in \Pi_N$ such that $K = 1$ so that the $\pi(N) = (\pi_1 = \pi_K = \{N\})$ is the grand coalition, $\Sigma(\pi(N)) = \Sigma(N)$ and any such GP-CM $\sigma^{\pi(N),f}$ is onto.

5. A GP-CM sequencing rule $\sigma^{\pi(N),f}$ is a constant sequencing rule if $\pi(N) = (\pi_1, \ldots, \pi_K)$ is such that $K = n$.

6. A GP-CM sequencing rule $\sigma^{\pi(N),f}$ gives the outcome efficient sequencing rule σ^* if $\pi(N) = (\{N\})$ and $f_j(\theta_j) = \theta_j$ for all $j \in N$.

7. A GP-CM sequencing rule $\sigma^{\pi(N),f}$ gives the just sequencing rule $\tilde{\sigma}$ if $\pi(N) = (\{N\})$ and $f_j(\theta_j) = (1/\prod_{k \in N \setminus \{j\}} s_k)\theta_j$ for all $j \in N$.

REMARK **3.3** For any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with the priority partition $\pi(N) \in \Pi_N$, *modified urgency index* $f_j(\theta_j)/s_j$ is used to determine the profile contingent order of serving the agents. Specifically, like Smith's (see Smith (1956)) rule for outcome efficient sequencing rule σ^*, for any GP-CM $\sigma^{\pi(N),f}$, the selected order $\sigma^{\pi(N),f}(\theta)$ satisfies the following condition.

$$\textbf{(GP-CM)} \qquad \text{For any } i, j \in \pi_k \in \pi(N),$$
$$(f_i(\theta_i)/s_i) \geq (f_j(\theta_j)/s_j) \Leftrightarrow \sigma_i^{\pi(N),f}(\theta) \leq \sigma_j^{\pi(N),f}(\theta).$$

Given the tie-breaking rule, this profile contingent selection $\sigma^{\pi(N),f}(\theta)$ is unique.

DEFINITION **3.11** For any GP-CM sequencing rule $\sigma^{\pi(N),f}$ with priority partition $\pi(N) \in \Pi_N$, a mechanism $(\sigma^{\pi(N),f}, \tau^{\pi(N),f})$ is a *GP-CM "Cut off" based mechanism* if the transfer rule is such that for any $\theta \in \Theta^n$ and any $i \in \pi_k \in \pi(N)$,

$$\tau_i^{\pi(N),f}(\theta) = \begin{cases} G_i(\theta_{-i}) & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k = \emptyset, \\ G_i(\theta_{-i}) - \sum\limits_{j \in P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k} s_j f_i^{-1}\left(\frac{s_i f_j(\theta_j)}{s_j}\right) & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k \neq \emptyset. \end{cases} \tag{3.13}$$

where the function $G_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary.

It is obvious that the incentive payment of any agent $i \in \pi_k \in \pi(N)$ under the GP-CM-"Cut off" based mechanism is the following:

$$I_i^{\pi(N),f}(\theta) = \begin{cases} 0 & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k = \emptyset, \\ \sum\limits_{j \in P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k} s_j f_i^{-1}\left(\frac{s_i f_j(\theta_j)}{s_j}\right) & \text{if } P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k \neq \emptyset. \end{cases}$$

The GP-CM-"Cut off" based transfers (3.13) specifies that for any $i \in \pi_k \in \pi(N)$ and any $\theta_{-i} \in \Theta^{N \setminus \{i\}}$, if θ_i is such that agent i is served last among the members of the group π_k in which he belongs for the order $\sigma^{\pi(N),f}(\theta_i, \theta_{-i})$, then $\tau_i^{\pi(N),f}(\theta_i, \theta_{-i}) = G_i(\theta_{-i})$. This part of the transfer is like the "Cut off" based transfer for the case where agent i's type is smaller than the smallest cut off point $\theta_i^{(T-1)}$ (see Proposition 3.2). If, however, θ_i' is such that agent i is not served last among the members of his group π_k under the order $\sigma^{\pi(N),f}(\theta_i', \theta_{-i})$, that is, if $P_i'(\sigma^{\pi(N),f}(\theta_i', \theta_{-i})) \cap \pi_k \neq \emptyset$, then agent i's transfer $\tau_i^{\pi(N),f}(\theta_i', \theta_{-i})$ not only has $G_i(\theta_{-i})$ but he also has to make an incentive payment $I_i(\theta_i', \theta_{-i})$. His incentive payment amount is the sum of cost that agent i inflicts on the followers from the members of his group π_k under the order $\sigma^{\pi(N),f}(\theta_i', \theta_{-i})$. This part of the incentive solving payment is nothing but the cost term $\sum_{r=t}^{T-1}\theta_i^{(r)}D_r(\theta_{-i})$ in the transfer under the "Cut off" based mechanism (see Proposition 3.2). In particular, given any $i \in \pi_k \in \pi(N)$ and given any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, the cut off points where the order of agent i changes are set of distinct elements from the collection $\{f_i^{-1}(s_i f_j(\theta_j)/s_j)\}_{j\in\pi_k\setminus\{i\}}$ and the absolute cost difference of agent i below and above any cut off point $f_i^{-1}(s_i f_j(\theta_j)/s_j)$ is given by $D_j(\theta_{-i}) = s_j$. Hence for each $j \in P_i'(\sigma^{\pi(N),f}(\theta)) \cap \pi_k$, the payment of i is $\left[f_i^{-1}(s_i f_j(\theta_j)/s_j)\right] D_j(\theta_{-i}) = f_i^{-1}(s_i f_j(\theta_j)/s_j)(s_j)$.

THEOREM **3.2** Consider any sequencing problem Ω_N^s with more than two agents. For any priority partition $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi_N$ and for any given set of functions $f = \{f_1, \ldots, f_n\}$ that are increasing and onto, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfers.

Till now we have obtained the following.

1. Any ACM sequencing rule $\sigma^{w,\kappa}$ such that there exists an agent $j \in N$ such that $w_j = 0$ is implementable with balanced transfers (see Remark 3.1).

2. Example 3.3 demonstrates the existence of an ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_i > 0$ for all $i \in N$ and κ-function is not a constant function which is not implementable with balanced transfers.

3. Any ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_i > 0$ for all $i \in N$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma'(N)$ and there exists a priority partition $\pi(N) \in \Pi_N$ such that $\Sigma'(N) = \Sigma(\pi(N))$ is a GP-CM sequencing rule and hence, by Theorem 3.2, is implementable with balanced transfers. Moreover, since the grand coalition $\pi(N) = (\pi_1 = \{N\})$ is also included in the set of priority partitions, any *onto* ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_i > 0$ for all $i \in N$ and $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma(N)$ is implementable with balanced transfers.

4. The non-affine cost minimizers NI sequencing rules included in GP-CM sequencing rules are a generalization of the affine cost minimizer sequencing rules where agents' waiting cost are replaced with a non-linear function of the waiting cost

which is increasing and onto. Theorem 3.2 shows that all these rules are also implementable with balanced transfers.

What can we say about any ACM sequencing rule $\sigma^{w,\kappa}$ such that (a) κ-function is a constant function, (b) $w_i > 0$ for all $i \in N$, and, yet, (c) it does not belong to the class of GP-CM sequencing rules? It is difficult to give a general answer and we provide one example of such an ACM sequencing rule which is not implementable with balanced transfers.

EXAMPLE **3.4** Consider any sequencing problem $\Omega^{(s_1,s_2,s_3)}_{\{1,2,3\}}$ with three agents. Consider the ACM sequencing rule $\sigma^{w,\kappa}$ such that $w_1 = w_2 = w_3 = 1 > 0$, $\kappa(\sigma) = 0$ for all $\sigma \in \Sigma'(\{1,2,3\})$ and $\Sigma'(\{1,2,3\}) = \{\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \Sigma(\{1,2,3\}) \mid \sigma_1 \neq 2\}$. From the discussion about the priority partitions $\Pi_{\{1,2,3\}}$ for the three agent case, that appears before the definition of GP-CM sequencing rules, it is easy to see that there does not exists a priority partition $\pi \in \Pi_{\{1,2,3\}}$ such that $\Sigma'(\{1,2,3\}) = \Sigma(\pi(\{1,2,3\})$ and hence this ACM sequencing rule is not in the family of GP-CM sequencing rules. Consider any two profiles $\theta = (\theta_1, \theta_2, \theta_3), \theta' = (\theta'_1, \theta'_2, \theta'_3) \in \Theta^3$ such that $\theta'_1 = s_1$, $\theta'_3 = 2s_3$, $\theta_3 = 3s_3$, $\theta'_2 = 3s_2$, $\theta_2 = 4s_2$, $\theta_1 = As_1$ and A is any number in the open interval $(2 + a, \min\{2 + 2a, 3\})$ where $a = s_2/(s_2 + s_3) \in (0, 1)$. Observe that $\theta_2/s_2 = 4 > \theta'_2/s_2 = \theta_3/s_3 = 3 > \theta_1/s_1 = A > \theta'_3/s_3 = 2 > \theta'_1/s_1 = 1$. Given the tie-breaking rule, we provide the chosen order and the incentive payments under the eight possible profiles.

(a) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and $I_1(\theta_1, \theta_2, \theta_3) = 0$. The relevant cut off points for agent 2 are $\theta^{(2)}_2 = (\theta_1(s_2 + s_3) - \theta_3 s_1)/s_1$ and $\theta^{(1)}_2 = (s_2\theta_3)/s_3$. Specifically, given (θ_1, θ_3), $\theta^{(2)}_2$ is that waiting cost of agent 2 for which the cost of selecting the order $(\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1)$ less the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 3, \sigma_3 = 2)$ equals zero. Hence his incentive payment is $I_2(\theta_1, \theta_2, \theta_3) = \theta_1(s_2 + s_3) - \theta_3 s_1 + s_2\theta_3$. The relevant cut off point for agent 3 is $\theta^{(2)}_3 = (\theta_1(s_2 + s_3) - \theta_2 s_1)/s_1$ and, given (θ_1, θ_2), $\theta^{(2)}_3$ is that waiting cost of agent 3 for which the cost of selecting the order $(\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ less the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ equals zero. Hence his incentive payment is $I_3(\theta_1, \theta_2, \theta_3) = \theta_1(s_2 + s_3) - \theta_2 s_1$. Therefore, $I(\theta_1, \theta_2, \theta_3) = \sum_{i \in \{1,2,3\}} I_i(\theta_1, \theta_2, \theta_3) = 2\theta_1(s_2 + s_3) - (\theta_2 + \theta_3)s_1 + \theta_3 s_2$.

(b) $\sigma^{w,\kappa}(\theta'_1, \theta_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and, like case (a), the aggregate incentive payment is $I(\theta'_1, \theta_2, \theta_3) = 2\theta'_1(s_2 + s_3) - (\theta_2 + \theta_3)s_1 + \theta_3 s_2$.

(c) $\sigma^{w,\kappa}(\theta_1, \theta'_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta_1, \theta'_2, \theta_3) = 2\theta_1(s_2 + s_3) - (\theta'_2 + \theta_3)s_1 + \theta_3 s_2$.

(d) $\sigma^{w,\kappa}(\theta_1, \theta_2, \theta'_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta_1, \theta_2, \theta'_3) = 2\theta_1(s_2 + s_3) - (\theta_2 + \theta'_3)s_1 + \theta'_3 s_2$.

(e) $\sigma^{w,\kappa}(\theta'_1, \theta'_2, \theta_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta'_1, \theta'_2, \theta_3) = 2\theta'_1(s_2 + s_3) - (\theta'_2 + \theta_3)s_1 + \theta_3 s_2$.

(f) $\sigma^{w,\kappa}(\theta_1, \theta'_2, \theta'_3) = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ and $I_3(\theta_1, \theta'_2, \theta'_3) = 0$. The only cut off point for agent 1 is $\theta_1^{(1)} = ((\theta'_2 + \theta'_3)s_1)/(s_2 + s_3)$ and, given (θ'_2, θ'_3), $\theta_1^{(1)}$ is that waiting cost of agent 1 for which the cost of selecting the order $(\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ less the cost of selecting the order $(\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3)$ equals zero. Hence $I_1(\theta_1, \theta'_2, \theta'_3) = (\theta'_2 + \theta'_3)s_1$. The relevant cut off point for agent 2 is $\theta_2^{(2)} = (s_2\theta'_3)/s_3$ and $I_2(\theta_1, \theta'_2, \theta'_3) = \theta'_3 s_2$. Hence, $I(\theta_1, \theta'_2, \theta'_3) = \sum_{i \in \{1,2,3\}} I_i(\theta_1, \theta'_2, \theta'_3) = (\theta'_2 + \theta'_3)s_1 + \theta'_3 s_2$.

(g) $\sigma^{w,\kappa}(\theta'_1, \theta_2, \theta'_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and the aggregate incentive payment is $I(\theta'_1, \theta_2, \theta'_3) = 2\theta'_1(s_2 + s_3) - (\theta_2 + \theta'_3)s_1 + \theta'_3 s_2$.

(h) $\sigma^{w,\kappa}(\theta'_1, \theta'_2, \theta'_3) = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2)$ and $I(\theta'_1, \theta'_2, \theta'_3) = 2\theta'_1(s_2 + s_3) - (\theta'_2 + \theta'_3)s_1 + \theta'_3 s_2$.

Taking the left hand side of condition (3.9) of Lemma 3.1 and then simplifying it using (a)-(h) above we get

$$\sum_{S \subseteq N} (-1)^{|S|} I(\theta(S)) = (s_1 + s_2)[s_2 + 2((s_2 + s_3) - As_1)]. \tag{3.14}$$

For condition (3.9) in Lemma 3.1 to hold, for any $A \in (2+a, \min\{2+2a, 3\})$ we must have that condition (3.14) must be equal to zero. This is not possible since the right hand side of condition (3.14) changes for different selections of A from the interval $(2+a, \min\{2+2a, 3\})$. In particular, if (3.14) is equal to zero for some selection $a \in A$, then (3.14) is not equal to zero for any selection $a + \epsilon \in A$ with $\epsilon > 0$. That we can always select two distinct numbers from the interval A is immediate. If $\min\{2+2a, 3\} = 2 + 2a$, then select $a_1 = 2 + ((3a)/2)$ and $a_1 + \epsilon = 2 + ((7a)/4)$. Note that $a_1, a_1 + \epsilon \in A$ and $\epsilon = a/4 > 0$. If $\min\{2+2a, 3\} = 3$, then select $b_1 = (5/2) + (a/2)$ and $b_1 + \epsilon' = (8/3) + (a/3)$. Note that $b_1, b_1 + \epsilon' \in A$ and $\epsilon' = (1/6)(1-a) > 0$. Therefore, this ACM sequencing rule $\sigma^{w,\kappa}$ is not implementable with balanced transfers.

Finally, there are NI sequencing rules, different both from GP-CM sequencing rules and from sequencing rules of Remark 3.1, that are implementable with balanced transfers. For example, consider any σ such that for each agent $j \in N \setminus \{1,2\}$, $\sigma_j(\theta) = k \in \{3, \ldots, n\}$ is fixed for all θ and for agents 1 and 2 we follow conditions specified by (T1a) for the TAB sequencing rules (ignoring the waiting costs of all other agents) to obtain their order. Clearly, this rule is implementable with balanced transfers by setting the transfer of all $j \in N \setminus \{1,2\}$ at zero, ceteris paribus. Hence, GP-CM sequencing rules and sequencing rules of Remark 3.1, taken together, is not necessary for implementability with balanced transfers.

REMARK **3.4** Our initial analysis (Section 3.3) shows that results on implementability of sequencing rules are consequences of results from a set of important papers from the dominant strategy mechanism design literature. However, this analysis is only a precursor to our main non-trivial contribution on balanced implementability. While we identify the entire class of balanced implementable sequencing rules with two agents, for sequencing problems with three or more agents we identify a natural class of priority based sequencing rules, that we face in our day to day life, that are implementable with balanced transfers. Therefore, in the presence of monetary transfers, our analysis concludes that many real-life priority based rules that can be implemented in a costless way as the transfers to implement such rules only requires intra-agent transfers (without inflicting any loss in society's resources). Needless to say costless implementability is an important aspect of a planner's mechanism problem under incomplete information though, in general, it is difficult to achieve. In that sense ours is a positive contribution to the literature on balanced implementability.

3.5 Conclusion

For sequencing problems with three or more agents we identify a natural class of priority based sequencing rules, that we face in our day to day life, that are implementable with balanced transfers (Section 3.4.2). We admit that it is hard to identify the complete class of rules that are implementable with balanced transfers and is an open question. Using the necessary condition of balanced implementability (Lemma 3.1), we identify the entire class of balanced implementable sequencing rules with two agents (Section 3.4.1). Therefore, in the presence of monetary transfers, our analysis concludes that many real-life priority based rules can be implemented in a costless way as the transfers to implement such rules only requires intra-agent transfers (without inflicting any gain or loss in society's resources). Needless to say costless implementability is an important aspect of a planner's mechanism problem under incomplete information though, in general, it is difficult to achieve (see Hurwicz and Walker (1990), Walker (1980) and Yenmez (2015)). In that sense ours is a positive contribution to the literature on balanced implementability.

In this paper we have assumed that the benefit derived from getting the service is sufficiently high for all agents so that participation (or individual rationality) constraints are not binding. Specifically, we have implicitly assumed that $\bar{U}_i(\sigma, \tau_i; \theta_i) = v_i - \theta_i S_i(\sigma) + \tau_i$ where v_i is large enough to ensure that $\bar{U}_i(\sigma, \tau_i; \theta_i) \geq 0$ under all the relevant sequencing rules and their associated "Cut off" based transfers. Hence, the benefit v_i that each agent i derives from getting the service never featured in our analysis and we could limit our attention to preferences of the form $U_i(\sigma, \tau_i; \theta_i) = -\theta_i S_i(\sigma) + \tau_i$. Implicit here is our assumption that $\bar{U}_i(\sigma, \tau_i; \theta_i) \geq 0$ is all that matters, that is, the outside option of each agent is zero utility. There may be other better ways of addressing individual rationality in this context with better profile contingent outside options. How to address this issue of individual rationality for the sequencing problem is another important direction for future research.

3.6 Appendix

Proof of Proposition 3.3: Consider any affine maximizer sequencing rule $\sigma^{w,\kappa}$. Let $T \subset N$ be such that $w_j = 0$ for all $j \in T$. Since, by affine maximization, for any profile θ, $\sigma_i^{w,\kappa}(\theta) < \sigma_j^{w,\kappa}(\theta)$ for any $j \in T$ and any $i \in N \setminus T$ and since the tie-breaking rule fixes the order of service across agents in T, it follows that for any $j \in T$, any θ_{-j}, $S_j(\sigma(\theta_j, \theta_{-j}))$ is a constant for all $\theta_j \in \Theta$. Hence for any agent $j \in T$, the completion time is fixed for all profiles that implies non-increasingness. Consider any agent i with $w_i > 0$, any profile $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, any $\theta'_i > \theta_i$ such that $\sigma(\theta_i, \theta_{-i}) := \sigma$, $\sigma(\theta'_i, \theta_{-i}) := \sigma'$ and $\sigma \neq \sigma'$. Using affine maximization we have the following:

(I) $w_i\theta_i S_i(\sigma) + \sum_{j \in N \setminus \{i\}} w_j \theta_j S_j(\sigma) + \kappa(\sigma) \leq w_i\theta_i S_i(\sigma') + \sum_{j \in N \setminus \{i\}} w_j \theta_j S_j(\sigma') + \kappa(\sigma')$, and

(II) $w_i\theta'_i S_i(\sigma) + \sum_{j \in N \setminus \{i\}} w_j \theta_j S_j(\sigma) + \kappa(\sigma) \geq w_i\theta'_i S_i(\sigma') + \sum_{j \in N \setminus \{i\}} w_j \theta_j S_j(\sigma') + \kappa(\sigma')$.

From (I) and (II) we get

(III) $w_i\theta_i[S_i(\sigma) - S_i(\sigma')] + \sum_{j \in N \setminus \{i\}} w_j\theta_j[S_j(\sigma) - S_j(\sigma')] \leq \kappa(\sigma') - \kappa(\sigma)$, and

(IV) $w_i\theta'_i[S_i(\sigma) - S_i(\sigma')] + \sum_{j \in N \setminus \{i\}} w_j\theta_j[S_j(\sigma) - S_j(\sigma')] \geq \kappa(\sigma') - \kappa(\sigma)$.

Using (III) and (IV) it easily follows that $w_i(\theta'_i - \theta_i)[S_i(\sigma) - S_j(\sigma')] \geq 0$. Given $\theta'_i > \theta_i$ and $w_i > 0$, it follows that $S_i(\sigma') = S_i(\sigma(\theta'_i, \theta_{-i})) \leq S_i(\sigma(\theta_i, \theta_{-i})) = S_i(\sigma)$ and we have non-increasingness.

To prove the final part we first prove that the NI sequencing rule σ^V (defined in Example 3.1) for any two-agent sequencing problem is not an affine cost minimizer. Suppose, to the contrary, that σ^V is an affine cost minimizer. Then for any θ' such that $\theta'_1 > a_1$, the affine cost minimization must give $w_1\theta'_1 s_1 + w_2\theta'_2(s_1+s_2) + \kappa(\sigma^1 = (\sigma_1 = 1, \sigma_2 = 2)) \leq w_1\theta'_1(s_1+s_2) + w_2\theta'_2 s_2 + \kappa(\sigma^2 = (\sigma_1 = 2, \sigma_2 = 1))$. Therefore, we must have (I) $w_2\theta'_2 s_1 + \kappa(\sigma^1) \leq w_1\theta'_1 s_2 + \kappa(\sigma^2)$ for any θ'_2. However, if $w_2 > 0$, then this is not possible as by keeping θ'_1 fixed and taking θ'_2 very large we can always have a violation of inequality (I). Hence, we must have $w_2 = 0$. But if $w_2 = 0$, then the sequencing rule σ^V is independent of the waiting cost of agent 2 which is not the case (since for any $\theta_1 < a_1$ the sequencing rule σ^V depends on whether θ_2 is greater than a_2 or not). Hence we have the required contradiction.

To complete the proof we show that for any sequencing problem with three or more agents, σ^{NA} (defined in Example 3.2) is not an affine cost minimizer. Suppose, to the contrary, that σ^{NA} is an affine cost minimizer. Since σ^{NA} has the property that for any $i \in N$, any given θ_{-i}, there exists $\theta'_i > \theta_i$ such that $S_i(\theta'_i, \theta_{-i}) < S_i(\theta_i, \theta_{-i})$, it is necessary that the affine cost minimizer must be such that $w_i > 0$ for all $i \in N$. Consider the profile $\theta = (\theta_1, \theta_2, \theta_3, \ldots, \theta_n)$ such that $\theta_2/s_2 > \theta_1/s_1 > \theta_3/s_3 > \ldots \geq \theta_n/s_n$. Then by σ^{NA}, the order selected is $\sigma = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3, \ldots, \sigma_n = n)$. Moreover affine cost minimization must rule out the order $\sigma' = (\sigma_1 = 3, \sigma_2 = 1, \sigma_3 = 2, \ldots, \sigma_n = n)$ and hence it follows that $w_1\theta_1 s_1 + w_2\theta_2(s_1+s_2) + w_3\theta_3(s_1+s_2+s_3) + \kappa(\sigma) \leq w_1\theta_1(s_1+s_2+s_3) + w_2\theta_2 s_2 + w_3\theta_3(s_2+s_3) + \kappa(\sigma')$. This inequality implies that $w_2\theta_2 s_1 + w_3\theta_3 s_1 \leq w_1\theta_1(s_2+s_3) + \kappa(\sigma') - \kappa(\sigma)$. By selecting a profile $\theta' = (x_2, \theta_{-2})$ such that $x_2 > \theta_2$ we continue to have $x_2/s_2 > \theta_1/s_1 > \theta_3/s_3 > \ldots > \theta_n/s_n$ and by σ^{NA}, the order selected continues to be $\sigma = (\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3, \ldots, \sigma_n = n)$. Hence for any $x_2 > \theta_2$ we must have

$$w_2 x_2 s_1 + w_3\theta_3 s_1 \leq w_1\theta_1(s_2+s_3) + \kappa(\sigma') - \kappa(\sigma). \tag{3.15}$$

But as x_2 increases, the left hand side of inequality (3.15) increases and the right hand side remains unchanged. Therefore, for a sufficiently large value of x_2, inequality (3.15) fails to hold and hence we have a contradiction to our assumption that σ^{NA} is an affine cost minimizer. ■

Proof of Lemma 3.2:

Proof of (B1): Without loss of generality, let $i = 1$ and let $X_0, X_1, X_{12} \in A_1$ and $X_2 \in A_2$. From the "Cut off" based mechanism we get $I_1(X_0) = I_1(X_1) = \theta_1^{(1)}(\theta_2)s_2$, $I_2(X_0) = I_2(X_1) = I_1(X_2) = I_2(X_{12}) = 0$, $I_2(X_2) = \theta_2^{(1)}(\theta_1)s_1$ and $I_1(X_{12}) = \theta_1^{(1)}(\theta_2')s_2$. By applying condition (3.9) of Lemma 3.1 we get $\sum_{S\subseteq\{1,2\}} I(\theta(S)) = \theta_2^{(1)}(\theta_1)s_1 - \theta_1^{(1)}(\theta_2')s_2 = 0$ and we get (B1) for $i = 1$.

Proof of (B2): Without loss of generality, let $i = 1$ and let $X_0, X_1 \in A_1$ and $X_2, X_{12} \in A_2$. From the "Cut off" based mechanism we get $I_1(X_0) = I_1(X_1) = \theta_1^{(1)}(\theta_2)s_2$, $I_2(X_0) = I_2(X_1) = I_1(X_2) = I_1(X_{12}) = 0$, $I_2(X_2) = \theta_2^{(1)}(\theta_1)s_1$ and $I_2(X_{12}) = \theta_2^{(1)}(\theta_1')s_1$. By applying condition (3.9) of Lemma 3.1 we get $\sum_{S\subseteq\{1,2\}} I(\theta(S)) = [\theta_2^{(1)}(\theta_1) - \theta_2^{(1)}(\theta_1')]s_1 = 0$ and we get (B2) for $i = 1$. ■

Proof of Theorem 3.1: We first show that any NI sequencing rule which is TAB is implementable with balanced transfers. If we have the TAB sequencing rule σ^{Tx} given by (T1a) with $k = 1$, then the "Cut off" based transfer of agents 1 and 2 are given by (3.3) and (3.4) respectively. If we set $h_2(\theta_1) = 0$ for all $\theta_1 \in \Theta$ and if we set

$$h_1(\theta_2) = \begin{cases} 0 & \text{if } \theta_2 \leq a_2, \\ a_2 s_1 & \text{if } \theta_2 > a_2, \end{cases} \tag{3.16}$$

then, using $a_1 s_2 = a_2 s_1$, we get

$$\tau_1(\theta) = -\tau_2(\theta) = \begin{cases} 0 & \text{if either } \theta_1 \geq a_1 \text{ or } \theta_2 \leq a_2, \\ a_2 s_1 & \text{if } \theta_1 < a_1 \text{ and } \theta_2 > a_2. \end{cases} \tag{3.17}$$

For the TAB sequencing rule (T1b) with $k = 1$, the argument is similar and hence omitted. Finally, if we have the TAB sequencing rule σ^{Tx} given by (T2a) and with $k = 1$, then, by Proposition 3.2, we get the following "Cut off" based transfers. For any $\theta \in \Theta^2$,

$$\tau_1(\theta) = \begin{cases} h_1(\theta_2) & \text{if } \theta_1 < a_1, \\ h_1(\theta_2) - a_1 s_2 & \text{if } \theta_1 \geq a_1, \end{cases} \tag{3.18}$$

and $\tau_2(\theta) = h_2(\theta_1)$. If we set $h_1(\theta_2) = 0$ for all $\theta_2 \in \Theta$, and if we set

$$h_2(\theta_1) = \begin{cases} 0 & \text{if } \theta_1 < a_1, \\ a_1 s_2 & \text{if } \theta_1 \geq a_1, \end{cases} \tag{3.19}$$

then we get implementability with balanced transfers. Specifically, we have

$$\tau_1(\theta) = -\tau_2(\theta) = \begin{cases} 0 & \text{if } \theta_1 < a_1, \\ -a_1 s_2 & \text{if } \theta_1 \geq a_1. \end{cases} \tag{3.20}$$

For the TAB sequencing rule (T2b) with $k = 1$, the argument is similar and hence omitted.

If we have the TAB sequencing rule given by (T3a), then with the following transfer rule we achieve budget-balance since $a_1 s_2 = a_2 s_1$,

$$\tau_1(\theta) = -\tau_2(\theta) = \begin{cases} 0 & \text{if either } \theta_1 \geq a_1 \text{ or } \theta_2 < a_2, \\ a_2 s_1 & \text{if } \theta_1 < a_1 \text{ and } \theta_2 \geq a_2. \end{cases} \tag{3.21}$$

If we have the TAB sequencing rule given by (T3b), then with the following transfer rule we can achieve budget-balance since $a_1 s_2 = a_2 s_1$,

$$\tau_1(\theta) = -\tau_2(\theta) = \begin{cases} 0 & \text{if either } \theta_1 > a_1 \text{ or } \theta_2 \leq a_2, \\ a_2 s_1 & \text{if } \theta_1 \leq a_1 \text{ and } \theta_2 > a_2. \end{cases} \tag{3.22}$$

Therefore, we get implementability with balanced transfers for TAB seqeuncing rules σ^{Tx}.

We now prove the converse, that is, if a non-constant NI sequencing rule is implementable with balanced transfers, then it must be a TAB sequencing rule. We prove this in two steps. Let σ be non-constant NI sequencing rule which is implementable with balanced transfers and satisfies the following property.

- **(P1)** There exists an $i \in \{1, 2\}$ such that we can find a pair $\theta, \theta' \in \Theta^2$ with the property that $\theta'_i > \theta_i$ and $X_0, X_i, X_{12} \in A_i(\sigma)$ and $X_j \in A_j(\sigma)$ where $X_0 = \theta$, $X_1 = (\theta'_1, \theta_2)$, $X_2 = (\theta_1, \theta'_2)$ and $X_{12} = \theta'$.

Step 1: If σ is a non-constant NI sequencing rule that is implementable with balanced transfers and for which **(P1)** holds, then σ must be a TAB sequencing rule of the form (T1).

Proof of Step 1: Suppose we have $X_0, X_i, X_{12} \in A_i(\sigma)$ and $X_j \in A_j(\sigma)$, for any $i, j \in \{1, 2\}$ with $i \neq j$. Without loss of generality, let $i = 1, j = 2$. Then by Lemma 3.2 (B1), the cut off points $\theta_1^{(1)}(\theta_2')$ and $\theta_2^{(1)}(\theta_1)$ are such that $\theta_1^{(1)}(\theta_2')s_2 = \theta_2^{(1)}(\theta_1)s_1$. Since $X_0 = \theta, X_1 = (\theta_1', \theta_2), X_{12} = (\theta_1', \theta_2') \in A_1(\sigma)$ we have $Q_1(\theta), Q_1(\theta_1', \theta_2), Q_1(\theta_1', \theta_2') \subset A_1(\sigma)$. Also we have $X_2 = (\theta_1, \theta_2') \in A_2(\sigma)$ hence $Q_2(\theta_1, \theta_2') \subset A_2(\sigma)$. Since $\theta_2^{(1)}(\theta_1)$ is the cut off point for agent 2 at θ_1, $Q_1'(F) \subset A_1(\sigma)$ and $Q_2'(F) \subset A_2(\sigma)$ where $F = (\theta_1, \theta_2^{(1)}(\theta_1))$. Similarly the cut off point for agent 1 at θ_2' is $\theta_1^{(1)}(\theta_2')$. Let $E = (\theta_1^{(1)}(\theta_2'), \theta_2')$, hence $Q_1'(E) \subset A_1(\sigma)$ and $Q_2'(E) \subset A_2(\sigma)$. Take any point $(\theta_1^P, \theta_2^P) := P \in T_1(E) := \{(\theta_1, \theta_2) \in \Theta^2 \mid \theta_1 > \theta_1^{(1)}(\theta_2') \,\&\, \theta_2 > \theta_2'\}$ and, if possible, assume $P \in A_2(\sigma)$. As shown in Figure 3, consider the points X_0, X_1', X_2', P where $X_0, X_1' \in A_1(\sigma)$ and $X_2', P \in A_2(\sigma)$. Then by Lemma 3.2 (B2) the cut off points for agent 2 at θ_1 and at θ_1^P are equal, that is, $\theta_2^{(1)}(\theta_1) = \theta_2^{(1)}(\theta_1^P)$. Given $(\theta_1^P, \theta_2^{(1)}(\theta_1)) \in A_1(\sigma)$, $\theta_2^{(1)}(\theta_1^P) > \theta_2^{(1)}(\theta_1)$. Hence our assumption that $P \in A_2(\sigma)$ is not correct. Therefore, $P \in A_1(\sigma)$ implying that $T_1(E) \subset A_1(\sigma)$. All these facts are represented in Figure 3, where the red coloured region denotes subsets of $A_2(\sigma)$ and the blue coloured region denotes subsets of $A_1(\sigma)$.

In Figure 4, let us consider any $B := (\theta_1^B, \theta_2^B) \in S_1 = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_1'' < \theta_1 \,\&\, \theta_2'' < \theta_2(\theta_1)\}$. If possible, assume $B \in A_2(\sigma)$. Consider the points B, R, X_{12}, M such that $B, M \in A_2(\sigma)$ and $R, X_{12} \in A_1(\sigma)$. Again, using Lemma 3.2 (B2), the cut off points for agent 1 for θ_2^B and θ_2', are equal, that is, $\theta_1^{(1)}(\theta_2^B) = \theta_1^{(1)}(\theta_2')$. However, this is not the case since $\theta_1^{(1)}(\theta_2^B) \leq \theta_1$ and $\theta_1^{(1)}(\theta_2') > \theta_1$ so that $\theta_1^{(1)}(\theta_2^B) \neq \theta_1^{(1)}(\theta_2')$. So $B \in A_1(\sigma)$. Finally, we now show that, for any point r in the rectangle X_2ECF (see Figure 4), $r \in A_2(\sigma)$. If possible, assume $r \in A_1(\sigma)$. We consider the points X_0, T, P, X_2 where $X_0, T \in A_1(\sigma)$ and $X_2, P \in A_2(\sigma)$. Since we have assumed $r \in A_1(\sigma)$, the cut off point for agent 2 at θ_1^r, that is, $\theta_2^{(1)}(\theta_1^r) \geq \theta_2^r$ and the cut off point for agent 2 at θ_1 is $\theta_2^{(1)}(\theta_1)$. Since $\theta_2^{(1)}(\theta_1^r) \geq \theta_2^r > \theta_2^{(1)}(\theta_1)$, Lemma 3.2 (B2) is violated. Hence for any r in the rectangle X_2ECF, $r \in A_2(\sigma)$. The final result of all these arguments is depicted in Figure 5.

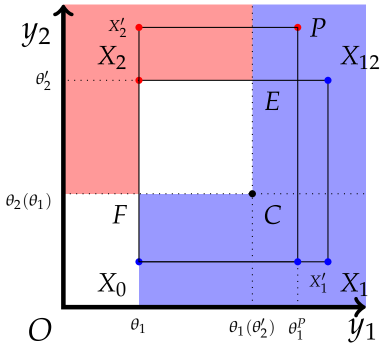


Figure 3

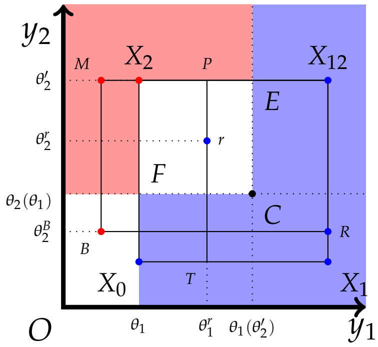


Figure 4

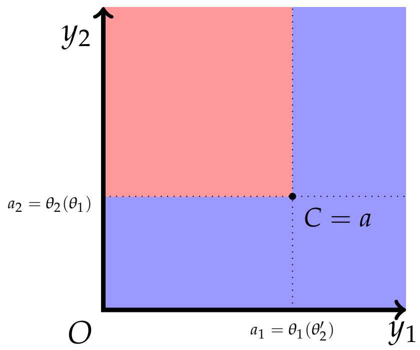


Figure 5

Define $a = (a_1 = \theta_1^{(1)}(\theta_2'), a_2 = \theta_2^{(1)}(\theta_1))$. Figure 5 shows that red coloured region, that is, the open set $Q_2'(a) \subseteq A_2(\sigma)$ and the blue coloured region, that is, the open set $\Theta^2 \setminus Q_2(a) \subseteq A_1(\sigma)$. If $a \in A_1(\sigma)$, then it can be easily shown that there are only these possibilities:

$A_1(\sigma) = \{\theta \in \Theta^2 \mid \textit{either } \theta_1 \geq a_1 \textit{ or } \theta_2 \leq a_2\}$, $A_1(\sigma) = \{\theta \in \Theta^2 \mid \textit{either } \theta_1 \geq a_1 \textit{ or } \theta_2 < a_2\}$ or $A_1(\sigma) = \{\theta \in \Theta^2 \mid \textit{either } \theta_1 > a_1 \textit{ or } \theta_2 \leq a_2\}$ which are the TAB sequencing rule with $k = 1$. If $a \in A_2(\sigma)$, then $A_1(\sigma) = \{\theta \in \Theta^2 \mid \textit{either } \theta_1 > a_1 \textit{ or } \theta_2 < a_2\}$ which is the TAB sequencing rule σ^{T1b}. Hence σ is a TAB sequencing rule of the form (T1 or T3) with $k = 1$ and $l = 2$. This proves Step 1. ;

Step 2: If σ is a non-constant NI sequencing rule that is implementable with balanced transfers and for which **(P1)** does not hold, then σ must be a TAB sequencing rule of the form (T2).

Proof of Step 2: Suppose σ is a non-constant NI sequencing rule that is implementable with balanced transfers and for which **(P1)** does not hold. Since σ is not a constant sequencing rule and satisfies NI, there exists $i \in \{1,2\}$ and a pair $\theta, \theta' \in \Theta^2$ such that $X_0, X_i \in A_i(\sigma)$ and $X_j, X_{12} \in A_j(\sigma)$ where $j \in \{1,2\}$, $j \neq i$, $X_0 = \theta$, $X_1 = (\theta_1', \theta_2)$,

$X_2 = (\theta_1, \theta_2'),\ X_{12} = \theta',\ \theta_1 < \theta_1'$ and $\theta_2 < \theta_2'$. Without loss of generality, let $i = 1, j = 2$ so that $X_0, X_1 \in A_1(\sigma)$ and $X_2, X_{12} \in A_2(\sigma)$. Then by Lemma 3.2 (B2), the cut off points $\theta_2^{(1)}(\theta_1)$ and $\theta_2^{(1)}(\theta_1')$ are such that $\theta_2^{(1)}(\theta_1) = \theta_2^{(1)}(\theta_1') := \bar{\theta}_2$. Consider any $\lambda \in (0,1)$ and the profile pair $(\theta_1(\lambda), \theta_2), \theta' \in \Theta^2$ where $\theta_1(\lambda) := \lambda\theta_1 + (1-\lambda)\theta_1'$ and define $X_0^\lambda = (\theta_1(\lambda), \theta_2)$, $X_1^\lambda = (\theta_1', \theta_2)$, $X_2^\lambda = (\theta_1(\lambda), \theta_2')$, $X_{12}^\lambda = \theta'$. Observe that $X_0^\lambda, X_1^\lambda \in Q_1(\theta_1, \theta_2) \subseteq A_1(\sigma)$ and $X_2^\lambda, X_{12}^\lambda \in Q_2(\theta_1', \theta_2') \subseteq A_2(\sigma)$. Hence by applying Lemma 3.2 (B2) we get the cut off points $\theta_2^{(1)}(\theta_1(\lambda))$ and $\theta_2^{(1)}(\theta_1')$ are such that $\theta_2^{(1)}(\theta_1(\lambda)) = \theta_2^{(1)}(\theta_1') = \bar{\theta}_2$ implying that $\theta_2^{(1)}(\theta_1(\lambda)) = \theta_2^{(1)}(\theta_1) = \theta_2^{(1)}(\theta_1') = \bar{\theta}_2$ for any $\lambda \in (0,1)$. Hence by applying non-increasingness of σ we have $Q_1'(\theta_1, \bar{\theta}_2) \subseteq A_1(\sigma)$ and $Q_2'(\theta_1', \bar{\theta}_2) \subseteq A_2(\sigma)$. This is depicted in Figure 6.

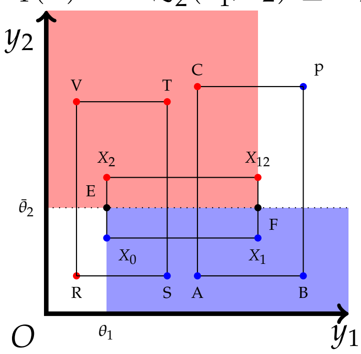


Figure 6

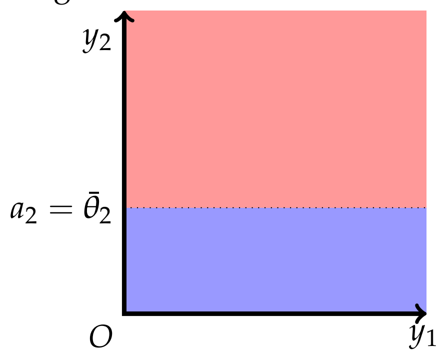


Figure 7

Consider the set $\bar{S}_2(\theta_1', \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_1'' > \theta_1' \ \&\ \theta_2'' > \bar{\theta}_2\}$ and any point $P \in \bar{S}_2(\theta_1', \bar{\theta}_2)$. If $P \in A_1(\sigma)$, then by selecting the rectangle $ABPC$ (see Figure 6) we find that $A, B, P \in A_1(\sigma)$ and $C \in A_2(\sigma)$ which is a violation of the fact that σ fails to satisfy Property **(P1)**. Hence $\bar{S}_2(\theta_1', \bar{\theta}_2) \subseteq A_2(\sigma)$. Similarly, consider the set $\bar{S}_1(\theta_1, \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_1'' < \theta_1 \ \&\ \theta_2'' < \bar{\theta}_2\}$ and any point $R \in \bar{S}_1(\theta_1, \bar{\theta}_2)$. If $R \in A_2(\sigma)$, then by selecting the rectangle $RSTV$ (see Figure 6) we find that $R, V, T \in A_2(\sigma)$ and $S \in A_1(\sigma)$ which is a violation of the fact that σ fails to satisfy Property **(P1)**. Hence $\bar{S}_1(\theta_1, \bar{\theta}_2) \subseteq A_1(\sigma)$. Therefore, we have obtained the following

(t1) $Q_1'(\theta_1, \bar{\theta}_2) \cup \bar{S}_1(\theta_1, \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_2'' < \bar{\theta}_2\} \subseteq A_1(\sigma)$ and

(t2) $Q_2'(\theta_1', \bar{\theta}_2) \cup \bar{S}_2(\theta_1', \bar{\theta}_2) = \{(\theta_1'', \theta_2'') \in \Theta^2 \mid \theta_2'' > \bar{\theta}_2\} \subseteq A_2(\sigma)$.

Cases (t1) and (t2) are depicted in Figure 7. By setting $\bar{\theta}_2 = a_2$, we get $\{\theta \in \Theta^2 \mid \theta_2 > a_2\} \subseteq A_2(\sigma)$ (from (t2) above) and $\Theta^2 \setminus \{\theta \in \Theta^2 \mid \theta_2 \geq a_2\} \subseteq A_1(\sigma)$ (from (t1) above).

What about points on the cut off line $y_2 = a_2 = \bar{\theta}_2$ that separates the two decision? Given non-increasingness of σ we have the following possibilities:

(i) All points of the line $y_2 = a_2$ are in $A_2(\sigma)$.

(ii) All points of the line $y_2 = a_2$ are in $A_1(\sigma)$.

(iii) There exists a $\theta_1^* > 0$ such that all points (θ_1, a_2) with $\theta_1 < \theta_1^*$ are in $A_2(\sigma)$, all points (θ_1', a_2) with $\theta_1' > \theta_1^*$ are in $A_1(\sigma)$ and (θ_1^*, a_2) belongs to either $A_1(\sigma)$ or $A_2(\sigma)$.

However, for case (iii), take the pair of profiles $\theta, \theta' \in \Theta^2$ such that $0 < \theta_1 < \theta_1^* < \theta_1'$ and $\theta_2 = a_2 < \theta_2'$ and define $X_0 = \theta$, $X_1 = (\theta_1', \theta_2)$, $X_2 = (\theta_1, \theta_2')$, $X_{12} = \theta'$. Then we have $X_0, X_2, X_{12} \in A_2(\sigma)$ and $X_1 \in A_1(\sigma)$. This violates our initial assumption that σ fails to satisfy Property **(P1)**. Hence on the cut off line $y_2 = a_2$ either case (i) holds or case (ii) holds. If case (i) holds, then we have σ^{T2a}, and, if case (ii) holds, then we have σ^{T2b}. Hence we have the TAB sequencing rule of the form (T2) with $k = 2$ and $l = 1$.

$\blacksquare$

Proof of Theorem 3.2: For any partition $\pi(N) = (\pi_1, \ldots, \pi_K) \in \Pi$ such that $K \geq 2$ and fix any set of increasing and onto functions $f = \{f_1, \ldots, f_n\}$ and consider the GP-CM sequencing rule $\sigma^{\pi(N),f}$. We show that any such GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfer by establishing that condition (3.7) holds, that is the profile contingent aggregate incentive payment is $(n-1)$ type separable. For any profile $\theta \in \Theta^n$ and any $\pi_r \in \pi(N)$, define the function $z_f(\theta; \pi_r) := \sum_{j \in \pi_r} I_j^{\pi(N),f}(\theta) = \sum_{j \in \pi_r} s_q \left(\sum_{q \in P_j'(\sigma^{\pi(N),f}(\theta)) \cap \pi_r} \left[f_j^{-1} \left(s_j f_q(\theta_q)/s_q \right) \right] \right)$. It is important to note that given any profile $\theta \in \Theta^n$, for any $\pi_r \in \pi(N)$, the sum of incentive payments of the group $\pi_r \in \pi(N)$ is $z_f(\theta; \pi_r)$ and it has the property that it is independent of the waiting costs of the agents $N \setminus \{\pi_r\}$. Hence for any $\theta \in \Theta^n$, for any $\pi_r \in \pi(N)$, we write $z_f(\theta; \pi_r) := z_f(\theta_{\pi_r})$. Consider the GP-CM-“Cut off" based mechanism with transfer (3.13) and select for any $\theta \in \Theta^n$ and any $i \in \pi_k \in \pi(N)$, $G_i(\theta_{-i}) = \sum_{\pi_r \in \pi(N) \setminus \{\pi_k\}} [z_f(\theta_{\pi_r})/(n - |\pi_r|)]$. Given this selection of $G_i(\cdot)$ functions we get

$$\sum_{i\in N}\tau_i^{\pi(N),f}(\theta)=\sum_{i\in N}G_i(\theta_{-i})-I^{\pi(N),f}(\theta)$$

$$=\sum_{i\in N}\left(\sum_{\pi_r\in\pi(N)\setminus\{\pi_k\}}\frac{z_f(\theta_{\pi_r})}{(n-|\pi_r|)}\right)-\sum_{\pi_k\in\pi(N)}\left(\sum_{j\in\pi_k}I_j^{\pi(N),f}(\theta)\right)$$

$$=\sum_{\pi_k\in\pi(N)}\left(\sum_{i\in N\setminus\{\pi_k\}}\frac{z_f(\theta_{\pi_k})}{(n-|\pi_k|)}\right)-\sum_{\pi_k\in\pi(N)}z_f(\theta_{\pi_k})$$

$$=\sum_{\pi_k\in\pi(N)}\left[\frac{z_f(\theta_{\pi_k})}{(n-|\pi_k|)}\left(\sum_{i\in N\setminus\{\pi_k\}}1\right)\right]-\sum_{\pi_k\in\pi(N)}z_f(\theta_{\pi_k})$$

$$=\sum_{\pi_k\in\pi(N)}\left[\frac{z_f(\theta_{\pi_k})}{(n-|\pi_k|)}(n-|\pi_k|)\right]-\sum_{\pi_k\in\pi(N)}z_f(\theta_{\pi_k})$$

$$=\sum_{\pi_k\in\pi(N)}z_f(\theta_{\pi_k})-\sum_{\pi_k\in\pi(N)}z_f(\theta_{\pi_k})=0.$$

Hence, for any given partition $\pi(N)=(\pi_1,\ldots,\pi_K)\in\Pi$ such that $K\geq 2$, for any set of increasing and onto functions $f=\{f_1,\ldots,f_n\}$, the GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfers.

For the partition $\pi(N)=(\pi_1=\pi_K=\{n\})\in\Pi$ such that $K=1$ and any set of increasing and onto functions $f=\{f_1,\ldots,f_n\}$, consider the GP-CM sequencing rule $\sigma^{\pi(N),f}$. We show that any such GP-CM sequencing rule $\sigma^{\pi(N),f}$ is implementable with balanced transfer by establishing that condition (3.7) holds, that is the profile contingent aggregate incentive payment is $(n-1)$ type separable.

Consider $\sigma^{\pi(N),f}(\theta)$ for the profile $\theta\in\Theta^n$ and consider agent i. Define $\Sigma^i(N)=\{\sigma\in\Sigma(N)\mid\sigma_i=n\}$ as the set of orders in $\Sigma(N)$ such that agent i is last in the order. We define the "induced" order $\sigma^{\pi(N),f}(\theta_{-i})\in\arg\min_{\sigma\in\Sigma^i(N)}\sum_{j\in N}f_j(\theta_j)S_j(\sigma)$. Given the profile $\theta\in\Theta^n$ and any agent i, the relation between the order $\sigma^{\pi(N),f}(\theta)$ and the induced order $\sigma^{\pi(N),f}(\theta_{-i})$ is as follows:

$$\sigma_j^{\pi(N),f}(\theta_{-i})=\begin{cases}\sigma_j^{\pi(N),f}(\theta)-1 & \text{if } j\in P_i'(\sigma^{\pi(N),f}(\theta)),\\ \sigma_j^{\pi(N),f}(\theta) & \text{if } j\in P_i(\sigma^{\pi(N),f}(\theta)).\end{cases}\tag{3.23}$$

In words, $\sigma^{\pi(N),f}(\theta_{-i})$ is generated from the order $\sigma^{\pi(N),f}(\theta)$ by moving agent i in the last position and moving all agents behind him up by one position. Consider the GP-CM-"Cut off" based mechanism with transfer (3.13) and select for any $\theta\in\Theta^n$

and any $i \in N$, $G_i(\theta_{-i}) = (1/(n-2)) \sum_{j\in N\setminus\{i\}} s_j \left(\sum_{k\in P_j(\sigma^{\pi(N),f}(\theta_{-i}))} X_{jk}(\theta_k)\right)$ where $X_{jk}(\theta_k) := f_k^{-1}\left(s_k f_j(\theta_j)/s_j\right)$. Given this selection of $G_i(.)$ functions we get

$$\sum_{i\in N} \tau_i^{\pi(N),f}(\theta) = \sum_{i\in N} G_i(\theta_{-i}) - I(\theta)$$

$$= \frac{1}{(n-2)} \sum_{i\in N} \sum_{j\in N\setminus\{i\}} s_j \left(\sum_{k\in P_j(\sigma^{\pi(N),f}(\theta_{-i}))} X_{jk}(\theta_k)\right) - I(\theta)$$

$$= \frac{1}{(n-2)} \sum_{i\in N} \left[\sum_{k\in N\setminus\{i\}} s_j \left(\sum_{j\in P'_k(\sigma^{\pi(N),f}(\theta))} X_{jk}(\theta_k)\right) - \sum_{k\in P_i(\sigma^{\pi(N),f}(\theta))} s_j X_{ik}(\theta_k)\right] - I(\theta)$$

$$= \frac{1}{(n-2)} \sum_{i\in N} \sum_{k\in N\setminus\{i\}} I_k(\theta) - \frac{1}{(n-2)} \sum_{i\in N} \left(\sum_{k\in P_i(\sigma^{\pi(N),f}(\theta))} s_j X_{ik}(\theta_k)\right) - I(\theta)$$

$$= \left(\frac{n-1}{n-2}\right) I(\theta) - \frac{1}{(n-2)} \sum_{i\in N} s_i \left(\sum_{k\in P_i(\sigma^{\pi(N),f}(\theta))} X_{ik}(\theta_k)\right) - I(\theta)$$

$$= \left(\frac{n-1}{n-2}\right) I(\theta) - \frac{1}{(n-2)} I(\theta) - I(\theta) = 0.$$

Thus, any GP-CM sequencing rule $\sigma^{\pi(N),f}$ which is onto is implementable with balanced transfers. ■

Chapter 4

Incentive and normative analysis on sequencing problems

4.1 Introduction

In this chapter, we analyze normative aspects of the sequencing problem. The main features of a sequencing problem are as follows: (1) there are n agents and a single server, (2) the server has multi-functional capability but can process one particular job at a time, that is the server can serve one agent at a time (3) jobs may not be identical across agents, so their job processing times may differ but are common knowledge. We assume agents have quasi-linear preferences over positions in queue and monetary transfers. Many real life phenomenon has this structure. A diagnostic center, installed with a multi-functional machine (due to space shortage), where a certain number of enlisted patients visits for diagnosis, software installation problem to PCs of a set of agents can be real life examples. Many other comparable situations can be found in Maniquet (2003), Hashimoto and Saitoh (2012), Mukherjee (2013).

In case of sequencing problem, the property, outcome efficiency, is a widely studied.[1]. In sequencing or queuing context outcome efficient allocation is the one, for which aggregate waiting cost is minimum. The seminal works of Vickrey (1961),

[1] See Suijs (1996), Mitra (2002), Maniquet (2003) in this context

Clarke (1971), Groves (1973) have shown that outcome efficiency can be harnessed with stratefyproofness or non-manipulability. Holmström (1979)'s result in context of sequencing problem implies a mechanism satisfies strategy-proofness and outcome efficiency if and only if it is a Vickrey-Clarke-Groves (VCG) mechanism [2]. We focus on the compatibility of a fairness axiom, egalitarian equivalence, introduced by Pazner and Schmeidler (1978) and get a subclass of Vickrey-Clarke-Groves (VCG) mechanism that satisfies egalitarian equivalence. Chun et al. (2014) has identified similar type of subclass of Vickrey-Clarke-Groves (VCG) mechanism for queuing problems. Their main assumption regarding the egalitarian equivalent reference bundle is; the reference position can only be an element from the set $\{1, 2, \ldots, n\}$. Hence, the reference waiting time is similarly restricted in their work. In our work, we have analyzed the situation where the only restriction on the reference waiting time is, it must be positive. We also have identified a constant reference waiting time function that, we argue, is very realistic from the practical point of view in context sequencing problems.

Mitra (2002) has shown among more general and natural class of sequencing problems, sequencing problems with linear cost structure is the only class for which outcome efficiency, budget balance and strategy-proofness (known as first best) can be achieved. Chun et al. (2014) has also attained feasibility along with egalitarian equivalent VCG mechanism in case of queuing problem when the reference position was the first position of the queue but we get an impossibility result in case of sequencing problem, that is, no mechanism is efficient, feasible, non-manipulable and egalitarian equivalent.

Sequencing problem have also been analyzed from the perspective of group manipulability . In this context we must mention the work of Mitra and Mutuswami (2011) that shows there does not exist any mechanism that satisfies outcome efficiency and strong group strategy-proofness in a single machine queuing context. Similarly in sequencing Kayi and Ramaekers (2008) has shown that no rule satisfies outcome efficiency and coalitional strategy-proofness. Whereas we show that, no

[2]See Vickrey (1961), Clarke (1971), Groves (1973)

mechanism satisfies outcome efficiency, egalitarian equivalence and pair-wise weak group strategy-proofness which is weaker than group strategy-proofness.

The idea of identical preference lower bound was first introduced by Moulin (1990) which he termed as egalitarian lower bound. This concept was applied in queuing problem by Maniquet (2003), Chun (2006) and others. We show that the two fairness notions identical preference lower bound and egalitarian equivalence are compatible. In case of two agents we have necessary and sufficient condition but for more than two agents we have a sufficient condition that ensures the compatibility between two fairness notions.

This chapter is arranged in the following way. In Section 4.2, we formally introduce the model and add necessary definitions. In Section 4.3, we state and prove characterization results regarding egalitarian equivalent VCG mechanism. In Section 4.4, we focus on feasibility and group staretegyproofness issues of egalitarian equivalent VCG mechanism . In Section 4.5, we analyze the possibility of identical preference lower bound property of egalitarian equivalent VCG mechanism. In Section 4.6, we again go back to analyze the compatibility of egalitarian equivalence and VCG mechanism when the reference waiting time is a non-constant function of the type profile of the agents. Lastly, in Section 4.7, we conclude the chapter.

4.2 The Model

We consider the set of agents $N = \{1, \ldots, n\}$ with a single machine. Each individual has a different kind of work to be executed by the machine. The machine can process one job at a time. Let $\forall i \in N$, $s_i \in \Re_{++}$ where s_i denotes the processing time of ith agent and we assume $s_1 \geq s_2 \geq \ldots \geq s_{n-1} \geq s_n$ without loss of generality. Each agent is identified with a positive waiting cost $\theta_i \in \Re_{++}$, the cost of waiting per unit of time. The profile of waiting costs of the set of all agents is typically denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Re^n_{++}$. For any $i \in N$, θ_{-i} denotes the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \Re^{n-1}_{++}$. Also $\mathbb{N}$ denotes the set of natural numbers.

A sequence is an onto function $\sigma : N \to \{1, \ldots, n\}$. An allocation of n jobs can be done in many ways. An allocation rule is a mapping $\sigma : \Re^n_{++} \to \Sigma(N)$ that specifies for each profile $\theta \in \Re^n_{++}$ an allocation (rank) vector $\sigma(\theta) \in \Sigma(N)$. Agent i's position is denoted by $\sigma_i(\theta)$ which is an input of the vector $\sigma(\theta)$. Let $\Sigma(N)$ denote the set of all possible sequence of agents in N. Given $\sigma \in \Sigma(N), \forall \in N, P_i(\sigma) = \{j \in N | \sigma_j(\theta) < \sigma_i(\theta)\}$ denotes the set of predecessors of i and similarly $P'_i(\sigma) = \{j \in N | \sigma_j(\theta) > \sigma_i(\theta)\}$ denotes the set of successors of i. Agent i's waiting time is denoted by $S_i(\sigma(\theta))$ and corresponding waiting cost is $S_i(\sigma(\theta))\theta_i$. A transfer rule is a mapping $t : \Re^n_{++} \to \Re^n$ that specifies for each profile $\theta \in \Re^n_{++}$ a transfer vector $t(\theta) = (t_i(\theta), \ldots, t_n(\theta)) \in \Re^n$. We assume that the utility function of each agent $i \in N$ is quasi-liner and is of the form $U_i(\sigma(\theta), t_i(\theta), \theta_i) = -S_i(\sigma(\theta)(\theta_i) + t_i(\theta)$, where $t_i(\theta)$ is the monetary transfer of agent to i.

DEFINITION **4.1** For all $\theta \in \Re^n_{++}$, a sequence $\sigma \in \Sigma(N)$ is outcome efficient if $\sigma \in E(\theta) = \operatorname{argmin}_{\sigma \in \Sigma(N)} \sum_{i=1}^{n} S_i(\sigma)\theta_i$.

The implication of outcome efficiency is that agents are ranked according to the non-increasing order of their relative waiting costs (that is, if $\theta_i/s_i \geq \theta_j/s_j$ under a profile θ, then $S_i(\sigma(\theta)) \leq S_i(\sigma(\theta))$). Moreover, there are profiles for which more than one rank vector is efficient. For example, in case of queuing problem if all agents have the same waiting cost, then all rank vectors are efficient. In the context of sequencing problem, if the profile of waiting cost is $(s_1, s_2, \ldots, s_n)$, then all agents have the same relative waiting cost. Therefore, we have an efficiency correspondence. In this paper, we choose a particular outcome efficient rule (that is, a single valued selection from the outcome efficiency correspondence) using a tie breaking rule. For our outcome efficient rule, we use the following tie breaking rule: if $i < j$ and $\theta_i/s_i = \theta_j/s_j$ then $S_i(\sigma(\theta)) < S_i(\sigma(\theta))$. This tie breaking rule guarantees that, given a profile $\theta \in \Re^n_{++}$, the efficient rule selects a single rank vector from $\Sigma(N)$.

A mechanism is (σ, t) constitutes of an allocation rule σ and a transfer rule t. We formally define the necessary concepts that we have already introduced in the introduction of this chapter.

DEFINITION **4.2** A mechanism(σ, t) is strategy-proof (SP) if for all $i \in N$, for all $\theta_i, \theta_i' \in \Re_{++}$ and for all $\theta_{-i} \in \Re_{++}^{(n-1)}$, $-S_i(\sigma(\theta_i, \theta_{-i}))\theta_i + t_i(\theta_i, \theta_{-i}) \geq -S_i(\sigma(\theta_i', \theta_{-i}))\theta_i + t_i(\theta_i', \theta_{-i})$.

It means, for every agents, reporting the true type weakly dominates reporting false type. Hence, stragyproofness restricts any kind of unilateral deviation.

DEFINITION **4.3** A mechanism(σ, t) is outcome efficient (OE) if for all announced profile $\theta \in \Re_{++}^n, \sigma(\theta) \in E(\theta) = \text{argmin}_{\sigma \in \Sigma(N)} \sum_{i=1}^n S_i(\sigma)\theta_i$.

The main results of this chapter that we derive in the next section is based on VCG mechanism that we define next.

DEFINITION **4.4** A mechanism (σ, t) is a VCG mechanism if for all $\theta, \sigma(\theta) \in E(\theta)$, and the transfers are given by,

$$\forall i \in N : t_i(\theta) = -\sum_{j \neq i} \theta_j S_j(\sigma(\theta)) + h_i(\theta_{-i}). \tag{4.1}$$

When the preferences are quasi-linear and the domain of type is convex then a mechanism is OE and SP if and only if it is a VCG mechanism (Holmström (1979)).

DEFINITION **4.5** A mechanism (σ, t) satisfies egalitarian equivalence (EE) if for all $\theta \in \Re_{++}^n$ there exist $(\bar{S}(\theta), t(\theta))$ such that for all $i \in N$, $-S_i(\sigma(\theta))\theta_i + t_i(\theta) = -\bar{S}(\theta)\theta_i + \bar{t}(\theta)$.

Here $(\bar{S}(\theta), \bar{t}(\theta))$ denotes the reference bundle, where $\bar{S}(\theta)$ is the reference waiting time and $\bar{t}(\theta)$ is the reference transfer. Egalitarian equivalence was introduced by Pazner and Schmeidler (1978) and is based on the idea that all individuals should be placed in a situation which is Pareto-indifferent to a perfectly egalitarian allocation. In case of sequencing problem $(\bar{S}(\theta), \bar{t}(\theta))$ is such a reference bundle, where, if the agent is placed remains indifferent to the original bundle that he receives under VCG mechanism.

DEFINITION **4.6** A mechanism (σ, t) satisfies budget balance (BB) if for all $\theta \in \Re^n_{++}$, $\sum_{i=1}^n t_i(\theta) = 0$.

DEFINITION **4.7** A mechanism (σ, t) satisfies feasibility (FSB) if for all $\theta \in \Re^n_{++}$, $\sum_{i=1}^n t_i(\theta) \leq 0$.

The profile θ and θ' are S-variants if for all $i \in N \setminus S$, $\theta_i = \theta'_i$.

DEFINITION **4.8** A mechanism (σ, t) is weakly group strategy-proof (WSP) if for all S-variants θ, θ' $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq U_i(\sigma(\theta'), t_i(\theta'), \theta_i)$ for at least one $i \in S$.

This implies as long as all the group member are not strictly better off by deviating from their true profile, such group will not be formed.

DEFINITION **4.9** A mechanism (σ, t) is pair-wise weakly group strategy-proof (PWSP) if for all *S-variants* θ, θ' where $|S| = 2$, $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq U_i(\sigma(\theta'), t_i(\theta'), \theta_i)$ for at least one $i \in S$.

This implies pair of agents deviates from their true profile by jointly misreporting if an only if they are both strictly better off from the situation when they truthfully reports.

Consider any agent $i \in N$. If the agent i a-priori perceives that he is not different form any agent $j(\neq i) \in N$ (in terms of relative waiting cost) then he will consider every feasible allocation $\sigma \in \Sigma(N)$ as a possible outcome. Hence, we need to consider the a-priori excepted cost perceived by agent i. Consider agent i at position r in the queue. For any $j(\neq i) \in N$, the waiting time imposed by agent j on agent i is $(r-1)(n-2)!s_j$. So in total any agent $j(\neq i) \in N$ imposes $\sum_{r=1}^n (r-1)(n-2)!s_j = n(n-1)!s_j/2$ amount of waiting time on agent i. Note that agent i can except all the $n!$ allocations. Hence, the average waiting cost perceived by agent i, imposed by all the other agents would be $(s_i + \sum_{j\neq i} s_j/2)\theta_i$ where θ_i is the per unit time waiting cost of agent i.

DEFINITION **4.10** A mechanism (σ, t) satisfies identical preference lower bound (IPLB) if for all $\theta \in \Re^n_{++}$, for all $i \in N$, $U_i(\sigma(\theta), t_i(\theta), \theta_i) \geq -(s_i + \sum_{j\neq i} s_j/2)\theta_i$.

This concept was first introduced by Moulin (1990) and is based on the idea that an agent's welfare is at least as that of consuming his equal share of resources. In the context of sequencing problem agents are considered identical as long as their relative waiting costs are same. That is for all $i, j \in N$, if $\theta_i/s_i = \theta_j/s_j$ then agents are considered identical. Identical preference lower bound implies that any agent's utility should be at least as that of average or expected utility of that agent when he/she perceives all the other agents as identical to himself.

4.3 Egalitarian equivalent VCG mechanism

In this section, we examine the implication of egalitarian equivalence on a strategy-proof mechanism that satisfies efficient allocation rule. We will use slightly different notation to refer an agent. An agent at i-th position in the queue is denoted as agent (i). So the true waiting cost profile is $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)})$ is such that $\lambda_{(1)} \geq \lambda_{(2)} \geq \ldots \geq \lambda_{(n)} > 0$ where $\forall i \in N$, $\lambda_{(i)} = \theta_{(i)}/s_{(i)}$. Hence, $\forall \theta = (\theta_{(i)}, \theta_{(-i)}) \in \Re^n_{++}$, $\forall i \in N$, $s_i \neq s_{(i)}$.

The crucial fact behind the idea of egalitarian equivalent allocation where everyone consumes the same "reference bundle" and derives same utility as they get with the initially allocated bundle. In case of queuing problem, Chun et al. (2014) have completely characterized EE, SP and OE mechanisms. They restricted the reference position, that can vary with type profile, on the set $\{1, 2, \ldots, n\}$ as these are the only positions available in queuing problem with $|N|$ agents. Hence, Chun et al. (2014) avoided any arbitrary reference waiting time to keep the analysis natural for queuing context.

Our modification in this context is the following: unlike queuing, in sequencing problem agents differs in job processing time simply because different agents have different jobs to process. Hence, it is not possible to contemplate all the position {1,2,. . . ,n} as a potential reference bundle. For example, when $N = \{1, 2, 3\}$, if we fix the reference position as the second position of the queue then $\bar{S}(\theta)$ is perceived differently by different agents as $\bar{S}(\theta) \in \{(s_1+s_2),(s_1+s_3),(s_2+s_3)\}$. Agent 1 in second position may face (s_1+s_2) or (s_1+s_3) as waiting time, agent 2 can face (s_1+s_2) or (s_2+s_3) and similarly agent 3 can perceive (s_1+s_3) and (s_2+s_3) . The only feasible reference position is the last position as whatever be the allocation and whoever is the agent, the reference waiting time for the last position is always $(s_1+s_2+s_3)$. Hence, to keep our analysis natural in sequencing context, we will assume the only feasible reference position is the last position of the queue. Therefore, we will have $\bar{S}(\theta) = \bar{S} = \sum_{i=1}^{n} s_i$. With this precondition== we get the following result:

PROPOSITION **4.1** A mechanism(σ, t) satisfies EE, SP and OE if and only if the reference bundle for the profile $\theta \in \mathfrak{R}^n_{++}$ (where θ is a non-zero profile) is of the form $(\bar{S}(\theta), \bar{t}(\theta))$ where $\forall \theta \in \mathfrak{R}_{++}$ and $\bar{t}(\theta) = \sum_{i \in N}\{\bar{S} - S_{(i)}(\sigma(\theta))\}\theta_{(i)} + \bar{k}$ when $\bar{S}(\theta) = \bar{S}$.

Proof: Let us consider an announcement profile $\theta = (\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(n)}) \in \mathfrak{R}^n_{++}$. Therefore, given the OE allocation rule and the tie breaking rule, we can arrange agents uniquely i.e. $\sigma_{(i)}(\theta) = i$. Since the domain of preference is quasi-linear and type spaces for the agents are convex, it follows from Hölmstrom's result on efficient and strategy-proof mechanisms that (σ, t) must be a VCG mechanism. This implies that the transfer is given by

$$\forall i \in N : t_{(i)}(\theta) = -\sum_{j \neq (i)} \theta_{(j)} S_{(j)}(\sigma(\theta)) + h_{(i)}(\theta_{-(i)}). \tag{4.2}$$

If we set $h_{(i)}(\theta_{-(i)}) = \sum_{j \neq (i)} S_{(j)}\theta_{(j)}(\sigma(\theta_{-(i)})) + g_{(i)}(\theta_{-(i)})$ in equation (4.2) we get

$$\forall i \in N : t_{(i)}(\theta) = -s_{(i)} \sum_{j \in P'_{(i)}(\sigma(\theta))} \theta_{(j)} + g_{(i)}(\theta_{-(i)}). \tag{4.3}$$

As the mechanism (σ, t) satisfies EE, SP and OE the following condition must hold

$$\forall i \in N : -\theta_{(i)} S_{(i)}(\sigma(\theta)) + t_{(i)}(\theta) = -\theta_{(i)}\bar{S} + \bar{t}(\theta).$$

Where the left side of the above equation is the utility from a VCG mechanism and the right hand side is the utility from EE requirement.The above expression can alternatively be written as

$$\bar{t}(\theta) = -\theta_{(i)} S_{(i)}(\sigma(\theta)) - s_{(i)} \sum_{j \in P'_{(i)}(\sigma(\theta))} \theta_{(j)} + g_{(i)}(\theta_{-(i)} + \theta_{(i)} \bar{S}. \tag{4.4}$$

Putting $i = 1$ into equation (4.4) we get,

$$\bar{t}(\theta) = -\theta_{(1)} S_{(1)}(\sigma(\theta)) - s_{(1)} \sum_{j \in P'_{(1)}(\sigma(\theta))} \theta_{(j)} + g_{(1)}(\theta_{-(1)}) + \theta_{(1)} \bar{S}.$$

Similarly for $i = 2$ we have,

$$\bar{t}(\theta) = -\theta_{(2)} S_{(2)}(\sigma(\theta)) - s_{(2)} \sum_{j \in P'_{(2)}(\sigma(\theta))} \theta_{(j)} + g_{(2)}(\theta_{-(2)}) + \theta_{(2)} \bar{S}.$$

Equating the expressions for $\bar{t}(\theta)$ we get, $-s_{(1)}\theta_{(1)} - s_{(1)}\theta_{(2)} + g_{(1)}(\theta_{-(1)}) = -\theta_{(2)}(s_{(1)} + s_{(2)}) + (s_{(1)} - s_{(2)}) \sum_{j \in P'_{(2)}(\sigma(\theta))} \theta_j + g_{(2)}(\theta_{-(2)}) - \bar{S}(\theta_{(1)} - \theta_{(2)})$.
Since $g_{(1)}(\theta_{-(1)})$ is independent of $\theta_{(1)}$ and $g_{(2)}(\theta_{-(2)})$ is independent of $\theta_{(2)}$ we get $g_{(1)}(\theta_{-(1)}) = (\bar{S} - s_{(2)})\theta_{(2)} + f_{(1)}(\theta_{N \setminus \{(1),(2)\}})$ and $g_{(2)}(\theta_{-(2)}) = (\bar{S} - s_{(1)})\theta_{(1)} + f_{(2)}(\theta_{N \setminus \{(1),(2)\}})$. Now comparing the expression for $\bar{t}(\theta)$ for $i = 1$ and $i = 3$ and using the expression of $g_{(1)}(\theta_{-(1)})$ we have,
$(\bar{S} - s_{(1)})\theta_{(1)} + (\bar{S} - s_{(2)})\theta_{(2)} + f_{(1)}(\theta_{N \setminus \{(1),(2)\}}) = -\theta_{(3)}(s_{(1)} + s_{(2)} + s_{(3)}) + s_{(1)}\theta_{(2)} + s_{(1)}\theta_{(3)} + (s_{(1)} - s_{(2)}) \sum_{j \in P'_{(3)}(\sigma(\theta))} \theta_{(j)} + g_{(3)}(\theta_{-(3)}) + \bar{S}\theta_{(3)}$.
Comparing the expressions on both sides in the similar fashion we get $g_{(1)}(\theta_{-(1)}) = (\bar{S} - s_{(2)})\theta_{(2)} + \{\bar{S} - (s_{(2)} + s_{(3)})\}\theta_{(3)} + f'_{(1)}(\theta_{N \setminus \{(1),(2),(3)\}})$ and $g_{(3)}(\theta_{-(3)}) = (\bar{S} - s_{(1)})\theta_1 + \{\bar{S} - (s_{(1)} + s_{(2)})\}\theta_{(2)} + f_{(3)}(\theta_{N \setminus \{(1),(2),(3)\}})$.
By using the same argument recursively we get

$$g_{(1)}(\theta_{-(1)}) = \sum_{j \neq (1)}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{N \setminus \{(1)\}}))\}\theta_{(j)} + k_{(1)}.$$

In fact (it can easily be shown that) the above expression, holds not only for $i = 1$ but for all $i \in N$;

$$g_{(i)}(\theta_{-(i)}) = \sum_{j \neq (i)}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{N\setminus\{(i)\}}))\}\theta_{(j)} + k_{(i)}.$$

Now we further get $\forall i, j \in N$, $k_i = k_j = \bar{k}$ by using the above expression of $g_{(i)}(\theta_{-(i)})$ into $\bar{t}(\theta)$ in equation (4.4) and equating them. Hence

$$\forall i \in N: \quad g_{(i)}(\theta_{-(i)}) = \sum_{j \neq (i)}^{n} \{\bar{S} - S_{(j)}(\sigma(\theta_{N\setminus\{(i)\}}))\}\theta_{(j)} + \bar{k}. \tag{4.5}$$

Using the above expression of $g_i(\theta_{(-i)})$ in equation (4.4) we have, $\bar{t}(\theta) = \sum_{i \in N}\{\bar{S}\} - S_{(i)}(\sigma(\theta))\}\theta_{(i)} + \bar{k}$.

Therefore it follows that when $\bar{S}(\theta) = \bar{S}$, a mechanism satisfies EE, SP and OE only if the reference bundle for the profile θ i.e. $(\bar{S}(\theta), t(\theta))$ is of the form $\bar{t}(\theta) = \sum_{i \in N}\{\bar{S} - S_i(\sigma(\theta))\}\theta_i + \bar{k}$.

Sufficiency is fairly obvious, hence omitted. ■

Using the expression of $g_{(i)}(\theta_{-(i)})$ given by equation (4.5), we get the expression of $t_{(i)}(\theta)$ as follows:

$$t_{(i)}(\theta) = \sum_{j \neq i}\{\bar{S} - S_{(j)}(\sigma(\theta))\}\theta_{(j)} + \bar{k}. \tag{4.6}$$

4.4 Feasibility and pair wise weakly group strategy-proofness

PROPOSITION **4.2** In a sequencing problem no mechanism satisfies OE, SP, EE and FSB.

Proof: For all $i \in N$ and for all $\theta = (\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(n)})$ we have $t_{(i)}(\theta) = \sum_{j \neq i}\{\bar{S} - S_{(j)}(\sigma(\theta))\}\theta_{(j)} + \bar{k}$. If FSB holds then for all $\theta = (\theta_{(1)}, \theta_{(2)}, \dots, \theta_{(n)})$, $\sum_{i \in N} t_{(i)}(\theta) \leq 0$. Therefore, we have the following: $\sum_{i \in N}\{S_{(i)}(\sigma(\theta)) - \bar{S}\}\theta_{(i)} \geq n\bar{k}/(n-1)$. If $\bar{k} \geq 0$, consider the profile $\theta = (\theta_{(n)}, \theta_{-(n)})$ such that for all $j \neq n, \theta_{(j)} = 1$. Since, $\bar{S} =$

$\sum_{i\in N} s_{(i)} = S_{(n)}$, we have $\sum_{i\in N}\{S_{(i)}(\sigma(\theta)) - \bar{S}\}\theta_{(i)} < n\bar{k}/(n-1)$. If $\bar{k} < 0$, consider the profile $\theta = (\theta_{(1)}, \theta_{-(1)})$ such that for all $j \neq 1, \theta_{(j)} = 1$ and $\theta_{(1)} = (1 + 2\bar{k}/(S_{(1)} - \bar{S}))$. Then as $\bar{S} = \sum_{i\in N} s_{(i)} = S_{(n)}$, we have $\sum_{i\in N}\{S_{(i)}(\sigma(\theta)) - \bar{S}\}\theta_{(i)} < n\bar{k}/(n-1)$.

Hence, FSB is violated. ■

REMARK **4.1** The consequence of the above proposition is, in case of sequencing problem, no mechanism satisfies OE, SP, EE and BB.

PROPOSITION **4.3** Consider a sequencing problem such that $|N| > 2$. Then no mechanism satisfies OE, PWSP, EE.

Proof: If a mechanism (σ, t) satisfies EE, SP and OE then for all $\theta \in \Re^n_{++}$ the allocation of an agent $i \in N$ is given by $(\sigma(\theta), t_{(i)}(\theta) = \sum_{j\neq i}\{\bar{S} - S_{(j)}(\sigma(\theta))\}\theta_{(j)})$. Suppose the true waiting cost profile is $\theta = (\theta_{(1)}, \theta_{(2)}, \ldots, \theta_{(n)})$ is such that $\lambda_{(1)} > \lambda_{(2)} > \ldots > \lambda_{(n)} > 0$ where for all $i \in N,\ \ \lambda_{(i)} = \theta_{(i)}/s_{(i)}$. Consider $\forall i \in N,\ \ \theta'_{(i)} = \theta_{(i)} + \epsilon_1$ such that $\epsilon_1 = \min(s_{(j)}\lambda_{(j-1)} - \theta_{(j)})/2,\ \ j \in \{2, 3, \ldots, n\}$. Let agents (1) and (2) jointly misreports as $\theta'_{(1)} = \theta_{(1)} + \epsilon_1$ and $\theta'_{(2)} = \theta_{(2)} + \epsilon_1$. The basic idea is to construct a new profile , such that, under this new misreported profile relative queue position is unaltered. Notice that, under this new profile $\theta^* = (\theta'_{(1)}, \theta'_{(2)}, \theta_{(3)}, \theta_{(4)}, \ldots, \theta_{(n)}),\ t_{(2)}(\theta^*) > t_{(2)}(\theta)$ and $t_{(1)}(\theta^*) > t_{(1)}(\theta)$ since $\epsilon_1 > 0$ by construction. Hence, profitable group deviation exists for agents (1) and (2). Therefore, PWSP is impossible along with OE and EE. ■

REMARK **4.2** In a sequencing problem with exactly two agents the class of mechanisms that satisfy EE, OE and SP is not PWSP. That is, group strategy-proofness is impossible in is case.

4.5 Identical preference lower bound(IPLB) and egalitarian equivalent VCG mechanism

PROPOSITION **4.4** In case of two agents consider a mechanism that satisfies OE,SP,EE then it also satisfies IPLB if and only if $\bar{k} \geq -s_{(2)}\theta_{(1)}/2$

Proof: In two-agent case a typical profile is $(\theta_{(1)}, \theta_{(2)}) \in \Re^2_{++}$. Hence, in the efficient allocation $\lambda_{(1)} \geq \lambda_{(2)}$, that is, $s_{(2)}\theta_{(1)} > s_{(1)}\theta_{(2)}$. For sequencing problem with two agents, the reference waiting time is $\bar{S} = s_{(1)} + s_{(2)}$.

IPLB is compatible with egalitarian equivalent VCG mechanism if for all $i \in N$ and for all $\theta = (\theta_{(1)}, \theta_{(2)}) \in \Re^2_{++}$, $U_{(i)}(\bar{S}, \bar{t}(\theta)) \geq C_{(i)}(\theta)$ or

$$-\bar{S}\theta_{(i)} + \sum_{j \in \{1,2\}} (\bar{S} - S_{(j)})\theta_{(j)} \geq -(s_{(i)} + \sum_{j \neq i} \frac{s_{(j)}}{2})\theta_{(i)}. \tag{4.7}$$

Consider, $i = 1$. Then following equation (4.7) we have, $-\{s_{(1)} + s_{(2)}\}\theta_{(1)} + \{(s_{(1)} + s_{(2)} - s_{(1)}\}\theta_{(1)} + \{s_{(1)} + s_{(2)} - s_{(1)} - s_{(2)}\}\theta_{(2)} \geq -\{s_{(1)} + s_{(2)}/2\}\theta_{(1)}$. Solving the above equation we get, (i).. $\bar{k} \geq -s_{(2)}\theta_{(1)}/2$.

Similarly, for $i = 2$, following equation (4.7) we have, $-\{s_{(1)} + s_{(2)}\}\theta_{(2)} + \{(s_{(1)} + s_{(2)} - s_{(1)}\}\theta_{(1)} + \{s_{(1)} + s_{(2)} - s_{(1)} - s_{(2)}\}\theta_{(2)} \geq -\{s_{(2)} + s_{(1)}/2\}\theta_{(2)}$. Solving the above equation we get,(ii).. $s_{(2)}\theta_{(1)} + \bar{k} \geq -s_{(1)}\theta_{(2)}/2$.

Notice that, if (i) holds then (ii) holds trivially. Therefore, condition (i), that is, $\bar{k} \geq -s_{(2)}\theta_{(1)}/2$[3] is necessary and sufficient condition for IPLB along with egalitarian equivalent VCG mechanism for two agents. ■

PROPOSITION **4.5** Consider a mechanism (σ, t) that satisfies OE, SP, EE if $\bar{k} \geq 0$ then it satisfies IPLB.

Proof: If a mechanism (σ, t) EE, OE, SP and IPLB then $\forall i \in N$, $\forall \theta \in \Re^n_{++}$ we have,

$$U_{(i)}(\sigma(\theta)) - C_{(i)}(\theta) = \left(\sum_{q \in P'_{(i)}(\sigma(\theta))} s_q - \sum_{r \in P_{(i)}(\sigma(\theta))} s_r \right) \theta_{(i)} + \sum_{j \neq i} (\bar{S} - S_{(j)})\theta_{(j)} + \bar{k} \geq 0. \tag{4.8}$$

Note that, $\sum_{j \neq i}(\bar{S} - S_{(j)})\theta_{(j)} = \sum_{r \in P_{(i)}(\sigma(\theta))}(\bar{S} - S_r)\theta_r + \sum_{q \in P'_{(i)}(\sigma(\theta))}(\bar{S} - S_q)\theta_q$. Also, $\sum_{r \in P_{(i)}(\sigma(\theta))}(\bar{S} - S_r)\theta_r = s_{(i)} \sum_{r \in P_{(i)}(\sigma(\theta))} \theta_r + \sum_{r \in P_{(i)}(\sigma(\theta))} \left(\sum_{q \in P'_{(i)}(\sigma(\theta))} s_q \right) \theta_r +$

[3] It is the average negative externality that is imposed on the first served agent by the last served agent.

$\sum_{r\in P_{(i)}(\sigma(\theta))}\left(\sum_{m=r+1}^{(i)-1} s_m\right)\theta_r$. But, $s_{(i)}\sum_{r\in P_{(i)}(\sigma(\theta))}\theta_r - \theta_r\sum_{r\in P_{(i)}(\sigma(\theta))} s_r \geq 0$, because agents with higher $\lambda_{(.)}$ are placed in the earlier positions of the queue (since we have outcome efficiency of allocation). Therefore, $\left(\sum_{q\in P'_{(i)}(\sigma(\theta))} s_q - \sum_{r\in P_{(i)}(\sigma(\theta))} s_r\right)\theta_{(i)} + \sum_{j\neq i}(\bar{S} - S_{(j)})\theta_{(j)} > 0$. Since $\bar{k} \geq 0$, therefore $U_{(i)}(\sigma(\theta)) - C_{(i)}(\theta) > 0$. Hence IPLB holds. ■

4.6 Egalitarian equivalent VCG mechanism revisited

This section is a diversion from the natural condition of sequencing problem. Ideally there should be an one to one correspondence between reference position and reference waiting time that had been in the context of queuing problem (See Chun et al. (2014)). The same is true with sequencing if and only if reference position is the last position of the queue. But now we assume any positive reference waiting time is possible. We have already seen that if reference position is assumed to be constant then egalitarian equivalent VCG mechanism is achievable. Now we ask the following question: what if the reference position is explicitly a function of the type profile θ where $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathfrak{R}^n_{++}$? The answer that we have found is a sufficient one, although not necessary. Our hunch about the necessary condition is that the reference position function $\bar{S}(\theta)$ should be symmetric in nature, that is, $\forall\theta, \theta'$ where θ' is some permutation of θ we need $\bar{S}(\theta) = \bar{S}(\theta')$ for symmetry.

LEMMA **4.1** A mechanism (σ, t) satisfies EE, SP and OE only if $\forall i, j (i \neq j) \in N$, $\forall\theta \in \mathfrak{R}^n_{++}$: $h_i(\theta_i) - h_j(\theta_j) = \bar{S}(\theta)(\theta_j - \theta_i)$.

Proof: The general form of VCG transfer is the followed form equation (4.2) and in this case is of the following form.

$$\forall i \in N : t_i(\theta) = -\sum_{j\neq i}\theta_j S_j(\sigma(\theta)) + h_i(\theta_{-i}). \tag{4.9}$$

So a VCG mechanism is egalitarian equivalent if $\forall\theta \in \mathfrak{R}^n_{++}$ the following holds:

$$\forall i \in N : -\theta_i S_i(\sigma(\theta)) + t_i(\theta) = -\theta_i \bar{S}(\theta) + \bar{t}(\theta). \tag{4.10}$$

Using equations (4.9) and (4.10) we have

$$\begin{aligned} -\theta_i S_i(\sigma(\theta)) - \sum_{j\neq i} \theta_j S_j(\sigma(\theta)) + h_i(\theta_{-i}) &= -\theta_i \bar{S}(\theta) + \bar{t}(\theta) \\ or \qquad C(\sigma(\theta)) + h_i(\theta_{-i}) &= -\theta_i \bar{S}(\theta) + \bar{t}(\theta) \end{aligned} \tag{4.11}$$

where $C(\sigma(\theta))$ denotes the cost under efficient allocation when the type profile is θ. For any $\theta \in \mathfrak{R}^n_{++}$ and any $i \neq j \in N$, using equation (4.11) we get

$$h_i(\theta_i) - h_j(\theta_j) = \bar{S}(\theta)(\theta_j - \theta_i). \tag{4.12}$$

■

PROPOSITION **4.6** If $N = \{1, 2\}$, a mechanism (σ, t) satisfies EE, SP and OE only if $\bar{S}(\theta)$ is symmetric.

Proof: Consider, $\theta = (\theta_1, \theta_2)$ and $\theta' = (\theta'_1, \theta'_2)$ where $\theta'_1 = \theta_2$ and $\theta'_2 = \theta_1$. Using equation (4.12) we get the following: When the type profile is $\theta = (\theta_1, \theta_2))$ then

$$h_1(\theta_2) - h_2(\theta_1) = \bar{S}(\theta_1, \theta_2)(\theta_2 - \theta_1) \text{ ...(I)}$$

and when the type profile is $\theta' = (\theta'_1, \theta'_2)$ then

$$h_1(\theta_1) - h_2(\theta_2) = \bar{S}(\theta_2, \theta_1)(\theta_1 - \theta_2)\text{...(II)}$$

Since equation (4.12) holds for all $\theta \in \mathfrak{R}^{++}_2$, from (I) and (II) we have $h_1(\theta_2) = h_2(\theta_1) = h(\bar{\theta})$ when $\theta_1 = \theta_2 = \bar{\theta}$. Hence, the functional form of $h_1(\cdot) = h_2(\cdot) = h(\cdot)$. Hence, the equation (4.12), in this case, can be rewritten as $h(\theta_2) - h(\theta_1) = \bar{S}(\theta_1, \theta_2)(\theta_2 - \theta_1)$...(1) when $\theta = (\theta_1, \theta_2)$. If $\theta' = (\theta'_1, \theta'_2)$ then $h(\theta_1) - h(\theta_2) = \bar{S}(\theta_2, \theta_1)(\theta_1 - \theta_2)$...(2). Form (1) and (2) we have, $\bar{S}(\theta) = \bar{S}(\theta')$. Hence, $\bar{S}(\theta)$ is symmetric. ■

PROPOSITION **4.7** If $\forall\theta \in \Re^n_{++}, \bar{S}(\theta) = \sum(\prod_{(1\leq t\leq n)} \theta_t^{k_t})$ where $\sum_{t=1}^n k_t = m \in \mathbb{N}$ and $h_i(\theta_{-i}) = \sum(\prod_{(t\neq i)} \theta_t^{k'_t})$ where $\sum_{t\neq i} k'_t = (m+1)$ then a mechanism mechanism (σ, t) satisfies EE, SP and OE.

Proof: If $\forall\theta \in \Re^n_{++}, \bar{S}(\theta) = \sum(\prod_{(1\leq t\leq n)} \theta_t^{k_t})$ where $\sum_{t=1}^n k_t = m \in \mathbb{N}$ and $h_i(\theta_{-i}) = \sum(\prod_{(t\neq i)} \theta_t^{k'_t})$ where $\sum_{t\neq i} k'_t = (m+1)$ then it can be easily verified that lemma (4.1) holds. Hence the proposition is proved. ■

REMARK **4.3** Consider $N = \{1,2\}$. Assume $\forall\theta \in \Re^2_{++}$, $\bar{S}(\theta_{(1)}, \theta_{(2)}) = \sum_{i=1}^m (\theta_{(1)}^{(m-i)} \theta_{(2)}^{(i-1)})$ where $m \in \mathbb{N}$. Then with $h_1(\theta_{-1}) = \theta_2^m$ and $h_2(\theta_{-2}) = \theta_1^m$ we can achieve egalitarian equivalent VCG mechanism.

Notice that, in particular, a sequencing problem with $s_1 = s_2 = s$ (=1 assumed in the literature of queuing) is also a queuing problem . Therefore, unlike Chun et al. (2014), in this situation $\bar{S}(\theta_1, \theta_2)$ can be of the above form that we have just mentioned and hence not of constant value.

4.7 Conclusion

In this chapter we have analyzed sequencing problem from both incentive and normative approaches. We have identified unique class of VCG mechanisms that ensures egalitarian equivalence and we also have shown the possibility result with identical preference lower bound in that unique class of VCG mechanisms. Sequencing game imposes a stronger restriction on the possible set of "reference position", compared to queuing game and that in turn results into the failure of having a feasible VCG mechanism along with egalitarian equivalence.

We have found the necessary and sufficient condition for the unique class of egalitarian equivalent VCG mechanism to satisfy identical preference lower bound when the number of participating agents is two, but the necessary condition for the same when the number of participating agent is more than two remains an open question.

We, in this chapter, also analyzed a situation where the restriction that sequencing problem imposes reference position is overlooked, that is, we assume almost no restriction on reference waiting time (except the fact that it must be positive) and identify the class of VCG mechanism that is egalitarian equivalent. Although the complete characterization in this case remains an open problem.

Feasibility which is very crucial from the mechanism designer's point of view, might be restored. In the analysis of this chapter we have shown the possibility of non-constant reference position that is profile dependent, for achieving egalitarian equivalent VCG mechanism. With this change, it would be matter of future research to check the possibility of feasibility.

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